Supersymmetry: Example Sheet 3

David Tong, February 2022

1. Show that the one-loop beta function of $SU(N_c)$ supersymmetric QCD with N_f flavours in the fundamental representation is $b_0 = 3N_c - N_f$

Show that the one-loop beta function vanishes for any $\mathcal{N} = 1$ supersymmetric gauge theory coupled to three chiral multiplets in the adjoint representation. (Note: if you add a suitable superpotential, this is $\mathcal{N} = 4$ super Yang-Mills and the beta function is exactly zero.)

2. Consider a $U(1)$ gauge theory coupled to N left-handed Weyl fermions, each with non-zero charge $q_i \in \mathbb{Z}$ with $i = 1, \ldots, N$.

i) Show that there is no consistent theory with $N = 3$. (You may quote well known mathematical results that you need without proof.)

ii) From his hospital bed, Ramanujan famously explained that Hardy's taxi number, 1729, was interesting. Why does Ramanujan's observation allows us to construct the simplest Abelian chiral gauge theory. Can this theory be coupled to gravity?

iii) Find an example of a consistent Abelian chiral gauge theory that can be coupled to gravity.

3*. A (non-supersymmetric) $SU(N)$ gauge theory is coupled to a single left-handed Weyl fermion χ in the symmetric $\Box\!\Box$ representation and p left-handed Weyl fermions ψ_i, $i = 1, \ldots, p$, in the anti-fundamental $\overline{\Box}$ representation. For what value of p is the quantum theory consistent?

Classically, the theory has a $G_F = SU(p)$ global symmetry that acts on the ψ_i. In addition, there are two $U(1)$ symmetries

$$\begin{aligned} U(1)_\chi : \qquad & \chi \to e^{i\alpha}\chi \quad \text{and} \quad \psi_i \to \psi_i \\ U(1)_\psi : \qquad & \chi \to \chi \quad \text{and} \quad \psi_i \to e^{i\beta}\psi_i \end{aligned}$$

Show that each of the $U(1)$ symmetries suffers a chiral anomaly and so is not a symmetry of the quantum theory. Find a linear combination of these symmetries that that does survive in the quantum theory.

Compute the $SU(p)^3$, $SU(p)^2\, U(1)$, $U(1)^3$ and $U(1)$ 't Hooft anomalies for the symmetries of the quantum theory.

It is conjectured that this theory confines without spontaneously breaking any global symmetry. The massless degrees fields are thought to be a collection of gauge singlet fermions

$$\lambda^{ij} = \psi^{[i}(\chi\psi^{j]})$$

transforming in the anti-symmetric $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ representation of the $SU(p)$ global symmetry. What is the $U(1)$ charge of this fermion? Show that the 't Hooft anomalies of the fermion λ match those of the original gauge theory.

[You will need the following data: For $SU(N)$, the fundamental $\square$, symmetric $\square\square$ and anti-symmetric $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ have the dimension, Dynkin index I and anomaly coefficient A given by

R	$\square$	$\square\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\dim(R)$	N	$\frac{1}{2}N(N+1)$	$\frac{1}{2}N(N-1)$
$I(R)$	1	$N+2$	$N-2$
$A(R)$	1	$N+4$	$N-4$

with $I(\bar{R}) = I(R)$ and $A(\bar{R}) = -A(R)$.]

4*. Confirm that the Bianchi identity

$$\mathcal{D}_\mu {}^\star F^{\mu\nu} = 0$$

holds for any non-Abelian gauge theory. Further show that, for a non-Abelian gauge theory,

$$\int d^4x \ \mathrm{Tr}\, F_{\mu\nu} {}^\star F^{\mu\nu} = 2 \int d^4x \ \partial_\mu K^\mu$$

with

$$K^\mu = \epsilon^{\mu\nu\rho\sigma}\ \mathrm{Tr}\left(A_\nu \partial_\rho A_\sigma - \frac{2i}{3} A_\nu A_\rho A_\sigma\right)$$

5. Consider the $SU(2)$ gauge configuration

$$A_\mu = \frac{1}{x^2+\rho^2}\,\eta^a_{\mu\nu} x^\nu \sigma^a \tag{1}$$

with ρ parameter and $\eta^a_{\mu\nu}$ a collection of 4×4 matrices given by

$$\eta^1_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ , \ \eta^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \ , \ \eta^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Show that the corresponding field strength is given by

$$F_{\mu\nu} = -\frac{2\rho^2}{(x^2+\rho^2)^2}\eta^a_{\mu\nu}\sigma^a$$

Why does this solve the Yang-Mills equation of motion $\mathcal{D}_\mu F^{\mu\nu}=0$? Compute the action

$$S = \frac{1}{2g^2}\int d^4x\ \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$$

Note: You will need the identity

$$\epsilon^{abc}\eta^a_{\mu\rho}\eta^b_{\nu\sigma} = \eta^c_{\mu\nu}\delta_{\rho\sigma} + \eta^c_{\rho\sigma}\delta_{\mu\nu} - \eta^c_{\mu\sigma}\delta_{\rho\nu} - \eta^c_{\rho\nu}\delta_{\mu\sigma}$$

6. Consider non-supersymmetric $SU(N)$ Yang-Mills (with $N>2$) coupled to a single *Dirac* fermion ψ in the symmetric $\square\!\square$ representation. What are the classical symmetries? What are the quantum symmetries?

This theory is expected to confine, develop a mass gap, and form a condensate

$$\langle\bar\psi\psi\rangle \sim \Lambda^3$$

How many ground states does the theory have?

How do your answers change if the Dirac fermion is in the anti-symmetric $\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$ representation of $SU(N)$?

Note: You may need to refer to the table in Question 3.