4 Supersymmetric Gauge Theories

Finally, we turn to the main subject of these lectures: supersymmetric gauge theory. In this section we will describe the classical structure of supersymmetric gauge theories. In Section 6 we turn to their quantum dynamics.

4.1 Abelian Gauge Theories

A gauge field A_μ sits inside a real superfield satisfying $V(x,\theta,\bar{\theta}) = V^\dagger(x,\theta,\bar{\theta})$. Expanding out such a superfield in components, we have

$$\begin{aligned} V(x,\theta,\bar{\theta}) = &\ C(x) + \theta\chi(x) + \bar{\theta}\bar{\chi}(x) + i\theta^2 M(x) - i\bar{\theta}^2 M^\dagger(x) + \theta\sigma^\mu\bar{\theta}\, A_\mu(x) \\ &+ \theta^2\bar{\theta}\left(\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) + \bar{\theta}^2\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) \\ &+ \frac{1}{2}\theta^2\bar{\theta}^2\left(D(x) - \frac{1}{2}\Box C(x)\right) \end{aligned} \tag{4.1}$$

The real superfield contains two real scalars, C and D, and a complex scalar M, together with two Weyl fermions χ_α and λ_α. Importantly, it also contains a real vector field A_μ. This will play the role of the gauge field in what follows. We've defined some of the components to include derivatives of others. This should simply be thought of as a redefinition of $D(x)$ and $\lambda(x)$, admittedly one that you wouldn't write down unless you had an inkling of what was coming.

If A_μ is to be a gauge field, then it must enjoy a gauge transformation. These too sit in superfield. We start by taking a chiral superfield Ω

$$\Omega = \omega + \sqrt{2}\theta\rho + \theta^2 G + i\theta\sigma^\mu\bar{\theta}\partial_\mu\omega - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\rho\,\sigma^\mu\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\Box\omega$$

then $i(\Omega - \Omega^\dagger)$ is a real superfield. Consider the generalised gauge transformation

$$V \to V + i(\Omega - \Omega^\dagger) \tag{4.2}$$

The vector component of the real superfield shifts as

$$A_\mu \to A_\mu - 2\partial_\mu(\mathrm{Re}\,\omega) := A_\mu + \partial_\mu\alpha \tag{4.3}$$

But this is precisely the form of a gauge transformation. But under this generalised gauge transformation, it's not just A_μ that shifts. The other fields in $V(x,\theta,\bar{\theta})$ also transform as

$$\begin{aligned} C &\to C - 2\,\mathrm{Im}\,\omega \\ \chi &\to \chi + \sqrt{2}i\rho \\ M &\to M + G \end{aligned}$$

Importantly, however, $\lambda \to \lambda$ and $D \to D$ remain unchanged. This can be traced to the extra derivative terms that we included in the superfield expansion (4.1) which were designed to soak up the shift by a chiral superfield.

We can now use this gauge transformation to simply set $C = \chi = M = 0$. This is known as *Wess-Zumino gauge*. Note that it's not a gauge choice that has done anything to fix A_μ. It's more a "super gauge choice" to fix the extraneous components in the superfield. In Wess-Zumino gauge, the superfield takes the simpler form

$$V_{WZ} = \theta\sigma^\mu\bar{\theta}\, A_\mu + \theta^2\, \bar{\theta}\bar{\lambda} + \bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D \tag{4.4}$$

It contains a gauge field A_μ, a Weyl fermion λ_α and an extra real scalar D that, as the top component of a superfield, will prove to be auxiliary. If we quantise A_μ and λ then we find the single-particle excitations of the gauge multiplet that we anticipated in Section 2.3.2.

If you act with a supersymmetry transformation on V_{WZ}, then it will take you out of Wess-Zumino gauge. This isn't a big headache; it just means that you have to do a compensating transformation to put yourself back in Wess-Zumino gauge afterwards. The supersymmetry transformations then act on the fields A_μ, λ and D as

$$\begin{aligned} \delta A_\mu &= \epsilon\sigma_\mu\bar{\lambda} + \lambda\sigma_\mu\bar{\epsilon} \\ \delta\lambda &= \epsilon D + (\sigma^{\mu\nu}\epsilon)F_{\mu\nu} \\ \delta D &= i\epsilon\sigma^\mu\partial_\mu\bar{\lambda} - i\partial_\mu\lambda\bar{\sigma}^\mu\bar{\epsilon} \end{aligned} \tag{4.5}$$

Note that the supersymmetry transformations (3.15) alone give us a term proportional to $\partial_\mu A_\nu$ in $\delta\lambda$. The compensating gauge transformation to take us back into Wess-Zumino gauge adds another term so this becomes the gauge invariant field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Finally, note that

$$V_{WZ}^2 = \frac{1}{2}\theta^2\bar{\theta}^2 A_\mu A^\mu \quad \text{and} \quad V_{WZ}^3 = 0 \tag{4.6}$$

This will be useful when constructing supersymmetric actions shortly.

4.1.1 The Field Strength and Action

We will build the action out of a field strength superfield, constructed from V by

$$W_\alpha = -\frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_\alpha V$$

This has some nice properties. First, it is a chiral superfield, obeying $\bar{\mathcal{D}}_{\dot{\alpha}} W_\alpha = 0$. This follows from the fact that $\bar{\mathcal{D}}^3 = 0$. Second, it is invariant under the superfield gauge symmetry (4.2): the $\Omega^\dagger$ term is killed immediately by $\mathcal{D}_\alpha \Omega^\dagger = 0$, while the two $\bar{\mathcal{D}}$'s contrive to kill the Ω term. (You need one $\bar{\mathcal{D}}$ to get past the $\mathcal{D}_\alpha$ and the other $\bar{\mathcal{D}}$ to kill Ω.) The upshot is that any action formed from W_α will be automatically gauge invariant.

Next, we compute the components of W_α. This is a straightforward calculation but the number of terms involved gets rather large. Happily, things are easier if we appreciate that W_α is a chiral superfield since this means we only need to worry about the θ terms, with the $\bar{\theta}$ terms following automatically from the expansion (3.19). In components, the field strength superfield reads

$$W_\alpha(x, \theta) = \lambda_\alpha(x) + \theta_\alpha D(x) + (\sigma^{\mu\nu}\theta_\alpha) F_{\mu\nu}(x) - i\theta^2 \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{\lambda}^{\dot{\alpha}}(x) + \ldots$$

The first component of the chiral superfield W_α is a spinor, rather than a scalar, reflecting the fact that W_α is itself a spinor chiral superfield. Importantly, W_α contains the field strength $F_{\mu\nu}$.

Since W_α is chiral, we can integrate it over half of superspace to get a supersymmetric action. We have

$$\int d^2\theta \; W^\alpha W_\alpha = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} F_{\mu\nu} {}^\star F^{\mu\nu} - 2i\lambda\sigma^\mu \partial_\mu \bar{\lambda} + D^2$$

where the second term involves the *dual field strength*

$${}^\star F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

This is like $F_{\mu\nu}$ but with the electric and magnetic fields swapped (one of them with a minus sign).

The term $iF_{\mu\nu}{}^\star F^{\mu\nu}$ is imaginary and so, at first glance, it looks like it will cancel when we add the hermitian conjugate $\int d^2\bar{\theta} \; W^\dagger_{\dot{\alpha}} W^{\dagger\,\dot{\alpha}}$. However, it turns out that this term plays an important role (at least this is true in the non-Abelian theories that we will discuss shortly) and we wish to keep it. This is achieved by introducing the gauge coupling constant e^2. Because this coupling constant sits in an F-term it is necessarily complex. We define

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{e^2}$$

And then write the Lagrangian

$$S_{\text{Maxwell}} = -\int d^4x \, \left[\int d^2\theta \frac{i\tau}{16\pi} W^\alpha W_\alpha + \text{h.c.}\right]$$
$$= \int d^4x \, \left[-\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{\vartheta}{32\pi^2} F_{\mu\nu}{}^\star F^{\mu\nu} - \frac{i}{e^2}\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2e^2}D^2\right] \tag{4.7}$$

This is the supersymmetric Maxwell action. The propagating degrees of freedom are the $U(1)$ gauge field and a fermion λ that, in this context, is called the *gaugino* or, more specifically, the *photino*. There is also a real, auxiliary field D.

The parameter e^2 is the coupling constant. It doesn't do anything in Maxwell theory, which is free, but will come into play when we add matter. Note that we're working in a convention where there is a factor of $1/e^2$ that sits in front of the Maxwell action. As we'll see, the gauge coupling doesn't then sit anywhere else. This differs from the convention that we first met in Quantum Field Theory where the Maxwell term was canonically normalised but there was a gauge coupling inside the covariant derivatives. The two conventions are related by a rescaling $A_\mu \to eA_\mu$. Note that the photino λ similarly has an unconventionally normalised kinetic term, with a $1/e^2$.

Finally, there is the parameter ϑ. This is known as the *theta angle*. (We've used calligraphic script ϑ to distinguish it from the superspace coordinate θ.) Classically, the theta angle doesn't do anything. This is because it multiplies a total derivative

$${}^\star F_{\mu\nu}F^{\mu\nu} = 2\partial_\mu(\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma)$$

However, things are more interesting in the quantum theory and the addition of such topological terms in the path integral can affect the dynamics. This is rather subtle for Maxwell theory, but underlies the story of 3d topological insulators. The effect is more pronounced in Yang-Mills theory and we'll discuss it further in Section 6. You can read (a lot) more about the theta angle in the lectures on Gauge Theory.

4.1.2 Supersymmetric QED

Next we add matter. This comes in the form of chiral multiplets Φ_i, where $i = 1, \dots, N$. We want these to be charged under the $U(1)$ gauge field so that under a gauge transformation

$$A_\mu \to A_\mu + \partial_\mu\alpha$$

The components of the chiral multiplet transform with charges $q_i \in \mathbb{Z}$. This means that the lowest components transform as

$$\phi_i \to e^{i\alpha q_i}\phi_i$$

By necessity, the fermions ψ_i and auxiliary fields F_i in the chiral multiplet Φ_i must have the same charge,

$$\psi_i \to e^{i\alpha q_i}\psi_i \quad \text{and} \quad F_i \to e^{i\alpha q_i} F_i$$

From (4.3), this gauge transformation sits within a larger superfield transformation, under which

$$\Phi_i \to \exp\left(-2iq_i\Omega\right)\Phi_i$$

This, however, means that the canonical Kähler potential that we've used so far is not gauge invariant:

$$\sum_{i=1}^{N} \Phi_i^\dagger \Phi_i \to \sum_{i=1}^{N} \exp\left(-2iq_i(\Omega - \Omega^\dagger)\right)\Phi_i\Phi_i^\dagger$$

However, it's simple to fix up. We simply need to use the new Kähler potential

$$K(\Phi_i, \Phi_i^\dagger, V) = \sum_{i=1}^{N} \ \Phi_i^\dagger e^{2q_i V}\Phi_i$$

with the transformation of V given in (4.2) rendering the whole expression gauge invariant. In Wess-Zumino gauge, the formulae (4.6) truncates at $e^{2qV} = 1 + 2qV + q^2V^2$. Integrating over superspace then gives

$$\int d^4\theta \ \Phi^\dagger e^{2qV}\Phi = \int d^4x \ \left[|\mathcal{D}_\mu\phi|^2 - i\bar{\psi}\bar{\sigma}^\mu\mathcal{D}_\mu\psi + |F|^2 - \sqrt{2}q\left(\phi\bar{\lambda}\bar{\psi} + \phi^\dagger\lambda\psi\right) + qD|\phi|^2\right]$$

Here the covariant derivatives are given by

$$\mathcal{D}_\mu\phi = \partial_\mu\phi - iqA_\mu\phi \quad \text{and} \quad \mathcal{D}_\mu\psi = \partial_\mu\psi - iqA_\mu\psi$$

The full action for an Abelian gauge theory then comes from combining the Maxwell action (4.7) with the matter fields. It is

$$\begin{aligned} S &= S_{\text{Maxwell}} + \sum_{i=1}^{N}\int d^4x \ d^4\theta \ \Phi_i^\dagger e^{2q_iV}\Phi_i \\ &= \int d^4x \ \Big[-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{\vartheta}{32\pi^2}F_{\mu\nu}{}^\star F^{\mu\nu} - \frac{i}{e^2}\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \sum_{i=1}^{N}\left(|\mathcal{D}_\mu\phi_i|^2 - i\bar{\psi}\bar{\sigma}^\mu\mathcal{D}_\mu\psi_i\right) \\ &\qquad + \frac{1}{2e^2}D^2 + \sum_{i=1}^{N}\left(|F_i|^2 - \sqrt{2}q_i\left(\phi_i\bar{\lambda}\bar{\psi}_i + \phi_i^\dagger\lambda\psi_i\right) + q_iD|\phi_i|^2\right)\Big] \end{aligned} \tag{4.8}$$

The first line contains the kinetic terms, the second the interactions. Note that there is a Yukawa coupling between the gaugino λ and the chiral multiplet fields, with $\phi^\dagger$ partnering ψ so that the Yukawa term is gauge invariant. In addition, there is a scalar potential that arises when we integrate out the auxiliary fields. The F terms don't do anything unless we also add a superpotential, while integrating out the D term results in the potential

$$V(\phi) = \frac{1}{2e^2} D^2 \quad \text{with} \quad D = e^2 \left(\sum_{i=1}^{N} q_i |\phi_i|^2 \right) \tag{4.9}$$

Provided that there are both positive and negative charges q_i (and there must be as we explain below) then the potential has flat directions in which

$$\sum_{i=1}^{N} q_i |\phi_i|^2 = 0 \tag{4.10}$$

The existence of a moduli space of vacua is an important feature of supersymmetric gauge theories. We will study it more closely in Section 4.3

A First Look at the Anomaly

There's nothing wrong with (4.8) as a classical theory. But, as a quantum theory, it has a problem. It turns out that for most choices of the charges q_i, the quantum theory is sick. It has an inconsistency that goes by the name of a *gauge anomaly*.

We will have a lot to say about anomalies, gauge and otherwise, later in these lectures. For now we simply mention that the quantum theory only makes sense if the charges satisfy the following two conditions

$$\sum_{i=1}^{N} q_i = \sum_{i=1}^{N} q_i^3 = 0 \tag{4.11}$$

These conditions are not special to supersymmetric theories. They hold for any theory that has Weyl fermions coupled to a $U(1)$ gauge group. We'll say more about where these conditions come from in Section 5.2. For now, note that they require us to have fields with both positive and negative charges which, in turn, ensures that there are solutions to (4.10) with $\phi_i \neq 0$.

There are non-trivial solutions to the consistency conditions (4.11) but, for the most part, we will work with trivial solutions in which chiral multiplets come in pairs so that for each Φ with charge q there is a second chiral multiplet that we call $\tilde{\Phi}$ with charge $-q$. The conditions (4.11) are then automatically satisfied. Each pair Φ and $\tilde{\Phi}$ is sometimes referred to as a *flavour*. If a flavour is said to have charge q, it means that Φ has charge q and $\tilde{\Phi}$ charge $-q$.

The simplest example comprises of a $U(1)$ gauge field interacting with N flavours (which means $2N$ chiral multiplets) of charge $+1$. This theory is known as *supersymmetric QED*, or SQED for short. The action is

$$\begin{aligned} S_{\text{SQED}} &= S_{\text{Maxwell}} + \sum_{i=1}^{N} \int d^4x \; d^4\theta \; \left(\Phi_i^\dagger e^{2_iV} \Phi_i + \tilde{\Phi}_i^\dagger e^{-2_iV} \tilde{\Phi}_i \right) \\ &= \int d^4x \; \Big[-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\vartheta}{32\pi^2} F_{\mu\nu} {}^\star F^{\mu\nu} - \frac{i}{e^2} \lambda \sigma^\mu \partial_\mu \bar{\lambda} \\ &\quad + \sum_{i=1}^{N} \left(|\mathcal{D}_\mu \phi_i|^2 + |\mathcal{D}_\mu \tilde{\phi}_i|^2 - i\bar{\psi} \bar{\sigma}^\mu \mathcal{D}_\mu \psi_i - i\bar{\tilde{\psi}}_i \bar{\sigma}^\mu \mathcal{D}_\mu \tilde{\psi}_i \right) \\ &\quad - \sqrt{2} \sum_{i=1}^{N} \left(\phi_i^\dagger \lambda \psi_i - \tilde{\phi}_i^\dagger \lambda \tilde{\psi}_i + \text{h.c.} \right) - \frac{e^2}{2} \left(\sum_{i=1}^{N} |\phi_i|^2 - |\tilde{\phi}_i|^2 \right)^2 \Big] \end{aligned} \quad (4.12)$$

where we've integrated out both D-term and F-terms so the scalar potential takes the form (4.9).

When we first met QED in the lectures on Quantum Field Theory, we coupled a Dirac fermion to a $U(1)$ gauge field. This Dirac fermion contains two chiral fermions, one left-handed ψ and one right-handed $\bar{\chi}$, both with the same charge. If we conjugate the right-handed fermion then it becomes a left-handed fermion χ. We now have two left-handed fermions with equal and opposite charges. That's precisely the fermionic matter content in each flavour in (4.12).

Adding Further Terms

There are further terms that we can add to the action (4.12) (or, indeed, to the more general action (4.8)). We can add any superpotential $W(\Phi)$ provided that it is gauge invariant. For example, we can always add to (4.12) the superpotential

$$W(\Phi, \tilde{\Phi}) = \sum_{i=1}^{N} m_i \tilde{\Phi}_i \Phi_i$$

This gives a mass $|m_i|$ to each chiral multiplet. In particular, the fermions get a Dirac mass. Note that such mass terms are only possible if there are pairs of chiral superfields with opposite charges.

There is one further, slightly curious term that we can add. This is known as the *Fayet-Iliopoulos term*,

$$\mathcal{L}_{\rm FI} = \int d^4\theta\ 2\zeta V = \zeta D \tag{4.13}$$

It is gauge invariant because D doesn't shift under the generalised gauge symmetry (4.2). Here $\zeta \in \mathbb{R}$ is the Fayet-Ilipoulos, or FI, parameter. Since this multiplies the D-term, it changes only the scalar potential (4.9) which becomes

$$V(\phi) = \frac{e^2}{2}\left(\sum_{i=1}^{N} q_i|\phi_i|^2 - \zeta\right)^2$$

In particular, supersymmetric vacua with $V(\phi) = 0$ now require some scalar field to get a non-vanishing expectation value which, in turn, breaks the $U(1)$ gauge symmetry.

4.2 Non-Abelian Gauge Theories

We can repeat everything above for non-Abelian gauge fields. We work with a gauge group G with Lie algebra

$$[T^A, T^B] = if^{ABC}T^C$$

The factor of i in the commutation relations ensures that the generators are Hermitian, so $(T^A)^\dagger = T^A$. We normalise the generators in the fundamental (i.e. minimal) representation as

$$\text{Tr}\, T^A T^B = \frac{1}{2}\delta^{AB} \tag{4.14}$$

In what follows, generators T^A will always be taken to be in the fundamental representation. If we need generators in other representations R then we will denote them as T^A_R. In these lectures we will mostly work with

$$G = SU(N_c)$$

with the subscript on N_c short for the number of "colours". We'll also mention results for other gauge groups as we go and, for now, keep things general.

4.2.1 Super Yang-Mills

Constructing supersymmetric Yang-Mills theory is slightly more fiddly version of what we did for Maxwell theory. We introduce a real superfield V in the adjoint of the gauge group. As usual, we can view an object in the adjoint representation as living in the Lie algebra by writing

$$V = V^A T^A \quad A = 1, \dots, \dim G$$

For $G = SU(N_c)$, if we take T^A to be in the fundamental representation then this means that V is an $N_c \times N_c$ matrix. In terms of the components, we have a gauge field, but this is now accompanied by a fermion λ and auxiliary field D, both of which must also sit in the adjoint representation. Equivalently, all of them naturally live in the Lie algebra

$$A_\mu = A_\mu^A T^A \quad , \quad \lambda_\alpha = \lambda_\alpha^A T^a \quad , \quad D = D^A T^A$$

Again, for $SU(N_c)$ this means that each of these should be thought of as an $N_c \times N_c$ matrix (in addition to any vector or spinor index they carry). The fermion is again called a *gaugino* or sometimes a *gluino*.

We again want to generalise the usual non-Abelian gauge symmetry to something that can act on a superfield. We do this by taking an adjoint valued chiral superfield

$$\Omega = \Omega^A T^A$$

Since Ω is in the Lie algebra, $e^{i\Omega} \in G$ and this acts on the real superfield as

$$e^{2V} \to e^{-i\Omega^\dagger} e^{2V} e^{i\Omega}$$

From the Baker-Cambell-Hausdorff formula, $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}$, we get the transformation law for the superfield itself

$$V \to V + \frac{i}{2}(\Omega - \Omega^\dagger) - \frac{i}{2}[V, \Omega + \Omega^\dagger] + \dots$$

We can use the shift that appears in the first term to once again go to Wess-Zumino gauge where the real superfield takes the form (4.4), now with all fields in the adjoint of G. You can check that the remaining gauge symmetry acts on A_μ in the usual way,

$$A_\mu \to U A_\mu U^{-1} + iU\partial_\mu U^{-1}$$

with $U \in G$. The field strength lives in a chiral multiplet, defined as

$$W_\alpha = -\frac{1}{8}\bar{\mathcal{D}}^2 \left(e^{-2V} \mathcal{D}_\alpha e^{2V}\right)$$

Evaluated in Wess-Zumino gauge, we use the fact that $V^3 = 0$, as in (4.6), to expand $e^{2V} = 1 + 2V + 2V^2$. A short calculation then shows that

$$
\begin{aligned}
W_\alpha(y,\theta) &= -\frac{1}{4}\bar{\mathcal{D}}^2\left(\mathcal{D}_\alpha V - [V, \mathcal{D}_\alpha V]\right) \\
&= \lambda_\alpha(y) + \theta_\alpha D(y) + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) - i\theta^2 \sigma^\mu_{\alpha\dot{\beta}} \mathcal{D}_\mu \bar{\lambda}^{\dot{\beta}}(y)
\end{aligned}
$$

with the non-Abelian field strength and covariant derivative defined by

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \quad \text{and} \quad \mathcal{D}_\mu\lambda = \partial_\mu - i[A_\mu, \lambda]
$$

To construct the action, we again define the complexified gauge coupling

$$
\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2} \tag{4.15}
$$

The action is then given by

$$
\begin{aligned}
S_{\text{SYM}} &= -\int d^4x \ \text{Tr}\left[\int d^2\theta \ \frac{i\tau}{8\pi} W^\alpha W_\alpha + \text{h.c.}\right] \\
&= \int d^4x \ \text{Tr}\Big[-\frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\vartheta}{16\pi^2} F_{\mu\nu}{}^\star F^{\mu\nu} - \frac{2i}{g^2}\lambda\sigma^\mu\mathcal{D}_\mu\bar{\lambda} + \frac{1}{g^2}D^2\Big]
\end{aligned} \tag{4.16}
$$

This is *super Yang-Mills*. After all that work, it's actually a very simple theory: just Yang-Mills coupled to a single, adjoint Weyl fermion. The factor of 2 differences compared to the Maxwell action (4.7) can be traced to the normalisation convention (4.14).

4.2.2 Supersymmetric QCD

We can add matter transforming in any representation R of the gauge group. The matter sits, as always, in a chiral superfield Φ that now transforms as

$$
\Phi \to \exp\left(-2i\Omega^A T^A_R\right)\Phi \tag{4.17}
$$

We construct a gauge invariant, supersymmetric action with the superfield expression

$$
\begin{aligned}
\int d^4x\, d^4\theta \ \Phi^\dagger e^{2V}\Phi = \ &\mathcal{D}_\mu\phi^\dagger\mathcal{D}^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\mathcal{D}_\mu\psi + F^\dagger F \\
&- \sqrt{2}(\bar{\psi}\lambda^A T^A_R\phi + \phi^\dagger\lambda^A T^A_R\psi) + \phi^\dagger D^A T^A_R\phi
\end{aligned}
$$

Here the covariant derivatives include the gauge field transforming in the appropriate representation R.

Again, various anomaly cancellation conditions must be satisfied when coupling Weyl fermions to non-Abelian gauge groups in complex representations. The simplest way forward is to work instead with Dirac fermions. This means that we take pairs of chiral superfields, Φ transforming in some representation R and $\tilde{\Phi}$ in the conjugate representation $\bar{R}$. (In much of the literature, these superfields are denoted Q and $\tilde{Q}$ but we'll stick with Φ and $\tilde{\Phi}$ to avoid any unnecessary confusion with the supercharges.)

The most common is to take R to be the fundamental representation. We could, for example, consider $G = SU(N_c)$ gauge group with N_f flavours of fermions, each in the fundamental representation. The action is then

$$\begin{aligned} S_{\rm SQCD} = \int d^4x \ {\rm Tr}\Big[-\frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\vartheta}{16\pi^2} F_{\mu\nu}{}^\star F^{\mu\nu} - \frac{2i}{g^2}\lambda\sigma^\mu\mathcal{D}_\mu\bar{\lambda}\Big] \\ + \sum_{i=1}^{N_f} \Big[|\mathcal{D}_\mu\phi_i|^2 + |\mathcal{D}_\mu\tilde{\phi}|^2 - i\bar{\psi}_i\bar{\sigma}^\mu\mathcal{D}_\mu\psi_i - i\tilde{\psi}_i\sigma^\mu\mathcal{D}_\mu\bar{\tilde{\psi}}_i\Big] \\ - \sqrt{2}\sum_{i=1}^{N_f}\Big[\phi_i^\dagger\lambda\psi_i - \tilde{\phi}_i\bar{\lambda}\bar{\tilde{\psi}}_i + \text{h.c.}\Big] - V(\phi,\tilde{\phi}) \end{aligned} \tag{4.18}$$

Here the covariant derivatives are

$$\mathcal{D}_\mu\phi = \partial_\mu\phi - iA_\mu\phi \quad \text{and} \quad \mathcal{D}_\mu\psi = \partial_\mu\psi - iA_\mu\psi$$

for the fields in the fundamental representation, and

$$\mathcal{D}_\mu\tilde{\phi} = \partial_\mu\tilde{\phi} + i\tilde{\phi}A_\mu \quad \text{and} \quad \mathcal{D}_\mu\tilde{\psi} = \partial_\mu\tilde{\psi} + i\tilde{\psi}A_\mu$$

for those in the anti-fundamental representation. Finally, the scalar potential is again given by the D-terms

$$V(\phi,\tilde{\phi}) = \frac{1}{2g^2}D^AD^A \quad \text{with} \quad D^A = -g^2\sum_{i=1}^{N_f}\Big(\phi_i^\dagger T^A\phi_i - \tilde{\phi}_iT^A\tilde{\phi}_i^\dagger\Big) \tag{4.19}$$

with T^A the $N_c \times N_c$ generators in the fundamental representation. This is the action of *supersymmetric QCD*, or SQCD for short. In a nod to the real world, we refer to the fermions ψ and $\tilde{\psi}$ as *quarks*. Their supersymmetric scalar partners ϕ and $\tilde{\phi}$ are called *squarks*.

Once again, we can also add masses for the quark multiplets by including the gauge invariant superpotential

$$\mathcal{W}(\Phi,\tilde{\Phi}) = \sum_{i=1}^{N_f} m_i\tilde{\Phi}_i\Phi_i$$

This gives an extra term to the scalar potential

$$\delta\mathcal{L}_{\rm mass} = -\sum_{i=1}^{N_f} |m_i|^2 \left(|\phi_i|^2 + |\tilde{\phi}_i|^2\right)$$

as well as Dirac masses for ψ_i and $\tilde{\psi}_i$.

There is no FI parameter that we can add for non-Abelian theories. The non-Abelian analog of (4.13) would involve $\mathrm{Tr}\, D$ but the trace of the generators of any non-Abelian Lie algebra always vanishes. Fayet-Iliopoulos terms can only be introduced for $U(1)$ gauge theories.

4.3 The Moduli Space of Vacua

In the absence of a superpotential, supersymmetric gauge theories do not have a unique ground state. Instead, the D-term potential has a flat direction with $V(\phi) = 0$. This is the moduli space of vacua. It will turn out that this moduli space holds the key to understanding the quantum dynamics of supersymmetric gauge theories. For this reason, we will spend some time studying its structure.

Consider, for example, $U(1)$ SQED with a single flavour. If we don't turn on a FI parameter then the D-term is (4.12)

$$D = -g^2(|\phi|^2 - |\tilde{\phi}|^2)$$

Clearly any solution with

$$|\phi|^2 = |\tilde{\phi}|^2 = v^2$$

has zero energy. To fully specify the classical theory, we must decide where on this moduli space we want to sit.

At all points on the moduli space, there are always massless particles. Indeed, the low-energy physics is dominated by the fluctuations along the moduli space, which always correspond to massless particles, together with their fermionic superpartners.

Meanwhile, the masses of heavy particles typically depend on where you sit on the moduli space which, in the current example, means that value of v^2. Because ϕ is charged under the $U(1)$ gauge field, when it gets an expectation value, the Higgs mechanism kicks in and the photon gets a mass of order

$$m_\gamma^2 \sim e^2 v^2$$

But the Yukawa terms in (4.12) mean that a particular combination of fermions also gets a mass, given by

$$m_{\text{fermion}} \sim ev$$

The fact that this is the same as m_γ is, of course, no coincidence: the photon, massive fermion and an additional massive scalar in the spectrum form a massive vector multiplet of the kind discussed in Section 2.3. The origin of the moduli space, at $\phi = \tilde{\phi} = 0$, is special because here the vector multiplet becomes massless.

The Geometry of Moduli Space

We denote the moduli space of vacua as $\mathcal{M}$. As we now explain, this manifold naturally comes with a number of interesting geometric structures.

First $\mathcal{M}$ is defined by the requirement that $V(\phi) = 0$. In the absence of a superpotential, this is equivalent to $D(\phi) = 0$. (Note that here ϕ denotes all chiral multiplet scalars and, for SQED and SQCD, this means both ϕ and $\tilde{\phi}$.). However, we should also remember that the gauge group G acts on these scalars. The gauge symmetry is not really a symmetry of the theory, but rather a redundancy in our description. This means that any two values of ϕ related by a gauge transformation should be viewed as physically equivalent. The upshot is that the vacuum moduli space $\mathcal{M}$ is defined as the quotient

$$\mathcal{M} = \{\phi \,|\, D(\phi) = 0\}/G \tag{4.20}$$

We have stumbled upon a construction known to mathematicians as the *symplectic reduction*. It's particularly natural because, as we've seen above, the D-term constraint $D(\phi) = 0$ is fully specified by the action of the group G. In this way, the group G gets to act twice: once as a constraint, and again as a quotient. Mathematicians call the constraint $D(\phi) = 0$ the *moment map*. If G includes an Abelian factor, the associated FI parameter is known as the *level*.

There are two, further ways to describe the moduli space $\mathcal{M}$. We will now describe these, but won't prove the equivalence with (4.20). Instead, we will content ourselves with some heuristic justification, followed by some examples[2].

[2] A full proof can be found in the paper by Marcus Luty and Wati Taylor, Varieties of vacua in classical supersymmetric gauge theories.

The fact that the group G "acts twice", is even more apparent if the second way of writing the moduli space: it is the holomorphic quotient

$$\mathcal{M} = \{ \phi \} / G_{\mathbb{C}} \tag{4.21}$$

with $G_{\mathbb{C}}$ the *complexified gauge group*. This means that we take the real parameters α that usually specify a gauge transformation – that is $\phi \to e^{iq\alpha}\phi$ for Abelian G or $\phi \to e^{i\alpha^a T^a_R}\phi$ for non-Abelian – and quotient by transformations with $\alpha \in \mathbb{C}$. You should think of the D-term constraint in (4.20) as like a gauge-fixing condition for the non-Hermitian part of the $G_{\mathbb{C}}$ transformations.

In fact, looking back at our construction of supersymmetric gauge theories, the gauge transformations started life in a chiral superfield Ω where everything was complex. They became real only after moving to Wess-Zumino gauge. From the perspective of supersymmetric gauge theory, the equivalence of (4.20) and (4.21) is best seen by looking at the more general gauge transformations before imposing Wess-Zumino gauge.

The final description of the moduli space will, in some circumstances, turn out to be the most useful. The manifold $\mathcal{M}$ can alternatively be viewed as

$$\mathcal{M} = \{\text{Gauge invariant, holomorphic monomials}\} \,/\, \{\text{Algebraic relations}\} \tag{4.22}$$

This is a description of $\mathcal{M}$ in terms of what mathematicians call an *algebraic variety*. This definition is best elucidated by examples that we will turn to below, but here we give the basic gist.

There are three key ideas that we need to explain in this definition: gauge invariant, holomorphic, and the algebraic relations. We cover each in turn:

- Because gauge symmetry is merely a redundancy in our choice of description, it should be possible to describe the dynamics of massless particles in terms of some gauge invariant fields. This is the basic idea underlying the characterisation (4.22)

- It's always possible to build such gauge invariant fields by taking combinations like $\phi^\dagger\phi$. These are invariant under G, but not invariant under the larger $G_{\mathbb{C}}$ that defines the moduli space according to (4.21). The need to impose invariance under $G_{\mathbb{C}}$, or equivalently the need to impose the D-term constraint $D = 0$, means that we should work with holomorphic gauge invariant combinations, meaning monomials that involve ϕ alone and not $\phi^\dagger$. Alternatively, and more physically, supersymmetry means that we should be able to describe the fields in terms of chiral multiplets, and these are necessarily holomorphic.

- Finally, it will turn out that, for some examples, not all of the gauge invariant combinations are independent. This is why there is the need to quotient by certain relations between them. This is best illustrated when we turn to examples below.

Mathematically, the equivalence between the quotient constructions (4.20) and (4.21) and the algebraic description (4.22) goes by the name of *geometric invariant theory.*

4.3.1 The Moduli Space of SQED

We'll start by looking at the simpler case of SQED. This is a $U(1)$ gauge theory coupled to N flavours. If we set the FI parameter to zero for now, then the D-term condition is (4.12)

$$\sum_{i=1}^{N} |\phi^i|^2 - |\tilde{\phi}_i|^2 = 0 \tag{4.23}$$

In addition, we should quotient by the $U(1)$ gauge action

$$\phi^i \to e^{i\beta}\phi^i \quad \text{and} \quad \tilde{\phi}_i \to e^{-i\beta}\tilde{\phi}_i \tag{4.24}$$

We started with $2N$ fields ϕ and $\tilde{\phi}$. There is one real constraint (4.23) which, together with the quotient (4.24) reduces the *complex* dimension of the vacuum moduli space by one. We then have

$$\dim \mathcal{M} = 2N - 1 \tag{4.25}$$

Let's see how to reproduce this counting when thinking of $\mathcal{M}$ as an algebraic variety defined by (4.22). The gauge invariant monomial are the bilinears

$$M_j{}^i = \tilde{\phi}_j \phi^i \tag{4.26}$$

We will refer to these, not entirely accurately, as "mesons". There are N^2 such fields and, at first glance, it looks like we have way too many. However, they are not all independent and this is where the algebraic relations in (4.22) come into play.

The meson matrix M is built from vectors ϕ and $\tilde{\phi}$ and so has, at most, rank 1. This means that there are $N-1$ eigenvalues that are guaranteed to vanish. In general, the determinant of an $N \times N$ matrix A can be written as

$$\epsilon_{i_1 \ldots i_N} A^{i_1}{}_{j_1} \ldots A^{i_N}{}_{j_N} = \det A \, \epsilon_{j_1 \ldots j_N}$$

The rank 1 matrix M therefore obeys

$$\epsilon_{i_1 \ldots i_N} (M_{j_1}{}^{i_1} - \lambda \delta_{j_1}{}^{i_1}) \ldots (M_{j_N}{}^{i_N} - \lambda \delta_{j_N}{}^{i_N}) = \det(M - \lambda)\, \epsilon_{j_1 \ldots j_N} = \lambda^{N-1}(\lambda - \lambda_0) \epsilon_{j_1 \ldots j_N}$$

This tells us that if we expand out the left-hand side, all terms of order λ^{N-2} and lower must vanish for a rank 1 matrix. In other words, we have the constraints

$$\epsilon_{i_1\ldots i_N} M_{j_1}^{\ i_1} M_{j_2}^{\ i_2} = 0 \tag{4.27}$$

with all other constraints following by contracting with further $M_j^{\ i}$. Our next task is to count how many independent constraints we have here. The $i_3, \ldots, i_N$ indices are left hanging so by picking these we can restrict i_1 and i_2 to run over any pair. But the resulting constraints aren't all independent. For example, there is a constraint that arises from $(i_1, i_2) = (1, 2)$ and another that arises from $(i_1, i_2) = (1, 3)$. But dividing the first constraint by the second, and rearranging, gives the constraint that arise from $(i_1, i_2) = (2, 3)$. In fact, it's not hard to convince yourself that the constraints that come from $(i_1, i_2) = (1, \text{anything but } 1)$ are independent and sufficient to give all others. Clearly there are $N - 1$ of these.

For each of these constraints, we still have the (j_1, j_2) indices hanging. These too are anti-symmetrised and the same argument that we gave above for (i_1, i_2) also holds for (j_1, j_2). This means that the total number of constraints from (4.27) is $(N-1)^2$. The algebraic variety $\mathcal{M}$, defined by all mesons (4.26) subject to the constraints (4.27) then has complex dimension

$$\dim \mathcal{M} = N^2 - (N-1)^2 = 2N - 1$$

in agreement with our earlier counting (4.25).

The Metric on the Vacuum Moduli Space

The vacuum moduli space inherits a natural metric. Indeed, if we restrict to very low energies the dynamics is that of the massless fields, corresponding to fluctuations along the moduli space. This is the realm of the non-linear sigma model that we discussed in Section 3.2.4. On general grounds, we know that not only is there a metric on $\mathcal{M}$ but this metric must be Kähler.

It is straightforward to compute this metric. Here we do it in two different ways for the simplest case of $N = 1$ flavour. The easiest way to proceed is to start with the Kähler potential

$$K = \phi^\dagger \phi + \tilde{\phi}^\dagger \tilde{\phi}$$

Note that the Kähler potential for a gauge theory involves terms like e^{2qV}, with V the real superfield, to ensure gauge invariance. We simply set the gauge fields to zero in the following calculation, so the Kähler potential is the canonical one above. Restricting to the moduli space (4.23), we have $|\phi|^2 = |\tilde{\phi}|^2$. Furthermore, if we work with the meson field $M = \tilde{\phi}\phi$, the Kähler potential becomes

$$K = 2|\phi|^2 = 2\sqrt{M^\dagger M} \tag{4.28}$$

The associated metric is just

$$ds^2 = \frac{|dM|^2}{2|M|} \tag{4.29}$$

We see immediately that the metric is singular at the origin $M = 0$. This singularity is telling us something important: when $\phi = \tilde{\phi} = 0$, there are new massless degrees of freedom. This is simply the photon and its superpartner which become massless at the origin because the Higgs mechanism turns off.

This is a lesson that we've seen before. When we integrated out heavy fields in Section 3.3, we found that the low-energy effective theory had singularities at points where the heavy fields became light. This is a general feature of low-energy effective theories, and one that will be important in Section 6 when we come to discuss the quantum dynamics of these theories. For now, the lesson is worth repeating one more time: singularities in the low-energy effective action signal the emergence of new, massless degrees of freedom.

There is a more prosaic way to do this same calculation that highlights our original quotient description of the vacuum moduli space (4.20). The general solution to the constraint (4.23) is

$$\phi = ve^{i\alpha}e^{i\beta} \quad \text{and} \quad \tilde{\phi} = ve^{i\alpha}e^{-i\beta}$$

with $v > 0$. The $e^{\pm i\beta}$ has been taken to coincide with the gauge action (4.24), so that v and α provide the coordinates on the moduli space $\mathcal{M}$.

At this point, there's an important factor of 2 that we have to take care of. The parameter β corresponding to the $U(1)$ gauge transformation has range $\beta \in [0, 2\pi)$. In contrast, we have $\alpha \in [0, \pi)$. This follows because we can always implement a gauge transformation with $\beta = \pi$ which flips the sign of ϕ and $\tilde{\phi}$ or, equivalently, takes $\alpha \to \alpha + \pi$.

The metric on $\mathcal{M}$ is inherited from the kinetic terms for the scalar fields. To this end, we promote v, α and β to fields that vary slowly over spacetime. The covariant derivatives are

$$\mathcal{D}_\mu \phi = (\partial_\mu v + iv(\partial_\mu \alpha + \partial_\mu \beta - A_\mu))\, e^{i(\alpha+\beta)}$$
$$\mathcal{D}_\mu \tilde{\phi} = (\partial_\mu v + iv(\partial_\mu \alpha - \partial_\mu \beta + A_\mu))\, e^{i(\alpha-\beta)}$$

We now choose $A_\mu = \partial_\mu \beta$ to absorb the variation of β. This how the quotient in (4.20) manifests itself in this calculation. The kinetic terms for the scalar fields, restricted to the vacuum moduli space, then become

$$\mathcal{L}_{\text{eff}} = |\mathcal{D}\phi|^2 + |\mathcal{D}\tilde{\phi}|^2 = 2\Big[\partial v^2 + v^2\, \partial\alpha^2\Big] \tag{4.30}$$

which we interpret as a metric like the non-linear sigma models (3.25) we discussed earlier. It's straightforward to check that this coincides with the metric (4.29) written in terms of the meson field.

At first glance, (4.30) looks like a flat metric. And, indeed, it is. But it's not the flat metric on $\mathbb{C}$ because the angular coordinate α doesn't have periodicity 2π. Instead, it's the flat metric on $\mathbb{C}/\mathbb{Z}_2$ and has a conical singularity at the origin $v = 0$. This how we see the emergence of the massless photon at this point.

Turning on the FI Parameter

A small variation on this calculation provides yet another perspective on the importance of singularities in the low-energy effective action. We again consider SQED with $N = 1$ flavour, but this time turn on a FI parameter. The D-term constraint now reads

$$|\phi|^2 - |\tilde{\phi}|^2 = \zeta \tag{4.31}$$

We assume that $\zeta \geq 0$. In the ground state, we necessarily have $|\phi|^2 \neq 0$ meaning that the photon now gets a mass on all points of the moduli space.

We can see how this manifests itself in the moduli space metric. The condition (4.31) is solved by

$$\phi = \sqrt{v^2 + \zeta}\, e^{i\alpha} e^{i\beta} \quad \text{and} \quad \tilde{\phi} = v e^{i\alpha} e^{-i\beta}$$

Our previous calculation to compute the metric on $\mathcal{M}$ is now a little more involved. The subtlety lies in figuring out what expression we should take for the gauge field A_μ. The answer can be found in its equation of motion. Or, more precisely, the equation of

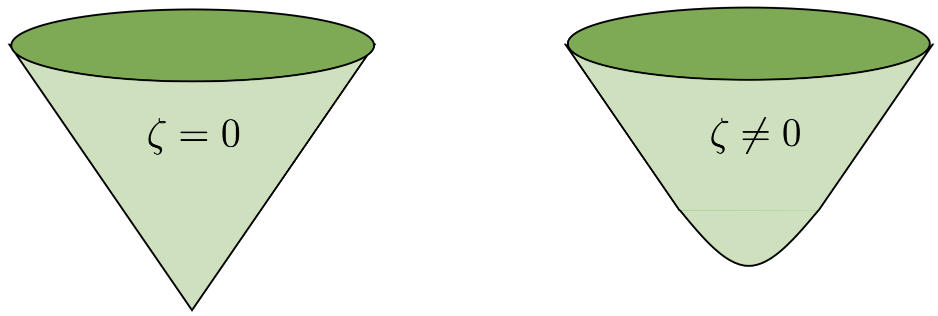

Figure 5. The moduli space of SQED. When $\zeta = 0$, the moduli space is the singular cone $\mathbb{C}/\mathbb{Z}_2$ shown on the left. The singularity at the origin reflects the existence of the massless photon. When $\zeta \neq 0$ the singularity is resolved and the moduli space is the smooth cone shown on the right. Now the photon is Higgsed everywhere on the moduli space.

motion in the limit $e^2 \to \infty$ where we neglect the Maxwell term. This is the appropriate limit when the gauge field responds immediately to fluctuations in the scalar and gives

$$A_\mu = \frac{\zeta}{2v^2 + \zeta}\,\partial_\mu \alpha + \partial_\mu \beta$$

It reduces to our previous, pure gauge, choice when $\zeta = 0$. Inserting this expression into the kinetic terms for ϕ and $\tilde{\phi}$, we compute the metric on the vacuum moduli space

$$\mathcal{L}_{\rm eff} = |\mathcal{D}\phi|^2 + |\mathcal{D}\tilde{\phi}|^2 = \frac{2v^2+\zeta}{v^2+\zeta}\left[\partial v^2 + \frac{4v^2(v^2+\zeta)^2}{(2v^2+\zeta)^2}\partial\alpha^2\right] \tag{4.32}$$

Importantly, as we approach the origin, $v^2 \to 0$, the metric is well approximated by

$$ds^2 \approx dv^2 + 4v^2 d\alpha^2 = dv^2 + v^2 d(2\alpha)^2$$

That extra factor of 2 makes all the difference! We now get the flat metric with the angular coordinate $2\alpha \in [0, 2\pi)$ which means that close to $v = 0$ the metric really does look like flat space. The resulting moduli space is sketched in Figure 5.

4.3.2 The Moduli Space of SQCD

We now play the same game for SQCD. We will take gauge group

$$G = SU(N_c)$$

coupled to N_f fundamental flavours, ϕ^i_a in the fundamental representation and $\tilde{\phi}^a_i$ in the anti-fundamental. Here $a = 1, \ldots, N_c$ labels is the gauge group index while $i = 1, \ldots N_f$ is the flavour index.

The generators $(T^A)^a{}_b$ in the fundamental representation are the set of Hermitian, traceless, complex $N_c \times N_c$ matrices. Meanwhile, the generators in the anti-fundamental representation are simply $\bar{T}^A = -T^A$. The $N_c^2 - 1$ D-term conditions (4.19) are then

$$\phi_i^\dagger T^A \phi^i - \tilde{\phi}_i T^A \tilde{\phi}^{\dagger\, i} = 0 \qquad A = 1, \ldots N_c^2 - 1$$

where there is an implicit sum over $i = 1, \ldots, N_f$. To get a better sense of these constraints, let us first relax the requirement that T^A is traceless. (This is what we would get if the gauge group was $U(N_c)$ rather than $SU(N_c)$.) In this case, the T^A provide a basis for all Hermitian matrices and the D-term condition is N_c^2 constraints

$$\phi_i^{\dagger\, a} \phi_b^i - \tilde{\phi}_i^a \tilde{\phi}_b^{\dagger\, i} = 0 \qquad a, b = 1, \ldots N_c \;\; \text{for } U(N_c)$$

But the fact that we're working with $SU(N_c)$ rather than $U(N_c)$ means that there's no reason to set the trace to zero. So our true D-term constraint is

$$\phi_i^{\dagger\, a} \phi_b^i - \tilde{\phi}_i^a \tilde{\phi}_b^{\dagger\, i} = \frac{1}{N_c} \left(\phi_i^{\dagger\, c} \phi_c^i - \tilde{\phi}_i^c \tilde{\phi}_c^{\dagger\, i} \right) \delta^a{}_b \qquad (4.33)$$

At first glance, this looks like it's still N_c^2 conditions. But if you take the trace then you find that both sides are trivially equal. This means that, in fact, it's only $N_c^2 - 1$ conditions, with no condition on the trace. This is what we wanted.

To understand the vacuum moduli space, we must first solve the equations (4.33). As we will now see, the nature of the solutions is different for $N_f < N_c$ and $N_f \geq N_c$. We deal with each in turn.

$\underline{N_f < N_c}$

We'd like to count the dimension of the moduli space $\mathcal{M}$, defined by (4.33) modulo gauge transformations. It's tempting to think that there are just $N_c^2 - 1$ constraints in (4.33) but how do we know that they are all independent? In fact, it's simple to see that these constraints cannot all be independent when $N_f < N_c$ because then we would have more constraints than degrees of freedom. Yet solutions to (4.33) certainly exist! To proceed, we use the fact that the D-terms and gauge symmetry are closely entwined. The D-terms only bite when the gauge symmetry does.

When $N_f < N_c$, we can always use an $SU(N_c)$ gauge transformations and $SU(N_f)$ flavour rotations to put the matrix ϕ in the block-diagonal form

$$\phi_a^i = \begin{pmatrix} v_1 & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & v_{N_f} \\ \hdashline 0 & \ldots & 0 \end{pmatrix} \qquad (4.34)$$

Here the columns have length N_c and the rows length N_f. We can then use the other $SU(N_f)$ to rotate $\tilde{\phi}$ to be in upper-diagonal form. (We can't make it fully diagonal because we've already used up the $SU(N_c)$ to diagonalise ϕ). However, now we invoke the D-term conditions (4.33). The only solutions to these conditions require that the off-diagonal terms in $\tilde{\phi}$ vanish. (You could check this for a simple case, say $N_c = 3$ and $N_f = 2$ to get a feel for why this is the case.) We're left with

$$\tilde{\phi}^{\dagger\, i}_a = \phi^i_a$$

As before, points on the moduli space related by a gauge transformation are to be physically identified. On a generic point on the moduli space (with $v_i \neq v_j \neq 0$ when $i \neq j$) the gauge group is broken to

$$SU(N_c) \to SU(N_c - N_f)$$

The number of broken gauge generators is then

$$\#\text{ broken generators} = (N_c^2 - 1) - ((N_c - N_f)^2 - 1)$$

Each of these is eaten by one of the original $2N_cN_f$ bosons ϕ and $\tilde{\phi}$. This means that the resulting vacuum moduli space has complex dimension

$$\dim \mathcal{M} = 2N_cN_f - [\#\text{ broken generators}] = N_f^2$$

Note that we only divide out by the points on the moduli space related by the $SU(N_c)$ gauge symmetry. There will still be points on the moduli space related by the flavour symmetry $SU(N_f)$ but these are physically distinct vacua.

We can also view the moduli space as an algebraic variety. Once again, the holomorphic monomials are the meson fields

$$M_j{}^i = \tilde{\phi}^a_j \phi^i_a \tag{4.35}$$

This time the name "meson" is more appropriate: we have contracted the gauge indices of ϕ and $\tilde{\phi}$ to form a gauge invariant composite. The mesons form N_f^2 fields but, in contrast to SQED, there is no constraint on M. The contracted gauge indices in (4.35) run over $a = 1, \ldots, N_c > N_f$ so there is no obstacle to M being maximal rank. We see immediately that $\dim \mathcal{M} = N_f^2$, in agreement with our result above.

We can compute the metric on $\mathcal{M}$ along the same lines as we saw for SQED. The Kähler potential is

$$K = \phi_i^{\dagger\,a}\phi_a^i + \tilde{\phi}_i^a\tilde{\phi}_a^{\dagger\,i}$$

We want to write this in terms of the meson field (4.35). To do this, first note that for $N_f < N_c$ the trace term on the right-hand side of the D-term (4.33) vanishes when restricted to the moduli space and we have

$$\phi_i^{\dagger\,a}\phi_b^i = \tilde{\phi}_i^a\tilde{\phi}_b^{\dagger\,i} \tag{4.36}$$

From this, we have

$$(M^\dagger M)_j^{\ i} = \tilde{\phi}_a^{\dagger\,i}\phi_k^{\dagger\,a}\phi_b^k\tilde{\phi}_j^b = (\tilde{\phi}_a^{\dagger\,i}\tilde{\phi}_k^a)(\tilde{\phi}_b^{\dagger\,k}\tilde{\phi}_j^b)$$

where, in the last equality, we've used (4.36). Taking the square root of this matrix equation tells us that $(\tilde{\phi}^\dagger\tilde{\phi})^i_{\ j} = (\sqrt{M^\dagger M})^i_{\ j}$, and so the Kähler potential is

$$K = 2\,\mathrm{Tr}\sqrt{M^\dagger M} \tag{4.37}$$

Just like the Kähler potential for SQED (4.28), the resulting metric will have singularities whenever M^{-1} ceases to exist. Again, these singularities correspond to new degrees of freedom becoming massless. At a generic point on the moduli space, there will be massless gauge bosons associated to the unbroken $SU(N_c - N_f)$ gauge symmetry. But along the loci on which M is not invertible we have an enhancement of the gauge group and new massless gauge bosons.

<u>$N_f \geq N_c$</u>

For $N_f \geq N_c$, the story is different. First, we can now use $SU(N_c)$ and $SU(N_f)$ transformations to find solutions to the D-term equations (4.33), again in block-diagonal form

$$\phi_a^i = \left(\begin{array}{ccc|c} v_1 & \ldots & 0 & 0 \\ & \ddots & & \vdots \\ 0 & \ldots & v_{N_c} & 0 \end{array}\right) \quad \text{and} \quad \tilde{\phi}_a^{\dagger\,i} = \left(\begin{array}{ccc|c} \tilde{v}_1 & \ldots & 0 & 0 \\ & \ddots & & \vdots \\ 0 & \ldots & \tilde{v}_{N_c} & 0 \end{array}\right)$$

with

$$|v_a|^2 = |\tilde{v}_a|^2 + \rho \qquad a = 1, \ldots, N_c$$

where ρ must be independent of a. This reflects the fact that the trace term on the right-hand side of (4.33) can now be non-zero.

At a generic point on $\mathcal{M}$, the $SU(N_c)$ gauge symmetry is completely broken. The complex dimension of the moduli space is therefore

$$\dim \mathcal{M} = 2N_c N_f - (N_c^2 - 1) \tag{4.38}$$

How can we describe this moduli space as an algebraic variety? The meson fields (4.35) provide N_f^2 degrees of freedom, but now there are constraints of the kind we met for SQED since M is at most rank N_c. In addition, there are also new gauge invariant fields. These are *baryons*, built from the totally anti-symmetric invariant tensor of $SU(N_c)$,

$$\begin{aligned} B^{i_1 \ldots i_{N_c}} &= \phi^{i_1}_{a_1} \ldots \phi^{i_{N_c}}_{a_{N_c}} \epsilon^{a_1 \ldots a_{N_c}} \\ \tilde{B}_{i_1 \ldots i_{N_c}} &= \tilde{\phi}^{a_1}_{i_1} \ldots \tilde{\phi}^{a_{N_c}}_{i_{N_c}} \epsilon_{a_1 \ldots a_{N_c}} \end{aligned}$$

Each of these is anti-symmetric in the N_c different flavour indices $i_1, \ldots, i_{N_c}$. There are then a bunch of further constraints between these baryons and mesons. Rather than doing this in full generality, we'll instead just describe how this works for the two cases that will prove most interesting in Section 6.

- $\underline{N_f = N_c}$: In this case, anti-symmetry properties mean that there is just a single baryon of each type

  $$B = \phi^1_{a_1} \ldots \phi^{N_c}_{a_{N_c}} \epsilon^{a_1 \ldots a_{N_c}} \quad \text{and} \quad \tilde{B} = \tilde{\phi}^{a_1}_1 \ldots \tilde{\phi}^{a_{N_c}}_{N_c} \epsilon_{a_1 \ldots a_{N_c}}$$

  The meson M can have rank N_f, so there are no constraints there. But there is a single relation between the mesons and baryons, given by

  $$\tilde{B}B = \det M \tag{4.39}$$

  This means that there are $N_f^2 + 2$ degrees of freedom in M, B and $\tilde{B}$ and a single relation, giving a moduli space of dimension $\dim \mathcal{M} = N_f^2 + 1$ in agreement with (4.38). The relation (4.39) will play a starring role when we come to consider the quantum theory in Section 6.3.

- $\underline{N_f = N_c + 1}$: Now there are N_f baryons of each type,

  $$B_j = \epsilon_{j i_1 \ldots i_{N_c}} B^{i_1 \ldots i_{N_c}} \quad \text{and} \quad \tilde{B}^j = \epsilon^{j i_1 \ldots i_{N_c}} \tilde{B}_{i_1 \ldots i_{N_c}}$$

  This time the constraints are less obvious, but they turn out to be

  $$\det M (M^{-1})^i{}_j = B^i \tilde{B}_j \quad \text{and} \quad M_j{}^i B^j = M_j{}^i \tilde{B}_i = 0 \tag{4.40}$$

At this point, things start to get a little messy! It turns out that not all these relations are independent, but there's no way to write them as a smaller set. Mathematicians say that the resulting variety is not a *complete intersection*. We'll simply duck the issue which, it turns out, will not hinder us from understanding the physics.

There is one sense in which the use of the words "mesons" and "baryons" might be misleading. In QCD, mesons and baryons are bound states of quarks, stuck together because of confinement. But confinement is a surprising and poorly understood property of the quantum theory. Here we are not invoking anything so dramatic. Indeed, we haven't yet discussed any quantum effects and what we've call SQCD might better be called SCCD for our current purposes. Instead, we're using meson and baryon fields simply because they are gauge invariant and so free of any gauge redundancy. We'll turn on the Q in SQCD in Section 6 where we'll see how this tallies with ideas of confinement.

4.3.3 Briefly, Gauged Linear Sigma Models in 2d

We've learned that we can construct interesting geometric spaces as the moduli spaces of vacua of supersymmetric gauge theories. This kind of construction goes by the name of *gauged linear sigma models*. It turns out that it's a particularly useful method when wielded in quantum field theories in $d = 1 + 1$ dimensions.

To see why, first consider the action for a non-linear sigma model in general d-dimensional spacetime

$$S = \int d^dx \; g_{ij}(\pi) \, \partial_\mu \pi^i \partial^\mu \pi^j \tag{4.41}$$

Here π^i are coordinates on a manifold $\mathcal{M}$ with metric g_{ij}.

When $d = 0 + 1$, we're dealing with the quantum mechanics of particle moving on $\mathcal{M}$. But we know what happens in this case: the wavefunction will spread over $\mathcal{M}$ and there will typically be a unique ground state.

This is conceptually very different from what happens in $d = 3 + 1$ dimensions. There, each point on $\mathcal{M}$ defines a different ground state of the system. There is no spread of the wavefunction.

The reason for this different behaviour can be traced to the long-distance property of the propagator. The propagator grows in $d = 0 + 1$ and $d = 1 + 1$ dimensions (logarithmically in the latter case) while it decays in $d = 2 + 1$ and higher. This fact is closely related to the *Mermin-Wagner theorem* which says that global symmetries cannot be spontaneously broken in $d = 0 + 1$ and $d = 1 + 1$ dimensions. (We met this theorem in the lectures on Statistical Field Theory and Gauge Theory.)

In the context of non-linear sigma models of the type (4.41), this long-distance behaviour of the propagator is telling us that $d = 0+1$ and $d = 1+1$ dimensions are special because the wavefunction spreads over the manifold $\mathcal{M}$. This means that the ground state of the system has a chance of knowing something about the global structure of the manifold $\mathcal{M}$, like its topology. Indeed, studying the dynamics of low-dimensional quantum systems on $\mathcal{M}$ has been a very fruitful source of developments in mathematics. This beginnings of this story are told in the lectures on Supersymmetric Quantum Mechanics.

The story is particularly rich for theories in $d = 1 + 1$ dimensions where, in addition to the wavefunction spreading over $\mathcal{M}$, the UV divergences of the quantum field theory mean that the metric on $\mathcal{M}$ is renormalised. At one-loop, the running is captured by the beautifully geometric RG equation

$$\mu \frac{\partial g_{ij}}{\partial \mu} = R_{ij} \tag{4.42}$$

where μ is the RG scale and R_{ij} the Ricci tensor. This formula is known as *Ricci flow*. It plays an important role in String Theory and has a number of applications in pure mathematics. Note that the flow stops only if the metric becomes Ricci flat, with $R_{ij} = 0$. At this point we have a 2d conformal field theory. However, not all manifolds admit such a Ricci flat metric.

Things become even more interesting when we throw supersymmetry into the mix. This is what we called $\mathcal{N} = (2, 2)$ supersymmetry in Section 2.4.3. It not only gives us an important level of control over the dynamics but, as we've seen already in these lectures, dovetails nicely with some interesting mathematical structures. It turns out that the gauge theory approach to realising non-linear sigma models as the vacuum moduli space is particularly powerful in this context. Here we just give a hint of how this works

First, the anomaly cancellation conditions (4.11) are for 4d quantum field theories and are not needed in two dimensions. (A 4d Weyl fermion reduces to a 2d Dirac fermion and so the theories we construct are not chiral in 2d.) This means that there is nothing to stop us considering $U(1)$ coupled to N chiral multiplets of charge $+1$ in

$d = 1 + 1$ dimensions. The D-term condition is

$$\sum_{i=1}^{N} |\phi_i|^2 = \zeta$$

where we turn on a FI parameter $\zeta > 0$. Taken on its own, this condition defines a sphere $\mathbf{S}^{2N-1}$. But we still have to quotient by the $U(1)$ action to get the vacuum moduli space and this gives

$$\mathcal{M} = \mathbf{S}^{2N-1}/U(1) = \mathbb{CP}^{N-1}$$

Here $\mathbb{CP}^{N-1}$ is complex projective space, defined as the space of complex lines in $\mathbb{C}^N$. This can also be seen in the definition (4.21) of the moduli space.

Things get more interesting if we add, in addition, a chiral superfield P with charge $-q$. The D-term condition is now

$$D = \sum_{i=1}^{N} |\phi_i|^2 - q|p|^2 - \zeta = 0$$

After quotienting by the $U(1)$ action, the vacuum moduli space is a non-compact manifold. But we now have the option of introducing a gauge invariant superpotential

$$W(P, \Phi) = PG(\Phi_1, \ldots, \Phi_N)$$

with G a homogeneous polynomial of degree q. The potential energy now also includes contributions from the F-terms

$$V_F = |p|^2 \sum_{i=1}^{N} \left| \frac{\partial G}{\partial \phi_i} \right|^2 + |G|^2$$

If we choose G to be *transverse*, meaning

$$\frac{\partial G}{\partial \phi_i} = 0 \ \ \forall i \ \Leftrightarrow \quad \phi_i = 0$$

then $V_F = 0$ only if $p = 0$ which means that we're back onto the $\mathbb{CP}^{N-1}$ vacuum manifold. But now, in addition, we must satisfy $G(\phi) = 0$. The resulting vacuum moduli space is now a compact manifold given by a degree q hypersurface, $\mathcal{M} \subset \mathbb{CP}^{N-1}$.

To give a sense of why the gauge theory description is useful in understanding the geometric properties of the vacuum manifold, here's a short anecdote. It turns out that the gauge theory flows to a conformal field theory only when $q = N$. (Only then does the FI parameter not run.) In this case, the vacuum moduli space X is a degree N hypersurface $\mathbb{CP}^{N-1}$. But it is known that such spaces defines what mathematicians call a *Calabi-Yau manifold*. One of the key properties of these spaces (conjectured by Calab and proven by Yau) is that they admit a Ricci flat metric. This ties in nicely with the gauge theory expectation because, as we have seen in (4.42), such a Ricci flat metric is necessary for conformal symmetry.

There are many more geometrical properties that can be extracted from a study of gauge theories in 2d dimensions, including mirror symmetry of Calabi-Yau manifolds[3].

4.4 Extended Supersymmetry

We discussed the representations of extended supersymmetry algebras in Section 2.4. For theories with $\mathcal{N} = 2$ supersymmetry (or eight supercharges) there are two different multiplets:

$$\begin{aligned}\mathcal{N} = 2 \text{ vector multiplet} = \mathcal{N} = 1 &\text{ vector multiplet } (A_\mu, \lambda_\alpha, D)\\ + \mathcal{N} = 1 &\text{ chiral multiplet } (\phi, \chi_\alpha, F)\end{aligned}$$

Here the chiral multiplet necessarily sits in the adjoint representation of the gauge group. There is also the $\mathcal{N} = 2$ matter multiplet

$$\begin{aligned}\mathcal{N} = 2 \text{ hypermultiplet} = \mathcal{N} = 1 &\text{ chiral multiplet } (q, \psi_\alpha, F)\\ + \mathcal{N} = 1 &\text{ chiral multiplet } (\tilde{q}, \tilde{\phi}_\alpha, \tilde{F})\end{aligned}$$

If the first of these transforms in the representation R of the gauge group then the second transforms in the conjugate representation $\bar{R}$. We can tune the matter content and interactions of $\mathcal{N} = 1$ theories to give theories with extended supersymmetry.

With $\mathcal{N} = 4$ there is just a single multiplet (at least restricting to non-gravitational theories) with content

$$\begin{aligned}\mathcal{N} = 4 \text{ vector multiplet} = \mathcal{N} = 1 &\text{ vector multiplet } (A_\mu, \lambda^1_\alpha, D)\\ + 3 \times \mathcal{N} = 1 &\text{ chiral multiplets } (\phi^i, \lambda^{i+1}_\alpha, F^i) \quad i =, 1, 2, 3\end{aligned}$$

In addition to the gauge field, we have three complex scalars and four Weyl fermions, all sitting in the adjoint representation of the gauge group.

[3]The use of gauge theories as a method to understand geometry was pioneered by Edward Witten in the paper Phases of $N = 2$ Theories. You can read more in Kentaro Hori's lecture notes which comprise Part 2 and Part 3 of the book Mirror Symmetry.

To construct theories with $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetry, we could try to build an extended superspace. It turns out that there is a superspace for $\mathcal{N}=2$ theories, known as harmonic superspace, but it's rather cumbersome to work with. In contrast, there is no superspace for $\mathcal{N}=4$ theories. Instead, we will build Lagrangians for both by tuning the interactions of $\mathcal{N}=1$ theories. The key is to get Lagrangians that exhibit larger R-symmetries.

4.4.1 $\mathcal{N}=2$ Theories

$\mathcal{N}=2$ super Yang-Mills comprises of a vector multiplet V and an adjoint chiral multiplet Φ. The $\mathcal{N}=2$ Lagrangian is constructed by simply turning off any superpotential for Φ. It is

$$\begin{aligned}\mathcal{L} &= -\mathrm{Tr}\left[\int d^2\theta\ \frac{i\tau}{8\pi}W^\alpha W_\alpha + \text{h.c.}\right] + \frac{1}{g^2}\int d^4\theta\ \Phi^\dagger e^{2V}\Phi \\ &= \frac{2}{g^2}\mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu\mathcal{D}_\mu\bar{\lambda} - i\chi\sigma^\mu\mathcal{D}_\mu\chi + \mathcal{D}_\mu\phi^\dagger\mathcal{D}^\mu\phi\right] + \frac{\vartheta}{16\pi^2}\mathrm{Tr}\,F_{\mu\nu}{}^\star F^{\mu\nu} \\ &\quad + \frac{2}{g^2}\,\mathrm{Tr}\Big[\sqrt{2}i\lambda[\phi^\dagger,\chi] + \sqrt{2}i\bar{\lambda}[\phi,\bar{\chi}] - \frac{1}{2}[\phi^\dagger,\phi]^2\Big]\end{aligned} \tag{4.43}$$

The potential term comes from integrating out the D-term from the $\mathcal{N}=1$ vector multiplet: we'll look more closely at the moduli space of vacua below.

Of more immediate importance are the fermion terms: the two Weyl fermions λ and χ sit on the same footing in the final Lagrangian, despite their origins in different $\mathcal{N}=1$ multiplets. This means that there is an $SU(2)$ symmetry that rotates them, under which they sit in a doublet $\mathbf{2}$. The bosonic field ϕ does not transform under this symmetry, which tells us that this must be an $SU(2)_R$ R-symmetry. This is the smoking gun for $\mathcal{N}=2$ supersymmetry. There is also a $U(1)_R$ symmetry, under which $R[\phi]=2$ and $R[\lambda]=R[\chi]=1$.

There is another way to derive the $\mathcal{N}=2$ Lagrangian. You can write down a minimal super Yang-Mills theory in $d=5+1$ dimensions, consisting of a gauge field coupled to a Weyl fermion. Upon dimensional reduction, this gives the Lagrangian (4.43).

We can couple matter to (4.43) in the form of hypermultiplets. These comprise of two chiral multiplet, Q and $\tilde{Q}$. (Note: until now the letter Q has always meant a supercharge, but it's not unusual to also use it to denote a chiral multiplet, with Q standing for "quark".) As we mentioned above, if Q sits in the representation R then $\tilde{Q}$ necessarily sits in the conjugate representation $\bar{R}$. This suffices to determine the interaction with the vector multiplet V,

$$\mathcal{L}_{\text{vector}} = \int d^4\theta \, \left[Q^\dagger e^{2V} Q + \tilde{Q}^\dagger e^{-2V} \tilde{Q}\right]$$

But in addition we should couple Q and $\tilde{Q}$ to the $\mathcal{N} = 2$ vector multiplet field Φ in such a way that the $SU(2)_R$ symmetry between λ and χ remains. This is achieved by the superpotential term

$$\mathcal{L}_{\text{chiral}} = \sqrt{2} \int d^2\theta \; \tilde{Q}\Phi Q + \text{h.c.}$$

The interactions between $\tilde{Q}$ and Q themselves are greatly limited by the extended supersymmetry: we can add only mass terms

$$W = \sqrt{2} m \tilde{Q} Q$$

A general $\mathcal{N} = 2$ theory is specified by the gauge group G and the representations R_i of any matter multiplets, together with their masses. (If G contains Abelian factors, we can also add FI terms. We will not include these in the following.) The scalar potential comes, as always, from integrating out D and F-terms. After some rearranging, the potential can be expressed as the sum of positive definite terms. For $SU(N_c)$, it is

$$\begin{aligned} V(\phi, q, \tilde{q}) = \frac{1}{g^2} \operatorname{Tr}[\phi^\dagger, \phi]^2 &+ \frac{g^2}{2} \sum_{A=1}^{\dim G} \left(\sum_i q_i^\dagger T_R^A q_i - \tilde{q}_i T_R^A \tilde{q}_i^\dagger \right)^2 + g^2 \sum_{A=1}^{\dim G} \left| \sum_i \tilde{q}_i T_R^A q_i \right|^2 \\ &+ \sum_i q_i^\dagger \{\phi^\dagger - m_i^\dagger, \phi - m_i\} q_i + \tilde{q}_i \{\phi^\dagger - m_i^\dagger, \phi - m_i\} \tilde{q}_i^\dagger \end{aligned} \tag{4.44}$$

(Initially, the D-term contains both ϕ and the q's and $\tilde{q}$'s. The first two terms on the first line both arise from this D-term, but the cross-term has sneaked into the third line, where it turns $\phi^\dagger \phi$ into the anti-commutator $\{\phi^\dagger, \phi\}$.)

The hypermultiplet scalars q and $\tilde{q}^\dagger$ transform as a doublet **2** under the $SU(2)_R$ symmetry. Conversely, their fermionic superpartners ψ and $\tilde{\psi}$ are singlets under $SU(2)_R$. The second and third terms in the potential (4.44) can be rewritten in way that makes the $SU(2)_R$ symmetry manifest. We introduce the doublet

$$\omega_i = \begin{pmatrix} q_i \\ \tilde{q}_i^\dagger \end{pmatrix}$$

The second term in (4.44) is a real D-term while the third is a complex F-term. But, with $\mathcal{N}=2$ supersymmetry they are better viewed as a potential $V = \frac{1}{g^2}\vec{D}^2$ arising from triplet of D-terms

$$\vec{D}^A = g^2 \sum_i \omega_i^\dagger T_R^A \vec{\sigma}\omega_i$$

where $\vec{\sigma}$ are the Pauli matrices. The triplet $\vec{D}$ transforms in the $\mathbf{3}$ of $SU(2)_R$.

The potential (4.44) has some interesting properties. Let's take the masses to vanish: $m_i = 0$. In this case, the second line takes the schematic form $|\phi|^2(|q|^2 + |\tilde{q}|^2)$. That means that if we're looking for vacuum states with $V(\phi, q, \tilde{q}) = 0$ then there are two possibilities: either $\phi = 0$ and the hypermultiplet scalars $q, \tilde{q}$ are turned on; or $\tilde{q} = q = 0$ and the vector multiplet scalar ϕ is turned on. Geometrically, this means that the vacuum moduli space factorises as

$$\mathcal{M} = \mathcal{M}_C \times \mathcal{M}_H$$

There are defined as follows:

- $\mathcal{M}_C$ is called the *Coulomb branch*. It is defined as the space $\tilde{q} = q = 0$ with ϕ restricted to obey

  $$[\phi^\dagger, \phi] = 0$$

  This is solved by ϕ sitting in the Cartan sub-algebra. For $G = SU(N_c)$. this means that $\phi = \text{diag}(\phi_1, \ldots, \phi_{N_c})$ with $\sum_a \phi_a = 0$. At a typical point, the gauge group is broken to the Cartan subalgebra with a bunch of surviving, massless photons. For example, for $G = SU(N_c)$, this means $G \to U(1)^{N_c - 1}$. At some special points, the surviving gauge group will be enhanced further.

  When the gauge group is broken to $U(1)$'s, all charged matter experiences a Coulomb force, hence the name of this branch of vacua.

- $\mathcal{M}_H$ is called the *Higgs branch*. It is defined as the space $\phi = 0$ with $\tilde{q}$ and q constrained to obey the conditions

  $$\vec{D}^A = 0$$

  In addition, we should quotient by the action of G. At a general point, the gauge group is completely Higgsed, hence the name of this branch of vacua.

The Higgs branch has real dimension that is a multiple of four and is a special case of a Kähler manifold, known as a *hyperKähler manifold*. (For what it's worth, a hyperKähler manifold has three independent complex structures while a Kähler manifold has just one.) The definition of the Higgs branch is an extension of the idea of symplectic reduction that gives a hyperKähler metric and is known as the *hyperKähler quotient construction*.

4.4.2 $\mathcal{N} = 4$ Theories

The more supersymmetry we have, the more restrictive the theory.

With $\mathcal{N} = 1$ supersymmetry, we are free to specify the gauge group and (chiral) matter content. In addition to the gauge coupling and masses, both suitably complexified, we can also introduce any superpotential interactions that we wish.

With $\mathcal{N} = 2$ supersymmetry, we are again free to specify the gauge group and (now non-chiral) matter content. But we have no freedom in the choice of interactions: the only arbitrary parameters are the gauge coupling and masses.

With $\mathcal{N} = 4$ supersymmetry, we get to specify only the gauge group and gauge coupling. All other terms in the Lagrangian are then dictated by supersymmetry.

There are a number of different ways to construct $\mathcal{N} = 4$ super Yang-Mills. It can be viewed as minimal super Yang-Mills in $d = 9 + 1$ dimensions, dimensionally reduced to $d = 3 + 1$. Alternatively, it can be viewed as an $\mathcal{N} = 2$ theory with a single adjoint hypermultiplet. The theory contains four adjoint Weyl fermions, transforming in the **4** of $SU(4)_R$ R-symmetry and six real scalars φ^i with $i = 1, \ldots, 6$, transforming in the **6**. The scalar potential is

$$V(\varphi) = -g^2 \sum_{i<j} [\varphi^i, \varphi^j]^2$$

There is now just a Coulomb branch, with G broken to the Cartan subalgebra at a generic point.