5 Boot Camp: Quantum Gauge Dynamics

Our ultimate aim in these lectures is to understand the quantum dynamics of supersymmetric gauge theories. But before we can appreciate this, we really need to understand something about the quantum dynamics of ordinary gauge theories. The purpose of this section is to provide the necessary background.

I should warn you that, in contrast to the rest of these lecture notes, we won't attempt to prove any of the statements made in this section. Indeed, some of them – like the phenomenon of confinement – can't currently be proven, although we do have overwhelming evidence that it takes place, both from numerics and from toy models, not least supersymmetric theories. (Not to mention experimental results like the fact that you are literally stuck together by confinement.) Other phenomena – like the one-loop beta function and the anomaly – have some technical calculations underlying them. Here we omit the technicalities and just state the relevant facts, meaning that you can relax and enjoy this section as something akin to the middle eight in a song. If you want to see the gory details that underlie these results then they can all be found in the lectures on Gauge Theory.

5.1 Strong Coupling

Our interest throughout this section will be on non-Abelian gauge theories. We start with Yang-Mills. The Lagrangian is

$$\mathcal{L}_{YM} = \int d^4x \; -\frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} \tag{5.1}$$

Here the field strength is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. As you can see, we work with the convention in which the coupling constant sits in front of the kinetic term.

5.1.1 The Beta Function

The key feature of Yang-Mills which makes it both subtle and hard is that the coupling g^2 runs under RG. At a scale μ the coupling is given by

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} - \frac{b_0}{(4\pi)^2}\log\frac{\Lambda_{UV}^2}{\mu^2} \tag{5.2}$$

where g_0^2 is the coupling constant evaluated at the cut-off scale Λ_{UV}. Here b_0 is the coefficient of the 1-loop beta function and, for pure Yang-Mills, is given by

$$b_0 = \frac{11}{6}I(\mathrm{adj})$$

G	$SU(N)$	$Sp(N)$	$SO(N)$	E_6	E_7	E_8	F_4	G_2
$I(\text{adj})$	$2N$	$2(N+1)$	$N-2$	4	3	1	3	4

Table 1. The quadratic Casimir $I(\text{adj})$ for all compact Lie groups.

This depends on a group theoretic factor $I(\text{adj})$, known as the *quadratic Casimir*. It has another avatar as the Dynkin index in the adjoint representation. (Note that we've defined $I(R)$ with a factor of 2 difference from the Gauge Theory lecture notes.) The quadratic Casimirs for the various compact Lie groups are shown in Table 1. In these lectures, we will focus almost exclusively on gauge group $G = SU(N)$.

The running of the coupling constant is often summarised in terms of the one-loop beta function

$$\beta(g) \equiv \mu \frac{dg}{d\mu} = -\frac{b_0}{(4\pi)^2} g^3 \tag{5.3}$$

whose solution gives the logarithmic behaviour (5.2).

The all-important feature of the beta function is the overall minus sign. This means that the theory is weakly coupled at high energies, a phenomenon known as *asymptotic freedom*. Conversely, it means that the theory is strongly coupled at low energies. It is this low-energy physics that we would like to understand.

What do we mean by low and high energy here? Where's the dividing line? The answer to this can be found within the formula (5.2). This is because we can construct a *strong coupling scale*

$$\Lambda = \mu \exp\left(-\frac{8\pi^2}{b_0 g^2(\mu)}\right) \tag{5.4}$$

This has the property that $d\Lambda/d\mu = 0$. In other words, it is an RG invariant. This is the scale at which the Yang-Mills theory becomes strong.

There's already something remarkable about the existence of the scale Λ. Classically, the Yang-Mills theory (5.1) has no dimensionful parameter. That means that there is nothing to set a scale. Instead, there is just a dimensionless coupling constant g^2. But the logarithmic running succeeds in turning this into a dimensionful parameter Λ! One way to see this is to note that to define the quantum theory, we necessarily had a dimensionful parameter lurking all along. This is the UV cut-off of the theory, Λ_{UV}. The strong coupling scale (5.4) is related to the UV cut-off by

$$\Lambda = \Lambda_{UV}\, e^{-8\pi^2/b_0 g_0^2}$$

This means that if the bare coupling is small, $g_0 \ll 1$, as it should be then the physical scale Λ is exponentially suppressed relative to the UV cut-off: $\Lambda \ll \Lambda_{UV}$.

5.1.2 Confinement and the Mass Gap

When the coupling is small, quantum field theories look similar to their classical counterparts. For example, classical Maxwell theory provides a decent guide to what you might expect from QED. In contrast, when the coupling is large, all bets are off. The quantum theory and classical theory may be completely different. Yang-Mills provides the archetypal example.

If you solve the classical Yang-Mills equations, you will find waves that propagate at the speed of light. This suggests that the quantum theory will give rise to a massless particle called a gluon, similar to the photon. Indeed, if you stare at the action there is no A_μ^2 term that might suggest a mass.

Nonetheless, we now know that quantum Yang-Mills contains no massless particles. We say that the theory is *gapped* which means that the first excited state has a finite energy above the ground state. This additional energy is, of course, just $E = mc^2$ where m is the mass of the lightest particle in the theory. The gap is of order the strong coupling scale, $m \sim \Lambda$.

We don't currently have the technology to prove the Yang-Mills mass gap. Indeed, it is generally considered one of the most important and challenging open problems in mathematical physics. We do, however, have very compelling numerical evidence that this occurs, together with some intuition built from various toy models and heuristic explanations for why it occurs. You can read about some of these in the lectures on Gauge Theory. We'll meet others later in these lectures.

In our world, the strong force is governed by an $SU(3)$ gauge theory known as QCD. The associated strong coupling scale is $\Lambda \approx 300$ MeV and is usually referred to as $\Lambda_{\rm QCD}$. No massless gluons are seen in Nature, but there is good evidence for states known as glueballs with masses around the scale Λ.

The existence of a mass gap goes hand in hand with another phenomenon: this is *confinement*. To explain this, consider placing two charged test particles in the Yang-Mills field. To be specific, we'll consider $G = SU(N)$ and take a quark in the fundamental representation $\mathbf{N}$ and an anti-quark in $\bar{\mathbf{N}}$. We simply ask: what force do they feel?

It's best to compute the potential energy between the two particles. You can first do this in the classical theory. There's a little bit of group theoretic fiddliness but the final result is very intuitive: the potential energy scales with the separation r between particles as

$$V(r) \sim \frac{g^2}{r} \tag{5.5}$$

This, of course, is the same scaling that we see in the Coulomb force of electromagnetism.

What about the quantum theory? If the separation between particles is small, meaning $r \ll 1/\Lambda$, you don't notice much difference. At these short distances the theory is weakly coupled and we again see the Coulomb-like potential (5.5) between test particles. We should replace the coupling constant in (5.5) with $g^2(\mu) = g^2(1/r)$ so it's more accurate to say that the potential scales as $V(r) \sim \log r/r$ but this is a mild correction to the physics.

In contrast, at large separation things are radically different. For distances $r \gg 1/\Lambda$, the potential between test particles takes the form

$$V(r) \sim \sigma r \tag{5.6}$$

The coefficient σ necessarily has dimension $[\sigma] = 2$ and this scale, like everything else in Yang-Mills, is set by $\sigma \sim \Lambda^2$. For reasons that we will explain shortly, σ is called the *string tension*. The force law (5.6) is, to put it mildly, a dramatic departure from what we're used to. The potential energy now *increases* with separation. Indeed, it costs an infinite amount of energy to pull the quark anti-quark pair to infinity. This kind of potential energy is said to be *confining*.

The phenomenon of confinement is, like the mass gap, something that we can't prove from first principles. Once again, however, there is clear numerical evidence together with a plethora of heuristic explanations.

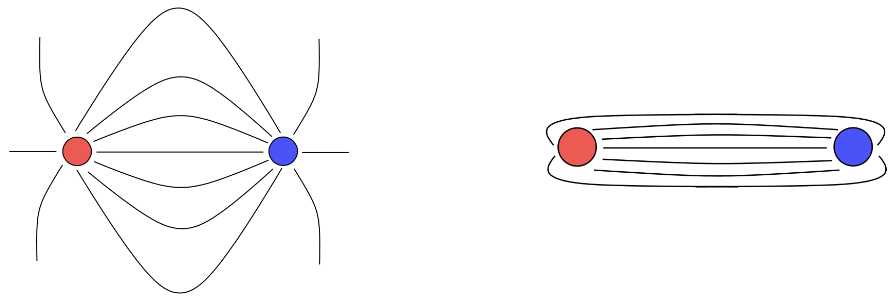


Figure 6. A rough sketch of the non-Abelian field lines in the Coulomb phase, on the left, and in the confining phase, on the right.

To get some very rough intuition for what's going on, we can repeat Faraday's old experiment (now in thought only!) and try to understand what the field lines look like. At short separation, in the Coulomb-like phase (5.5), the field lines form the familiar pattern, first spreading out radially before they bend over to combine with those emitted by the anti-particle. This is shown on the left-hand side of Figure 6. However, as the particles are separated to larger distances, the fact that the gauge field is massive makes itself known. The field lines no longer spread out, but instead lie closely together to form a collimated flux tube. This flux tube acts very much like a string, connecting the two quarks. If its tension, or energy per unit length, is σ then it gives rise to a confining force law like (5.6).

The above description of confinement should be taken with something of a pinch of salt. After all, we are in a strongly interacting quantum field theory and there is no single field configuration that governs the physics. Instead, there are many fields configurations that we should sum over that contribute to the path integral. The discussion above should be understood to mean that those field configurations that resemble the flux tube dominate.

The story above was told in terms of test particles. When we introduce dynamical matter fields into the theory, one would naively expect the associated particles to bind together like the test particles above. And, roughly speaking, this is indeed what happens, at least if the number of light species is small enough. (We'll flesh out this statement shortly.) For example, in QCD the quarks bind together into mesons and baryons. Mesons contain a quark anti-quark pair while baryons contain three quarks and are a colour singlet by dint of the ϵ^{abc} invariant tensor. For $G = SU(N)$ we would get mesons which again contain a quark anti-quark pair and baryons containing N quarks.

There is much more to say about confinement. In particular, the correct, mathematical description of the confining phase lies involves a non-local operator known as the Wilson loop

$$W[C] = \mathrm{Tr}\, \mathcal{P} \exp\left(i \oint_C A\right)$$

Here C is a closed curve in spacetime, while $\mathcal{P}$ stands for "path ordering". In a Coulomb-like phase, the expectation value scales as $\langle W[C]\rangle \sim \exp(-L[C])$ where $L[C]$ is the length of the perimeter of C. Meanwhile, in the confining phase the expectation value scales as $\langle W[C]\rangle \sim \exp(-A[C])$ where $A[C]$ is the area spanned by the curve C. An explanation of why this is the right diagnostic, together with its significance, can be found in the lectures on Gauge Theory.

5.1.3 Adding Matter

Until now, we've considered pure Yang-Mills and its response to test particles. Now we wish to add dynamical matter. The first thing that this does is change the beta function.

Suppose that we have a bunch of Weyl fermions transforming in some representations R_f and a bunch of scalars transforming in some representation R_s. Then the one-loop beta function (5.2) becomes

$$b_0 = \frac{11}{6} I(\text{adj}) - \frac{2}{6} \sum_{\text{fermions}} I(R_f) - \frac{1}{6} \sum_{\text{scalars}} I(R_s) \tag{5.7}$$

Here the group theoretical factors are Dynkin indices. For the representation R, the Dynkin index $I(R)$ is defined by the normalisation of the trace

$$\mathrm{Tr}\; T^A_R\, T^B_R = \frac{1}{2} I(R)\, \delta^{AB} \tag{5.8}$$

Our previous normalisation (4.17) means that we're taking the fundamental representation to have $I(\text{fund}) = 1$. Some examples of $I(R)$ for $SU(N)$ representations are collected in Table 2.

Strictly speaking, the beta function takes the form (5.7) only if the matter is massless. If the matter has some mass m, then the beta function runs like (5.7) for energies $\mu > m$, but as we drop below the mass scale m the matter decouples and its contribution to the one-loop beta function is removed.

Irrep	$\square$	adj	$\square\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
dim	N	N^2-1	$\frac{1}{2}N(N+1)$	$\frac{1}{2}N(N-1)$
$I(R)$	1	$2N$	$N+2$	$N-2$
$A(R)$	1	0	$N+4$	$N-4$

Table 2. Some group theoretic properties of $SU(N)$ representations. Here $\square\square$ is the symmetric representation and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ the anti-symmetric. Conjugate representations have $I(\bar{R}) = I(R)$ and $A(\bar{R}) = -A(R)$.

Again, the first thing to notice is the signs. Both fermions and scalars give a contribution to the beta function that has the opposite sign to the gauge bosons. This means that if we have too much matter then we will have $b_0 < 0$ and, correspondingly, $\beta(g) > 0$ and the theory will be weakly coupled in the infra-red. In this case, the quantum theory looks very much like classical Yang-Mills at low energies, with massless gauge bosons. Here we would like to understand what happens when $b_0 > 0$ and the theory is strongly coupled.

To illustrate this, we will consider a specific set of matter particles. We take

$$G = SU(N_c)$$

with N_f flavours of quarks in the fundamental representation. This means that we have a collection of left-handed Weyl spinors $\psi^a_{\alpha i}$ and $\tilde{\psi}^i_{\alpha a}$. Here $a = 1, \ldots, N_c$ is the gauge index and $i = 1, \ldots, N_f$ the flavour index. We take ψ to transform in the fundamental $\mathbf{N}_c$ representation and $\tilde{\psi}$ in anti-fundamental representation $\bar{\mathbf{N}}_c$ representation. (If we take the complex conjugate of $\tilde{\psi}$, we get a Dirac spinor in the $\mathbf{N}_c$ representation.) The action is

$$\mathcal{L}_{QCD} = -\frac{1}{2g^2}\,\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} - \sum_{i=1}^{N_f}\left[i\bar{\psi}_i\bar{\sigma}^\mu\mathcal{D}_\mu\psi_i + i\bar{\tilde{\psi}}^i\bar{\sigma}^\mu\mathcal{D}_\mu\tilde{\psi}^i\right] \tag{5.9}$$

with

$$\mathcal{D}_\mu\psi = \partial_\mu\psi - iA_\mu\psi \quad \text{and} \quad \mathcal{D}_\mu\tilde{\psi} = \partial_\mu\tilde{\psi} + i\tilde{\psi}A_\mu$$

We could add a mass for the quarks by introducing terms like

$$\mathcal{L}_{\text{mass}} = \sum_{i=1}^{N_f} m_i\tilde{\psi}^i\psi_i + \text{h.c.}$$

However, our interest will be on the case with massless quarks, with $m_i = 0$.

You might wonder why this is interesting. After all, the quarks in our world aren't massless. But they are *almost* massless! The up and down quarks have masses of a few MeV, much less than the relevant scale $\Lambda_{\rm QCD} \approx 300$ MeV. Meanwhile, the strange quark has a mass $m_{\rm strange} \approx 95$ MeV, still smaller than $\Lambda_{\rm QCD}$ although not by much. This means that understanding the behaviour of massless QCD is not a bad starting point for understanding the full theory.

5.1.4 Chiral Symmetry Breaking

The important observation is that massless QCD (5.9) has an extra symmetry that the massive theory doesn't have, under which the ψ and $\tilde{\psi}$ fermions rotate independently. The global symmetry includes

$$G_F = SU(N_f)_L \times SU(N_f)_R \tag{5.10}$$

Here $SU(N_f)_L$ acts on the ψ while $SU(N_f)_R$ acts on the $\tilde{\psi}$,

$$\psi_i \to (L^\star)_i^{\ j}\psi_j \quad \text{and} \quad \tilde{\psi}^i \to R^i_{\ j}\tilde{\psi}^j \tag{5.11}$$

with $L \in SU(N_f)_L$ and $R \in SU(N_f)_R$. (In fact, the full symmetry of the classical theory is $U(N_f)_L \times U(N_f)_R$; we'll discuss these additional $U(1)$ factors in Section 5.2.) The group G_F is known as the *chiral symmetry*, chiral because it acts on Weyl spinors rather than Dirac spinors. This kind of symmetry only exists when the masses $m_i = 0$.

The question that we want to ask is: what becomes of this chiral symmetry? The answer to this depends on the number of flavours N_f in a way that is not fully understand. However, for suitably small N_f the theory develops a vacuum expectation value

$$\langle \tilde{\psi}^i \psi_j \rangle \sim \Lambda^3 \delta^i_{\ j}$$

The formation of this condensate is a strong coupling effect and, like confinement, poorly understood. In contrast, the consequence of the condensate is both well understood and dramatic. First, note that the condensate does not preserve the chiral symmetry (5.11). Indeed, it transforms as

$$\langle \tilde{\psi}^i \psi_j \rangle \to \Lambda^3\, R^i_{\ k}(L^\dagger)^k_{\ j} \tag{5.12}$$

This is the phenomenon of *chiral symmetry breaking*, sometimes shortened to χSB. The surviving subgroup requires us to set $L = R$ in (5.11), meaning

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_{\rm diag}$$

The spontaneous breaking of chiral symmetry means that massless QCD actually has a moduli space of vacua, since each choice of $L \neq R$ in (5.12) gives a different, equally valid, ground state, albeit one that is entirely equivalent to the original because they are related by a global symmetry. The vacuum moduli space is the coset

$$\mathcal{M} = \left[SU(N_f)_L \times SU(N_f)_R\right]/SU(N_f)_{\rm diag} \tag{5.13}$$

with dimension

$$\dim \mathcal{M} = N_f^2 - 1$$

There is an important difference between this vacuum moduli space and those that arise in supersymmetric theories. All points on $\mathcal{M}$ in QCD are equivalent because any point is related to any other by the action of a symmetry. This is not the case for the supersymmetric moduli space.

Nonetheless, there is one important feature that is common whenever we have flat directions and this is the importance of massless particles, corresponding to fluctuations along $\mathcal{M}$. When the flat directions arise from broken symmetries, as in the present case, these massless particles are *Goldstone bosons*.

We learn something interesting. Yang-Mills theory has a mass gap. But massless QCD, at least for $N_f > 1$, does not. Even if the theory confines, giving massive baryons and glueballs, chiral symmetry breaking means that there are massless Goldstone bosons. These can be identified with certain meson states called *pions*.

Of course, in our world the pions are not massless. But this is because the constituent quarks are not exactly massless so the chiral symmetry is not exact. Nonetheless, the chiral symmetry is an approximate symmetry which, in turn, means that the would-be Goldstone bosons are light, but not exactly massless. Indeed, the pions are notably lighter than all other hadrons in QCD.

5.1.5 Phases of Massless QCD

We're now in a position to describe the different phases of massless QCD as we vary N_c and N_f. There is much that we don't yet understand (here "we" means everyone, not just those following these lectures!) and there are a few subtleties that I will sweep under the carpet. But, with broad brush, we we can sketch the different phases of the theory.

We start with low N_f:

- When $N_f = 0$, we have pure Yang-Mills. The theory sits in the confining phase, with a mass gap.

- When $N_f = 1$, there is no chiral symmetry group (5.10) and so no chiral symmetry breaking. The theory is again thought to have a mass gap, with quarks bound in mesons and baryons.

- When $2 \leq N_f \leq N^\star$ the theory confines and exhibits chiral symmetry breaking. This means that the low energy theory consists of freely interacting Goldstone bosons, parameterising the moduli space (5.13).

  The big question here is: what is the maximum value $N^\star$ for which chiral symmetry breaking occurs? We don't know the answer to this. Various approaches, including numerics, suggest that it is somewhere around

$$N^\star \approx 4N_c$$

  Our lack of knowledge of this simple question highlights just how poorly we understand strongly interacting field theories.

Now let's jump to high values of N_f and we'll then try to fill in the details in the middle.

- When $N_f \geq \frac{11}{2}N_c$, the beta function is positive. You can see this from the general expression (5.7) which, for massless QCD, becomes

$$b_0 = \frac{11}{3}N_c - \frac{2}{3}N_f \tag{5.14}$$

  This means that theory is weakly coupled in the infra-red: the low-energy physics consists of massless gluons, weakly interacting with massless quarks. As we go to smaller and smaller energies, the interactions become weaker and weaker. Strictly speaking, in the far IR, the physics is free.

  On the flip side, these become arbitrarily strongly coupled in the UV, with the gauge coupling diverging at some very high scale. This doesn't mean that we should discard them, but they don't make sense at arbitrarily high energies scales. Said another way, we can't take the UV cut-off Λ_{UV} to infinity while keeping any low-energy interactions. Nonetheless, it's quite possible that these theories may arise as the low-energy limit of some other theory. We will see examples in Section 6 when we discuss supersymmetric extensions of QCD.

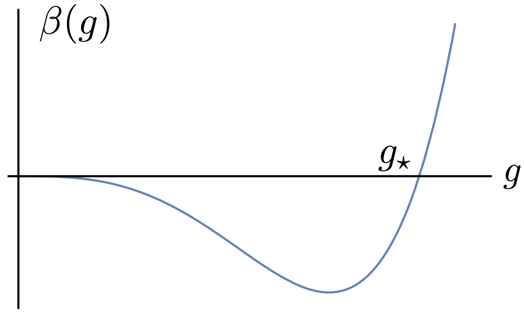


Figure 7. The beta function for N_f slightly below the asymptotic freedom bound has a zero which indicates the existence of an interacting conformal field theory.

That leaves us with the physics in the middle region. We'll keep working down from the asymptotic freedom bound $11N_c/2$.

- When $N^{\star\star} < N_f < \frac{11}{2}N_c$, things are more interesting. To see what happens, we need the two-loop beta function

$$\beta(g) = -\frac{b_0}{(4\pi)^2}g^3 - \frac{b_1}{(4\pi)^4}g^5 + \dots$$

with the one-loop coefficient b_0 given in (5.14) and the two-loop coefficient

$$b_1 = \frac{34N_c^2}{3} - \frac{N_f(N_c^2-1)}{N_c} - \frac{10N_fN_c}{3}$$

In the window of interest, $b_0 > 0$ and $b_1 < 0$, so we can play the one-loop contribution against the two-loop contribution to find a zero of the beta function

$$g_\star^2 = -(4\pi)^2\frac{b_0}{b_1}$$

with $\beta(g_\star) = 0$. The beta function is shown in Figure 7. The existence of such a fixed point is telling us that we have an interacting conformal field theory: there are massless modes, but they are no longer free in the infra-red. This is known as the *Banks-Zaks fixed point*.

Importantly, when N_f lies just below the asymptotic freedom bound, so $\frac{N_f}{N_c} = \frac{11}{2} - \epsilon$, this fixed point lies at $g_\star \ll 1$ which means that we can trust the analysis without having to worry about higher order corrections. Moreover, because $g_\star$ is small we can use perturbation theory to calculate anything that we want.

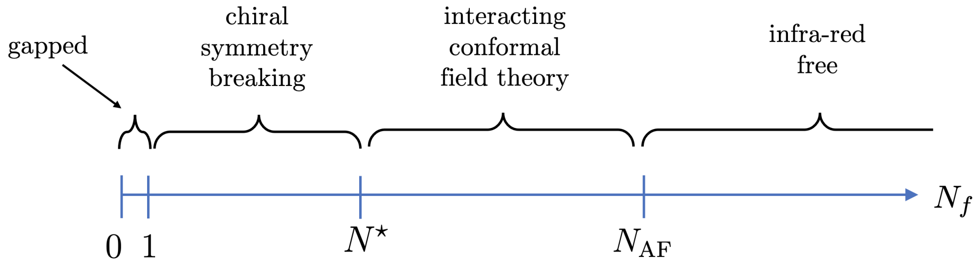


Figure 8. The expected phases of massless QCD. The asymptotic freedom bound is $N_f = \frac{11}{2}N_c$. The lower edge of the conformal window is not known but is expected to be somewhere around $N_f \approx 4N_c$.

> However, as N_f decreases, the value of the fixed point $g_\star$ increases until we can no longer trust the analysis above. The expectation is that we get a conformal field theory only for some range of N_f, lying within $N^{\star\star} < N_f < \frac{11}{2}N_c$. This is known as the *conformal window*. We don't currently know the value of $N^{\star\star}$.

That leaves us with understanding what happens in the middle when $N^\star < N_f \leq N^{\star\star}$. Our best guess is that there is no such regime, and the upper edge of the chiral symmetry breaking phase coincides with the lower edge of the conformal window,

$$N^{\star\star} = N^\star$$

This guess is motivated partly by numerics and partly by a lack of any compelling alternative. For us, the lesson to take away is that strongly interacting quantum field theories are hard and even the most basic questions are beyond our current abilities. A summary of the expected behaviour of massless QCD is shown in Figure 8.

5.2 Anomalies

The next topic that we need to cover is *anomalies*. This is a beautiful subject and, in many ways, the place in which quantum field theory intersects most cleanly with topics in mathematics. Here we won't describe any of these mathematical underpinnings, but instead just cover the minimum material necessary for our later applications.

The main idea is to understand how certain symmetries manifest themselves in quantum field theory. To this, end consider a single left-handed Weyl fermion in $d = 3+1$ dimensions. The action is

$$S = \int d^4x \; i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi$$

This action is clearly invariant under the $U(1)$ global symmetry $\psi \to e^{i\alpha\psi}$, with the corresponding current $j^\mu = \bar{\psi}\sigma^\mu\psi$. To illustrate the anomaly, we will couple this current to a gauge field A_μ with charge $q \in \mathbb{Z}$. The action is now

$$S = \int d^4x \; i\bar{\psi}\bar{\sigma}^\mu \mathcal{D}_\mu \psi$$

where the covariant derivative contains the new coupling $\mathcal{D}_\mu\psi = \partial_\mu\psi - iqA_\mu\psi$. This is action is now invariant under the gauge symmetry

$$\psi \to e^{iq\alpha(x)}\psi \quad \text{and} \quad A_\mu \to A_\mu + \partial_\mu\alpha \tag{5.15}$$

Before we proceed, I should mention that there are two distinct ways to think about the gauge field A_μ and this distinction will be important when we come to look at the various implications of anomalies. They are:

- A_μ could be a *dynamical gauge field*. In the classical theory, this means that we treat it as a dynamical variable, with its own equation of motion, typically after adding a Maxwell term to the action. In the quantum theory, it means that we integrate over A_μ in the path integral.
- A_μ could be a *background gauge field*. This means that it is something fixed, under our control, and should be viewed as a parameter of the theory. Turning it on typically breaks Lorentz symmetry, but could be useful to explore how our system responds to the presence of an electric or magnetic field. In the quantum theory, A_μ appears as a source on which the partition function depends.

We will consider gauge fields of both types in what follows. However, for now, we will consider A_μ to be a background gauge field, something that is under our control.

While the classical theory is clearly invariant under the gauge transformation (5.15), the question that we really want to ask is: what about the quantum theory? For this, we should turn to the path integral, with the partition function in Euclidean space defined as

$$Z[A] = \int D\psi D\bar{\psi} \; \exp\left(-\int d^4x \; i\bar{\psi}\bar{\sigma}^\mu\mathcal{D}_\mu\psi\right)$$

Clearly the action in the exponent remains invariant under gauge transformations. But now we must also worry about the measure in the path integral, and this takes some care to define. The statement of the anomaly is that the measure is *not* invariant under gauge transformations. Instead, it turns out that the measure, and hence the partition function, changes by a phase

$$Z[A] \to \exp\left(\frac{iq^3}{32\pi^2}\int d^4x\ \alpha F_{\mu\nu}{}^\star F^{\mu\nu}\right)\ Z[A] \tag{5.16}$$

with ${}^\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

This subtlety only happens for fermions. If we have scalar fields charged under a symmetry, then the measure is perfectly invariant. At heart, this is related to the fact that there is no difficulty in giving masses to scalar fields while preserving symmetries, but giving masses for fermions necessarily breaks certain symmetries.

The purpose of this section is to understand the implication of this calculation and a number of variants. As we now explain, there are three different avatars of the anomaly. We deal with them each in turn.

5.2.1 Gauge Anomalies

The first implication of the anomaly (5.16) is that it is an obstruction to gauging. Although the action is invariant under the gauge symmetry, the measure is not and neither is the partition function. That means that we cannot promote the gauge field A_μ to a dynamical field, where we integrate over it in the path integral. If we attempted to do this, we would get a sick theory.

There are a number of ways to see why the theory is sick but here is a simple one. Recall that when we first attempted to quantise the gauge field A_μ in the lectures on Quantum Field Theory we had some work to do to decouple the negative norm states that arise from quantising A_0. That work ultimately boiled down to using the gauge invariance to remove these states. But in an anomalous theory, we no longer have that gauge invariance at our disposal and the Hilbert space will involve negative norm states. That's bad.

The upshot is that a $U(1)$ gauge theory, coupled to a single Weyl fermion, is not consistent. To proceed, we must have multiple, left-handed Weyl fermions ψ_i, each with some charge q_i. (If we have right-handed fermions, simply conjugate them to make them left-handed.) The phase in (5.16) is then proportional to the sum of q_i^3. The gauge theory is consistent only if

$$\sum_i q_i^3 = 0 \tag{5.17}$$

This was one of the conditions that we met previously in (4.14). This condition is sometimes written in a different way. One, very simple way to solve this constraint is to take pairs of Weyl fermions with charges $\pm q$. If we conjugate one of them to become a right-handed Weyl fermion, we then have a single Dirac fermion with charge q. These are called vector-like theories and QED is the most familiar example.

There are, however, more interesting solutions to (5.17) that do involve $\pm$ pairs. These are known as *chiral gauge theories*.

The discussion above holds for an Abelian gauge symmetry. There is a similar story for a non-Abelian gauge symmetry G. For a single Weyl fermion, transforming in the representation R of G, the anomaly is proportional to the group theoretic factor $A(R)$. For the fundamental representation, $A(R) = 1$. For other representations, it is given by

$$\mathrm{Tr}\, T_R^A \{T_R^B, T_R^C\} = A(R)\, \mathrm{Tr}\, T^A \{T^B, T^C\}$$

Some examples of $A(R)$ for $SU(N)$ representations are collected in Table 2. To be consistent, a non-Abelian gauge theory coupled to a bunch of left-handed Weyl fermions must obey

$$\sum_i A(R_i) = 0 \tag{5.18}$$

which is the non-Abelian version of (5.17). If R is a complex representation, then it's simple to show that $A(\bar{R}) = -A(R)$. This means that we can again always satisfy (5.18) by taking Dirac fermions, rather than Weyl fermions, since these have a left-handed fermion in a representation R and another in $\bar{R}$.

One consequence of the relation $A(\bar{R}) = -A(R)$ is that $A(R) = 0$ for any real representation. This means that there is no obstacle to coupling a single Weyl fermion in a real representation to a non-Abelian gauge group. Indeed, we've seen this already in these lectures: pure super-Yang-Mills has a single adjoint Weyl fermion, but the adjoint representation is real so there is no problem.

Relatedly, here's a comment that will prove useful shortly: only massless fermions contribute to the anomaly. If you have a Weyl fermion ψ in a complex representation R of a group G, then to give it a mass preserving G you need a second Weyl fermion $\tilde{\psi}$ in representation $\bar{R}$. You can then write down a Dirac mass term $m\tilde{\psi}\psi$. But the two Weyl fermions $\tilde{\psi}$ and ψ cancel in their contribution to the anomaly. Alternatively, you can write down as Majorana mass $m\psi\psi$ for any fermion in a real representation of G

but, as we have seen, there is no contribution to the anomaly from fermions in a real representation. This means that only fermions that cannot get a mass preserving G contribute to the anomaly for G.

When we previously discussed the requirements of anomaly cancellation in (4.14), we gave a further condition on $U(1)$ gauge theories. We asked that they also satisfy

$$\sum_i q_i = 0 \tag{5.19}$$

This, it turns out, is a little more subtle and it follows from the requirement that the theory can be consistently coupled to gravity. There is no corresponding requirement for non-Abelian gauge theories (essentially because $\mathrm{Tr}\, T^A = 0$ for any generator of a simply connected Lie algebra).

The upshot is that if you want to have a theory with a dynamical gauge field, them you better make sure that the anomaly (5.17) or (5.18) cancels. Furthermore, if you want your theory to be compatible with gravity, then you have one further hoop (5.19) to jump through.

5.2.2 Chiral (or ABJ) Anomalies

Here is a slight variant on the same calculation that leads to a physically very different conclusion. Again, consider a single Weyl fermion, now coupled to a background non-Abelian gauge field A in some representation R of the global symmetry G. It's useful to think of $G = SU(N)$, and R either the fundamental or adjoint representation. We can construct the partition function

$$Z[A] = \int D\psi D\bar{\psi}\ \exp\left(-\int d^4x\ i\bar{\psi}\bar{\sigma}^\mu \mathcal{D}_\mu \psi\right)$$

now with $\mathcal{D}_\mu\psi = \partial_\mu\psi - iA_\mu^A T_R^A \psi$. We know that the partition function isn't invariant under gauge transformations of G. But here we instead ask a different question: is it invariant under $U(1)$ rotations of the fermion?

$$\psi \to e^{iq\alpha}\psi \tag{5.20}$$

The answer is again no, with the partition function transforming as

$$Z[A] \to \exp\left(\frac{iqI(R)}{16\pi^2}\int d^4x\ \alpha \mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu}\right)\ Z[A] \tag{5.21}$$

with $I(R)$ the Dynkin index defined previously in (5.8). This looks very similar to our previous result, but it should now be thought of a mixed anomaly between the $U(1)$ symmetry (5.20) and the non-Abelian symmetry G. This can be seen in the coefficient $qI(R)$ which is still cubic but now a mix of Abelian and non-Abelian generators.

An interesting consequence of this is that, in the presence of background gauge fields for G, the $U(1)$ symmetry is no longer conserved. If we repeat Noether's theorem, including the anomaly (5.21), we find that the $U(1)$ current associated to the symmetry (5.20) now obeys

$$\partial_\mu j^\mu = \frac{qI(R)}{32\pi^2}\,\mathrm{Tr}\,F_{\mu\nu}{}^\star F^{\mu\nu} \tag{5.22}$$

When the right-hand side is non-zero, the current is no longer conserved.

An important example of this occurs in the theory of massless QCD that we introduced in the last section. The gauge group is $G = SU(N_c)$ and the Lagrangian is (5.9),

$$\mathcal{L}_{QCD} = -\frac{1}{2g^2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} - \sum_{i=1}^{N_f}\left[i\bar{\psi}_i\bar{\sigma}^\mu\mathcal{D}_\mu\psi_i + i\bar{\tilde{\psi}}^i\bar{\sigma}^\mu\mathcal{D}_\mu\tilde{\psi}^i\right] \tag{5.23}$$

We have added extra fermions to cancel the gauge anomaly in G, as we should. But, as we will see, a mixed anomaly of the type (5.21) remains.

Classically, the theory (5.23) has a $U(N_f)_L \times U(N_f)_R$ global symmetry, with each factor rotating ψ and $\tilde{\psi}$ independently. We studied the $SU(N_f)_L \times SU(N_f)_R$ subgroup in some detail in the previous section, but didn't mention the two $U(1)$ factors. These are usually written as

$$\begin{aligned} U(1)_B:\ \ \psi_i \to e^{i\beta}\psi_i \quad &\text{and} \quad \tilde{\psi}^i \to e^{-i\beta}\tilde{\psi}^i \\ U(1)_A:\ \ \psi_i \to e^{i\alpha}\psi_i \quad &\text{and} \quad \tilde{\psi}^i \to e^{i\alpha}\tilde{\psi}^i \end{aligned} \tag{5.24}$$

The subscript B stands for "baryon" since this is the vector-like symmetry under which baryons are charged. Since ψ and $\tilde{\psi}$ have opposite charges under $U(1)_B$, there is no obstacle to gauging it should we wish. Moreover, the $\pm$ charges also cancel on the right-hand side of (5.22), and the $U(1)_B$ current is conserved in the quantum theory.

In contrast, the *axial symmetry* $U(1)_A$ has the same charges for ψ and $\tilde{\psi}$. This means that the associated current is, following (5.22), no longer conserved. Instead, it obeys

$$\partial_\mu j^\mu_A = \frac{N_f}{16\pi^2}\,\mathrm{Tr}\,F_{\mu\nu}{}^\star F^{\mu\nu} \tag{5.25}$$

Note that the gauge fields on the right-hand side are now dynamical $SU(N_c)$ gauge fields that fluctuate. There is now no way to set them to zero. There is no axial $U(1)_A$ symmetry in the quantum theory.

This also explains why we didn't include $U(1)_A$ when discussing chiral symmetry breaking in the previous section. Since it is not a symmetry, there is no corresponding Goldstone boson. (In the real world, the meson associated to $U(1)_A$ is called the η' and is significantly heavier than the pion Goldstone bosons.)

This, then, is the second avatar of the anomaly. It manifests itself as a symmetry of the classical theory that does not survive the quantisation procedure. In fact, this is how the anomaly was first discovered. In this context, it usually goes by the name of the *chiral anomaly*, or the *ABJ anomaly* after Adler, Bell and Jackiw who first uncovered this subtle effect of quantum field theory. (Yes, *that* Bell.)

There is one further way to think about the chiral anomaly. Non-Abelian gauge theories have an additional, topological term

$$S_\vartheta = \frac{\vartheta}{16\pi^2}\int d^4x \ \mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu}$$

This is the *theta term*. We already met it when constructing super Yang-Mills theory in (4.19). Comparing with the form of the mixed anomaly (5.21), we see that axial transformation (5.24) can be thought of as shifting the theta angle

$$U(1)_A: \ \ \vartheta \to \vartheta + 2\alpha \tag{5.26}$$

We've met this kind of idea previously in Section 3.3, where we found it useful to think of parameters – supurions – transforming under symmetries (which, of course, means that the symmetries aren't actually symmetries). In Section 6, we'll learn how we can combine the shift of the ϑ angle with holomorphy in supersymmetric theories.

5.2.3 't Hooft Anomalies

So far we have discussed two manifestations of the anomaly:

- For a gauge symmetry, the anomaly better cancel. Or else.
- A mixed anomaly between a global symmetry and gauge symmetry means that the global symmetry isn't.

But what if we have an anomaly just for a global symmetry? What are the consequences? From what we've discussed above, we know that the symmetry isn't conserved if we couple it to background gauge fields. But nothing compels us to do so. So what else can we learn from this?

The answer is both subtle and powerful. An anomaly for a purely global symmetry puts strong constraints on the low-energy dynamics of the theory. The anomaly should be thought of as a robust way of characterising the theory, and this characterisation cannot change under RG flow, now under any other deformation of the theory, providing that the symmetry remains unchanged. Such anomalies in global symmetries are referred to as *'t Hooft anomaly*.

We will first explain the basic idea and then give a concrete example. Suppose that we have some quantum field theory – typically a non-Abelian gauge theory – that is weakly coupled in the UV, but flows to strong coupling in the IR. We will abstractly call the UV theory $\mathcal{T}_{UV}$. We assume that it has some global symmetry G_F. This should be a true symmetry of the quantum theory, meaning that it has no mixed anomalies with the gauge symmetry.

This UV theory may have an anomaly for G_F. If G_F is Abelian, anomaly is simply $\sum q^3$ as in (5.17); if it is non-Abelian the anomaly is $\sum A(R)$ as in (5.18). Either way, we will denote this anomaly as $\mathcal{A}_{UV}$ and assume $\mathcal{A}_{UV} \neq 0$,

The theory now flows under RG to a theory $\mathcal{T}_{IR}$ in the IR which, as we've seen, will typically be very different. We have the following result:

Claim: Either the symmetry G_F is spontaneously broken, or the anomalies match meaning

$$\mathcal{A}_{UV} = \mathcal{A}_{IR}$$

This is a wonderfully powerful result. If G_F is spontaneously broken then we necessarily have massless Goldstone bosons. But if G_F is unbroken then we must have massless fermions that reproduce the anomaly. This is known as *'t Hooft anomaly matching*.

Proof: The argument for 't Hooft anomaly matching is very slick. Suppose that $\mathcal{A}_{UV} \neq 0$ then we know from the discussion above that we're not allowed to couple G_F to dynamical gauge fields. That would lead to a sick theory.

To proceed, we introduce a bunch of extra massless Weyl fermions transforming under G_F. We call these *spectator fermions*. These won't interact directly with our original fields in $\mathcal{T}_{UV}$, but they are designed so that the total anomaly of the original fields and these new fermions vanishes:

$$\mathcal{A}_{UV} + \mathcal{A}_{\text{spectator}} = 0$$

Now there's nothing to stop us introducing dynamical gauge fields for G_F. We do so, but with a very very small coupling constant. We'll see the importance of this shortly.

Now let's go back to our original theory $\mathcal{T}_{UV}$. It will flow to strong coupling at some scale Λ and we'd like to understand the physics $\mathcal{T}_{IR}$ below this scale. If the gauge coupling for G_F is small enough, then this RG flow takes place entirely unaffected by the presence of the G_F gauge fields. This means that one of two things could have happened. It may be that the strong coupling dynamics of $\mathcal{T}_{UV}$ spontaneously breaks the symmetry G_F. (For example, as we've seen, this is expected to happen if we take G_F to be the chiral symmetry of QCD.) This was the first possibility of our claim. Alternatively, G_F may be unbroken at low-energies. In this case, we're left with $\mathcal{T}_{IR}$, together with the spectator fermions, all coupled to the G_F gauge fields. But this can only be consistent if

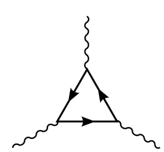
$$\mathcal{A}_{IR} + \mathcal{A}_{\text{spectator}} = 0$$

Clearly, this is only consistent if $\mathcal{A}_{IR} = \mathcal{A}_{UV}$. $\square$

Triangle Diagrams

Until now, we've explained the anomaly as a transformation of the fermion measure in the path integral. However, the anomalies also show up in perturbation theory when computing corrections to Ward identities like (5.25). In this way of looking at things, one has to compute so called *triangle diagrams*. Schematically, these take the form

$$\text{Anomaly} = \sum_{\text{fermions}}$$



where you sum over all Weyl fermions running in loops. The outer legs are currents, either gauge or global. The fact that there are three legs reflects the fact that the anomalies are always proportional to the cube of generators. Our three kinds of anomalies are related to the different types of currents on the legs

- Gauge^3: This is a gauge anomaly.
- Global $\times$ Gauge^2: This is the chiral anomaly.
- Global^3: This is the 't Hooft anomaly.

An Application: Confinement Implies Chiral Symmetry Breaking

We saw in the last section that massless QCD exhibits two, distinct strong coupling phenomena: confinement and chiral symmetry breaking. We will now show that they're not quite as unrelated as they first appear.

As we've seen, the $U(1)_A$ symmetry of massless QCD is anomalous. The true symmetry group is therefore

$$G_F = U(1)_B \times SU(N_f)_L \times SU(N_f)_R$$

Let's first compute the 't Hooft anomalies in the ultra-violet, where the quarks contribute. There is no 't Hooft anomaly for $U(1)_B^3$ because this is a vector-like symmetry. In contrast, there is a 't Hooft anomaly associated to the chiral, $SU(N_f)$ factors. In fact, there are two. The first is the purely non-Abelian anomaly

$$[SU(N_f)_L]^3 : \quad \mathcal{A} = \sum A(\overline{\Box}) = -N_c$$

Here the anomaly $\mathcal{A}$ arises because each quark ψ carries a colour index $a = 1, \ldots, N_c$. The ψ fermions transform in the $\overline{\Box}$ of $SU(N_f)_L$ and $A(\overline{\Box}) = -1$. But there are N_c such fermions. Hence the result $N_c A(\overline{\Box}) = -1$. There is a similar anomaly for $SU(N_f)_R$. In addition, there is a mixed 't Hooft anomaly between $U(1)_B$ and $SU(N_f)$. This is

$$[SU(N_f)_L]^2 \times U(1)_B : \quad \mathcal{A}' = \sum qI(\overline{\Box}) = N_c$$

which again simply counts the number of quarks.

Now the question is: what happens in the infra-red? For suitably low N_f, we've already explained the chiral symmetry G_F is expected to be broken down to $U(1)_B \times SU(N_f)_{\text{diag}}$, but we didn't give any justification for this. The idea of 't Hooft anomaly matching goes some way to help.

Here is the idea. We will assume that the theory confines and, moreover, that in the infra-red, the physics is described by weakly interacting mesons and baryons. (This is in contrast to the conformal field theories that we see at larger N_f.) In such a situation, 't Hooft anomaly matching shows that the chiral symmetry *must* be broken.

Here is the argument. Suppose that G_F is unbroken in the infra-red. Then they must be massless fermions around that can reproduce the anomalies $\mathcal{A}$ and $\mathcal{A}'$. Moreover, by assumption, these massless fermions must be bound states of quarks, either mesons or baryons.

Mesons certainly can't do the job because these are bosons. Baryons, meanwhile, contain N_c quarks so these too are bosons when N_c is even. This is telling us that when N_c is even, a confining theory contains no fermions at low-energies and so certainly can't reproduce the anomalies. We learn that chiral symmetry breaking must occur when N_c is even.

What about N_c odd? Now baryons are fermions. Is it possible that some of these baryons could be massless and reproduce the 't Hooft anomalies? This time we have something of a calculation to do. First, you have to figure out what representations of G_F the baryons sit in. Then you have to figure out what combination of massless baryons could match the anomalies $\mathcal{A}$ and $\mathcal{A}'$. It takes some work, but the answer is that the baryons can never reproduce the anomalies. (You can find the calculation in Section 5.6 of the lectures on Gauge Theory.) This means that if QCD confines into weakly interacting colour singlets, then chiral symmetry is necessarily broken.

5.3 Instantons

One of the new ingredients in these lectures is the Yang-Mills theta angle

$$S_\vartheta = \frac{\vartheta}{16\pi^2}\int d^4x\ \mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu}$$

This deserves some explanation.

First, the theta term is a total derivative,

$$S_\vartheta = \frac{\theta}{8\pi^2}\int d^4x\ \partial_\mu K^\mu \quad \text{with} \quad K^\mu = \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(A_\nu\partial_\rho A_\sigma - \frac{2i}{3}A_\nu A_\rho A_\sigma\right)$$

This means that it does not affect the classical equations of motion. Nonetheless, it can affect the quantum dynamics of gauge theories. This arises because the path integral receives contributions from field configurations that have something interesting going on at infinity so that the boundary term S_ϑ is non-vanishing. This something interesting can be found in the topology of the gauge group.

To explain this, we first Wick rotate so that we work in Euclidean spacetime $\mathbb{R}^4$. Configurations that have a finite action from the Yang-Mills term must asymptote to pure gauge,

$$A_\mu \to i\Omega\partial_\mu\Omega^{-1} \quad \text{as } x\to\infty \tag{5.27}$$

with $\Omega \in G$. This means that finite action, Euclidean field configurations involve a map

$$\Omega(x) : \mathbf{S}^3_\infty \mapsto G$$

with $\mathbf{S}^3_\infty = \partial\mathbb{R}^4$. Maps of this kind fall into disjoint classes. This arises because the gauge transformations can "wind" around the spatial $\mathbf{S}^3$ in such a way that one gauge transformation cannot be continuously transformed into another. Such winding is characterised by *homotopy theory*. In the present case, the maps are labelled by an element of the homotopy group which is

$$\Pi_3(G) = \mathbb{Z}$$

for all simple, compact Lie groups G. In words, this means that the winding of gauge transformations (5.27) at infinity is classified by an integer n.

This statement is most intuitive for $G = SU(2)$ since $SU(2) \cong \mathbf{S}^3$ and the homotopy group counts the winding from one $\mathbf{S}^3$ to another. For higher dimensional G, it turns out that it's sufficient to pick an $SU(2)$ subgroup of G and consider maps which wind within that. You then need to check that these maps cannot be unwound within the larger G.

It can be shown that, in general, the winding $n \in \mathbb{Z}$ is computed by

$$n(\Omega) = \frac{1}{24\pi^2}\int_{\mathbf{S}^3_\infty} d^3S\ \epsilon^{ijk}\mathrm{Tr}\,(\Omega\partial_{\mathrm{i}}\Omega^{-1})(\Omega\partial_{\mathrm{j}}\Omega^{-1})(\Omega\partial_{\mathrm{k}}\Omega^{-1}) \tag{5.28}$$

Evaluated on any configuration, the theta term becomes (5.27)

$$S_\vartheta = \vartheta n \tag{5.29}$$

It is the contribution from configurations with $n \neq 0$ in the path integral that means that observables in quantum gauge theories can depend on ϑ.

We can say more if we work in a regime in which the theory is weakly coupled. Here the path integral is dominated by the saddle points, which are solutions to the classical equations of motion. This means that any ϑ dependence should come from field equations that wind at infinity, so $n \neq 0$, and solve the classical equations of motion,

$$\mathcal{D}_\mu F^{\mu\nu} = 0 \tag{5.30}$$

There is a cute way of finding solutions to this equation. The Yang-Mills action is

$$S_{YM} = \frac{1}{2g^2}\int d^4x\ \mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}$$

Note that in Euclidean space, the action comes with a + sign. This is to be contrasted with the Minkowski space action (5.1) which comes with a minus sign. We can write this as

$$S_{YM} = \frac{1}{4g^2}\int d^4x \ \mathrm{tr}\,(F_{\mu\nu} \mp {}^\star F_{\mu\nu})^2 \pm \frac{1}{2g^2}\int d^4x \ \mathrm{tr}\,F_{\mu\nu}{}^\star F^{\mu\nu} \geq \frac{8\pi^2}{g^2}|n|$$

where, in the last line, we've used the result (5.29). We learn that in the sector with winding n, the Yang-Mills action is bounded by $8\pi^2 n/g^2$. The action is minimised when the bound is saturated. This occurs when

$$F_{\mu\nu} = \pm^\star F_{\mu\nu} \tag{5.31}$$

These are the (anti) self-dual Yang-Mills equations. The argument above shows that solutions to these first order equations necessarily minimise the action is a given topological sector and so must solve the equations of motion (5.30). In fact, it's straightforward to see that this is the case since it follows immediately from the Bianchi identity $\mathcal{D}_\mu{}^\star F^{\mu\nu} = 0$.

Solutions to the (anti) self-dual Yang-Mills equations (5.31) have finite action, which means that any deviation from the vacuum must occur localised in Euclidean spacetime. In other words, they are point-like objects in $\mathbb{R}^4$. Because they occur for just an "instant of time" they are known as *instantons.*

There is much to say about instantons. You can read about the role they play in quantum Yang-Mills in the lectures on Gauge Theory and more about the structure of the solutions to (5.31) in the lectures on Solitons. For our purposes, it will suffice to point out that the contributions of instantons to any quantity comes with the characteristic factor

$$e^{-S_{\text{instanton}}} = e^{-8\pi^2|n|/g^2}e^{i\vartheta n} \tag{5.32}$$

Famously, the function $e^{-8\pi^2/g^2}$ has vanishing Taylor expansion about the origin $g^2 = 0$. This is telling us that effects due to instantons are smaller than any perturbative contribution, which takes the form g^{2n}. Nonetheless, that doesn't mean that instantons are useless since they can contribute to quantities that apparently vanish in perturbation theory.

The theta dependence $e^{i\vartheta n}$ associated to an instanton is also interesting. It is a complex phase. The fact that it is complex can be traced to the $\epsilon^{\mu\nu\rho\sigma}$ tensor in S_ϑ. This means that S_θ contains a single time derivative and so, upon Wick rotation, still sits in the path integral with a factor of i. The fact that $n \in \mathbb{Z}$ means that ϑ is a periodic variable, with

$$\vartheta \in [0, 2\pi)$$

Instantons are usually referred to as *non-perturbative* effects. This is a little bit of a misnomer. The use of instantons requires weak coupling $g^2 \ll 1$, so in this sense they are just as perturbative as usual perturbation theory. The name *non-perturbative* really means "not perturbative around the vacuum". Instead, the perturbation theory occurs around the instanton solution.

This also means that the theta dependence (5.32) is only expected at weak coupling $g^2 \ll 1$. As we've seen, in the far infra-red non-Abelian gauge theories are typically strongly coupled and the theta dependence of quantities can take a different form. We'll see examples in what follows.

An Example: An Instanton in $SU(2)$

It is fairly straightforward to write down the instanton solutions with winding $n = 1$. For $SU(2)$, such a configuration is given by

$$A_\mu = \frac{1}{x^2 + \rho^2}\,\eta^a_{\mu\nu} x^\nu \sigma^a \tag{5.33}$$

Here ρ is a parameter whose role we will describe shortly. The $\eta^a_{\mu\nu}$ are usually referred to as 't Hooft matrices. They are three 4×4 matrices which provide an irreducible representation of the $su(2)$ Lie algebra. They are given by

$$\eta^1_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ , \ \ \eta^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \ , \ \ \eta^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

These matrices are self-dual: they obey $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\eta^i_{\rho\sigma} = \eta^i_{\mu\nu}$. (Note that we're not being careful about indices up vs down as we are in Euclidean space with no troublesome minus signs.) In the solution (5.33), the 't Hooft matrices intertwine the $su(2)$ group index $a = 1, 2, 3$ with the spacetime index μ and this implements the asymptotic winding of the gauge fields.

The associated field strength is given by

$$F_{\mu\nu} = -\frac{2\rho^2}{(x^2 + \rho^2)^2}\eta^a_{\mu\nu}\sigma^a$$

This inherits its self-duality from the 't Hooft matrices: $F_{\mu\nu} = {}^\star F_{\mu\nu}$ and therefore solves the Yang-Mills equations of motion, $\mathcal{D}_\mu F_{\mu\nu} = 0$.

We can get some sense of the form of this solution. First, the non-zero field strength is localised around the origin $x = 0$. (By translational invariance, we can shift $x^\mu \to x^\mu - X^\mu$ to construct a solution localised at any other point X^μ.) The solution depends on a parameter ρ which can be thought of as the size of the instanton lump. The fact that the instanton has an arbitrary size follows from the classical conformal invariance of the Yang-Mills action.