3 Chiral Superfields

In the previous section we've understood how supersymmetry acts on single particles states in the Hilbert space. But, ultimately, we want to write down field theories that are invariant under supersymmetry. Part of this requires understanding how supersymmetry acts on fields.

We've already seen a taster of this in the introduction. The action (1.1) was given by

$$S = \int d^4x \left[\partial_\mu \phi^\dagger \partial^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - \left| \frac{\partial W}{\partial \phi} \right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial \phi^2}\psi\psi - \frac{1}{2}\frac{\partial^2 W^\dagger}{\partial \phi^{\dagger\, 2}}\bar{\psi}\bar{\psi} \right] \tag{3.1}$$

This involves a complex scalar ϕ and a single Weyl fermion ψ_α. After our discussion in the last section, we now recognise this as the fields corresponding to a chiral multiplet. We claimed in the introduction that this action is invariant under the transformation

$$\delta\phi = \sqrt{2}\epsilon\psi \quad \text{and} \quad \delta\psi_\alpha = \sqrt{2}i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\,\partial_\mu\phi - \sqrt{2}\epsilon_\alpha \frac{\partial W^\dagger}{\partial \phi^\dagger} \tag{3.2}$$

There are a few questions that we'd like to ask. First: how can we construct actions like (3.1)? After all, it's not like we can just stare at the action and see that it's invariant under the transformations (3.2). It takes a bit of work to show this. Secondly, how are the transformations (3.2) related to the supercharges and supersymmetry algebra that we met in the previous section.

The purpose of this section is to answer these questions. In particular, we'll see how we can rewrite the action (3.1) in a way that the supersymmetry is manifest. The trick to doing this is to combine the bosonic field ϕ and the femionic field ψ_α into a single object known as a *superfield*.

3.1 Superspace

Usually, fields are functions of x^μ, the coordinates of Minkowski space. But, as we've seen, supersymmetry is an extension of the Poincaré group. Correspondingly, superfields live not on Minkowski space, but on an extension of Minkowski space known as *superspace*.

The coordinates of superspace are

$$x^\mu \ , \quad \theta_\alpha \ , \quad \bar{\theta}^{\dot{\alpha}}$$

Here x^μ, with $\mu = 0, 1, 2, 3$ are the coordinates of Minkowski space. In superspace these are augmented with Grassmann-valued spinors θ_α and $\bar{\theta}^{\dot{\alpha}}$. In other words, superspace is not a regular manifold of the kind that we know and love from courses on differential geometry. Instead it is an example of a supermanifold, with both commuting and anti-commuting dimensions.

3.1.1 The Geometry of Superspace

In what follows, we'll explore the idea of fields on superspace and see how they encapsulate a collection of fields that transform into each other under supersymmetry. However, we could reasonably ask: how did we come up with the idea of superspace in the first place? There is, it turns out, a group theoretic answer to this.

In general, if we're given a Lie group G, we might want to know what manifolds $\mathcal{M}$ accommodate a natural action of G.

One obvious choice is to take the manifold to be the group itself: $\mathcal{M} = G$. In this case, each element $g \in G$ gives us natural map $\mathcal{M} \mapsto \mathcal{M}$ given by $g' \in \mathcal{M} \mapsto g \cdot g'$.

A slightly less obvious choice is to take a *coset* space. This is the manifold $\mathcal{M} = G/H$ where $H \subset G$ is a subgroup of G. A point $\{g\}$ in the coset G/H is defined by the equivalence relation among elements of G

$$g \equiv g \cdot h \quad \text{for all } h \in H$$

Again, any element $g \in G$ gives us a natural map $\mathcal{M} = G/H \mapsto G/H$ defined by $\{g'\} \in \mathcal{M} \mapsto \{g \cdot g'\}$.

For example, the group $G = SU(2)$ is, as a manifold, $G = \mathbf{S}^3$. We can consider the subgroup $H = U(1) \subset SU(2)$ to get the coset $SU(2)/U(1) \cong \mathbf{S}^2$. (Mathematically, this is known as the *Hopf fibration*.) Obviously there is a natural action of $SO(3) \cong SU(2)/\mathbf{Z}_2$ on $\mathbf{S}^2$.

This, somewhat abstract, way of thinking gives us a new perspective on Minkowski space itself. It can be viewed as the coset

$$\mathbb{R}^{1,3} = G/H = \frac{\text{Poincaré Group}}{\text{Lorentz Group}}$$

Here a general element of the Poincaré group G is comprised of Lorentz boosts, generated by $M^{\mu\nu}$, and translations generated by P^μ. We write this as

$$g(\omega, a) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + ia_\mu P^\mu\right)$$

Meanwhile, the Lorentz group H consists only of Lorentz boosts. This means that coset space can be parameterised just by a^μ which we can equivalently think of as coordinates $x^\mu = a^\mu$ on Minkowski space. The fact that Minkowski space can be viewed as a coset merely confirms something that we knew already: there is an action of the Poincaré group on Minkowski space.

Now, however, we would like to construct a space on which the group of supersymmetry transformations naturally acts. These are given by

$$g(\omega, a, \theta, \bar{\theta}) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + ia_\mu P^\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right) \tag{3.3}$$

with Q_α and $\bar{Q}^{\dot{\alpha}}$ the supersymmetry generators that we met in the previous section. The spinors θ^α and $\bar{\theta}_{\dot{\alpha}}$ should be viewed as parameterising the "amount" of supersymmetry transformation that we're doing, albeit with the "amount" now somewhat harder to quantify as it's a Grassmann valued object. With Grassmann elements of this kind, g is an element of a *super Lie group* which, in this case, is known as the *super-Poincaré group*. The coset construction continues to work in the same way and we define superspace to be

$$\text{Superspace} = G/H = \frac{\text{Super-Poincaré Group}}{\text{Lorentz Group}}$$

A point in superspace is now parameterised by $x^\mu = a^\mu$ and the Grassmann-valued spinors θ_α and $\bar{\theta}^{\dot{\alpha}}$ as advertised above.

Before we go on, a quick comment on nomenclature. The Lorentz group is, of course, $SO(1,3)$. (Actually, strictly speaking if we want to include spinor representations it is $SL(2,\mathbb{C}) = \text{Spin}(1,3)$ but we'll ignore this double cover subtlety.) The Poincaré group is the semi-direct product $ISO(1,3) = SO(1,3) \ltimes \mathbb{R}^4$ and Minkowski space is $\mathbb{R}^{1,3} = ISO(1,3)/SO(1,3)$. Meanwhile, the super-Poincaré group is usually written as $ISO(1,3|1)$ with the additional "bar 1" or "slash 1" telling us that we have $\mathcal{N} = 1$ supersymmetry. Superspace is then the "4+4" dimensional supermanifold $\mathbb{R}^{1,3|4} = ISO(1,3|1)/SO(1,3)$. We'll have no need for any of this notation in these lectures.

The Action on Superspace

The whole point of the coset construction of superspace is that it tells us how the supergroup acts. This will be important in what follows so let's flesh it out a little. First, we write the general element of the supergroup (3.3) as

$$g(\omega, x, \theta, \bar{\theta}) = \tilde{g}(x, \theta, \bar{\theta})h(\omega)$$

where $h(\omega)$ is a Lorentz transformation and $\tilde{g}(a,\theta,\bar{\theta})$ is the representative of the coset

$$\tilde{g}(x,\theta,\bar{\theta}) = \exp\left(ix_\mu P^\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right)$$

This specifies a point $(x,\theta,\bar{\theta})$ in superspace,

We now want to see how the momentum operator P and supercharges Q and $\bar{Q}$ shift the point $(x,\theta,\bar{\theta})$ in superspace. Let's start with the momentum operator. We introduce the supergroup element

$$U(a) = \exp\left(ia_\mu P^\mu\right)$$

Then we have

$$U(a)\,\tilde{g}(x,\theta,\bar{\theta}) = e^{iaP}e^{ixP+i\theta Q+i\bar{\theta}\bar{Q}} = e^{i(x+a)P+i\theta Q+i\bar{\theta}\bar{Q}} = \tilde{g}(x+a,\theta,\bar{\theta})$$

This gives us a familiar result: momentum generates translations,

$$x^\mu \to x^\mu + a^\mu$$

Now we do the same for the supercharges. This time we will find a small twist to the story. We introduce the supergroup element

$$V(\epsilon,\bar{\epsilon}) = \exp\left(i\epsilon^\alpha Q_\alpha + i\bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right)$$

Note that ϵ^α and $\bar{\epsilon}_{\dot{\alpha}}$ are Grassmann-valued spinors. They shouldn't be confused with the anti-symmetric $\epsilon^{\alpha\beta}$ matrices that we met earlier. (Sorry!) Now the action on superspace is given by

$$V(\epsilon,\bar{\epsilon})\,\tilde{g}(x,\theta,\bar{\theta}) = e^{i\epsilon Q+i\bar{\epsilon}\bar{Q}}e^{ixP+i\theta Q+i\bar{\theta}\bar{Q}} \tag{3.4}$$

The small twist is that Q and $\bar{Q}$ do not anti-commute with each other. In fact, now that we've multiplied the supercharges with anti-commuting spinors ϵ and θ, we can talk about commutation relations rather than anti-commutation relations. We have

$$\begin{aligned} Q_\alpha\bar{Q}_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}}Q_\alpha = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu \quad &\Rightarrow \quad \epsilon^\alpha\left(Q_\alpha\bar{Q}_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}}Q_\alpha\right)\bar{\theta}^{\dot{\alpha}} = 2(\epsilon^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}})P_\mu \\ &\Rightarrow \quad [\bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}, \epsilon_\alpha Q^\alpha] = 2(\epsilon\sigma^\mu\bar{\theta})P_\mu \end{aligned} \tag{3.5}$$

where the Grassmann nature of $\bar{\theta}$, ϵ, Q and $\bar{Q}$ means that we pick up a minus sign in going from the first line to the second, turning $\{\ ,\ \}$ into $[\ ,\ ]$.

We now evaluate (3.4) using the BCH formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots}$$

The commutator (3.5), together with the fact that the higher commutator terms ... in the BCH formula all vanish in the present case, gives us the result

$$\begin{aligned} V(\epsilon,\bar{\epsilon})\,\tilde{g}(x,\theta,\bar{\theta}) &= e^{ixP+i(\theta+\epsilon)Q+i(\bar{\theta}+\bar{\epsilon})\bar{Q}+(\epsilon\sigma\bar{\theta})P-(\theta\sigma\bar{\epsilon})P} \\ &= \tilde{g}(x+i\theta\sigma\bar{\epsilon}-i\epsilon\sigma\bar{\theta},\theta+\epsilon,\bar{\theta}+\bar{\epsilon}) \end{aligned}$$

Here we see the twist. The supercharges shift the Grassmann coordinate in superspace as we might have anticipated. But, at the same time, they also shift the point in Minkowski space by a Grassmann bilinear

$$\begin{aligned} x^\mu &\to x^\mu + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta} \\ \theta &\to \theta+\epsilon \\ \bar{\theta} &\to \bar{\theta}+\bar{\epsilon} \end{aligned} \tag{3.6}$$

Note that the shift in x^μ due to the Grassmann bilinear can't be thought of as normal translation by some number. Instead, it's a more formal expression. Ultimately, we'll see how this manifests itself in terms of the superfields and their more familiar components.

3.1.2 Superfields

A superfield is a function on superspace, $Y = Y(x,\theta,\bar{\theta})$. To start, we take this to be a complex-valued function on superspace.

In principle, the superfield could transform in some non-trivial representation of the Lorentz group. For example it could carry a vector index μ or a spinor index α. However, rather remarkably, we will find all the fields that we need – scalar, spinor and vector – lurking within the simplest scalar superfield. (We will, however, come across superfields carrying spinor indices in Section 4.)

To see this, we Taylor expand the superfield in θ and $\bar{\theta}$. But this is easy because θ and $\bar{\theta}$ are Grassmann valued objects obeying, for example,

$$\theta_\alpha\theta_\beta = -\theta_\beta\theta_\alpha$$

This means that the Taylor expansion truncates after some finite length. In particular we have $\theta_\alpha\theta_\beta\theta_\gamma = 0$. So the Taylor expansion of $Y(x,\theta,\bar{\theta})$ stops after terms quadratic in θ and $\bar{\theta}$. Expanding the superfield out in this way then reveals a bunch of more familiar fields lurking within,

$$Y(x,\theta,\bar{\theta}) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}(x) + \theta^2 M(x) + \bar{\theta}^2 N(x) \\ + \theta^\alpha \bar{\theta}^{\dot{\alpha}} V_{\alpha\dot{\alpha}}(x) + \theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}^2 \theta^\alpha \rho_\alpha(x) + \theta^2 \bar{\theta}^2 D(x) \qquad (3.7)$$

Here $\theta^2 = \theta^\alpha \theta_\alpha$ and $\bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}$.

There are a few things to say about this. First, note that the superfield does indeed contain all the fields that we usually care about: there are four complex scalars ϕ, M, N and D, two left-handed spinors ψ and ρ, two right-handed spinors $\bar{\chi}$ and $\bar{\lambda}$ and a vector $V_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} V_\mu$.

Second, note that it contains many more fields that we might have thought from our analysis in the previous section! The representations on single particle states suggested that there should be a chiral multiplet containing a single complex scalar and a Weyl fermion and a vector multiplet containing a gauge field and a Weyl fermion. Yet the superfield Y contains a plethora of such fields. We will shortly see how we can impose further restrictions on Y that truncate the number of fields lying within to match our earlier expectation.

Our next task is to understand how superfields transform under supersymmetry transformations. We'll again start with translations $x^\mu \to x^\mu + a^\mu$ which, as we have seen, are generated by the unitary operator

$$U = \exp\left(i a^\mu P_\mu\right)$$

Previously, we viewed this as a group element acting on superspace. But in quantum field theory, it has another avatar as an operator acting on the Hilbert space. The fields in quantum field theory are, of course, also operators and the superfield is no different. The action of U on such operators enacts the translation, meaning

$$U Y(x,\theta,\bar{\theta}) U^\dagger = Y(x+a,\theta,\bar{\theta})$$

For infinitesimal a^μ, we expand $U = e^{iaP} = 1 + i a_\mu P^\mu + \mathcal{O}(a)^2$. We also Taylor expand the field, $Y(x+a) = Y(x) + a^\mu \partial_\mu Y(x) + \mathcal{O}(a^2)$. Equating the terms linear in a we see that the translations are captured in the commutation relation on fields

$$[P_\mu, Y] = -i\partial_\mu Y \qquad (3.8)$$

We can treat the action of the supercharges in a similar fashion. We again have the unitary operator

$$V(\epsilon,\bar{\epsilon}) = \exp\left(i\epsilon^\alpha Q_\alpha + i\bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}\right)$$

Acting on superfields, this gives

$$VY(x,\theta,\bar{\theta})V^\dagger = Y(x + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}, \theta+\epsilon, \bar{\theta}+\bar{\epsilon})$$

where we've invoked the transformation of the superspace coordinate (3.6). If we now treat ϵ_α as an infinitesimal spinor and work to leading order in ϵ, we find the commutation relations

$$[Q_\alpha, Y] = \left(-i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right)Y \tag{3.9}$$

$$[\bar{Q}_{\dot{\alpha}}, Y] = \left(+i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\right)Y \tag{3.10}$$

In this expression, the derivatives with respect to Grassmann coordinates are defined by

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha} \quad \text{with} \quad \partial_\alpha\theta^\beta = \delta_\alpha^\beta \;\; \text{and} \;\; \partial_\alpha\bar{\theta}_{\dot{\beta}} = 0$$

$$\bar{\partial}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \quad \text{with} \quad \bar{\partial}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}^{\dot{\beta}} \;\; \text{and} \;\; \bar{\partial}_{\dot{\alpha}}\theta_\beta = 0$$

These Grassmann derivatives are themselves Grassmann. This means that they pick up a minus sign when they pass through other Grassmann variables. So, for example, if you wish to differentiate $\chi^\beta\theta^\gamma$, where both χ and θ are Grassmann variables, then you have

$$\frac{\partial}{\partial\chi^\alpha}(\chi^\beta\theta^\gamma) = \delta_\alpha^\beta\theta^\gamma \quad \text{and} \quad \frac{\partial}{\partial\theta^\alpha}(\chi^\beta\theta^\gamma) = -\delta_\alpha^\gamma\chi^\beta$$

where that extra minus sign in the second expression comes from dragging the $\partial/\partial\theta^\alpha$ through the χ^β before it gets to attack its prey.

It's useful to define differential operators associated to the right-hand sides of (3.8), (3.9) and (3.10). To this end, we write

$$\begin{aligned} \mathcal{P}_\mu &= -i\partial_\mu \\ \mathcal{Q}_\alpha &= -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \\ \bar{\mathcal{Q}}_{\dot{\alpha}} &= +i\bar{\partial}_{\dot{\alpha}} + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \end{aligned} \tag{3.11}$$

Be warned: these differ from the operators P_μ, Q_α and $\bar{Q}_{\dot{\alpha}}$ only by the use of curly calligraphic script. You can check that anti-commutation relation of these differential operators is something familiar

$$\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}\mathcal{P}_\mu$$

together with $\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = \{\bar{\mathcal{Q}}_{\dot{\alpha}}, \bar{\mathcal{Q}}_{\dot{\beta}}\} = 0$. This is telling us that $\mathcal{P}$, $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_{\dot{\alpha}}$ also furnish a representation of the supersymmetry algebra, now acting on fields on superspace

Supersymmetry Transformation of Fields

We can unpack the supersymmetry transformations (3.9) and (3.10) to see how it acts on the individual fields sitting with Y. The infinitesimal change of the superfield is defined to be

$$\delta Y = i[\epsilon Q + \bar{\epsilon}\bar{Q}, Y] = i(\epsilon \mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})Y \tag{3.12}$$

Expanding out Y in terms of the components (3.7), the operators $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ act on each term. $\mathcal{Q}$ removes a θ (where there is one) and adds a $\bar{\theta}\partial_\mu$ (where there aren't too many $\bar{\theta}$'s already) Obviously $\bar{\mathcal{Q}}$ is the conjugate. We then compare the various θ and $\bar{\theta}$ and terms.

For example, the lowest term in Y is the scalar $\phi(x)$. To compute its variation, we look for the term in δY with neither θ's nor $\bar{\theta}$'s. This comes from ∂_α acting on the term $\theta\psi$ and $\bar{\partial}_{\dot{\alpha}}$ acting on $\bar{\theta}\bar{\chi}$. The result is

$$\delta\phi = \epsilon\psi + \bar{\epsilon}\bar{\chi} \tag{3.13}$$

Meanwhile, the highest term in Y is the scalar $D(x)$. To compute its variation, we find the term in δY that comes with the full complement of $\theta^2\bar{\theta}^2$. This happens comes from the $\bar{\theta}\partial_\mu$ term in $\mathcal{Q}$ and the $\theta\partial_\mu$ term in $\bar{\mathcal{Q}}$. The net effect is that the variation of $D(x)$ is a total derivative

$$\delta D = \frac{i}{2}\partial_\mu(\epsilon\sigma^\mu\bar{\lambda} - \rho\sigma^\mu\bar{\epsilon}) \tag{3.14}$$

This will prove to be part of the story as we proceed.

It takes a bit of work to get the transformation of all the remaining component fields in (3.7). You'll have the pleasure of doing this work in the first examples sheet. The answer turns out to be

$$\begin{aligned}
\delta\psi &= 2\epsilon M + (\sigma^\mu\bar{\epsilon})(i\partial_\mu\phi + V_\mu) \\
\delta\bar{\chi} &= 2\bar{\epsilon}N - (\epsilon\sigma^\mu)(i\partial_\mu\phi - V_\mu) \\
\delta M &= \bar{\epsilon}\bar{\lambda} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon} \\
\delta N &= \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\chi} \\
\delta V_\mu &= \epsilon\sigma_\mu\bar{\lambda} + \rho\sigma_\mu\bar{\epsilon} + \frac{i}{2}\left(\partial^\nu\psi\sigma_\mu\bar{\sigma}_\nu\epsilon - \bar{\epsilon}\bar{\sigma}_\nu\sigma_\mu\partial^\nu\bar{\chi}\right) \\
\delta\bar{\lambda} &= 2\bar{\epsilon}D + \frac{i}{2}\bar{\sigma}^\nu\sigma^\mu\bar{\epsilon}\,\partial_\mu V_\nu + i\bar{\sigma}^\mu\epsilon\,\partial_\mu M \\
\delta\rho &= 2\epsilon D - \frac{i}{2}\sigma^\nu\bar{\sigma}^\mu\epsilon\,\partial_\mu V_\nu + i\sigma^\mu\bar{\epsilon}\,\partial_\mu N
\end{aligned} \tag{3.15}$$

The variation of each has at least two terms, one with a derivative ∂_μ and one without.

3.1.3 Constraining Superfields

As we already commented, the superfield Y is too big. It has way more fields than we expect from the representation theory of Section 2.3. This is because Y is not an irreducible representation. It can be reduced to something smaller. The question is: how?

We want to impose constraints on Y such that it remains a superfield. That means that whatever object we have after the constraint should also transform as (3.9) and (3.10) under supersymmetry transformations. So our first step to understanding the possible constraints is to figure out what kind of operations we can perform on superfields that keep them as superfields.

There are some obvious operations, albeit ones that won't help with our constraint. If we have two superfields Y_1 and Y_2 then αY_1 is a superfield for any $\alpha \in \mathbb{C}$, as is $Y_1 + Y_2$ and $Y_1 Y_2$. For example, to see that $Y_1 Y_2$ is a superfield, we need to note that

$$[Q_\alpha, Y_1 Y_2] = [Q_\alpha, Y_1]Y_2 + Y_1[Q_\alpha, Y_2] = (\mathcal{Q}_\alpha Y_1)Y_2 + Y_1(\mathcal{Q}_\alpha Y_2) = \mathcal{Q}_\alpha(Y_1 Y_2)$$

as required.

More pertinent for our purposes, if Y is a superfield then so too is $\partial_\mu Y$. However, crucially, neither $\partial_\alpha Y$ nor $\bar{\partial}_{\dot\alpha} Y$ are superfields. Algebraically, this is because

$$[\epsilon^\alpha Q_\alpha, \bar{\partial}_{\dot\alpha}] = \epsilon^\alpha \sigma^\mu_{\alpha\dot\alpha}\partial_\mu \neq 0$$

To build some intuition for what's going on, note that $\bar{\partial}_{\dot\alpha} Y$ doesn't include, for example, the highest component $\theta^2\bar{\theta}^2 D$ term; there was such a term in Y but one of the $\bar{\theta}$'s is removed after acting with $\bar{\partial}_{\dot\alpha}$. However, acting with a supercharge Q_α will generate such a term. In other words, it's not consistent with supersymmetry to simply state by fiat that the last term vanishes, $D(x) = 0$. Act with a supersymmetry transformation and this will no longer be true. It's analogous to setting $A_3 = 0$ in a vector field A_μ and thinking that you've found an object with just three components, only to realise that A_3 gets resurrected after a rotation.

However, there is a way forward. We define the *covariant derivatives*

$$\begin{aligned} \mathcal{D}_\alpha &= \partial_\alpha + i\sigma^\mu_{\alpha\dot\alpha}\bar{\theta}^{\dot\alpha}\partial_\mu \\ \bar{\mathcal{D}}_{\dot\alpha} &= -\bar{\partial}_{\dot\alpha} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \end{aligned}$$

These are very similar to the $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_{\dot\alpha}$ differential operators defined in (3.11), but with a relative minus sign difference (and an overall factor of i difference). Their key property is that they anti-commute with $\mathcal{Q}$ and $\bar{\mathcal{Q}}$

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_{\dot\beta}\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{Q}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{Q}}_{\dot\beta}\} = 0 \tag{3.16}$$

The covariant derivatives also obey

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}\mathcal{P}_\mu \tag{3.17}$$

together with $\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 0$.

From (3.16), we have

$$[\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}}\,,\, \mathcal{D}_\alpha] = [\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}}\,,\, \bar{\mathcal{D}}_{\dot\alpha}] = 0$$

This tells us that both $\mathcal{D}_\alpha Y$ and $\bar{\mathcal{D}}_{\dot\alpha} Y$ are superfields. For example, under the supersymmetry transformation (3.12), we have

$$\delta Y = i(\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})Y \quad \Rightarrow \quad \delta(\mathcal{D}_\alpha Y) = i(\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})\mathcal{D}_\alpha Y$$

Now we can discuss the various constraints that we can place on a superfield Y. There are four of interest (of which, only three will play a major role in these lectures).

- A *chiral superfield* Φ is defined by the constraint

  $$\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0$$

- An *anti-chiral superfield* Ψ is defined by the constraint

  $$\mathcal{D}_\alpha\Psi = 0$$

  Note that you can't impose both chiral and anti-chiral conditions since the anti-commutator (3.17) would then require that the superfield is actually constant. Moreover, if Φ is a chiral superfield then $\bar\Phi = \Phi^\dagger$ is an anti-chiral superfield. (I give a simple way to see this at the end of Section 3.1.4.) The fact that we can't impose both conditions simultaneously means that we can't take Φ to be real: chiral superfields are necessarily complex. We will see that chiral superfields correspond to the chiral multiplets that we met in Section 2.3.

- A *real superfield* V is defined by the simple requirement that

  $$V = V^\dagger$$

  We will postpone our discussion of real superfields to Section 4. There we will see that the real superfields correspond to the vector multiplet that we met in Section 2.3.

- Finally, a *linear superfield* J is defined

$$J = J^\dagger \quad \text{and} \quad \mathcal{D}^2 J = \bar{\mathcal{D}}^2 J = 0$$

These play a slightly less prominent role than the (anti)-chiral and real superfields. In particular, we won't build supersymmetry actions out of linear superfields. However, it turns out that they are useful homes for certain composite operators in quantum field theory, most notably Noether currents associated to global symmetries.

We will spend the rest of this section studying the properties of chiral superfields.

3.1.4 Chiral Superfields

A chiral superfield obeys the constraint

$$\bar{\mathcal{D}}_{\dot{\alpha}} \Phi = 0 \tag{3.18}$$

We will first solve this equation to understand what it means for the superfield Φ.

There's a useful trick here. We introduce the coordinate

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$$

The advantage of this coordinate is that we have

$$\bar{\mathcal{D}}_{\dot{\alpha}} y^\mu = \left(-\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha \sigma^\nu_{\alpha\dot{\alpha}}\partial_\nu\right)\left(x^\mu + i\theta^\beta\sigma^\mu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}\right) = -i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}} - i\bar{\partial}_{\dot{\alpha}}(\theta^\beta\sigma^\mu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}}) = 0$$

where to see that the two terms cancel, you have to remember that you pick up an extra minus sign as the $\bar{\partial}_{\dot{\alpha}}$ passes through the θ^β. In addition, we have

$$\bar{\mathcal{D}}_{\dot{\alpha}}\theta_\beta = 0$$

This means that if we view a general superfield as a function of $\Phi = \Phi(y, \theta, \bar{\theta})$ then, of the three arguments, only $\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} \neq 0$ and the condition (3.18) tells us

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0 \quad \Rightarrow \quad \Phi = \Phi(y, \theta)$$

In other words Φ is almost a function only of θ and not of $\bar{\theta}$, the "almost" because there is in fact a $\bar{\theta}$ buried in the y^μ. This means that we can expand in components

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y)$$

where the $\sqrt{2}$ is a convention. We can then further Taylor expand the y^μ to get the expression for a chiral superfield in components

$$\Phi(x,\theta,\bar\theta) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) \\ + i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\,\sigma^\mu\bar\theta - \frac{1}{4}\theta^2\bar\theta^2\Box\phi(x) \qquad (3.19)$$

with $\Box = \partial_\mu\partial^\mu$. We see that the chiral superfield contains just three component fields: a complex scalar ϕ, a Weyl spinor ψ and another complex scalar F. The higher components of $\Phi(x)$ are simply derivatives of the first two fields.

This is much closer to what we expected based on our analysis in Section 2.3. There we found a chiral multiplet consists of single particle states associated to a complex scalar ϕ and a Weyl fermion ψ. However, we've also got a second complex scalar F. We will see later that this is an object known as an *auxiliary field*. For now it's worth noticing that, in contrast to ϕ and ψ, there are no terms in the chiral superfield with ∂F. This will be important as we proceed.

The supersymmetry transformations of the chiral multiplet are

$$\begin{aligned} \delta\phi &= \sqrt{2}\epsilon\psi \\ \delta\psi &= \sqrt{2}i\sigma^\mu\bar\epsilon\,\partial_\mu\phi + \sqrt{2}\epsilon F \\ \delta F &= \sqrt{2}i\bar\epsilon\bar\sigma^\mu\partial_\mu\psi \end{aligned} \qquad (3.20)$$

Note that F transforms as a total derivative, just like D in the original unconstrained superfield (3.14). We'll see the relevance of this shortly.

There is a very similar story for the anti-chiral superfields. As we mentioned previously, these can be viewed as the complex conjugate of a chiral superfield. To see this, note that if a chiral superfield $\Phi(y,\theta)$ is function of y^μ and θ, then its conjugate $\Phi^\dagger(\bar y,\bar\theta)$ is a function of $\bar y^\mu = x^\mu - i\theta\sigma^\mu\bar\theta$ and $\bar\theta$. But it's simple to check that $\mathcal{D}_\alpha\bar y^\mu = \mathcal{D}_\alpha\bar\theta^{\dot\alpha} = 0$ and so $\Phi^\dagger$ is indeed an anti-chiral superfield obeying $\mathcal{D}_\alpha\Phi^\dagger = 0$. In components, we have

$$\Phi^\dagger(\bar y,\bar\theta) = \phi^\dagger(\bar y) + \sqrt{2}\bar\theta\bar\psi(\bar y) + \bar\theta^2 F^\dagger(\bar y)$$

We can then further expand out $\bar y$ further if we wish to get an expression analogous to (3.19),

$$\Phi^\dagger(x,\theta,\bar\theta) = \phi^\dagger(x) + \sqrt{2}\bar\theta\bar\psi(x) + \bar\theta^2 F^\dagger(x) \\ - i\theta\sigma^\mu\bar\theta\partial_\mu\phi^\dagger(x) + \frac{i}{\sqrt{2}}\bar\theta^2\theta\sigma^\mu\partial_\mu\bar\psi(x) - \frac{1}{4}\theta^2\bar\theta^2\Box\phi^\dagger(x)$$

3.2 And... Action

To construct actions that are invariant under Poincaré group, we take suitable Lagrangian densities of fields and integrate them over spacetime. Analogously, to construct actions that are invariant under supersymmetry, we take suitable Lagrangian densities of superfields and integrate them over superspace.

3.2.1 Integrating Over Superspace

First, let's remind ourselves how Grassmann integration works. (It is, happily, much easier than normal integration!) If we have a single Grassmann variable θ then

$$\int d\theta\ 1 = 0 \quad \text{and} \quad \int d\theta\ \theta = 1$$

This means that if we have a function $f(x,\theta) = f_0(x) + \theta f_1(x)$, then Grassmann integration picks out the component multiplying θ,

$$\int d\theta\ f(x,\theta) = f_1(x)$$

In this manner, integration over Grassmann variables is the same thing as differentiation: $\int d\theta = \partial/\partial\theta$. In particular, we have a Grassmann version of the fundamental theorem of calculus

$$\int d\theta\ \frac{\partial f}{\partial\theta} = \int d\theta\ f_0(x) = 0 \tag{3.21}$$

Here we will need to integrate over superspace, parameterised by θ_α and $\bar{\theta}^{\dot{\alpha}}$. We define

$$\int d^2\theta = \frac{1}{2}\int d\theta^1 d\theta^2 \quad \text{and} \quad \int d^2\bar{\theta} = -\frac{1}{2}\int d\bar{\theta}^1 d\bar{\theta}^2$$

Those strange factors of $\frac{1}{2}$ are because $\theta^2 = \theta^\alpha\theta_\alpha = -2\theta^1\theta^2$. We then have

$$\int d^2\theta\ \theta^2 = -\int d\theta^1 d\theta^2\ (\theta^1\theta^2) = 1$$

where the minus sign disappears when $d\theta^2$ moves past θ^1. Note that the measure $d^2\bar{\theta}$ comes with an extra minus sign but this cancels the corresponding minus sign in $\bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = +2\bar{\theta}^1\bar{\theta}^2$. Once again, we have $\int d^2\bar{\theta}\ \bar{\theta}^2$. Finally, we also use the (not entirely logical) notation

$$\int d^4\theta = \int d^2\theta\, d^2\bar{\theta}$$

Now suppose that we build an action out of some function of superfields. That function will itself be a superfield that we will call $K(x, \theta, \bar{\theta})$ but, in contrast to what we've discussed so far, we'll view K as a composite superfield whose component are functions of other fields. We the construct the action of the form

$$S = \int d^4x \, d^4\theta \; K(x, \theta, \bar{\theta}) \tag{3.22}$$

The action is real if K is a real superfield, obeying $K = K^\dagger$. As we saw above, this is a valid constraint on a superfield. Under a supersymmetry transformation, we have

$$\delta S = \int d^4x \, d^4\theta \; \delta K$$

where any superfield K must change as (3.12). This means that we have

$$\delta K = \epsilon^\alpha (\partial_\alpha K - i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu K) + (-\bar{\partial}_{\dot{\alpha}} K + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu K)\bar{\epsilon}^{\dot{\alpha}}$$

But each of these terms involves a derivative. Those terms that are differentiated with respect to a Grassmann coordinate automatically vanish when integrated over superspace by virtue of (3.21). Meanwhile, those terms that involve a differential ∂_μ give at most a boundary term which, if fields drop off suitably quickly asymptotically, also vanishes. We learn that any action of the form (3.22) is necessarily invariant under supersymmetry:

$$\delta S = 0$$

In fact, we can give an expression for the action. The superfield K has an expansion

$$K(x, \theta, \bar{\theta}) = K_{\text{first}}(x) + \ldots + \theta^2 \bar{\theta}^2 \, K_{\text{last}}(x)$$

The action (3.22) simply picks up the last of these terms

$$S = \int d^4x \; K_{\text{last}}(x)$$

We refer to terms in the action that come from integrating over all of superspace as *D-terms*. The name isn't a great one but comes from the fact that the last component in a real superfield is usually denoted D.

In anticipation of this, in the general expansion of the superfield (3.7) we called the final term D. We also saw that it transforms as a total derivative under a supersymmetry transformation (3.14). This gives another way of seeing the result above: any Lagrangian given by a D-term transforms as a total derivative and so the action is invariant.

3.2.2 The Action for Chiral Superfields

What does this mean for our chiral superfield Φ? As with any other field, we have a choice of what action to build. But, typically in quantum field theory, the simplest possibilities are the most interesting.

Because Φ is complex, we also necessarily have the anti-chiral superfield $\Phi^\dagger$ to play with. Multiplying these together gives a real superfield $\Phi^\dagger\Phi$ that we can integrate over superspace to get the action,

$$S_{\text{chiral}} = \int d^4x\, d^4\theta\ \Phi^\dagger\Phi$$

This means that the action is given by the D-term of $\Phi^\dagger\Phi$. A short calculation, and some integration by parts, shows that the action becomes

$$S_{\text{chiral}} = \int d^4x\ \left[\partial_\mu\phi^\dagger\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^\dagger F\right]$$

where we have thrown away some total derivatives. These are just the standard kinetic terms for a complex scalar ϕ and Weyl fermion ψ. But now we see that there's something special about F: it doesn't have any kinetic terms. Moreover, this will continue to be true as we write down further supersymmetric interactions. This is what it means to be an auxiliary field.

Because there are no kinetic terms for F, it has no propagating degrees of freedom and, when quantised, doesn't give rise to any particle states. That's why it didn't appear in our representation theory analysis of Section 2.3. Nonetheless, there is a good reason that F appears in the chiral superfield.

When looking at single particle states, we previously argued that there have to be equal number of bosonic and fermionic degrees of freedom. And there are. But now we're looking at the action, we can ask two variants of this question. First, we can insist that the number of physical propagating degrees of freedom match. In the context of field theory, these are said to be "on-shell" degrees of freedom. This means that we count the degrees of freedom *after* imposing the equations of motion. The complex scalar field ϕ has two degrees of freedom, while the non-propagating scalar F has none. Meanwhile, the Weyl fermion ψ_α has two complex components but obeys a first order, rather than second order equation of motion which means that ψ_α counts both "position" and "momentum". So the equation of motion cuts the number of on-shell degrees of freedom, giving two. This, of course, matches the degrees of freedom of ϕ.

However, we require the action to be invariant under supersymmetry for all field configurations, not just those that obey the equations of motion. And this motivates us to count the "off-shell" degrees of freedom, meaning the number of fields before equations of motion are imposed. The two complex scalars ϕ and F have two each, while the Weyl spinor ψ_α has four off-shell degrees of freedom because it contains two complex components. The presence of the auxiliary field F is required to match these off-shell degrees of freedom.

Next we want to write down supersymmetric masses and Yukawa-type interactions for these fields. These don't arise from D-terms. Indeed, you could try writing down a more general function $K(\Phi, \Phi^\dagger)$ and integrating over $\int d^4\theta$ but you'll find that it doesn't generate the kind of interactions we want. (We'll see what it does generate in Section 3.2.4.) Instead we have to do something different.

This something different is an option that arises only for chiral superfields. Roughly speaking, because a chiral superfield depends on only half of superspace, we can get a supersymmetric action by integrating it over only half of superspace.

More precisely, given a chiral superfield Φ the function $W(\Phi)$ is also a chiral superfield. In components it reads

$$W(\Phi) = W(\phi) + \sqrt{2}\frac{\partial W}{\partial \phi}\,\theta\psi + \theta^2\left(\frac{\partial W}{\partial \phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial \phi^2}\psi\psi\right) + \ldots$$

where the $+\ldots$ are the extra terms on the second line of (3.19) that include a $\bar{\theta}$ term. But, as you can see in (3.19), each of these is a total derivative and so will not contribute to the action. This means that, for the purposes of building an action, we can think of $W(\Phi)$ as a function only of θ and not of $\bar{\theta}$. This means that we can construct a supersymmetric action by integrating over only half of superspace

$$S_W = \int d^4x \left[\int d^2\theta\; W(\Phi) + \int d^2\bar{\theta}\; W^\dagger(\Phi^\dagger)\right]$$

where the second term is the Hermitian conjugate of the first and is needed to make the action real. This action picks out the θ^2 term in $W(\Phi)$ and is known as an *F-term*, so named because the auxiliary field in a chiral multiplet is usually called F.

We see in (3.20) that the F field (and, by extension any F term that multiplies θ^2 in a chiral multiplet) transforms as a total derivative under supersymmetry. This gives us another way to see that the action S_W is indeed invariant under supersymmetry.

Putting together the D-term and F-term contributions, we get our final supersymmetric action

$$S = S_{\text{chiral}} + S_W = \int d^4x \left[\partial_\mu \phi^\dagger \partial^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + F^\dagger F + \left(F\frac{\partial W}{\partial \phi} - \frac{1}{2}\frac{\partial^2 W}{\partial \phi^2}\psi\psi + \text{h.c.}\right)\right]$$

This is known as the *Wess-Zumino action*. The function $W(\Phi)$ is called the *superpotential*.

(An aside: There is a completely different object that is also called the Wess-Zumino action, or sometimes the Wess-Zumino-Witten or WZW action. This is a topological term that involves an integral over a higher dimensional space. It has nothing to do with supersymmetry. You can read about it in the lectures on Gauge Theory.)

As promised, the auxiliary field F appears only algebraically in the action. For such fields, it is legitimate to eliminate it by the equation of motion which, in this case, reads simply

$$F + \frac{\partial W^\dagger}{\partial \phi^\dagger} = 0 \quad \text{and} \quad F^\dagger + \frac{\partial W}{\partial \phi} = 0$$

Putting this back into the action gives us an action just in terms of those fields that have propagating degrees of freedom,

$$S = \int d^4x \left[\partial_\mu \phi^\dagger \partial^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - \left|\frac{\partial W}{\partial \phi}\right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial \phi^2}\psi\psi - \frac{1}{2}\frac{\partial^2 W^\dagger}{\partial \phi^{\dagger\,2}}\bar{\psi}\bar{\psi}\right]$$

This is the form of the action that we met back in the introduction in (1.1). We see that the scalar potential is positive definite and takes the form

$$V(\phi, \phi^\dagger) = \left|\frac{\partial W}{\partial \phi}\right|^2$$

We still have to specify the form of the superpotential. In general, this can be any holomorphic function of ϕ. If want to restrict ourselves to theories that are renormalisable then we should take a superpotential that is no greater than cubic. For example, we could take

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \tag{3.23}$$

In general, both m and λ can be complex. This gives the potential

$$V = \left|m\phi + \lambda\phi^2\right|^2$$

After expanding this out, the mass of the scalar field is $|m|$. Note that, in addition to the $|\phi|^4$ term, there are also cubic terms $\phi^2\phi^\dagger$ and $\phi^{\dagger\,2}\phi$. These give Feynman diagrams in which a single ϕ particle splits into two others which means that particle number is not conserved in the Wess-Zumino model and, relatedly, there is no way to distinguish particles from anti-particles. This is related to the fact the theory does *not* have a $U(1)$ global symmetry in the presence of the general superpotential (3.23) with $m, \lambda \neq 0$.

With a cubic superpotential, the equation of motion for the Weyl fermion is

$$i\bar{\sigma}^\mu\partial_\mu\psi + m^\star\bar{\psi} = -2\lambda^\star\phi^\dagger\bar{\psi}$$

The fermion also has mass $|m|$. There is no $U(1)$ symmetry associated to this fermion and the mass is an example of a Majorana mass. Note also that the Yukawa term on the right-hand side specifies the interaction between the fermion and scalar and is characterised by the same coupling λ that determines the self-interaction of the scalar. This will have important consequences when we turn to the quantum theory.

Multiple Chiral Superfields

There is a straightforward generalisation of the Wess-Zumino action to multiple chiral superfields Φ_i. We now take the action

$$S = \int d^4x\, d^4\theta\, \sum_i \Phi_i^\dagger\Phi_i + \int d^4x \left[\int d^2\theta\; W(\Phi) + \text{h.c.}\right] \tag{3.24}$$

where if we wish the theory to be renormalisable we should again restrict to a cubic superpotential

$$W(\Phi) = \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}\lambda_{ijk}\Phi_i\Phi_j\Phi_k$$

The resulting potential is

$$V(\phi) = \sum_i \left|\frac{\partial W}{\partial\phi_i}\right|^2$$

Again, this is positive definite as it must be in a supersymmetric theory since the energy is necessarily positive.

As we have seen, for a single massive chiral multiplet the Weyl fermion necessarily has a Majorana mass. With two chiral multiplets, we may have a Dirac mass. Let's call the chiral multiplets Φ and $\tilde{\Phi}$. Then the simple superpotential

$$\mathcal{W} = m\tilde{\Phi}\Phi$$

gives rise to two Weyl equations, each of which mixes the spinors ψ and $\tilde{\psi}$,

$$i\bar{\sigma}^\mu \partial_\mu \psi + m^\star \bar{\tilde{\psi}} = 0 \quad \text{and} \quad i\bar{\sigma}^\mu \partial_\mu \tilde{\psi} + m^\star \bar{\psi} = 0$$

This is the Dirac equation, decomposed into two Weyl pieces. (Sorry for the ugliness of piling a bar on top of a tilde.) Note that it now has a $U(1)$ symmetry, under which ψ and $\tilde{\psi}$ (or, equivalently the superfields Φ and $\tilde{\Phi}$) rotate with opposite charges.

3.2.3 Supersymmetry of the Wess-Zumino Model Revisited

It's worth pausing for a recap. We've derived the Wess-Zumino model which, for a single chiral superfield, before integrating out F, is given by

$$S = \int d^4x \, \left[\partial_\mu \phi^\dagger \partial^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + F^\dagger F + \left(F\frac{\partial W}{\partial \phi} - \frac{1}{2}\frac{\partial^2 W}{\partial \phi^2}\psi\psi + \text{h.c.}\right)\right]$$

Our arguments involving superspace have told us that this action is invariant under the supersymmetry transformations (3.20).

$$\begin{aligned} \delta\phi &= \sqrt{2}\epsilon\psi \\ \delta\psi &= \sqrt{2}i\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi + \sqrt{2}\epsilon F \\ \delta F &= \sqrt{2}i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi \end{aligned}$$

together with the hermitian conjugate transformations

$$\begin{aligned} \delta\phi^\dagger &= \sqrt{2}\bar{\epsilon}\bar{\psi} \\ \delta\bar{\psi} &= -\sqrt{2}i\epsilon\sigma^\mu\partial_\mu\phi^\dagger + \sqrt{2}\bar{\epsilon}F^\dagger \\ \delta F^\dagger &= \sqrt{2}i\epsilon\sigma^\mu\partial_\mu\bar{\psi} \end{aligned}$$

But this is something that we can just check. It's a little tedious but, given the importance of this result, it's worth doing. From our discussion above, we know that the kinetic terms and the superpotential terms should be independently invariant. We can check each in turn. First the kinetic terms. We have

$$\delta S_{\text{chiral}} = \int d^4x \, \left[\partial^\mu\phi^\dagger\partial_\mu\delta\phi - i\delta\bar{\psi}\,\bar{\sigma}^\mu\partial_\mu\psi + F^\dagger\delta F + \text{h.c.}\right]$$

We've kept only half the terms, the other half buried in the hermitian conjugate. (Admittedly, there was some forethought involved in which terms to keep to ensure that they cancel among themselves.) Using the supersymmetry transformations above, we have

$$\delta S_{\rm chiral} = \sqrt{2}\int d^4x\ \left[\partial^\mu\phi^\dagger\epsilon\partial_\mu\psi - \partial_\nu\phi^\dagger\epsilon\sigma^\nu\bar{\sigma}^\mu\partial_\mu\psi - iF^\dagger\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi + iF^\dagger\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi + \text{h.c.}\right]$$

We see that the two terms with $F^\dagger$ cancel immediately. For the other two terms we have a little bit of work to do. Note that, by integrating by parts twice, we can symmetrise over $(\mu\nu)$ in the second term. But you can check that $\sigma^{(\nu}\bar{\sigma}^{\mu)} = \eta^{\mu\nu}$ which then ensures that the first two terms also cancel and $\delta S_{\rm chiral} = 0$.

For the superpotential terms we have

$$\delta S_W = \int d^4x\ \left[\delta F\frac{\partial W}{\partial\phi} + F\frac{\partial^2 W}{\partial\phi^2}\delta\phi - \frac{\partial^2 W}{\partial\phi^2}\psi\,\delta\psi - \frac{1}{2}\frac{\partial^3 W}{\partial\phi^3}\psi\psi\,\delta\phi + \text{h.c.}\right]$$

The final $\partial^3 W/\partial\phi^3$ term multiplies ψ^3 and so vanishes because ψ is a 2-component Grassmann field. We're then left with

$$\delta S_W = \sqrt{2}\int d^4x\ \left[i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi\frac{\partial W}{\partial\phi} + F\frac{\partial^2 W}{\partial\phi^2}\epsilon\psi - i\frac{\partial^2 W}{\partial\phi^2}\psi\sigma^\mu\bar{\epsilon}\partial_\mu\phi - \frac{\partial^2 W}{\partial\phi^2}F\epsilon\psi + \text{h.c.}\right]$$

The $F\epsilon\psi$ terms cancel immediately. The other two cancel after an integration by parts, together with the fact that $\psi\sigma^\mu\bar{\epsilon} = -\bar{\epsilon}\bar{\sigma}^\mu\psi$. We then have $\delta S_W = 0$ as promised.

There is also a version of this calculation after we have integrated out the auxiliary field F, replacing it with its equation of motion $F = -\partial W^\dagger/\partial\phi^\dagger$. As we've seen, the Wess-Zumino action becomes

$$S = \int d^4x\ \left[\partial_\mu\phi^\dagger\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi - \left|\frac{\partial W}{\partial\phi}\right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi\psi - \frac{1}{2}\frac{\partial^2 W^\dagger}{\partial\phi^{\dagger\,2}}\bar{\psi}\bar{\psi}\right]$$

We can also replace F in the supersymmetry transformations. These become

$$\delta\phi = \sqrt{2}\epsilon\psi \quad \text{and} \quad \delta\psi = \sqrt{2}i\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi - \sqrt{2}\epsilon\frac{\partial W^\dagger}{\partial\phi^\dagger}$$

The calculation described above goes through with only minor modifications (although you can no longer treat the kinetic and superpotential terms independently). This is the supersymmetry invariance of the Wess-Zumino model that we promised back in the introduction.

3.2.4 Non-Linear Sigma Models

The restriction to a cubic superpotential above is motivated by the requirement that the theory be renormalisable. But for theories of scalars, this requirement isn't always at the top of our list. The reason is that these theories may arise as the low-energy description of something more interesting. In this situation, there's no reason to think that the low-energy description should be valid at arbitrarily high-energy scales and so no reason to impose renormalisability.

An illustrative analogy can be found in QCD. At high energies this is a theory of quarks and gluons but at low energies, after confinement has imposed itself on the dynamics, it is a theory of light scalar particles called pions. We denote these fields as $\pi^i(x)$ with i labelling the different pion fields. (For what it's worth, $i = 1, \ldots, 8$ in QCD if we include mesons that contain up, down and strange quarks.) The low-energy dynamics of pions takes the form

$$S_{\rm NLSM} = \int d^4x \; g_{ij}(\pi)\, \partial_\mu \pi^i \partial^\mu \pi^j \tag{3.25}$$

Theories of this kind go by the unhelpful name of *non-linear sigma models*. The fields π^i can be thought of as coordinates on some manifold $\mathcal{M}$ that is called the *target space*. The interactions are hiding in the derivative terms and are packaged into a collection of functions $g_{ij}(\pi)$ that can be viewed as a metric on $\mathcal{M}$. The action (3.25) describes massless scalar fields, although it is always possible to add mass terms if necessary.

Actions of the type (3.25) arise in many places in physics. We first meet them in General Relativity as the action for particles (rather than fields) moving in a curved space or spacetime. But they also occur in many places in condensed matter physics and statistical physics. (The $O(N)$ models discussed in the lectures on Statistical Field Theory are an example.) You can learn more about the specific metric $g_{ij}(\pi)$ that describes pion dynamics in Section 5 of the lectures on Gauge Theory. Here, our interest is in writing down supersymmetric versions of non-linear sigma models.

We can achieve this simply by introducing more interesting D-terms. We consider n chiral superfields Φ^i with $i = 1, \ldots, n$. We'll denote the anti-chiral superfields as $\bar{\Phi}^{\bar{i}}$ with the $\bar{i} = 1, \ldots, n$ index a useful reminder that these label anti-chiral fields. We then consider the action

$$S = \int d^4x\, d^4\theta \; K(\Phi, \bar{\Phi}) \tag{3.26}$$

with $K(\Phi, \bar{\Phi})$ any real function of these superfields. This function is known as the *Kähler potential*.

Previously, we took

$$K = \sum_i \Phi^{\dagger\,\bar{i}}\Phi^i$$

We will refer to this as the *canonical Kähler potential*. It is the form that we must take if we want our theory to renormalisable. But if we're willing to entertain low-energy effective theories then we can take a general, real function K. To compute the resulting action, we simply need to compute the D-term of $K(\Phi, \Phi^\dagger)$. This calculation is a little laborious but the result is quite beautiful. The supersymmetric non-linear sigma model takes the form

$$\begin{aligned} S = \int d^4x \ \ \Big[g_{i\bar{j}} &\left(\partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{j}} + \frac{i}{2}\partial_\mu \psi^i \sigma^\mu \bar{\psi}^{\bar{j}} - \frac{i}{2}\psi^i\sigma^\mu\partial_\mu \bar{\psi}^{\bar{j}} + F^i \bar{F}^{\bar{j}}\right) \\ &+ \frac{1}{2}\frac{\partial g_{i\bar{j}}}{\partial \phi^k}\left(\psi^k\psi^i\bar{F}^{\bar{j}} - i\bar{\psi}^{\bar{j}}\sigma^\mu\psi^i\,\partial_\mu\phi^k\right) + \text{h.c.} \\ &+ \frac{1}{4}\frac{\partial^2 g_{i\bar{j}}}{\partial\phi^k\partial\bar{\phi}^{\bar{l}}}(\psi^i\psi^k)(\bar{\psi}^{\bar{j}}\bar{\psi}^{\bar{l}})\Big] \end{aligned} \tag{3.27}$$

where the metric $g_{i\bar{j}}$ is related to the Kähler potential as

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial\phi^i\bar{\phi}^{\bar{j}}} \tag{3.28}$$

Note that this metric only has components with one holomorphic and one anti-holomorphic index. We can eliminate the auxiliary field F through its equation of motion

$$g_{i\bar{j}}F^i + \frac{1}{2}\frac{\partial g_{i\bar{j}}}{\partial\phi^k}\,\psi^k\psi^i = 0 \quad \text{and} \quad g_{i\bar{j}}\bar{F}^{\bar{j}} + \frac{1}{2}\frac{\partial g_{i\bar{j}}}{\partial\bar{\phi}^{\bar{l}}}\,\bar{\psi}^{\bar{l}}\bar{\psi}^{\bar{j}} = 0$$

Substituting this back into the action, we find

$$S = \int d^4x \left[g_{i\bar{j}}\left(\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{j}} + \frac{i}{2}\mathcal{D}_\mu\psi^i\sigma^\mu\bar{\psi}^{\bar{j}} - \frac{i}{2}\psi^i\sigma^\mu\mathcal{D}_\mu\bar{\psi}^{\bar{j}}\right) + \frac{1}{4}R_{i\bar{j}k\bar{l}}(\psi^i\psi^k)(\bar{\psi}^{\bar{j}}\bar{\psi}^{\bar{l}})\right]$$

Rather wonderfully, all the terms now take a nice geometrical form. The kinetic term for the fermion involves a kind of covariant derivative, defined by

$$D_\mu\psi^i = \partial_\mu\psi^i + \Gamma^i_{\ jk}\psi^j\partial_\mu\phi^k$$

where, for a metric given by (3.28), the Christoffel symbol is given by

$$\Gamma^i_{\ jk} = g^{i\bar{l}}\,\frac{\partial g_{k\bar{l}}}{\partial\phi^j}$$

Meanwhile, the four-fermion interaction terms comes multiplying the Riemann tensor. For a metric given by (3.28), this too takes a special form

$$R_{i\bar{j}k\bar{l}} = g_{m\bar{j}} \frac{\partial \Gamma^m_{ik}}{\partial \bar{\phi}^{\bar{l}}} = \frac{\partial^2 g_{i\bar{j}}}{\partial \phi^k \bar{\phi}^{\bar{l}}} - g^{m\bar{n}} \frac{\partial g_{i\bar{n}}}{\partial \phi^k} \frac{\partial g_{m\bar{j}}}{\partial \bar{\phi}^{\bar{l}}}$$

We have stumbled upon the mathematical framework of *Kähler geometry*. This is a particular form of complex geometry that can be placed on manifolds that are even dimensional and can be endowed with complex coordinates, like the ϕ^i and above. A *Kähler manifold* is a manifold that is endowed with a *Kähler two-form*

$$\Omega = 2ig_{i\bar{j}} d\phi^i \wedge d\bar{\phi}^{\bar{j}}$$

such that

$$d\Omega = 0$$

This requires that the $g_{i\bar{j}}$ satisfies

$$\frac{\partial g_{i\bar{j}}}{\partial \phi^k} = \frac{\partial g_{k\bar{j}}}{\partial \phi^i} \quad \text{and} \quad \frac{\partial g_{i\bar{j}}}{\partial \bar{\phi}^{\bar{l}}} = \frac{\partial g_{i\bar{l}}}{\partial \bar{\phi}^{\bar{j}}}$$

This condition is locally equivalent to the existence of a Kähler potential $K(\phi, \bar{\phi})$, with the metric given by (3.28).

Finally, note that the Kähler potential is not unique. The action (3.26) is invariant under any shift

$$K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi})$$

where $\Lambda(\Phi)$ is any holomorphic function of Φ^i. This is because $\Lambda(\Phi)$ is a chiral superfield and necessarily vanishes when integrated over all of superspace. These shifts are called *Kähler transformations*.

Supersymmetry has led us to the mathematical framework of Kähler geometry. This is just one of many close connections between supersymmetry and interesting geometric structures. Some of these connections are explored further in the lectures on Supersymmetric Quantum Mechanics.

Adding a Superpotential

The supersymmetric non-linear sigma model (3.27) describes massless fields. We can always add an additional superpotential $W(\Phi)$ to the action. We won't write down the full action, but simply comment that the scalar potential now takes the form

$$V(\phi, \bar{\phi}) = g^{i\bar{j}} \frac{\partial W}{\partial \phi^i} \frac{\partial W^\dagger}{\partial \bar{\phi}^{\bar{j}}} \tag{3.29}$$

with $g^{i\bar{j}}$ the inverse metric.

A Comment on Supergravity

Throughout these lectures we will restrict ourselves to theories with global, or rigid, supersymmetry. As we've mentioned previously, if one extends supersymmetry to a gauge symmetry, making it local, then the resulting theory necessarily includes gravity. This is supergravity. In this case, the scalar potential for a bunch of chiral multiplets again has a fixed form, depending only on the Kähler potential K and superpotential W. It is

$$V(\phi,\bar{\phi}) = e^{K/M_{\rm pl}^2}\left(g^{i\bar{j}}D_iW D_{\bar{j}}W^\dagger - 3\frac{|W|^2}{M_{\rm pl}^2}\right) \tag{3.30}$$

where

$$D_iW = \frac{\partial W}{\partial \phi^i} + \frac{1}{M_{\rm pl}^2}\frac{\partial K}{\partial \phi^i}W$$

Here $M_{\rm pl}$ is the Planck mass. In the limit that $M_{\rm pl} \to \infty$, gravity becomes arbitrarily weak and the potential (3.30) reduces to our previous potential (3.29).

Perhaps surprisingly, the supergravity potential is *not* positive definite. This is related to the fact that supersymmetric theories can exist in anti-de Sitter spacetimes with a negative cosmological constant.

3.3 Non-Renormalisation Theorems

So far our discussion of supersymmetric theories has been entirely classical. But the great advantage of supersymmetry is that it allows us to gain control over the quantum dynamics of the theory. We can start to understand this already just with chiral multiplets. In this section we will show that the superpotential does not receive quantum corrections at any order in perturbation theory. This is known as a *non-renormalisation theorem*. In contrast, all bets are off with the Kähler potential: it is no more constrained than the kinetic terms in any other quantum field theory.

The original proof of the non-renormalisation theorem used Feynman diagrams for superfields. This means that we write down a diagram in which, say, the propagators correspond to superfields. These "super-Feynman diagrams" then encode a number of normal Feynman diagrams, some with bosons running in loops and others with fermions running in loops. One can then show that the most general super-Feynman diagram doesn't contribute to the superpotential.

In these lectures, we're not going to develop the machinery of superfield Feynman diagrams. Instead, we will give a much simpler argument that uses only the symmetries of the problem.

Before we get going, an important comment. Throughout these lectures, theories of chiral superfields will typically be viewed as low-energy effective actions. More precisely, they will be viewed as *Wilsonian low-energy effective actions*. This means that they describe physics only on some suitably large length scale, or equivalently at energies less than some UV cut-off, $E \leq \Lambda_{UV}$. All short distance, or high energy, degrees of freedom have been integrated out but may, in some cases, leave an imprint on the low-energy degrees of freedom. We'll see examples of this as we proceed.

A Wilsonian effective action already takes into account any quantum effects above the cut-off Λ_{UV}. But not those below. You need to use the action to compute, for example, loop diagrams to understand the low-energy quantum dynamics. But there are no UV divergences because the action comes equipped with an explicit cut-off.

There is another, more formal kind of effective action that is common in quantum field theory. This is the *one particle irreducible*, better known as *1PI*, effective action. It arises as the Legendre transform of the (log of) the partition function. In contrast to the Wilsonian effective action, the 1PI effective action is best viewed as a classical action, with all quantum effects already taken into account. This can be problematic in the presence of massless particles since the 1PI effective action may have IR singularities. In contrast, there is no such problem with the Wilsonian effective action.

3.3.1 R-Symmetry Revisited

Given a quantum field theory, one of the first things we should do is understand its symmetries. The kind of Wess-Zumino models (or, more generally non-linear sigma models) that we've described above could have many different Abelian or non-Abelian global symmetries acting on the chiral superfields Φ^i. However, there is one that is of particular importance. This is the $U(1)$ *R-symmetry*. It is special because it does not commute with supersymmetry. Instead, as we saw in (2.25), it obeys

$$[R, Q_\alpha] = -Q_\alpha \quad \text{and} \quad [R, \bar{Q}_{\dot{\alpha}}] = +\bar{Q}_{\dot{\alpha}}$$

This means that the R-charge of the scalar ϕ and fermion ψ in a chiral superfield necessarily differ. If the scalar has charge r, then the other members of the multiplet have

$$R[\phi] = r \quad \Rightarrow \quad R[\psi] = r - 1 \quad \text{and} \quad R[F] = r - 2 \tag{3.31}$$

Another way of saying this is to return to the expansion of a chiral superfield (3.19),

$$\Phi = \phi + \sqrt{2}\theta\psi + \theta^2 F + \ldots$$

We endow the supercoordinate θ with an R-charge

$$R[\theta] = +1$$

This tallies with our expression (3.11) for the supercharge $\mathcal{Q} \sim \partial/\partial\theta + \ldots$ which tells us that $\mathcal{Q}$ and θ have opposite charges. The upshot is that if the superfield has R-charge $R[\Phi] = r$, then the other charges in (3.31) follow.

So when do theories enjoy an R-symmetry? Let's consider the simplest Wess-Zumino model (3.24) for a single chiral superfield. The D-term, which gives the kinetic terms, is clearly invariant under any R-symmetry. That leaves the superpotential. This multiplies $d^2\theta$ but Grassmann integration acts in the same way as differentiation which means that the measure has charge

$$R[d^2\theta] = -2$$

We see that the action is invariant under R-symmetry only if we can assign charges to the superfield such that the superpotential has charge

$$R[W] = +2 \tag{3.32}$$

When we have just a single superfield Φ, this is rather limiting. It holds only if the superpotential is a monomial

$$W(\Phi) = \Phi^n$$

in which case we can assign $R[\Phi] = 2/n$. For example, if we take $W(\phi) = \frac{1}{2}m\phi^2$ then the Lagrangian has an R-symmetry under which $\phi \to e^{i\alpha}\phi$ and $\psi \to \psi$. This case is a little boring because there are no interaction terms between ϕ and ψ so obviously we can rotate them independently. We could, however, take $W(\phi) = \frac{1}{3}\lambda\phi^3$ in which case we have the Yukawa term $\phi\psi\psi$ which is invariant under the R-symmetry $\phi \to e^{2i\alpha/3}$ and $\psi \to e^{-i\alpha/3}\psi$. However, if we include both mass and Yukawa terms, there is no R-symmetry. The surprise, as we will now see, is that the lack of an R-symmetry doesn't stop it being useful!

3.3.2 The Power of Holomorphy

We will now see what the R-symmetry has to do with the non-renormalisation of the superpotential. I should warn you that the argument that follows, originally due to Seiberg, is extremely slick and was developed only after a more nuts and bolts argument using Feynman diagrams had been found. But the symmetry argument is both easier and, ultimately, more powerful.

There are a number of conceptual steps that we need to take before the non-renormalisation theorem becomes clear. These are all related to the parameters that appear in the superpotential, things like the mass m and Yukawa coupling λ in (3.23). Each of these parameters is naturally complex. Moreover, like the chiral superfields themselves, the superpotential must be a *holomorphic* function of these parameters.

Of course, as written in (3.23), the superpotential is, by definition, a holomorphic function of parameters. There's an m that sits in the first term and a λ in the second and these are complex. However, the point is that any quantum corrections to the superpotential must also be holomorphic in parameters. This greatly restrains the allowed quantum corrections.

There are two ways to argue that the superpotential must be holomorphic in parameters. The first is direct, but convoluted, and invokes a kind of supersymmetric Ward identity. The second way is to say a bunch of words that hopefully makes it obvious. We're going to adopt the second way.

In any quantum field theory, we can view parameters as arising from some fixed, *background scalar fields*. This means that the parameters may come from some dynamical, but very heavy, scalar field with a potential that pins the value of the scalar to that of the parameter. If this is the case, we wouldn't notice any difference at low energies because these new fields are so heavy. We would see the fluctuations of the parameter only at high energies.

This idea is realised in our world: in the Standard Model the scale of the masses of all elementary particles is set by the expectation value of the Higgs boson. It's an idea that is extended dramatically in string theory where all dimensionless parameters of a low-energy theory also arise as the expectation value of some scalar. However, it is a way of thinking that has proven to be useful in many other arenas including, as we will now see, in supersymmetric theories. The new fields that replace the parameters are sometimes called *spurions*.

This change of perspective from parameters to spurions doesn't change the low-energy behaviour of the theory. But, remarkably, it does allow us to put constraints on what this low-energy behaviour can be. These constraints are especially strong in supersymmetric theories because the spurion must be the lowest component of a chiral superfield. And, as such, the parameters must appear holomorphically in the superpotential.

To understand what this buys us, let's return to the simple case of a single chiral superfield with superpotential

$$W_{\text{tree}} = \frac{1}{2} m\Phi^2 + \frac{1}{3}\lambda\Phi^3 \tag{3.33}$$

We refer to this as the tree-level superpotential. Our goal is to understand how it is changed by quantum corrections.

As we've seen above, this theory does not have an R-symmetry. Nonetheless, thinking of the parameters as spurions suggests that we could think of enlarged symmetries under which the parameters also transform. In this larger framework, the theory has two symmetries: one R-symmetry that we call $U(1)_R$ and one global symmetry that commutes with supersymmetry that we call $U(1)_F$. The charges are

	$U(1)_R$	$U(1)_F$
Φ	1	1
m	0	-2
λ	-1	-3

All components of the superfield have the same charge under $U(1)_F$, while the charge under $U(1)_R$ tells us how the lowest scalar component of the superfield transforms, with other components given by (3.31). Relatedly, the superpotential is invariant under $U(1)_F$ but has charge +2 under $U(1)_R$, as in (3.32).

I stress again that neither $U(1)_R$ nor $U(1)_F$ are symmetries of our theory since a true symmetry isn't allowed to change parameters of the theory. Said another way, non-vanishing charges for m and λ are telling us that these symmetries are explicitly broken. Nonetheless, the spurions give a useful book-keeping device to characterise exactly how the symmetry is broken. Moreover, as we will now see, they also place strong constraints on the quantum corrections to theory.

Any quantum corrections to the superpotential must be consistent with the two symmetries $U(1)_R$ and $U(1)_F$. Combined with holomorphy, this becomes a very powerful constraint on what can appear. We can form a single, dimensionless combination of superfields that carries no charge at all: this is $\lambda\Phi/m$. (The superfield has the same dimension as a scalar, namely $[\Phi] = 1$. Meanwhile the mass and Yukawa coupling have dimensions $[m] = 1$ and $[\lambda] = 0$.) The only kinds of superpotentials that we can write down consistent with the symmetries are then of the form

$$W_{\text{eff}} = m\Phi^2 \, f\left(\frac{\lambda\Phi}{m}\right)$$

Note that holomorphy was key here. In most situations assigning a charge to a complex parameter isn't particularly restrictive since, say, $|\lambda|^2$ carries no charges and so can appear anywhere. But the fact that only holomorphic quantities can appear in the superpotential is a game changer.

We still have an arbitrary function $f(\lambda\Phi/m)$ that can appear. But this can be pinned down by studying the theory in different limits. First, for $\lambda \ll 1$, we are in the weakly coupled limit. This means that for small λ we should reproduce the tree level superpotential (3.33), perhaps with corrections at order λ^2 or higher coming from loop diagrams. In other words, the expansion of $f(x)$ about $x = 0$ must take the form

$$f(x) = \frac{1}{2} + \frac{1}{3}x + \mathcal{O}(x^2)$$

However, should also have a well defined superpotential in the limit $m \to 0$ in which we have massless particles. This tells us that we must have $f(x) = \frac{1}{2} + \frac{1}{3}x$ or, equivalently,

$$W_{\text{eff}} = \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3 = W_{\text{tree}}$$

This is the result we promised: the superpotential receives no quantum corrections to any order in perturbation theory in λ.

(Looking forward: in Section 6, we will study the quantum dynamics of supersymmetric gauge theories. There we will find that superpotentials are, in some circumstances, dynamically generated. But even there they will not be perturbative effects. The superpotentials will arise either by some strong coupling effect or by an instanton effect.)

While the superpotential is immune to quantum corrections, this is not true of the Kähler potential. There are now no holomorphy restrictions and nothing to prohibit corrections of order λ^2 and higher. This means that the physical masses and Yukawa couplings do, in fact, receive quantum corrections. To see this, note that typically the Kähler potential will pick up quantum correction of the form

$$K(\Phi, \Phi^\dagger) = \Phi^\dagger\Phi \to \mathcal{Z}\Phi^\dagger\Phi$$

where $\mathcal{Z} = 1 + \mathcal{O}(\lambda^2)$ is sometimes, inappropriately, called the *wavefunction renormalisation.* This renormalisation factor will have a characteristic logarithmic form

$$\mathcal{Z} = 1 + c|\lambda|^2 \log\left|\frac{\Lambda_{\text{UV}}}{m}\right|^2 + \ldots \tag{3.34}$$

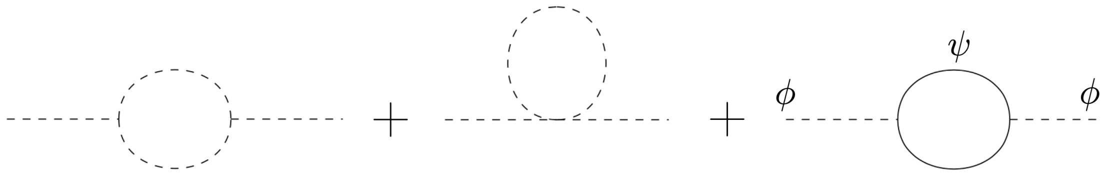

Figure 1. The three one-loop diagrams contributing to the mass of the scalar ϕ. As shown in the last diagram, the dotted line denotes the scalar ϕ and the solid line the fermion ψ.

Here c is a constant whose exact value can be calculated but isn't of interest for our purposes and ... refers to higher loop corrections. This renormalisation changes the kinetic terms for each of the fields and the action is now

$$S = \int d^4x\, d^4\theta\ \mathcal{Z}\Phi^\dagger\Phi + \int d^4x\, d^2\theta\ \left[\frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3\right] + \text{h.c.}$$

Importantly, supersymmetry ensures that there is just a single renormalisation $\mathcal{Z}$ for the superfield, meaning that each of the component fields ϕ, ψ and F experiences the same $\mathcal{Z}$. In such a situation, we should work with the canonically normalised field $\hat{\Phi} = \mathcal{Z}^{1/2}\,\Phi$ and the action becomes

$$S = \int d^4x\, d^4\theta\ \hat{\Phi}^\dagger\hat{\Phi} + \int d^4x\, d^2\theta\ \left[\frac{1}{2}\frac{m}{\mathcal{Z}}\hat{\Phi}^2 + \frac{1}{3}\frac{\lambda}{\mathcal{Z}^{3/2}}\hat{\Phi}^3\right] + \text{h.c.}$$

In this way, the non-renormalisation of the superpotential is not enough to protect the physical mass and Yukawa coupling, which are $m_{\text{phys}} = m/\mathcal{Z}$ and $\lambda_{\text{phys}} = \lambda/\mathcal{Z}^{3/2}$ respectively.

This may seem like a disappointing end to our non-renormalisation claim: the superpotential doesn't change, but the physical parameters sitting within it do. Nonetheless, there's something important going on here. That's because supersymmetry has ensured that the mass m^2_{phys} picks up only a multiplicative renormalisation.

This contrasts strongly with the mass renormalisation expected of a scalar field in a typical quantum field theory. Typically, this mass renormalisation is additive. In particular, any one of the three diagrams shown in Figure 1 would give a contribution of the form

$$m^2_{\text{phys}} \sim m^2 + |\lambda|^2\Lambda^2_{\text{UV}}$$

This is the statement that quantum fluctuations tend to push the mass of scalar fields up to the cut-off scale. In the absence of fine tuning (or some other explanation like symmetry breaking) scalars in quantum field theory are typically heavy. Yet this doesn't happen in supersymmetric theories: miraculously, the additive renormalisation cancels between each of the diagrams above. This occurs because, as we have seen, the same coupling λ appears in the Yukawa coupling to the fermions and in the 3-point and 4-point vertices of the scalars. The result is that, in supersymmetric theories, there is no difficulty with the masses of scalars being small. In particular, if we choose to set $m = 0$ in the superpotential so that the chiral multiplet is massless then quantum corrections do not change this.

This is the key reason that supersymmetry has attracted the interest of phenomenologists. The mass of the Higgs boson is seemingly much lighter than the cut-off scale of the Standard Model, an issue referred to as the *hierarchy problem*. (See the lectures on Particle Physics for a non-technical account of this.) The existence of supersymmetry at, say, the TeV scale would provide a natural explanation of this. Sadly, there is no evidence that this is the explanation favoured by nature.

3.3.3 Integrating Out Heavy Fields

We may sometimes find ourselves in situations in which our theory has two or more fields with different masses. In this case, we can integrate out the heavier fields, leaving ourselves with an action just for the lighter ones. This will be an important tool for us later, so we pause here to see how it works.

Consider the theory of two chiral superfields Φ and Z, both with canonical Kähler potential $K = \Phi^\dagger\Phi + Z^\dagger Z$, and with superpotential

$$W = \frac{1}{2}MZ^2 + \frac{1}{2}\lambda\Phi^2 Z \tag{3.35}$$

In this example, Z is the heavy field with mass M while Φ is massless, but interacts with Z. If we care only about physics at energies $E \ll M$, we can simply integrate out Z to leave ourselves with a theory for Φ.

Usually in quantum field theory, integrating out fields requires us to evaluate some complicated functional determinants or Feynman diagrams. But, at the level of the superpotential, things are straightforward. For a field configuration Φ, the heavy field will rapidly arrange itself to minimise its energy which it does by adjusting to

$$\frac{\partial W}{\partial Z} = 0 \quad \Rightarrow \quad Z = -\frac{\lambda}{2M}\Phi^2$$

Substituting this back into the superpotential gives our effective superpotential

$$W = -\frac{1}{8}\frac{\lambda^2}{M}\Phi^4$$

This results in a ϕ^6 interaction for the scalar, together with the Yukawa-like interaction for the fermion.

We can also reach the same conclusion by analysing the (spurious) symmetries of the theory. This time there are two global symmetries, $U(1)_\Phi$ and $U(1)_Z$ in addition to the R-symmetry. The charges of various fields and parameters are

	$U(1)_R$	$U(1)_\Phi$	$U(1)_Z$
Φ	1	1	0
Z	0	0	1
M	2	0	-2
λ	0	-2	-1

The unique superpotential consistent with these symmetries that does not involve Z is

$$W \sim \frac{\lambda^2}{M}\Phi^4 \tag{3.36}$$

This symmetry argument doesn't give the overall constant $-1/8$ but, as we've seen above, that's not difficult to get by simply solving the equation of motion.

Note that there's a different philosophy at play here from when we showed the non-renormalisation of the superpotential (3.33). In the earlier case we insisted that the superpotential was well behaved as $m \to 0$. However, in the present case the superpotential clearly diverges as $M \to 0$. But this is to be expected: the theory involving Φ alone is only supposed to make sense at energies $E \ll M$. The fact that the superpotential diverges as $M \to 0$ is telling us something physical: that we shouldn't have discarded the field Z in this limit since it wasn't heavy. This is a lesson that we will see several times as these lectures progress: our low-energy theory will break down in any limit where some field that we have ignored becomes massless.

There's also a terminological issue here. Physicists refer to the superpotential (3.36) as "holomorphic" in Φ, λ and M. Strictly speaking it's not holomorphic in M, but instead *meromorphic* because of the pole. As we explained above, the pole certainly has physical consequence, but we won't belabour the point and will continue to take about holomorphy rather than the more accurate meromorphy.

3.3.4 A Moduli Space of Vacua

We can see a twist on this same theme if we study the superpotential (3.35) in the limit $M = 0$. We have

$$W = \frac{1}{2}\lambda\Phi^2 Z \tag{3.37}$$

This theory has a feature that will become increasingly important as these lectures develop: there is not a unique ground state, or even a finite number of isolated ground states. Instead the potential energy is given by

$$V(\phi, z) = \left|\frac{\partial W}{\partial \phi}\right|^2 + \left|\frac{\partial W}{\partial z}\right|^2 = |\lambda \phi z|^2 + \frac{1}{4}\left|\lambda \phi^2\right|^2$$

We've now resorted to our earlier notation of referring to the lowest scalar component of the superfields Φ and Z by the lower case letter ϕ and z respectively. The minima of the potential are given by

$$V(\phi, z) = 0 \quad \Leftrightarrow \quad \phi = 0 \quad \text{and} \quad z = \text{anything}$$

This means that the potential has a flat direction. Provided that $\phi = 0$, there is no energy cost to turning on z. We say that there is a *moduli space of vacua*. In such a situation, the choice of ground state z is not determined dynamically. Instead, to fully specify the theory, we must also state the expectation value of the field z. Importantly, different choices of z give rise to different theories. For example, we can see immediately from the potential that the mass of ϕ is $m_\phi = |\lambda z|$. In other words, this is moduli space of inequivalent vacua.

Now the roles of z and ϕ are reversed! Provided that $z \neq 0$, the ϕ field is massive while z is massless. We can again play the kind of game that we saw above: is there a superpotential $W(Z)$ that we can write down that might arise after Φ is integrated out? It's simple to see that the answer is no. Everywhere along the moduli space, we have

$$W(Z) = 0$$

This is important. Had we found $W(Z) \neq 0$, it would have meant that there was a quantum generated potential that lifts the flat direction and that the true quantum theory has a preferred ground state. But the non-renormalisation theorem tells us that no such potential is generated. Instead we learn that the moduli space of ground states survives in the quantum theory.

The existence of a moduli space of inequivalent vacua is commonplace in supersymmetric theories but never happens in the absence of supersymmetry. In any non-supersymmetric theory, quantum corrections always generate a potential on the would-be moduli space. This is known as the *Coleman-Weinberg potential* and it picks the true ground state of the system, typically pushing the scalar either to $z = 0$ or to $z = \infty$.

We can get some intuition for the Coleman-Weinberg in a simple quantum mechanics example. Suppose that we have a quantum particle that can move in the (x, y) plane but with a potential that we take to be

$$V_{\text{toy model}} = x^2 y^2$$

The classical system has two flat directions: $x = 0$ and $y =$ anything; or $y = 0$ and $x =$ anything. Suppose that we sit at some $y \neq 0$ but classically set $x = 0$. We then look at the quantum system by supposing that y is constant and quantising the x degree of freedom. But this is just a quantum harmonic oscillator with frequency given by $\omega = y$. And the ground state energy of the quantum harmonic oscillator is $E \sim \hbar\omega = \hbar y$. In this way, the quantisation of x gives rise to an energy that pushes y back towards the origin. Indeed, this quantum mechanical system has a unique ground state, localised around the origin.

The Coleman-Weinberg potential is the analogous phenomenon in quantum field theory. It is generic but is avoided in supersymmetric theories due to a delicate cancellation between bosons and fermions, very similar to those at play in the loop diagrams in Figure 1. We'll be meeting many different vacuum moduli spaces as these lectures progress. Indeed, one of the emerging themes of these lectures is that the geometry of these moduli spaces contains important clues to the underlying physics.

For now, let's go back to our field theory (3.37) and ask: what happens to the moduli space at $z = 0$? Here the ϕ field also becomes massless and it should no longer be valid to ignore it. But how do we see this if we're focussed on the dynamics of z alone? The answer to this can be found in the Kähler potential. Classically, this takes the canonical form $K = Z^\dagger Z$, corresponding to to a flat metric

$$ds^2 = \frac{\partial^2 K}{\partial z \partial \bar{z}}\, dz\, d\bar{z} = d\bar{z} dz$$

However, as we saw above, when we integrate out the massive Φ field the Kähler potential receives a one-loop quantum correction (3.34) and becomes

$$K = Z^\dagger Z \left(1 + c|\lambda|^2 \log \left| \frac{\Lambda_{UV}}{Z} \right|^2 + \ldots \right) \tag{3.38}$$

where $|Z|$ appears in the argument of the logarithm courtesy of the role it plays as the mass of Φ. This results in a metric on the moduli space given by

$$ds^2 = \frac{\partial^2 K}{\partial z \partial \bar{z}}\, dz\, d\bar{z} = \left(-c|\lambda|^2 \log \left(\frac{\bar{z} z}{\Lambda_{UV}^2} \right) + \text{constant} + \ldots \right) d\bar{z} dz$$

We see that distances diverge as we approach $z \to 0$. The log singularity at $z = 0$ is the sign that we have attempted to integrate out a massless particle at that point.

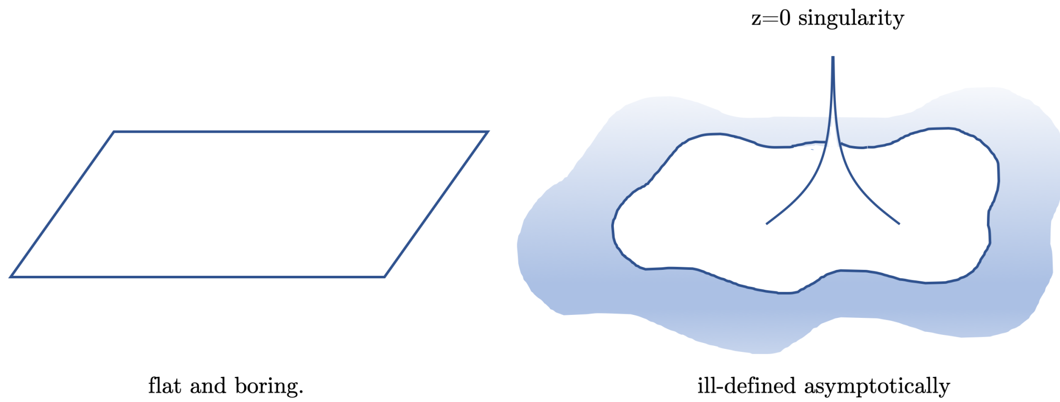

Figure 2. The classical moduli space on the left and the quantum corrected moduli space on the right, with it's singularity at $z = 0$ revealing the massless particle and its negative signature at large z showing that the quantum theory is ill-defined.

There is also some strange behaviour for large $|z|$. When $|z| \gg \Lambda_{UV}$, the first term is negative and, for large enough $|z|$, will overwhelm the constant term, giving us a negative metric. This, of course, is nonsensical. It's telling us that our scalar theory doesn't make sense at very high expectation values or, equivalently at very high energies. In other words, it is capturing the phenomenon of the Landau pole in ϕ^4 theory, but now in a novel geometric fashion. A depiction of the classical and quantum moduli spaces is shown in Figure 2.

3.4 A First Look at Supersymmetry Breaking

A symmetry is said to be *spontaneously broken* if it acts non-trivially on the ground state. This means that the Noether charge Q for the symmetry fails to annihilate the vacuum,

$$Q|0\rangle \neq 0$$

Broken symmetries have important consequences. If a discrete symmetry is spontaneously broken then it implies the existence of multiple, isolated ground states. If a continuous symmetry is spontaneously broken then it implies the existence of a massless particle called a *Goldstone boson*. These ideas underlie Landau's classification of phases of matter and were discussed in some detail in the lectures on Statistical Field Theory and the lectures on Gauge Theory. In this section, we will make a first pass at understanding when supersymmetry may be spontaneously broken and what the consequences are.

First, some basics. From the supersymmetry algebra $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu$ we can derive an expression for the Hamiltonian

$$H = P^0 = \frac{1}{4}\{Q_1^\dagger, Q_1\} + \frac{1}{4}\{Q_2^\dagger, Q_2\}$$

We already noted in Section 2.2.2 that this implies that all states in a supersymmetric theory necessarily have energy $E \geq 0$. This means that any state with $E = 0$ must be a ground state. These states obey

$$E_{\text{ground}} = \langle 0|H|0\rangle = 0 \quad \Leftrightarrow \quad Q_\alpha|0\rangle = 0$$

In this case the supercharges annihilate the ground state which means that supersymmetry is unbroken. Conversely, supersymmetry is spontaneously broken if and only if the energy of the ground state is non-vanishing

$$E_{\text{ground}} = \langle 0|H|0\rangle > 0 \quad \Leftrightarrow \quad Q_\alpha|0\rangle \neq 0$$

In other words, the ground state energy E_{ground} is the order parameter for broken supersymmetry.

There is another way of looking at this. In theories of chiral multiplets (with a canonical Kähler potential) the potential energy is given by (3.29)

$$V(\phi, \bar{\phi}) = \sum_i |F_i|^2 = \sum_i \left|\frac{\partial W}{\partial \phi^i}\right|^2$$

The ground state energy is non-zero if and only if the F-term gets an expectation value in the vacuum

$$F_i = -\frac{\partial W^\dagger}{\partial \bar{\phi}^i} \neq 0$$

This is known as *F-term supersymmetry breaking*. (There is another option that involves vector multiplets known as *D-term supersymmetry breaking*.)

3.4.1 The Goldstino

If a normal continuous symmetry is spontaneously broken, it results in a massless particle known as a Goldstone boson. If supersymmetry is spontaneously broken, it results in a massless fermion that we call a *Goldstino*.

First, some intuition. When a normal, continuous symmetry is spontaneously broken, the symmetry sweeps out a manifold of equivalent ground states. The canonical example is the breaking of a $U(1)$ symmetry that gives rise to the $\mathbf{S}^1$ rim of the Mexican hat potential. The massless Goldstone mode then arises from fluctuations along this flat direction.

Something similar happens for supersymmetry. From the supersymmetry transformations (3.20), we see that when $F^i \neq 0$, a supersymmetry transformation acting on the vacuum turns on a linear combination of fermions

$$\delta\psi^i = \sqrt{2}\epsilon F^i$$

This is the Goldstino.

There is a simple, hands-on way to see the existence of this massless fermion within the class of theories that we're discussing here. The ground state of the system, whether supersymmetric or not, sits at

$$\frac{\partial V}{\partial \phi^i} = 0 \quad \Rightarrow \quad \sum_j \frac{\partial^2 W}{\partial \phi^i \partial \phi^j} \frac{\partial W^\dagger}{\partial \bar{\phi}^j} = -\sum_j \frac{\partial^2 W}{\partial \phi^i \phi^j} F_j = 0$$

If supersymmetry is broken then $F_j \neq 0$ for some j and the equation above then tells us that the matrix $\partial^2 W/\partial\phi^i\partial\phi^j$ necessarily has an eigenvector with vanishing eigenvalue. But $\partial^2 W/\partial\phi^i\partial\phi^j$ is the fermion mass matrix in our theory. So we learn that when supersymmetry is broken there is at least one massless fermion.

There is a more powerful, general approach to show the existence of the Goldstino that holds for the strongly coupled theories that we will discuss later. This is in close analogy to the original proof of Goldstone's theorem and we just give a bare bones sketch here. The idea is to first construct the *supercurrent* S^μ_α. This is the conserved current associated to supersymmetry transformations and, like any other conserved current, obeys $\partial_\mu S^\mu_\alpha = 0$. The supercharge Q_α arises from this current in the usual way:

$$Q_\alpha = \int d^3x \; S^0_\alpha$$

The supercurrent obeys the algebra

$$\{Q_\alpha, \bar{S}^\mu_{\dot{\alpha}}\} = 2\sigma^\nu_{\alpha\dot{\alpha}} T^\mu_{\ \nu}$$

with $T_{\mu\nu}$ the energy-momentum tensor. This reproduces the usual supersymmetry algebra (2.21) when integrated over space. The proof of the existence of a massless Goldstino then proceeds by computing the two-point function

$$p^\mu \langle S_{\mu\alpha}(p)\, \bar{S}_{\nu\dot{\alpha}}(-p)\rangle = -2\sigma^\mu_{\alpha\dot{\alpha}}\eta_{\mu\nu}E_0$$

with E_0 the ground state energy. This tells us that whenever $E_0 \neq 0$ there is a pole in the $\langle S\bar{S}\rangle$ 2-point function at $p = 0$. This pole corresponds to a massless fermion, the Goldstino.

These lectures are very much focussed on more formal aspects of supersymmetry rather than any possible application to our world. Nonetheless, the existence of the Goldstino raises a puzzle. Clearly we don't see supersymmetry at the energies we have explored so far, which is roughly speaking $E \lesssim 100$ GeV or so. That, in itself, is not such a big issue since it may well be that supersymmetry is broken at some higher energy scale. But, in that case the argument above suggests that we would expect to see a massless Goldstino in our world and no such particle exists. (You might wonder if perhaps the neutrino could act as a Goldstino. This isn't possible because the Goldstino is created from the vacuum and so should share its quantum numbers, while the neutrino carries electroweak charge.)

The resolution to this lies in supergravity. Recall that supergravity involves a local, or gauged, version of supersymmetry. When a normal gauge symmetry is broken, the would-be massless Goldstone boson is "eaten" by the Higgs mechanism and becomes massive. The same is true of gauged supersymmetry. In the context of supergravity, the would-be Goldstino is eaten by the gravitino and both become massive with mass of order E_0, the supersymmetry breaking scale.

3.4.2 The Witten Index

Not all theories can spontaneously break supersymmetry. There is a topological obstruction that they must overcome. This obstruction is the *Witten index*.

We met the Witten index briefly back in Section 2.3. It defined as the sum over all states

$$\text{Tr}(-1)^F e^{-\beta H} \tag{3.39}$$

The trace is taken over the infinite number of states in the quantum field theory Fock space. Here F is the fermion number, so that the Witten index counts bosonic states with a $+1$ and fermionic states with a -1. In contrast to the discussion in Section

2.3, we've now included a factor of $e^{-\beta H}$, where H is the Hamiltonian. This acts as a regulator on the very high energy states. But, as we'll now show, these high energy states don't in fact contribute to the Witten index.

To make the discussion precise, we should really work on a compact space, like $\mathbf{T}^3$. This ensures that momentum is quantised and, correspondingly, the energy spectrum is discrete. There are then no subtleties in taking the trace.

The key fact about the Witten index is that any states with energy $E > 0$ necessarily come in boson-fermion pairs. This follows from the kind of representation theory that we did in Section 2.3. More precisely, if we define the combination of supercharges

$$\mathbb{Q} = Q_1 + Q_2^\dagger$$

then, from the supersymmetry algebra (2.21), it is simple to see that these obey

$$\{\mathbb{Q}, \mathbb{Q}^\dagger\} = 4H$$

Consider the action of this operator on a state with energy $H|\phi\rangle = E|\phi\rangle$ with $E \neq 0$. We can then define the fermionic creation and annihilation operators

$$a = \frac{\mathbb{Q}}{2\sqrt{E}} \quad \Rightarrow \quad \{a, a^\dagger\} = 1$$

This algebra has a two-dimensional irreducible representation $|\phi\rangle$ and $a^\dagger|\phi\rangle$, both with energy E. One of these states is bosonic and the other fermionic, ensuring that they cancel in their contribution to the Witten index.

Note that the degeneracy of $E > 0$ states is true whether or not supersymmetry is broken. If supersymmetry is unbroken, it arises because of mass degeneracy of particles in a supermultiplet. If supersymmetry is broken then the degeneracy arises simply from the addition of a zero energy Goldstino mode. (More precisely, on a compact space it arises from the quantisation of the Goldstino zero mode.) In this case, there is no need for the masses of bosonic and fermionic particles to be equal.

This argument for the degeneracy of the spectrum breaks down for states of zero energy. For such supersymmetric ground states there is no obstacle to having just a single state obeying

$$Q_\alpha|0\rangle = Q_\alpha^\dagger|0\rangle = 0$$

More generally, it may well be the case that a theory has multiple ground states. In this case, each ground state could be bosonic or fermionic. Here a "fermionic" ground state is nothing exotic: it just means that it sits in the sector of the Hilbert space with $(-1)^F|0\rangle = -|0\rangle$ rather than $(-1)^F|0\rangle = +|0\rangle$.

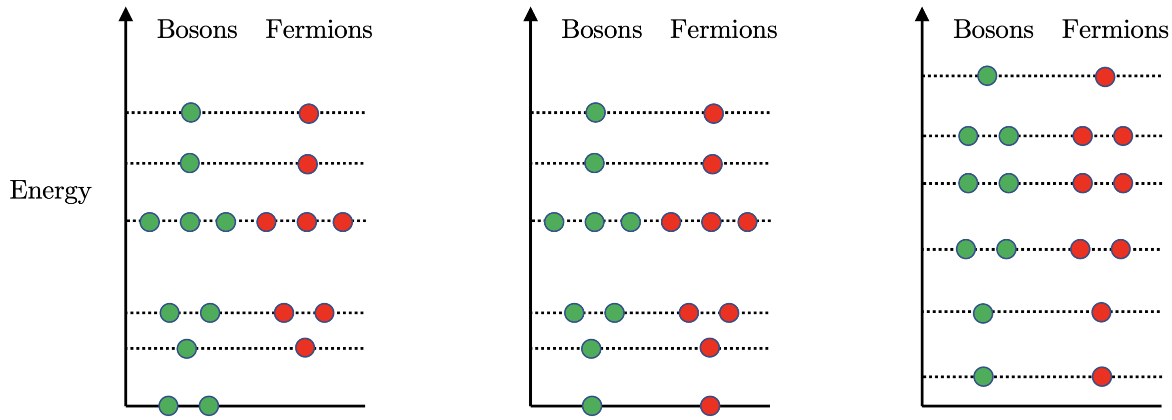

Figure 3. The spectrum on the left has $\mathrm{Tr}(-1)^F e^{-\beta H} = 2$ and cannot break supersymmetry as parameters are changed. The one in the middle has $\mathrm{Tr}(-1)^F e^{-\beta H} = 0$. It does not break supersymmetry but as parameters are varied there is nothing to protect it from turning into the spectrum on the right which does break supersymmetry.

The upshot is that the Witten index (3.39) actually counts the difference in the number of $E = 0$ ground states

$$\mathrm{Tr}(-1)^F e^{-\beta H} = n_B(E = 0) - n_F(E = 0)$$

In particular, the Witten index is independent of the value of β. Moreover, it is actually independent of any other parameter in the theory. To see this, consider a generic spectrum of a supersymmetric theory as shown in Figure 3. All $E \neq 0$ states come in pairs, while $E = 0$ states may be unpaired. As we vary parameters in the theory, some of the $E = 0$ ground states may get lifted and get non-zero energy. But they can only be lifted in pairs and the Witten index remains unchanged. In this sense, the Witten index provides a topological classification of theory.

(Actually, this last statement is only true providing that asymptotic nature of the potential does not change. We'll see an example below.)

All of this means that supersymmetry can only be spontaneously broken in theories with $\mathrm{Tr}(-1)^F = 0$. In contrast, if $\mathrm{Tr}(-1)^F \neq 0$ for some choice of parameters then the theory cannot break supersymmetry as the parameters are changed and this remains true even as the dynamics becomes strongly coupled.

An Example

All of the theories that we will explore in this section are weakly coupled and we can tell whether supersymmetry is broken simply by looking at the potential. This means that we don't really have any need for the Witten index. It starts to show its teeth only for the strongly interacting theories that we will meet in Section 6. Nonetheless, it's useful to get a feeling for how supersymmetric ground states are robust.

Consider a Wess-Zumino model with a single chiral superfield Φ with a superpotential that is a polynomial of degree $p+1$,

$$W(\phi) = a_{p+1}\phi^{p+1} + a_p\phi^p + \ldots + a_1\phi$$

A supersymmetric ground state exists if there are solutions to the equation

$$\frac{\partial W}{\partial \phi} = 0 \tag{3.40}$$

But there's always a solution to this equation because we're solving a polynomial over the complex numbers. In fact, there are always p such solutions (counted with multiplicity). As we vary the coefficients a_i the ground states move around, but they are never lifted. This reflects the fact that this theory has $\text{Tr}(-1)^F e^{-\beta H} = p$. It's a little fiddly to show that all ground states contribute the same $+1$ to the Witten index, rather than with different signs. You can find the argument in the lectures on Supersymmetric Quantum Mechanics where the Witten index plays a central role throughout.

There is, however, an important caveat to the statement that the theory always has p ground states. If we set $a_{p+1} = 0$ then the superpotential becomes a polynomial of degree p and the theory has $p-1$ ground states. It's simple to see what happens here: as we take the limit $a_{p+1} \to 0$, one of the ground states starts heading off to infinity in field space $\phi \to \infty$. This provides a salutary lesson: the Witten index can change if we change how the theory behaves in the asymptotic region of field space. We will see other examples below where, as we vary parameters, a moduli space of ground states emerges then disappears again. This also provides a scenario where the Witten index can jump.

3.4.3 The O'Raifeartaigh Model

The Witten index argument, together with some basics facts about roots of polynomials, means that you have to strive to write down theories that break supersymmetry. Nonetheless, it's not too difficult to achieve. The first model was constructed in 1975 by O'Raifeartaigh. It contains three chiral superfields that we call Y, Z and Φ with the superpotential

$$W = \frac{h}{2} Y(\Phi^2 - \mu^2) + mZ\Phi \tag{3.41}$$

We take all fields to have a canonical Kähler potential so the theory is renormalisable. (We will relax this assumption below.) The parameter h is dimensionless, while $[\mu] = [m] = 1$. It's useful to note that the potential has an R-symmetry (a real one, not a spurious one) under which $R[Y] = R[Z] = 2$ and $R[\Phi] = 0$.

The fields Y and Z act like Lagrange multipliers in the superpotential, setting

$$\frac{\partial W}{\partial Y} = \frac{h}{2}\left(\Phi^2 - \mu^2\right) = 0 \quad \text{and} \quad \frac{\partial W}{\partial Z} = m\Phi = 0$$

Clearly there's no way to set both of these to zero so supersymmetry is spontaneously broken.

The potential of this model is given by

$$V(y, z, \phi) = \frac{1}{4}\left|h\phi^2 - h\mu^2\right|^2 + |m\phi|^2 + |hy\phi + mz|^2$$

Note that y and z are just names of scalar fields here; they are not to be confused with coordinates on spacetime. The minima of the potential always sits at $z = hy\phi/m$ so the final term vanishes. What happens next depends on the ratio of parameters

$$\alpha = \left|\frac{h\mu}{m}\right|$$

If $\alpha < 1$ then the minima is at $\phi = z = 0$. If $\alpha > 1$ then this minima splits into two minima at $\phi = \pm\,$something and a saddle. Importantly, in either case y is arbitrary: it is a flat direction.

It is simple to check that the whole superfield Y is massless. The fermion is the Goldstino while the phase of y is a Goldstone boson associated to a broken R-symmetry. The surprise is that $|y|$ is also massless, with no symmetry reason to protect it. As we now explain, the classical moduli space parameterised by $|y|$ doesn't survive in the full quantum theory.

The Quantum Generated Potential

Importantly, the mass spectrum of the O'Raifeartaigh model depends on the the value of $|y|$: each point on this moduli space describes different physics. Furthermore, and in contrast to our earlier supersymmetric models, the masses of the bosons and fermions are different. This is important because it means that when we integrate out these heavy fields they will induce a Coleman-Weinberg potential on the moduli space parameterised by $|y|$. Here we give some general comments on the form of this potential.

Integrating out heavy fields in a 4d quantum field theory usually give three kinds of divergences: quartic, quadratic and logarithmic. In each case, bosons give rise to a positive potential and fermions a negative potential. In a supersymmetric theory, these exactly cancel which is the reason that moduli space of vacua are not lifted when supersymmetry is broken. As we now explain, when supersymmetry is spontaneously broken some, but not all, of this cancellation remains.

First the quartic divergences. These are given by

$$V_{\text{eff}} \sim \text{Str}\,\Lambda_{UV}^4$$

where Λ_{UV} is the UV cut-off and Str is the *supertrace* which means that we sum over all complex bosonic fields minus the sum over all fermionic fields. (Note that we're summing over the different fields of the theory here. This contrasts with the Witten index where we were performing the much larger sum over all states in the Hilbert space.) But supersymmetric theories have an equal number of bosonic and fermionic fields so all quartic divergences disappear regardless of whether supersymmetry is spontaneously broken or not.

Next up are the quadratic divergences. These take the form

$$V_{\text{eff}} \sim \Lambda_{UV}^2 \text{Str}\,\mathcal{M}^2 = \Lambda_{UV}^2 \left(\text{Tr}\,\mathcal{M}_B^2 - \text{Tr}\,\mathcal{M}_F^2\right)$$

Here $\mathcal{M}$ is the tree-level mass matrix, including both bosons and fermions. In the second equality we've written it in terms of a sum over bosonic and fermionic fields with their appropriate mass matrices $\mathcal{M}_B$ and $\mathcal{M}_F$. Clearly this too vanishes when there is a degeneracy of masses. But a rather nice result says that it also vanishes when supersymmetry is spontaneously broken:

Claim: $\text{Str}\,\mathcal{M}^2 = 0$ for F-term supersymmetry breaking.

Proof: This holds generally in any theory with N superfields and a canonical Kähler potential. The proof involves just a little bit of algebra. First, the $N \times N$ mass matrix for a Weyl fermion is

$$(\mathcal{M}_F)_{ij} = \frac{\partial^2 W}{\partial \phi^i \phi^j}$$

We write this in terms of the auxiliary field $\bar{F}_{\bar{i}} = -\partial W / \partial \phi^i$ as $(\mathcal{M}_F)_{ij} = -\bar{F}_{\bar{i}j}$. The mass-squared matrix that appears in the supertrace formula is the Hermitian matrix

$$(\mathcal{M}_F)^2 = (\mathcal{M}_F)_{\bar{i}j} (\mathcal{M}_F)^\dagger_{j\bar{k}} = \bar{F}_{\bar{i}j} F_{j\bar{k}}$$

Meanwhile, we have to be a little more careful with the bosons because after supersymmetry breaking the real and complex parts of the scalar will typically have different mass. (This happens, for example, in the O'Raifeartaigh Model.) This means that we should break the bosons into real and imaginary pieces and consider the $2N \times 2N$ mass matrix

$$\mathcal{M}_B^2 = \begin{pmatrix} \frac{\partial^2 V}{\partial \phi^i \bar{\phi}^{\bar{j}}} & \frac{\partial^2 V}{\partial \phi^i \phi^l} \\ \frac{\partial^2 V}{\partial \bar{\phi}^{\bar{j}} \bar{\phi}^{\bar{k}}} & \frac{\partial^2 V}{\partial \bar{\phi}^{\bar{j}} \phi^l} \end{pmatrix}$$

But $V = F_i \bar{F}_{\bar{i}}$. Plugging this expression into $\mathcal{M}_B^2$ above and taking the trace (remembering that there's a factor of $\frac{1}{2}$ because we're now working with real fields rather than complex) gives the claimed result. □

All of which means that in a theory with spontaneously broken supersymmetry, the only contribution to the effective potential comes from the logarithmic divergences. It can be shown that these too take the form a supertrace over the mass matrix

$$V_{\text{eff}} = \frac{1}{64\pi^2} \, \text{Str} \, \mathcal{M}^4 \log \left(\frac{\mathcal{M}}{\Lambda_{UV}} \right)^2$$

Again, this vanishes if supersymmetry is unbroken. But now it does not vanish if supersymmetry is spontaneously broken. This gives the quantum potential that lifts flat directions in this case.

The mass matrix $\mathcal{M}$ depends on the value of the field y, and hence V_{eff} should be viewed as a potential that lifts this flat direction. In any theory with a flat direction, quantum generated potentials typically push the field to one end or another. Computing the masses shows that here the true ground state of the system sits at $y = 0$. This is the unique ground state with spontaneously broken supersymmetry.

3.4.4 R-symmetry and the Nelson-Seiberg Argument

We could continue exploring different models (and we will below!) but it is useful to first stop and try to understand some general features of supersymmetry breaking. To this end, let's first look at a small extension of the O'Raifeartaigh model,

$$W = \frac{h}{2}Y(\Phi^2 - \mu^2) + mZ\Phi + \frac{\nu}{2}\Phi^2 + \frac{\epsilon}{2}Y^2 \tag{3.42}$$

This differs from the O'Raifeartaigh model by the addition of the last two terms. Note that these two terms break the R-symmetry and this will be important shortly. For now, we can simply study the scalar potential arising from this superpotential. It is

$$V(y,z,\phi) = \frac{1}{4}\left|h\phi^2 - h\mu^2 + 2\epsilon y\right|^2 + |m\phi|^2 + |hy\phi + mz + \nu\phi|^2$$

Now the theory does have a supersymmetric ground state, sitting at $z = \phi = 0$ and $y = h\mu^2/2\epsilon$.

If, however, we now take $\epsilon \to 0$ to remove the last term in (3.42), then the supersymmetric vacuum moves off to infinity in field space $y \to \infty$ and we once again find ourselves with a theory that breaks supersymmetry, one that appears to be very similar to the original O'Raifeartaigh model. However, in one way there is a key difference between them. To describe this difference we first need to explain what it means for theories to be "generic".

All the theories we're discussing in this section should be viewed as low-energy effective theories, coming from some unknown UV physics. But there is a mantra that can be applied to such low-energy theories: anything that is not forbidden is mandatory. This means that quantum effects will conspire to generate all possible terms in the potential provided that they are consistent with the symmetries of the theory. A low energy effective theory that includes all such terms, with no particular fine tuning of the coefficients, will be said to be "generic".

In this sense, the O'Raifeartaigh model (3.41) is generic. It has an R-symmetry and there are no further terms that one can add consistent with this symmetry.

In contrast, the extension of the O'Raifeartaigh model (3.42) is not generic. It no longer has an R-symmetry, but we have not included Z^2 terms nor Φ^3 terms nor many other terms that we could write down. Despite this, it turns out that the behaviour we have seen – namely the existence of a supersymmetric ground state – persists if we add all these extra terms. So it is sufficient for our discussion.

However, among this large class of theories that do not have an R-symmetry, we only find one that breaks supersymmetry if we set one of the coefficients to vanish: $\epsilon = 0$. This is a very particular choice of coefficient. If the theory (3.42) arose as the low-energy limit of some other theory — one which itself did not have an R-symmetry — then there would be no reason to expect that $\epsilon = 0$. For this reason, it's unlikely that the supersymmetry breaking we've found in this model is actually useful.

In fact, one can make these kind of arguments more generally. Consider a theory with N chiral superfields Φ^i and a potential $W(\phi)$. A supersymmetric ground state obeys

$$\frac{\partial W}{\partial \phi^i} = 0 \tag{3.43}$$

Supersymmetry is broken if we can cook up a superpotential for which there are no solutions to this equation. But these are N equations in N variables and for a generic W they always have a solution. That means that a supersymmetric ground state can always be found.

It is, however, appropriate to restrict W by symmetry arguments and we might wonder if that will help us find a generic W that breaks supersymmetry. For example, suppose that W is invariant under a $U(1)$ global symmetry under which the superfield Φ_i transforms with charge q_i,

$$\Phi_i \to e^{i\alpha q_i}\Phi_i$$

In this case the superpotential can always be written as a function of $W = W(X_i)$ with X_i the invariant ratios

$$X_i = \frac{\Phi_i}{\Phi_1^{q_i/q_1}} \quad i = 2, \ldots, N$$

But now the conditions for a supersymmetric ground state are just $\partial W/\partial X_i = 0$ for $i = 2, \ldots, N$ which are $N-1$ conditions for $N-1$ variables. Again, for a generic W there will be a solution. We see that imposing global symmetries doesn't help us in finding supersymmetry breaking potentials.

However, the story is different if there is an R-symmetry. We take the superfields to transform with charges r_i,

$$\Phi_i \to e^{i\alpha r_i}\Phi_i$$

We again form the invariant ratios

$$\tilde{X}_i = \frac{\Phi_i}{\Phi_1^{r_i/r_1}} \quad i = 2, \ldots, N$$

The key difference is that the superpotential must have R-charge +2. This means that it takes the form

$$W(\Phi_1, \tilde{X}_i) = \Phi_1^{2/r_1} \, \tilde{W}(\tilde{X}_i)$$

The conditions for a supersymmetric ground state are now $\partial \tilde{W}/\partial \tilde{X}_i = 0$. But, as long as $\Phi_1^{2/r_1} \neq 0$, we must also have $\tilde{W}(\tilde{X}) = 0$. This is now N conditions on $N-1$ variables $\tilde{X}_i$ and generically there will *not* be a solution.

This is the *Nelson-Seiberg argument*. It says that models of supersymmetry breaking with generic superpotentials should have an R-symmetry. This is indeed true of the O'Raifeartaigh model.

Our main interest in these lectures is not to construct realistic supersymmetric theories, but rather to explore the strong coupling dynamics of quantum field theories. Nonetheless, it's worth mentioning that the argument for the existence of an R-symmetry causes something of a headache if you're trying to build realistic models in which supersymmetry is spontaneously broken. In some models, like the O'Raifeartaigh model, the non-supersymmetric ground state preserves the R-symmetry (recall that, ultimately, the quantum potential pushes us to $y = 0$.). But this causes problems further down the line because, as we will see in Section 4, an R-symmetry prohibits masses for the superpartners of gauge fields, known as *gauginos*. But these must be heavy in any realistic theory.

Alternatively, we could cook up models in which both supersymmetry and the R-symmetry are spontaneously broken. But this then leads to a light Goldstone boson known as the *R-axion*. Again, we must find a way to give this a mass.

3.4.5 More Ways to (Not) Break Supersymmetry

In the remainder of this section, we briefly discuss a number of other simple models that illustrate different ways in which supersymmetry can be broken.

Runaway Potentials

Here is a model that looks like it breaks supersymmetry but, on closer inspection, does something different. It consists of two fields, Z and Φ, with superpotential

$$W = \frac{h}{2} Z \Phi^2 - \lambda \Phi$$

It has an R-symmetry with $R[\Phi] = 2$ and $R[Z] = -2$ and a scalar potential given by

$$V = \frac{1}{4}|h\phi^2|^2 + |hz\phi - \lambda|^2$$

Clearly there is no way to set both terms to zero so we seem to again have a situation in which supersymmetry is broken. However, instead something slightly different is happening and the potential slopes to zero asymptotically. To see this, look at the direction with $\phi = \lambda/hz$ for which the potential is given by

$$V(z) = \left|\frac{\lambda^2}{2hz^2}\right|^2$$

Clearly $V(z) \to 0$ as $z \to \infty$. So it is better to say that this theory has no stable ground state at all: the field is pushed to $z \to \infty$ where supersymmetry is restored. We will see behaviour like this emerging dynamically in Section 6.

Metastable Supersymmetry Breaking

Let's now consider a slightly different variant of the model (3.42) that broken R-symmetry. We take the superpotential

$$W = \frac{h}{2}Y(\Phi^2 - \mu^2) + mZ\Phi + \frac{\epsilon}{2}Z^2$$

The potential is

$$V(y, z, \phi) = \frac{1}{4}\left|h\phi^2 - h\mu^2\right|^2 + |m\phi + \epsilon z|^2 + |hy\phi + mz|^2$$

This breaks R-symmetry and so, on general grounds, we might expect it to have a supersymmetric vacuum (provided that we have taken the superpotential to be suitably generic). This is indeed the case: the supersymmetric ground state is given by $\phi^2 = \mu$ and $z = -m\phi/\epsilon$ and $y = m^2/h\epsilon$.

For ϵ very small, this ground state sits a long way from the origin of field space. Moreover, if we look close to the origin, $y = 0$, then the potential is very similar to the original O'Raifeartaigh model. In particular, when $\phi = z = 0$ there is a flat direction along y, albeit one that is not a global minimum of the the potential. When we include quantum corrections, this will be lifted and, for suitable values of the parameters, we will find a local, supersymmetry breaking vacuum at the origin. A schematic sketch of this situation is shown in Figure 4.

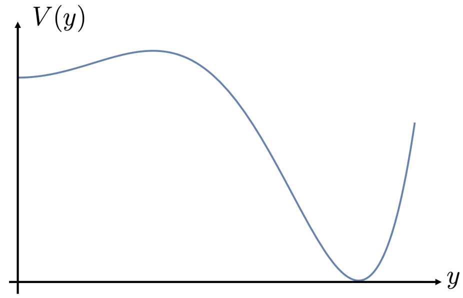

Figure 4. A schematic sketch of the metastable minima at $y = 0$ that breaks supersymmetry and the global, supersymmetric ground state at $y \sim 1/\epsilon$. (The actual potential should be plotted in higher dimensions.)

In a quantum field theory, any local minima of a potential that is not the global minimum is a metastable state, with a finite lifetime. This means that if we initially sit in the supersymmetry breaking minimum, we will eventually tunnel out into the supersymmetric ground state. Nonetheless, it is possible to use such metastable minima to build phenomenologically viable models. You just need to make sure that "eventually" $\gg$ 100 billion years (or whatever allows you to sleep easy at night).

Playing with the Kähler Potential

So far we haven't discussed the simplest theory that breaks supersymmetry. This is a single chiral multiplet with superpotential

$$W = \mu^2 \Phi$$

Clearly $\partial W/\partial\phi = \mu^2 \neq 0$. But this feels too cheap. The ground state energy may be non-zero, but the theory is just a free massless fermion (the Goldstino!) and a free complex scalar. It's hard to argue that there's any deep physics in there.

Things change however if we consider a more general Kähler potential $K = K(\phi^\dagger\phi)$. The fermion remains massless but a potential is now generated for the scalar, given by

$$V(\phi) = |\mu|^4 \left(\frac{\partial^2 K}{\partial\phi\partial\phi^\dagger}\right)^{-1}$$

The price that we pay is that the theory is no longer renormalisable. Of course, as we've stressed above, given that we view these scalar field theory as low energy effective theories, that is not necessarily a bad thing.

For example, suppose that, when expanded around the origin, the Kähler potential takes the form

$$K(\phi, \phi^\dagger) = |\phi|^2 - \frac{1}{M^2}|\phi|^4 + \ldots$$

This kind of behaviour can arise from integrating out heavy particles of mass M. (We found a log correction to the Kähler potential from integrating out particles in (3.38), but other interactions can give the power-law above.) We should view M as the UV cut-off of the theory. Other energy scales in the game should necessarily be much smaller than the cut-off which, for us, means $\mu \ll M$.

With such a Kähler potential, the actual potential energy reads

$$V(\phi, \phi^\dagger) = |\mu|^4 \left(1 + \frac{4}{M^2}|\phi|^2 + \ldots\right)$$

This now has a minima at $\phi = 0$. The net result is that the scalar ϕ has a mass $m_\phi = 2\mu^2/M^2$.

A comment on the scales here. As we've mentioned repeatedly, all the theories in this section should be viewed as low-energy effective theories arising from some high energy completion. In the present case, our theory is valid at energy scales $\sim \mu$. We have integrated out stuff at the much higher scale $M \gg \mu$ and this is what gives rise to the correction to the Kähler potential. It's necessary that there is a separation of scales here. Although the scalar ϕ is not massless, it is light in the sense that $2\mu^2/M \ll \mu$.

Different Kähler potentials can give the different kinds of behaviour that we saw above, including runaway potentials and metastable vacua.