# Supersymmetric Quantum Mechanics

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**Recommended Books and Resources**

There are very few decent textbooks that cover the material of these lectures. The handful of textbooks that exist with titles like "supersymmetric quantum mechanics" tend to focus on the slightly dull topics of exact solutions, rather than on the connections to geometry that we care about here. Nonetheless, there are two books that will be useful for what lies ahead:

- Nakahara "*Geometry, Topology and Physics*"

This book covers homology, cohomology, index theorems and Kähler manifolds, which is much of the mathematics you'll need in these lectures. Later editions of the book also cover supersymmetric quantum mechanics towards the end although, in contrast to the rest of the book, the presentation of this material isn't particularly good.

- Kentaro Hori, in the Clay Mathematics Monograph "*Mirror Symmetry*".

This book is something of a mixed bag, with contributions from many authors. But the sections written by Kentaro Hori, which comprise Part 2 and Part 3 of the book (pages 143 to 480) are spectacularly good. Our lectures will largely follow the first few steps along the path laid down by Kentaro although I suspect that he would disapprove of the times I replace his rigorous statements with wild, but enthusiastic, handwaving. You can download the book directly from the Clay Mathematics Institute.

While decent books on the subject are in short supply, there is one resource that I strongly recommend. A remarkably large fraction of these lectures (not to mention subsequent developments in the field) is due to Edward Witten. His papers are not only brimming with beautiful physics, but are also models of scientific writing. If you want to learn large swathes of modern physics, you could do worse than turn to Witten's papers. Those from the late 1970s and early 1980s are particularly accessible. Much of what we cover in these lectures can be found in the papers "Constraints on Supersymmetry Breaking" and "Supersymmetry and Morse Theory". These, and a number of further resources, can be found on the course webpage.

## Contents

**Acknowledgements**

This course is aimed at beginning graduate students. It assumes a good background in quantum mechanics and path integrals and a basic knowledge of differential geometry, say at the level of my lecture notes on General Relativity. The course is based largely on lectures by Kentaro Hori and David Skinner. I'm particularly grateful to David Skinner and Oscar Randal-Williams for patiently explaining a number of issues to me. I also thank Andy Zhao for helping me get minus signs right in the susy variation of the sigma model action.

# 0 Introduction

It will come as no surprise to hear that there is a close relationship between mathematics and physics. Yet, for many centuries, the relationship was more than a little one sided. There was, in the language of marriage counsellors, a lack of equitable reciprocity. Physicists took, but gave little in return. Admittedly there were exceptions, some of them rather important like Newton's development of calculus. Nonetheless, it remains true that mathematics is a tool that us physicists cannot live without, while many mathematicians have no more use of physics than they do of chemistry or botany.

In the last few decades, this narrative has started to change. Physicists have been giving back. As our understanding of quantum field theories has grown, we have uncovered increasingly sophisticated mathematical structures lurking within. These are largely, but not exclusively, the structures that arise in geometry and topology. Using physicist's methods and techniques to solve quantum fields theories has revealed connections to these mathematical ideas. Initially this gave new ways of deriving results well known to mathematicians. But, as the quantum field theories became more involved, so too did the mathematics until physicists were able to discover new results that came as a complete surprise to mathematicians. Prominent among these is an idea called *mirror symmetry*, a novel relationship between different manifolds.

You might reasonably wonder what advantage physicists have over mathematicians in this game. After all, we're certainly not smarter. (At least, not most of us.) And yet, there are times when we are able to leapfrog mathematicians and then turn around and present them with new results that sit firmly within their area of expertise. This seems unfair, like physicists have some kind of secret weapon that mathematicians are unable to wield. And we do. In fact, we have two. The first is the path integral. The second, a wilful disregard for rigour.

These two weapons are not unrelated. The path integral approach to quantum field theory has so far evaded attempts to be placed on a rigorous footing, at least beyond quantum mechanics. This means that most often the physicist's approach to these questions does not meet the mathematician's bar for proof. Physics is perhaps better thought of as an idea generating machine, giving new insights into areas of mathematics that can subsequently be proven using more traditional methods. Happily, in most cases, these subsequent proofs have turned out to be much more than an exercise in dotting i's and crossing $\hbar$'s. Mathematicians take their own path to a problem, developing new ideas along the way, and these then feed back into our understanding of quantum field theory. Over the past few decades this process has resulted in a harmonious and extraordinarily fruitful relationship between communities of physicists and mathematicians.

This interaction has revolutionised certain areas of mathematics. For example, it's difficult to envisage a thriving field of symplectic geometry without mirror symmetry. But it has also changed what we mean by "mathematical physics". Towards the end of the $20^{\text{th}}$ century, this was viewed as a rather a dry subject and mostly involved bringing a mathematician's level of pedantry to bear on problems that physicists care about, but with little insight flowing back into the underlying physics. Now, this situation has been reversed, with interesting and exciting ideas flowing in both directions. To emphasise the shift of focus, this new activity is sometimes rebranded "physical mathematics".

Much of this interplay between physics and mathematics takes place in the arena of supersymmetric field theories. (There are important exceptions, Witten's Fields medal winning work on knot polynomials in Chern Simons theory among them.) Supersymmetric theories are a class of quantum field theories that have a symmetry relating bosons and fermions. There is, so far, no experimental evidence that supersymmetry is a symmetry of our world. But supersymmetric theories have a number of special properties that allow us to make much more progress in solving them than would otherwise be possible. It is often in these solutions to supersymmetric field theories that we find results of interest to mathematicians.

The purpose of these lectures is to take the first first few steps along this journey. Sadly we will not reach the heights of the subject like mirror symmetry or knot invariants, both of which require quantum field theories in higher dimensions ($d = 1 + 1$ and $d = 2 + 1$ respectively). Instead, we will restrict ourselves to $d = 0 + 1$ dimensional quantum field theories, also known as quantum mechanics. We will study a number of examples of supersymmetric quantum mechanics and, in solving them, recover some of the highlights of $20^{\text{th}}$ century geometry, including ideas of de Rham, Hodge, Morse, Atiyah and Singer.

I should warn you that the level of rigour when addressing the more mathematical aspect of these lectures will be mediocre at best. Anyone with a real interest in these ideas is encouraged to learn both the underlying mathematics and physics to truly appreciate how the two connect. But that is not the path we will take here. Instead, these lectures will assume only a basic knowledge in differential geometry (at the level, say, of my lectures on General Relativity.) We will then use supersymmetric quantum mechanics as a vehicle to take us deeper into the mathematician's territory, allowing us to take a peek at some of the beautiful vistas that await.

# 1 Introducing Supersymmetric Quantum Mechanics

In this section, we discuss some basic facts about supersymmetric quantum mechanics. Our focus will be on a simple class of quantum mechanical systems that, while they have a certain elegance, won't exhibit any deep mathematics. Instead, we will treat them as a proving ground, allowing us to build some intuition for supersymmetry while developing a number of useful calculational techniques. We'll then bring these to bear on problems with a deeper mathematical pedigree in Section 3.

## 1.1 Supersymmetry Algebra

Supersymmetric quantum mechanics is the name given to a class of Hamiltonians $H$ that can be written as

$$H = \frac{1}{2}\{Q, Q^\dagger\} \quad \text{with} \quad Q^2 = 0 \tag{1.1}$$

Here $\{A, B\} = AB + BA$ is the anti-commutator. The operator $Q$ is called the *supercharge* and, as you can see, is something like the square root of the Hamiltonian. Equation (1.1) is called the *supersymmetry algebra*. As we will see, Hamiltonians that can be written in this way enjoy many special properties.

### 1.1.1 A First Look at the Energy Spectrum

The first property is straightforward: the energy of any state is necessarily non-negative. To see this, we just take the usual expectation value in a state $|\psi\rangle$,

$$\begin{aligned} 2\langle\psi|H|\psi\rangle &= \langle\psi|Q^\dagger Q + QQ^\dagger|\psi\rangle \\ &= |Q|\psi\rangle|^2 + |Q^\dagger|\psi\rangle|^2 \geq 0 \end{aligned}$$

Furthermore, we see that energy $E$ is only zero for states $|\psi\rangle$ that are annihilated by both the supercharge and its adjoint

$$E = 0 \quad \Leftrightarrow \quad Q|\psi\rangle = Q^\dagger|\psi\rangle = 0 \tag{1.2}$$

Already, the statement that we have a positive definite spectrum is slightly surprising. Usually in quantum mechanics, we don't care about the overall energy of states since we can always add a constant to the Hamiltonian without changing the physics. But that's not the case for supersymmetric quantum mechanics (nor, indeed, for supersymmetric quantum field theories). The requirement that $E \geq 0$ also rules out some very familiar quantum mechanical potentials, like $V = -1/r$ of the hydrogen atom. The potential in supersymmetric quantum mechanics must always be positive definite.

As an aside: there's only one other place in physics where we care about the overall value of the ground state energy, and that's the cosmological constant in general relativity. So far, sadly, no plausible link has been found between the value of the cosmological constant and the supersymmetry algebra.

We can learn more from the supersymmetry algebra. The energy eigenstates of supersymmetric quantum mechanics are *almost* always degenerate. Consider the set of states with some fixed energy $E$,

$$H|\psi\rangle = E|\psi\rangle$$

It's simple to check from the supersymmetry algebra (1.1) that $[H, Q] = [H, Q^\dagger] = 0$, facts which require us to also use $Q^2 = Q^{\dagger\,2} = 0$. This means that the operators $Q$ and $Q^\dagger$ act within an energy eigenspace. If the energy is $E \neq 0$, we have

$$\{Q, Q^\dagger\} = 2E \quad \Rightarrow \quad \{c, c^\dagger\} = 1 \;\; \text{with } c = \frac{Q}{\sqrt{2E}} \tag{1.3}$$

We also have $c^2 = c^{\dagger\,2} = 0$. This is the same algebra that is formed by fermionic creation and annihilation operators. The algebra has a two-dimensional irreducible representation spanned by the states $|0\rangle$ and $|1\rangle$ with the properties that

$$c|0\rangle = 0 \quad \text{and} \quad |1\rangle = c^\dagger|0\rangle$$

Equivalently we have $c^\dagger|1\rangle = 0$ and $|0\rangle = c|1\rangle$. The algebra is telling us that all energy eigenstates with $E \neq 0$ states must come in pairs. Of course, there could be a still bigger degeneracy, with several pairs all having the same energy. But, at each energy level, the number of states must be even.

The one exception is when we have states with energy $E = 0$. As we've seen, if such states exist then they are necessarily the ground states. Importantly, the argument above that enforces the degeneracy of the spectrum fails: it is quite possible to have a lone ground state $|\Omega\rangle$ because, as we can see in (1.2), any such ground state necessarily obeys $Q|\Omega\rangle = Q^\dagger|\Omega\rangle = 0$. Again, it's quite possible to have more than one ground state. But if that's the case, they're not related by the action of $Q$ or $Q^\dagger$.

Finally, there is a slightly more formal way of viewing the story above. Inspired by the connection to fermionic creation operators, we define the "fermion number operator"

$$F = c^\dagger c$$

This obeys $[F, Q] = -Q$ and $[F, Q^\dagger] = Q^\dagger$ and $[F, H] = 0$. Clearly this operator is well defined only on states with energy $E \neq 0$, where it acts as $F|0\rangle = 0$ and $F|1\rangle = |1\rangle$.

Correspondingly, the Hilbert space decomposes into "bosonic states" with $F = 0$ and "fermionic states" with $F = 1$,

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F \tag{1.4}$$

We say that there is a $\mathbb{Z}_2$ *grading* of the Hilbert space. The $E \neq 0$ pairs have one state in $\mathcal{H}_B$ and one in $\mathcal{H}_F$. As it stands, it's not clear which of these Hilbert spaces we should assign the $E = 0$ states to. This will become clearer when we turn to specific examples below.

Finally, one last piece of terminology. If a ground state with energy $E = 0$ exists, then we say that supersymmetry is *unbroken*. If the ground state has energy $E > 0$ then we say that supersymmetry is *broken*. This language is really adopted from higher dimensions where symmetries that do not leave the vacuum invariant are said to be "spontaneously broken". In the present context we say that supersymmetry is broken if the vacuum is not annihilated by the supercharges: the connection to symmetries will become clearer as we proceed.

### 1.2 A Particle in a Potential

An abstract algebra like (1.1) is all well and good, but to build intuition we really need a concrete example that realises this algebra. Happily such an example exists: we consider the quantum mechanics of a particle moving on a line. The only small novelty is that the particle has an internal degree of freedom, like spin, that can take two different values. The Hilbert space is

$$\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$$

where the $L^2(\mathbb{R})$ means normalisable functions on the real line $\mathbb{R}$ which, of course, is simply the Hilbert space for a particle on a line. Meanwhile the $\mathbb{C}^2$ factor is the internal degree of freedom. In keeping with the notation of the previous section, we'll take the internal states of the $\mathbb{C}^2$ factor to be spanned by $|0\rangle$ and $|1\rangle$. The Hilbert space then decomposes into our "fermionic" and "bosonic" pieces,

$$\mathcal{H} = L^2(\mathbb{R})|0\rangle \oplus L^2(\mathbb{R})|1\rangle = \mathcal{H}_B \oplus \mathcal{H}_F$$

In this context, it might be better to think of $|0\rangle$ and $|1\rangle$ as a spin degree of freedom, with $\mathcal{H}_B$ and $\mathcal{H}_F$ the "spin down" and "spin up" components of the Hilbert space. On the other hand, it might be confusing to think of "spin up" as a fermion and "spin down" as a boson so I stress that these are just names at this stage and don't come with any other fermionic/bosonic connotations. We'll use both pieces of terminology in what follows.

For our supercharge $Q$, we take

$$Q = (p - ih'(x)) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.5}$$

Here $p = -id/dx$ is the usual momentum operator (in units where $\hbar = 1$) and $h(x)$ is a real function. We have $Q^2 = 0$ because the $2 \times 2$ matrix squares to zero. Taking the conjugate gives

$$Q^\dagger = (p + ih'(x)) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1.6}$$

and so

$$H = \frac{1}{2}(QQ^\dagger + Q^\dagger Q) = \frac{1}{2}(p^2 + h'^2)\, \mathbb{1} - \frac{1}{2} h'' \sigma^3 \tag{1.7}$$

The first factor is the familiar Hamiltonian for a particle with unit mass moving on a line with potential

$$V(x) = \frac{1}{2}\left(\frac{dh}{dx}\right)^2 \tag{1.8}$$

This term comes with the $2 \times 2$ unit matrix $\mathbb{1}$ and so doesn't care about the spin of the particle. In contrast, the second term comes with the Pauli matrix

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This term acts like a magnetic field, distinguishing spin up and spin down by the minus sign.

The operator $F$ that distinguishes spin up from spin down is simply

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.9}$$

This tells us that the "bosonic" or "spin down" part of the Hilbert space $\mathcal{H}_B$ is composed of states of the form $\psi(x)|0\rangle = \psi(x)\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and the "fermionic" or "spin up" part of the Hilbert space $\mathcal{H}_F$ is composed of states of the form $\psi(x)|1\rangle = \psi(x)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Note that the definition of $F$ now happily extends to the zero energy states as well.

### 1.2.1 Ground States

Usually, it is challenging to find the exact ground states of any quantum mechanical potential. One of the rather pretty features of supersymmetric quantum mechanics is that we can sometimes find exact expressions for the ground states.

To kick things off, let's look at the semi-classical ground states. The potential energy (1.8) is positive definite and has a minimum whenever there is a critical point of $h$,

$$V(x) = 0 \quad \Leftrightarrow \quad h'(x) = 0$$

If we Taylor expand around such a critical point $x = x_0$, we have

$$h(x) \approx h(x_0) + \frac{1}{2}\omega(x - x_0)^2 + \ldots$$

This gives a potential energy (1.8) that is, to leading order, a harmonic oscillator, $V(x) = \frac{1}{2}\omega^2(x - x_0)^2 + \ldots$. While the classical ground state energy of a harmonic oscillator vanishes, quantum mechanically we have $E_0 = \frac{1}{2}|\omega|$ (working in units with $\hbar = 1$.) But the supersymmetric system also gets a contribution only from the spin-dependent term in (1.7) which, at leading order, is

$$\Delta E = \pm\frac{1}{2}|\omega| \tag{1.10}$$

If we take the minus sign, this precisely cancels the contribution from the harmonic oscillator ground state energy, giving us a total, semi-classical energy $E = 0$. This simple minded analysis shows that it's quite plausible that zero energy ground states exist in this system.

Let's now look more closely at the full quantum problem and, in particular, the question of whether $E = 0$ ground states exist. A general state takes the form

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \phi(x) \end{pmatrix} \tag{1.11}$$

But to qualify as an $E = 0$ ground state, this must be annihilated by both the supercharges, $Q\Psi = Q^\dagger\Psi = 0$, meaning that

$$-i\left(\frac{d}{dx} + h'\right)\psi = 0 \quad \text{and} \quad -i\left(\frac{d}{dx} - h'\right)\phi = 0$$

The magic of supersymmetry means that, at least for the ground state, the Schrödinger equation has morphed from a challenging second order differential equation into a pair of decoupled, first order differential equations. Note that this same trick doesn't work to figure out the excited states of the theory. We can't solve for the whole spectrum. But we can solve for the ground state.

Indeed, the equations are straightforward to solve. We have

$$\psi(x) = e^{-h} \quad \text{and} \quad \phi(x) = e^{+h} \tag{1.12}$$

There is, as always in quantum mechanics, one last criterion: we need to determine if these states are normalisable. This clearly depends on the form of $h(x)$ which, in turn, determines the potential energy (1.8). There are three possibilities

- If $h \to +\infty$ as $|x| \to +\infty$ then $\psi(x)$ is normalisable and we must have $\phi(x) = 0$. In this case there is a unique ground state that sits in the "fermionic" or "spin up" part of the Hilbert space $\mathcal{H}_F$.
- If $h \to -\infty$ as $|x| \to +\infty$ then $\phi(x)$ is normalisable and we must have $\psi(x) = 0$. In this case there is a unique ground state that sits in the "bosonic" or "spin down" part of the Hilbert space $\mathcal{H}_B$.
- If $h$ has neither of these properties, then there is no $E = 0$ ground state and supersymmetry is broken. In this case the ground state necessarily has $E \neq 0$ and is degenerate.

To get a better sense of what's going on, let's look at some simple examples.

Example 1: Quadratic $h$

To start, we take $h = \frac{1}{2}\omega x^2$ with $\omega > 0$. In this case, we just have a harmonic oscillator with potential energy (1.8) given by $V(x) = \frac{1}{2}\omega^2 x^2$. The additional spin-dependent term in the Hamiltonian (1.7) just shifts the spectrum up or down by $\frac{1}{2}\omega$. The upshot is that the "fermionic" or "spin up" spectrum in $\mathcal{H}_F$ takes the form

$$E_F = \omega n \quad n = 0, 1, 2 \ldots$$

Here we find the unique ground state. Meanwhile, the "bosonic" or "spin down" spectrum in $\mathcal{H}_B$ takes the form

$$E_B = \omega n \quad n = 1, 2, \ldots$$

As promised, all excited states with $E > 0$ are paired, but there is a single unpaired ground state at $E = 0$.

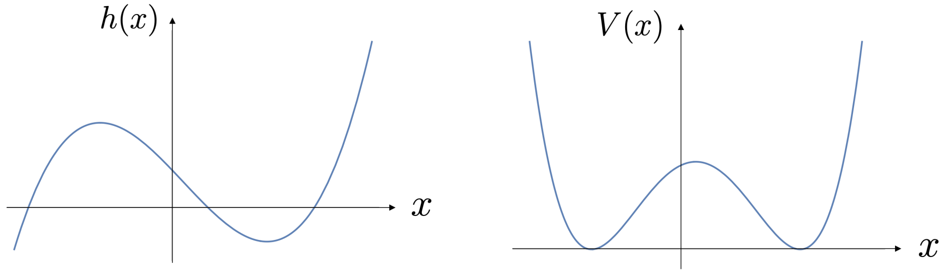



**Figure 1**. The potential for a cubic $h$ has two classical $E = 0$ ground states.

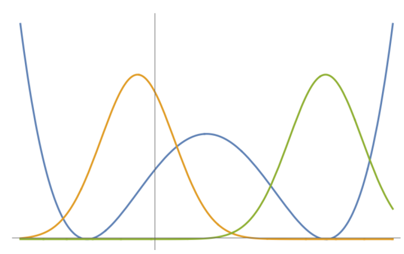
Note that, had we chosen $\omega < 0$, the situation would be reversed with the ground state living in $\mathcal{H}_B$.

Example 2: Cubic $h$

When $h$ is a polynomial of degree higher than two, we can't solve the entire spectrum. But we can get a good understanding of the ground states. Suppose that we take

$$h = \lambda x^3 + \ldots$$

where $\ldots$ are lower order monomials. As we have seen, in this case there can be no $E = 0$ ground states.



A typical form of $h$ is shown on the left of Figure 1, with the corresponding potential $V(x)$ on the right. If we neglected the spin degree of freedom, we would have the familiar double well potential of quantum mechanics and we have some intuition about what happens in this case. Clearly there are two classical minima with $V(x) = 0$ and we can construct an approximation to the ground state with a Gaussian wavefunction that is localised at one, or other, of the minima, as shown on the right, with the orange and green curves each showing different candidate ground state wavefunctions.

Since $h'' < 0$ near the left-hand minimum we expect that this wavefunction can lower its energy by sitting in the "spin down" part of the Hilbert space $\mathcal{H}_B$. Similarly, $h'' > 0$

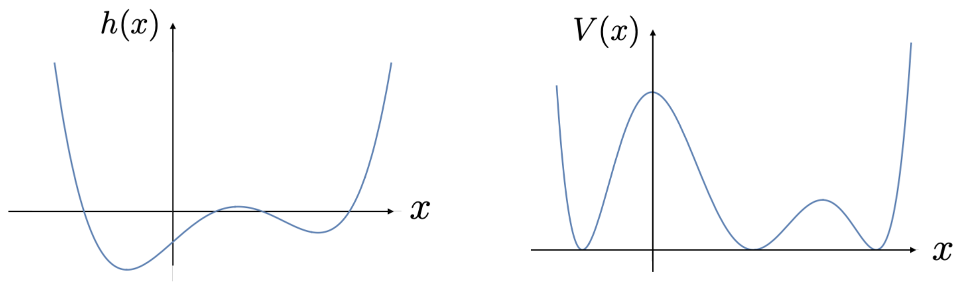



**Figure 2**. The potential for a quartic $h$ has three classical $E = 0$ ground states.

near the right-hand minimum so we expect that it's energetically preferable for this wavefunction to sit in $\mathcal{H}_F$.

Usually in a double well potential, the particle can lower its energy by tunnelling through the barrier and sitting in a superposition of both states. But that's not the case here because the two wavefunctions live in different components of spin space. This kills the possibility for tunnelling. Instead, the supersymmetric set-up is closer to our naive, classical guess of the ground states, with a Gaussian around each minima giving a good approximation to the ground state. Our arguments above tell us that the energy of this two-fold degenerate ground state is necessarily $E > 0$ We will say more about tunnelling in this system and how to compute the actual energy in Section 2.2.

Example 3: Quartic $h$

Next consider $h(x)$ of the form

$$h = \lambda x^4 + \ldots \tag{1.13}$$

where $\ldots$ are terms of order cubic and lower. We pick $\lambda > 0$ so that $h \to +\infty$ as $|x| \to \infty$.

A typical $h(x)$ and the associated potential $V(x)$ are shown in Figure 2. There are now three classical ground states and a naive semi-classical approach would suggest that we can approximate the true ground states as Gaussians localised around any of these three minima. The two outside minima have $h'' > 0$ and so the lowest energy wavefunctions live in $\mathcal{H}_F$, while the middle minimum has $h'' < 0$ and so the lowest energy state sits in $\mathcal{H}_B$.

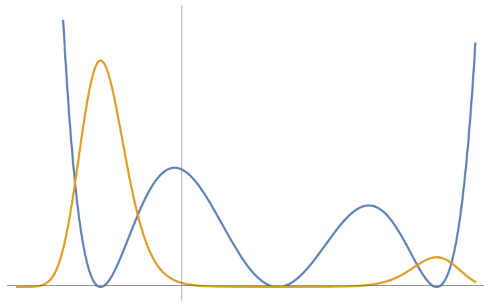


**Figure 3**. The exact $E = 0$ wavefunction is localised only on the outer minima.

However, this time we know the exact ground state: it is given by $\psi(x) = e^{-h(x)}$ and lives in $\mathcal{H}_F$. This is plotted in orange, superposed on the potential, in Figure 3. The wavefunction is peaked on those places where $h < 0$ which, in this case, means that two outer minima. This clearly demonstrates the tunnelling phenomena, in which the true ground state sits in a superposition of minima but, as you can see, there is not necessarily a symmetric distribution between the two vacua.

We started with three states that we thought had the smallest energy – one for each minima – but only one survives as the true $E = 0$ ground state. The other two states must have some small, but non-zero energy. These states are the Gaussian localised in the middle vacuum, and the combination of states localised on the outside minima that is orthogonal to the ground state. Although it is far from obvious from staring at the potential, supersymmetry tells us that the energies of these states must be degenerate.

As we vary the parameters in the function $h(x)$, the energy spectrum of the theory will change. However, the energy of the ground state remains pinned to $E = 0$. The one exception to this statement occurs if we sent $\lambda \to 0$. In this case, one of the minima of the potential runs off to infinity, as $x \sim 1/\lambda$, and carries the $E = 0$ ground state wavefunction with it. In this case, we go over to the situation of a cubic $h(x)$ described above in which there are two ground states, both with $E > 0$.

### 1.2.2 The Witten Index

The robustness of supersymmetric ground states can be formulated more generally using the *Witten index*. As we've seen, the Hilbert space decomposes into two pieces

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$$

These two pieces are characterised by the "fermion number operator" $F$ which has eigenvalues 0 or 1. It is often more useful to consider the operator $(-1)^F$, sometimes called *fermion parity*, that takes eigenvalues $+1$ on states in $\mathcal{H}_B$ and $-1$ on states in $\mathcal{H}_F$. For our example of a particle on a line, it's simple to check that $(-1)^F = -\sigma^3$.

The Witten index is defined as

$$\mathcal{I} = \text{Tr}\,(-1)^F\, e^{-\beta H}$$

Here the trace is taken over all states in the Hilbert space. The parameter $\beta$ plays a role like inverse temperature $\beta = 1/k_B T$ in statistical mechanics. The Witten index differs from the usual statistical mechanical partition function by the signs $(-1)^F$. Importantly, as we will now argue, in supersymmetric theories the Witten index is actually independent of $\beta$

$$\frac{d\mathcal{I}}{d\beta} = 0$$

This follows because, as we have seen, the spectrum of supersymmetric quantum mechanics is degenerate for any state with $E > 0$. Formally, there is an isomorphism between $\mathcal{H}_B$ and $\mathcal{H}_F$,

$$\mathcal{H}_B\Big|_{E>0} \cong \mathcal{H}_F\Big|_{E>0}$$

This means that the trace over any state with $E > 0$ simply cancels out in the Witten index: for every $+e^{-\beta E}$ from $\mathcal{H}_B$ there is a corresponding $-e^{-\beta E}$ from $\mathcal{H}_F$. This means that the Witten index only receives contributions from the zero energy states which, as we've seen, need not be duplicated in both $\mathcal{H}_B$ and $\mathcal{H}_F$. In other words, the Witten index really counts the difference in the number of ground states in each sector,

$$\mathcal{I} = \dim \mathcal{H}_{0,B} - \dim \mathcal{H}_{0,F}$$

where $\mathcal{H}_{0,B}$ is the space of $E = 0$ bosonic ground states, and similar for $\mathcal{H}_{0,F}$.

Before we proceed, a few comments. Since $\mathcal{I}$ doesn't depend on $\beta$, you might wonder why we don't just set $\beta = 0$ and consider $\text{Tr}\,(-1)^F$. Indeed, often the Witten index is written in this way as shorthand, but it's a dangerous thing to do. The quantity $\text{Tr}(-1)^F$ is an infinite series of $+1$ and $-1$ and by pairing terms together in various ways you can get any answer that you like. Including $e^{-\beta H}$ in the definition acts as a regulator for this sum, rendering it finite. Of course, it's a familiar regulator because it also appears in the partition function in statistical mechanics.

The same arguments that show $d\mathcal{I}/d\beta = 0$ also show that $\mathcal{I}$ is independent of the parameters of the Hamiltonian $H$. This was demonstrated in the examples above although, as we also saw, it comes with a caveat: if you change the Hamiltonian too dramatically then you can lose states in your Hilbert space and this will change $\mathcal{I}$. This happens for the particle on a line whenever we change the power of the leading term in $h(x)$.

The Witten index counts the difference between the bosonic and fermionic $E = 0$ states. However, in the simple examples considered above, it actually counts the number of $E = 0$ states, positive if they're bosonic, negative if they're fermionic. One might wonder if, in practice, it always does this. Indeed, there is some intuition that suggests this is the case. If there's no good reason for pairs of states to be stuck at $E = 0$ then, as you vary parameters in the potential, it's tempting to think that they will be lifted to $E > 0$.

However, it's not difficult to exhibit examples where, for example, $\mathcal{I} = 0$ but there are a pair of bosonic and fermionic $E = 0$ states. A particularly simple example arises from particle moving on a circle $\mathbf{S}^1$ of radius $R$. The supercharge (1.5) and Hamiltonian (1.7) take the same form as before and are characterised by a periodic function $h(x) = h(x+2\pi R)$. We can follow our earlier footsteps to find a two parameter family of ground states labelled by $\alpha, \beta \in \mathbb{C}$,

$$\Psi(x) = \alpha \begin{pmatrix} e^{-h} \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ e^{+h} \end{pmatrix}$$

This time, because the particle lives on a circle, there is no issue with the normalisability of the wavefunction. We see that the system has two linearly independent $E = 0$ ground states for any choice of $h$. Yet, because one ground states lives in $\mathcal{H}_B$ and the other in $\mathcal{H}_F$, the Witten index of this system is $\mathcal{I} = 0$. The potential (in blue) and wavefunctions (in orange and green) for $h(x) = \sin(x/R)$ are shown in Figure 4.

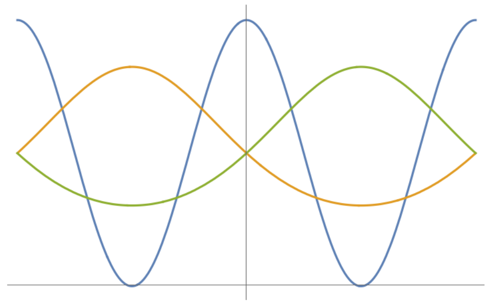


**Figure 4**. Two $E = 0$ ground states for the double well potential on a circle.

For this particle on the circle, the pair of states sticks at $E = 0$ as we change the parameters of $h$, even though these ground states are not protected by the Witten index. One might wonder if there's a deeper reason for this. There is and it's related to the deeper mathematical concept of *cohomology*. We'll look at this further in Section 3.

Finally, one last comment before we move on. The manipulations of the Witten index rely on the discreteness of the energy spectrum. There are more subtle situations, where a particle moves on a non-compact space without a potential, where the energy spectrum is continuous and, despite the bose-fermi degeneracy in the spectrum, strange things can happen that mean that $\mathcal{I}$ does, in fact, depend on $\beta$. We will not encounter situations of this kind in these lectures.

### 1.3 The Supersymmetric Action

There is one fairly large omission in our discussion so far. As presented above, supersymmetric Hamiltonians have a nice algebraic structure. But we have no inkling of why supersymmetry has anything to do with symmetry!

A clue is to be found in the commutation relations

$$[H, Q] = [H, Q^\dagger] = 0 \quad \Rightarrow \quad [H, Q + Q^\dagger] = 0$$

Usually in quantum mechanics, Hermitian operators that commute with the Hamiltonian correspond to conserved quantities and conserved quantities come, via Noether's theorem, from symmetries. This suggests that perhaps $Q+Q^\dagger$ is somehow the conserved charge associated to a symmetry. But what symmetry?

Often the Lagrangian framework is a better starting point when looking for symmetries. To this end, we would like to introduce a Lagrangian for our supersymmetric theory of a particle on a line. We know well how to think of position and momentum in the Lagrangian setting. But how do we incorporate the discrete $\mathbb{C}^2$ factor in the Hilbert space that gave us the all-important $\mathbb{Z}_2$ grading?

The answer is that we should turn to fermions. In higher dimensions, adding a fermion to a Lagrangian gives another field. But in quantum mechanics, fermions simply offer a different way of describing some discrete aspect of the physics.

To illustrate this, consider the action

$$S = \int dt\ L = \int dt\ \left[\frac{1}{2}\dot{x}^2 + i\psi^\dagger\dot{\psi} - \frac{1}{2}h'^2 + h''\psi^\dagger\psi\right] \tag{1.14}$$

where $\psi$ and its conjugate $\psi^\dagger$ are Grassmann variables. Note that their kinetic terms are first order, like the Dirac action that we met in Quantum Field Theory, albeit without the intricacies of gamma matrices. We will first show that this action is equivalent to the supersymmetric Hamiltonian (1.7) describing a particle with an internal degree of freedom moving on a line. We'll then understand how to think of the supercharges $Q$ in the Lagrangian formulation.

To construct the Hamiltonian from a Lagrangian, we proceed in the usual manner. We first introduce the conjugate momentum for both bosonic and fermionic degrees of freedom

$$p(t) = \frac{\delta S}{\delta \dot{x}(t)} = \dot{x}(t) \quad \text{and} \quad \pi(t) = \frac{\delta S}{\delta\dot{\psi}(t)} = i\psi^\dagger(t)$$

In the quantum theory, these obey the canonical (anti)-commutation relations

$$[x,p] = i \quad \text{and} \quad \{\psi,\psi^\dagger\} = 1 \tag{1.15}$$

which, in the Heisenberg picture, hold at a fixed time $t$. The Hamiltonian is then the Legendre transform

$$H = p\dot{x} + \pi\dot{\psi} - L$$

There is, however, a small subtlety awaiting us. We think of the Lagrangian as a classical object in which $x$ and $\dot{x} = p$ be placed in any order. Relatedly, $\psi$ and $\psi^\dagger$ are viewed as "classical Grassmann variables" in the action, which means that if one moves past the other then we just pick up a minus sign. But in the Hamiltionian, these are all to be thought of as quantum operators and, because of the commutation relations (1.15), ordering matters. Which ordering should we take?

This kind of ordering ambiguity is not uncommon when going from classical to quantum systems. In the present situation we don't have to worry about $x$ and $p$ (although we will later in these lectures) but only about the ordering of $\psi$ and $\psi^\dagger$. In the action, it doesn't matter whether we write the last term as $h''\psi^\dagger\psi$ or $-h''\psi\psi^\dagger$: they are the same. But in the Hamiltonian, they differ by a constant because, when viewed as quantum operators, $\psi^\dagger\psi = -\psi\psi^\dagger + 1$. In most other contexts, there is no way to fix this ambiguity and it reflects the fact that there are different ways to quantise a classical theory. However, for us, we do have a way to fix the ambiguity since the resulting Hamiltonian should be supersymmetric. The correct answer, as we will see, is to take

$$H = \frac{1}{2}(p^2 + h'^2) - \frac{1}{2}h''(\psi^\dagger\psi - \psi\psi^\dagger) \tag{1.16}$$

where, in the final term, we've split the difference and treated $\psi^\dagger\psi$ and $\psi\psi^\dagger$ in a symmetric fashion.

To make contact with our previous notation, we just need to appreciate that, due to their Grassmann nature, $\{\psi,\psi\} = \{\psi^\dagger,\psi^\dagger\} = 0$ which, in conjunction with the anti-commutation relation (1.15), has a two-dimensional real representation. Indeed, we met this before when discussing the energy spectrum of supersymmetric quantum mechanics in (1.3). The representation can be thought of as simply replacing the Grassmann variables with $2\times 2$ matrices,

$$\psi \;\rightarrow\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \psi^\dagger \;\rightarrow\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This then gives $\psi^\dagger\psi - \psi\psi^\dagger = \sigma^3$, and the Hamiltonian (1.16) coincides with our previous result (1.7).

Written in terms of the fermionic degrees of freedom, the supercharges (1.5) and (1.6) take the form

$$Q = (p - ih')\psi \quad \text{and} \quad Q^\dagger = (p + ih')\psi^\dagger \tag{1.17}$$

The (anti)-commutation relations of $Q$ with the various fields are

$$\begin{aligned} [Q, x] = -i\psi \quad &, \quad [Q^\dagger, x] = -i\psi^\dagger \\ \{Q, \psi\} = 0 \quad &, \quad \{Q^\dagger, \psi\} = p + ih' \\ \{Q, \psi^\dagger\} = p - ih' \quad &, \quad \{Q^\dagger, \psi^\dagger\} = 0 \end{aligned} \tag{1.18}$$

You can check that these commutation relations give $\{Q, Q^\dagger\} = 2H$, with $H$ given in (1.16), as they should.

### 1.3.1 Supersymmetry as a Fermionic Symmetry

Now we can see what this has to do with symmetry. The action (1.14) has the special property that it is invariant under the following *supersymmetry transformations*

$$\begin{aligned} \delta x &= \epsilon^\dagger \psi - \epsilon \psi^\dagger \\ \delta \psi &= \epsilon(-i\dot{x} + h') \\ \delta \psi^\dagger &= \epsilon^\dagger (i\dot{x} + h') \end{aligned} \tag{1.19}$$

Note that these swap bosonic fields $x$ for fermionic fields $\psi$. This is the characteristic feature of supersymmetry that distinguishes it from other symmetries. For this to make sense, the infinitesimal transformation parameter $\epsilon$ must be a Grassmann valued object.

Let's first check that the action (1.14) is indeed invariant under the supersymmetry transformations as claimed. A generic variation of the action gives

$$\delta S = \int dt \; \left[\dot{x}\delta\dot{x} + i\delta\psi^\dagger \,\dot{\psi} + i\psi^\dagger\delta\dot{\psi} - h'h''\delta x + h'''\delta x\, \psi^\dagger\psi + h''(\delta\psi^\dagger\psi + \psi^\dagger\delta\psi)\right]$$

Now we substitute in the particular supersymmetry transformation (1.19). We collate the $\epsilon$ and $\epsilon^\dagger$ terms on different lines to find

$$\begin{aligned} \delta S = \int dt \; \epsilon &\left[-\dot{x}\dot{\psi}^\dagger - i\psi^\dagger \frac{d}{dt}(-i\dot{x} + h') + h'h''\psi^\dagger - h'''\psi^\dagger\psi^\dagger\psi - h''\psi^\dagger(-i\dot{x} + h')\right] \\ &+ \epsilon^\dagger \left[\dot{x}\dot{\psi} + i(i\dot{x} + h')\dot{\psi} - h'h''\psi + h'''\psi\psi^\dagger\psi + h''(i\dot{x} + h')\psi\right] \end{aligned}$$

There are some minus signs to ensare the unwary: these arise in moving the $\epsilon$ parameters past other Grassmann objects.

We can immediately discard many terms. First, the $\dot{x}\dot{\psi}$ and $\dot{x}\dot{\psi}^\dagger$ terms cancel (for the latter, after an integration by parts). Second the $h'''$ terms disappear on Grassmann grounds. We're left with

$$\begin{aligned} \delta S &= \int dt \; \epsilon \left[i\dot{\psi}^\dagger h' + h'h''\psi^\dagger - h''\psi^\dagger(-i\dot{x} + h')\right] \\ &\qquad + \epsilon^\dagger \left[ih'\dot{\psi} - h'h''\psi + h''(i\dot{x} + h')\psi\right] \\ &= \int dt \; i\epsilon \frac{d}{dt}\left(h'\psi^\dagger\right) + i\epsilon^\dagger \frac{d}{dt}\left(h'\psi\right) \end{aligned}$$

But this is a total derivative and so we have

$$\delta S = 0 \tag{1.20}$$

as advertised.

Before we go on, it will be useful to present this result in a slightly different way. We can think of the transformations (1.19) as generated by the following fermionic operators

$$\mathcal{Q} = \int dt\ \left[\psi(t)\frac{\delta}{\delta x(t)} + (i\dot{x} + h')\frac{\delta}{\delta\psi^\dagger(t)}\right]$$

$$\mathcal{Q}^\dagger = \int dt\ \left[-\psi^\dagger(t)\frac{\delta}{\delta x(t)} - (i\dot{x} - h')\frac{\delta}{\delta\psi(t)}\right] \tag{1.21}$$

Here, the functional derivates act as

$$\frac{\delta}{\delta x(t)}x(t') = \delta(t-t') \quad \text{and} \quad \frac{\delta}{\delta x(t)}\psi(t') = \frac{\delta}{\delta x(t)}\psi^\dagger(t') = 0$$

with similar expressions for the fermions. The supersymmetry transformations (1.19) can then be written as, for example, $\delta x(t) = (\epsilon^\dagger\mathcal{Q} + \epsilon\mathcal{Q}^\dagger)x(t)$. The invariance of the action (1.20) becomes simply

$$\mathcal{Q}S = \mathcal{Q}^\dagger S = 0 \tag{1.22}$$

This form of the generators will be useful in Section 2 when we discuss the path integral formulation of supersymmetric quantum mechanics.

### The Supercharge is a Noether Charge

Finally, we can make good on our promise and see that the supercharges $Q$ and $Q^\dagger$ are indeed Noether charges for supersymmetry. Usually when the action has a symmetry, we can construct the Noether charge by allowing the transformation parameter to depend on time. Things are no different here. We vary the action with $\epsilon = \epsilon(t)$. There are two steps where things differ from our previous calculation: first when we vary the kinetic terms, and again at the last where we see that the variation of the Lagrangian is a total derivative which requires an integration by parts. We end up with

$$\delta S = \int dt\ \dot{\epsilon}^\dagger Q$$

where the Noether charge $Q$ in this calculation coincides with our previous expression (1.17) for the supercharge: $Q = (\dot{x} - ih')\psi = (p - ih')\psi$.

It's slightly odd that the variation of the action involves $\dot{\epsilon}^\dagger$ but not $\dot{\epsilon}$. We can trace this to our choice of fermion kinetic term $\psi^\dagger\dot{\psi}$, which is asymmetric between $\psi$ and $\psi^\dagger$. We could instead start with the more symmetric choice

$$S = \int dt\ L = \int dt\ \left[\frac{1}{2}\dot{x}^2 + \frac{i}{2}\left(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi\right) - \frac{1}{2}h'^2 + h''\psi^\dagger\psi\right]$$

Clearly this is equivalent to our original action (1.14) after an integration by parts, but it's sometimes best to work with these kinds of symmetric kinetic terms for fermions, and computing the Noether charge is one such place. We now find

$$\delta S = \int dt\ \frac{1}{2}\dot{\epsilon}^\dagger Q - \frac{1}{2}\dot{\epsilon} Q^\dagger$$

where $Q^\dagger = (p + ih')\psi^\dagger$.

We can now go full circle. In the operator framework of quantum mechanics, the Noether charges generate the symmetry. Again, supersymmetry is no different. The transformation of any field is given as

$$\delta\,(\text{field}) = i[\epsilon^\dagger Q - \epsilon Q^\dagger, \text{field}]$$

where, as before, $\epsilon$ is a Grassmann valued parameter. The minus sign in the expression above ensures that $\epsilon^\dagger Q - \epsilon Q^\dagger$ is Hermitian (because $(\epsilon^\dagger Q)^\dagger = Q^\dagger \epsilon = -\epsilon Q^\dagger$) and the overall factor of $i$ ensures that $\delta x$ is Hermitian. Using the commutation relations (1.18), you can check that we recover the supersymmetry transformations (1.19) as promised.

## 1.4 A Particle Moving in Higher Dimensions

There are some straightforward generalisations of the supersymmetric theories of a particle moving on a line that we considered in the last section. These will bring out a number of new themes that we will return to as these lectures progress.

### 1.4.1 A First Look at Morse Theory

We start with a direct generalisation of our earlier supersymmetric system to a particle moving in $\mathbb{R}^n$, parameterised by coordinates $x^i$ with $i = 1, \ldots, n$.

The observation that supersymmetry relates bosonic to fermionic fields suggests that we should also introduce $n$ Grassmann valued fields $\psi^i$, with $i = 1, \ldots, n$. These obey the anti-commutation relations

$$\{\psi^i, \psi^j\} = \{\psi^{\dagger\, i}, \psi^{\dagger\, j}\} = 0 \quad \text{and} \quad \{\psi^i, \psi^{\dagger\, j}\} = \delta^{ij} \tag{1.23}$$

As in our previous discussion, these fermionic fields should be viewed as operators acting on some internal, finite dimensional Hilbert space. To construct a representation we introduce the "Fock vacuum" state $|0\rangle$ that obeys

$$\psi^i|0\rangle = 0 \quad \text{for all } i = 1, \ldots, n$$

We can then build up the space of states by acting on $|0\rangle$ with $\psi^{\dagger\,i}$, recalling that you only get to act with a given $\psi^\dagger$ once. This means that the spectrum of internal states takes the form

$$\begin{gathered} |0\rangle \\ \psi^{\dagger\,i}|0\rangle \\ \psi^{\dagger\,i}\psi^{\dagger\,j}|0\rangle \\ \vdots \\ \psi^{\dagger\,1}\ldots\psi^{\dagger\,n}|0\rangle \end{gathered}$$

There are $\binom{n}{p}$ states in the sector where we act with $p$ different $\psi^\dagger$'s. The total number of states is then

$$\sum_{p=0}^{n}\binom{n}{p} = (1+1)^n = 2^n$$

where you should expand out the $(1+1)^n$ in the middle using the binomial theorem to get the sum on the left. This means that our supersymmetric quantum mechanics will describe a particle moving in $\mathbb{R}^n$ with $2^n$ internal states.

There's a useful geometrical way to think about these states. At the top of the pyramid depicted above we have wavefunctions that look like $\phi(x)|0\rangle$: these are just functions over $\mathbb{R}^n$.

At the next level, the wavefunctions look like $\phi(x)\psi^{\dagger\,i}|0\rangle$ and come with an internal index $i = 1,\ldots,n$. We usually think of objects on $\mathbb{R}^n$ that carry such an index as vectors. However, as we now explain, the anti-symmetric nature of the Grassmann variable means that it's much more natural to think about these states as one-forms on $\mathbb{R}^n$.

We really see why it's useful to think of these states as forms when we get to the second level. Here wavefunctions look like $\phi(x)\psi^{\dagger\,i}\psi^{\dagger\,j}|0\rangle = -\phi(x)\psi^{\dagger\,j}\psi^{\dagger\,i}|0\rangle$, with the $i,j$ index necessarily anti-symmetric. But this is precisely the definition of a two-form in differential geometry. This then continues until we reach the unique top form $\psi^{\dagger\,1}\ldots\psi^{\dagger\,n}|0\rangle$. All of this suggests that we should make the identification between Grassmann variables and forms

$$\psi^{\dagger\,i} \;\longleftrightarrow\; dx^i\wedge$$

On $\mathbb{R}^n$, there's little advantage to be had in working with $p$-forms rather than just sticking with Grassmann variables and, for the rest of this section, we'll use the latter notation. However, this relationship to $p$-forms will be of crucial importance when we turn to more geometrical settings in Section 3.

The supersymmetric quantum mechanics also has a fermion parity operator $(-1)^F$ which simply counts the number of excited fermions mod 2. By convention, we take $F|0\rangle = 0$ so $(-1)^F|0\rangle = +1$. Then if $|p\rangle$ denotes a state in the sector with $p$ excited fermions, we have

$$(-1)^F|p\rangle = (-1)^p|p\rangle \tag{1.24}$$

In other words, $(-1)^F$ counts the degree of a $p$-form, mod 2.

### The Supersymmetric Hamiltonian

The supersymmetric quantum mechanics for a particle moving in $\mathbb{R}^N$ involves a real function $h(x^i)$ and the Hamilton

$$H = \frac{1}{2}\sum_{i=1}^{n}\left(p_i^2 + (\partial_i h)^2\right) - \frac{1}{2}\partial_i\partial_j h[\psi^{\dagger\, i}, \psi^j] \tag{1.25}$$

It's not difficult to check that this can be written in the defining way $H = \frac{1}{2}\{Q, Q^\dagger\}$ with the standard (anti)-commutation relations and the supercharges

$$Q = (p_i - i\partial_i h)\psi^i \quad \text{and} \quad Q^\dagger = (p_i + i\partial_i h)\psi^{\dagger\, i} \tag{1.26}$$

where summation convention is used, both for the supercharges and the final term of the Hamiltonian.

We can compute the Witten index by looking at the semi-classical ground states. The bosonic part of the Hamiltonian has a ground state at any critical point $x = X$,

$$\partial_i h(X) = 0 \quad \text{ for all } i = 1, \ldots, n$$

But where does this ground state sit in the internal space? First recall what happened in the simpler case where we just had one fermion and, correspondingly, two states $|0\rangle$ and $|1\rangle$ with $\psi|0\rangle = 0$ and $|1\rangle = \psi^\dagger|0\rangle$. In that case, the final term in the Hamiltonian was

$$H_{\text{Fermi}} = -\frac{1}{2}h''[\psi^\dagger, \psi]$$

So acting on the two states, we had

$$\begin{aligned} H_{\text{Fermi}}|0\rangle &= +\frac{1}{2}h''|0\rangle \\ H_{\text{Fermi}}|1\rangle &= -\frac{1}{2}h''|1\rangle \end{aligned}$$

So in that simpler case, if $h'' > 0$ at the critical point, then we lower the energy by sitting in the state $|1\rangle$, while if $h'' < 0$ then we should sit in $|0\rangle$. This can be seen in the various examples that we explored in the previous section.

Now let's return to the multi-fermion case, with

$$H_{\text{Fermi}} = -\frac{1}{2}\partial_i\partial_j h[\psi^{\dagger\, i}, \psi^j]$$

At each critical point $x = X$, we should think of the Hessian $\partial_i\partial_j h(X)$ as a matrix, with a collection of eigenvectors $e_a^j$ and eigenvalues $\lambda_a$. In fact, to align with other conventions, it turns out to be best to think of the eigenvalue equation of the matrix $-\partial_i\partial_j h$,

$$-(\partial_i\partial_j h)\, e_k^j = \lambda_k\, e_k^j \tag{1.27}$$

where $k = 1, \ldots, n$ labels the different eigenthings and shouldn't be summed over. The generalisation of the story above is now the following: for each negative eigenvalue $\lambda_k < 0$, we should excite the corresponding collection of fermions $e_k^j \psi^{\dagger\, j}$. Meanwhile, for each positive eigenvalue $\lambda_k > 0$, we should just leave well alone: we're better off in the unexcited state. At a given critical point $x = X$, the semi-classical ground state then sits in the part of the Hilbert space given by

$$|\text{ground}\rangle \sim \prod_{k \text{ with } \lambda_k < 0} (e_k^j \psi^{\dagger\, j})|0\rangle$$

We define the *Morse index* to be

$$\mu(X) = \text{The number of negative eigenvalues of } -\partial_i\partial_j h(X) \tag{1.28}$$

(We picked the eigenvalues of $-\partial_i\partial_j h$ rather than $+\partial_i\partial_j h$ so that this definition of the Morse index, in terms of negative rather than positive eigenvalues, is the standard one.) The ground state around the critical point $X$ sits in the sector with $\mu(X)$ excited fermions. In the geometrical language, this means that the ground state wavefunction is a $p$-form, where $p = \mu(X)$ is the Morse index.

Now we can put everything together. We know that the Witten index only receives contributions from the ground states, and we now know that these are associated to critical points $X$ of $h$, and live in the sector with $\mu(X)$ excited fermions. We will assume that $h(x)$ is chosen to be suitably generic so that there are no degenerate critical points. Then, using our previous result (1.24), we have

$$\text{Tr}\,(-1)^F e^{-\beta H} = \sum_X (-1)^{\mu(X)} \tag{1.29}$$

where the sum is over all critical points $X$ of $h$.

Note that we're not assuming that all critical points of $h$ correspond to true $E = 0$ ground states of the theory. It may well be that some get lifted to non-zero energy and, later in these lectures, we'll put in some effort to understand when this happens. But that's not relevant for computing the Witten index since any such states must get lifted in pairs and so cancel out.

The same formula (1.29) also holds for our earlier model with a single $x$ and $\psi$. There a maximum of $h$ was necessarily followed by a minimum, so the sum over critical points could never exceed $+1$ or drop below $-1$. Now, however, we could have multiple ground states. For example, we could have a situation where all the critical points $X$ have $\mu(X)$ even. In this case, they all contribute $+1$ to the Witten index and each of them must correspond to a true, $E = 0$ ground state of the system.

### 1.4.2 More Supersymmetry and Holomorphy

It is quite possible for a quantum system to be invariant under more than one supersymmetry transformation. The extended supersymmetry algebra replaces (1.1) with

$$\frac{1}{2}\{Q_\alpha, Q_\beta^\dagger\} = H\delta_{\alpha\beta} \quad \text{and} \quad \{Q_\alpha, Q_\beta\} = \{Q_\alpha^\dagger, Q_\beta^\dagger\} = 0 \tag{1.30}$$

with $\alpha, \beta = 1, \ldots, q$. A Hamiltonian that can be written in this form is said to have $N = 2q$ supersymmetries, with the 2 because each $Q$ is complex. In this convention, the kind of quantum mechanics that we considered up until now is said to have $N = 2$ supersymmetry. (I should warn you that the nomenclature for counting supersymmetry generators in quantum mechanics is not completely standard: things settle down once we go to higher dimensional quantum field theories.)

In this section, we'll construct a quantum mechanical model that has $N = 4$ supersymmetry, meaning two complex supercharges $Q_1$ and $Q_2$ and their conjugates. Our strategy is to start with the Hamiltonian (1.25) with $2n$ degrees of freedom. We'll split these up into two groups of $n$, related by supersymmetry as

$$\begin{aligned} x^i &\longleftrightarrow \psi_x^i \\ y^i &\longleftrightarrow \psi_y^i \end{aligned}$$

with $i = 1, \ldots, n.$ . This supersymmetry is generated by the supercharge (1.26) which takes the form

$$\begin{aligned} Q_1 &= \left(p_{xi} - i\frac{\partial h}{\partial x^i}\right)\psi_x^i + \left(p_{yi} - i\frac{\partial h}{\partial y^i}\right)\psi_y^i \\ Q_1^\dagger &= \left(p_{xi} + i\frac{\partial h}{\partial x^i}\right)\psi_x^{\dagger\, i} + \left(p_{yi} + i\frac{\partial h}{\partial y^i}\right)\psi_y^{\dagger\, i} \end{aligned}$$

The supercharge depends on a single real function $h(x,y)$. The idea is to introduce a second supercharge that will relate the degrees of freedom in a different way, namely

$$\begin{aligned} x^i &\longleftrightarrow \psi_y^i \\ y^i &\longleftrightarrow \psi_x^i \end{aligned}$$

It takes some messing around to get the minus signs right, but it turns out that the following supercharge does the job

$$\begin{aligned} Q_2 &= \left(p_{xi} + i\frac{\partial h}{\partial x^i}\right)\psi_y^i - \left(p_{yi} + i\frac{\partial h}{\partial y^i}\right)\psi_x^i \\ Q_2^\dagger &= \left(p_{xi} - i\frac{\partial h}{\partial x^i}\right)\psi_y^{\dagger\, i} - \left(p_{yi} - i\frac{\partial h}{\partial y^i}\right)\psi_x^{\dagger\, i} \end{aligned}$$

The two supercharges $Q_1$ and $Q_2$ obey the algebra (1.30) but only if the function $h(x,y)$ has some special properties. For example, we can compute the Hamiltonian in two different ways,

$$\begin{aligned} \{Q_1, Q_1^\dagger\} = &\sum_{i=1}^n \left(p_{xi}^2 + p_{yi}^2 + \left|\frac{\partial h}{\partial x^i}\right|^2 + \left|\frac{\partial h}{\partial y^i}\right|^2\right) \\ &- \sum_{i,j=1}^n \left(\frac{\partial^2 h}{\partial x^i \partial x^j}[\psi_x^{\dagger\, i}, \psi_x^j] + \frac{\partial^2 h}{\partial y^i \partial y^j}[\psi_y^{\dagger\, i}, \psi_y^j] + \frac{\partial^2 h}{\partial x^i \partial y^j}([\psi_x^{\dagger\, i}, \psi_y^j] + [\psi_y^{\dagger\, j}, \psi_x^i])\right) \end{aligned}$$

Alternatively, we have

$$\begin{aligned} \{Q_2, Q_2^\dagger\} = &\sum_{i=1}^n \left(p_{xi}^2 + p_{yi}^2 + \left|\frac{\partial h}{\partial x^i}\right|^2 + \left|\frac{\partial h}{\partial y^i}\right|^2\right) \\ &+ \sum_{i,j=1}^n \left(\frac{\partial^2 h}{\partial x^i \partial x^j}[\psi_y^{\dagger\, i}, \psi_y^j] + \frac{\partial^2 h}{\partial y^i \partial y^j}[\psi_x^{\dagger\, i}, \psi_x^j] - \frac{\partial^2 h}{\partial x^i \partial y^j}([\psi_y^{\dagger\, i}, \psi_x^j] + [\psi_x^{\dagger\, j}, \psi_y^i])\right) \end{aligned}$$

The difference lies in the second line, where the $\psi_x$ and $\psi_y$ fermions are exchanged, together with some minus signs. At first glance, it looks like these are simply different Hamiltonians. However, all is not lost: these two Hamiltonians coincide if the function $h(x,y)$ obeys

$$\frac{\partial^2 h}{\partial x^i \partial x^j} = -\frac{\partial^2 h}{\partial y^i \partial y^j} \quad \text{and} \quad \frac{\partial^2 h}{\partial x^i \partial y^j} = \frac{\partial^2 h}{\partial y^i \partial x^j} \tag{1.31}$$

There's a much nicer way of writing these conditions: as we will now see, they are telling us that $h(x,y)$ is related to a holomorphic function.

**Complex Variables**

We introduce the complex coordinates

$$z^i = x^i + iy^i \quad \text{and} \quad \bar{z}^{\bar{i}} = x^i - iy^i$$

Notice the extra bar on the $\bar{i} = 1, \ldots, n$ index on the conjugate $z^\dagger = \bar{z}$; it's a fairly common notation when dealing with complex coordinates. The corresponding derivative operators are

$$\partial_i = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial y^i}\right) \quad \text{and} \quad \bar{\partial}_{\bar{i}} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i}\right)$$

which obeys $\partial_i z^j = \delta_i^j$ and $\bar{\partial}_{\bar{i}} \bar{z}^{\bar{j}} = \delta_{\bar{i}}^{\bar{j}}$ and $\partial_i \bar{z}^{\bar{j}} = \bar{\partial}_{\bar{i}} z^j = 0$.

Now consider a holomorphic function $W(z)$ which depends only on the $z^i$ and not on $\bar{z}^{\bar{i}}$. If we decompose this in terms of a real and imaginary piece

$$W(z) = -h(x,y) - ig(x,y) \tag{1.32}$$

then the Cauchy-Riemann equations read

$$\frac{\partial W}{\partial \bar{z}^{\bar{i}}} = 0 \quad \Rightarrow \quad \frac{\partial h}{\partial x^i} = \frac{\partial g}{\partial y^i} \quad \text{and} \quad \frac{\partial h}{\partial y^i} = -\frac{\partial g}{\partial x^i}$$

It is simple to show that these then imply the requirements (1.31).

This motivates us to frame the theory with $N = 4$ supersymmetry in terms of complex variables rather than real variables. In addition to the complex coordinates $z^i$, we also introduce complex momenta

$$p_i = \frac{1}{2}\left(p_{xi} - ip_{yi}\right) \quad \text{and} \quad \bar{p}_{\bar{i}} = \frac{1}{2}\left(p_{xi} + ip_{yi}\right)$$

as well as "complex" Grassmann variables. Here the word "complex" is in inverted commas because our original Grassmann variables were already complex; we just introduce different linear combinations

$$\Psi^i = \psi_x^i + i\psi_y^i \quad \text{and} \quad \bar{\Psi}^{\bar{i}} = \psi^{\dagger\, i} - i\psi^{\dagger\, i}$$
$$\tilde{\Psi}^i = \psi_x^{\dagger\, i} + i\psi_y^{\dagger\, i} \quad \text{and} \quad \bar{\tilde{\Psi}}^{\bar{i}} = \psi_x^i - i\psi_y^i$$

We've now abandoned the $\dagger$ notation for complex conjugation and resorted instead to the barred notation. (If nothing else, it is easier to write bars when doing long calculations.) Finally, we have combinations of supercharges

$$\begin{aligned}
Q_+ &= \frac{1}{2}(Q_1 + iQ_2) = p_i\Psi^i - \frac{i}{2}\bar{\partial}_{\bar{i}}\bar{W}\,\bar{\tilde{\Psi}}^{\bar{i}} \\
Q_+^\dagger &= \frac{1}{2}(Q_1^\dagger - iQ_2^\dagger) = \bar{p}_{\bar{i}}\bar{\Psi}^{\bar{i}} + \frac{i}{2}\partial_i W\tilde{\Psi}^i \\
Q_- &= \frac{1}{2}(Q_1 - iQ_2) = \bar{p}_{\bar{i}}\bar{\tilde{\Psi}}^{\bar{i}} - \frac{i}{2}\partial_i W\Psi^i \\
Q_-^\dagger &= \frac{1}{2}(Q_1 + iQ_2) = p_i\tilde{\Psi}^i + \frac{i}{2}\bar{\partial}_{\bar{i}}\bar{W}\bar{\Psi}^{\bar{i}}
\end{aligned}$$

These obey the extended supersymmetry algebra (1.30), now with $\alpha, \beta = +, -$.

The flurry of complexified definitions conspire to make the theory look somewhat simpler. In the Lagrangian picture, it takes the form

$$\begin{aligned}
L = \sum_{i=1}^{n} \left( |\dot{z}_i|^2 + i\tilde{\Psi}^i\partial_t\bar{\tilde{\Psi}}^i + i\bar{\Psi}^{\bar{i}}\partial_t\Psi^i - \frac{1}{4}|\partial_i W|^2 \right) \\
- \frac{1}{2}\sum_{i,j}\left(\partial_i\partial_j W\,\Psi^i\tilde{\Psi}^j + \bar{\partial}_{\bar{i}}\bar{\partial}_{\bar{j}}\bar{W}\,\bar{\tilde{\Psi}}^{\bar{i}}\bar{\Psi}^{\bar{j}}\right)
\end{aligned} \tag{1.33}$$

Supersymmetric Lagrangians of this kind, involving complex scalar fields and fermions, are usually referred to as *Landau-Ginzburg* theories. This is a nod to the Landau-Ginzburg theories that we met when discussing phase transitions in Statistical Physics. But it's not a very good nod. In particular, the theory (1.25) with just a single supersymmetry is just as much related to the kinds of models that Landau and Ginzburg considered but is never given this name in the context of supersymmetry. It's best to think of the name "Landau-Ginzburg" for the Lagrangian (1.33) as merely a quirk of history and forget that the term is also used elsewhere in physics.

The Landau-Ginzburg Lagrangian depends on a single holomorphic function $W(z)$. This is known as the *superpotential*. The fact that extended supersymmetry comes hand in hand with holomorphy and associated ideas in complex analysis is extremely important. We will not discuss quantum mechanics with $N = 4$ supersymmetry in these lectures, but it's not for want of interesting content. In particular, there is a beautiful relationship to a form of complex geometry known as "Kähler geometry" that underlies many of the most interesting results in this subject.

Furthermore, when we go to higher dimensional field theories, supersymmetry generators are associated to spinors and these necessarily have more than one component. This means that in, for example, $d = 3 + 1$ dimensions, the simplest supersymmetric theories have the form (1.33) and are based on complex, rather than real variables. In that context, the holomorphy of the superpotential goes a long way towards allowing us to solve some complicated features of supersymmetric quantum field theories. This is covered in some detail in the lectures on Supersymmetric Field Theory.

#### The Ground States

Finally, we can turn to some physics of the theory (1.33). As previously, we can ask how many ground states the theory has. The semi-classical ground states are associated to critical points of the superpotential,

$$\partial_i W = 0 \quad \text{for all } i = 1, \ldots, n$$

We know from our discussion in Section 1.4.1 what we should do next: we compute the Morse index for each critical point, meaning the number of positive eigenvalues of the Hessian of $h$. But this is trivial for a holomorphic function $W(z)$. For example, if there is a critical point near the origin, we can expand (after a suitable diagonalisation)

$$W(z) \approx \sum_i \lambda_i (z^i)^2 + \ldots$$

In terms of our real variables, $(z^i)^2 = (x^i)^2 - (y^i)^2 + 2ix^iy^i$, while our original function $h(x,y)$ is given, from (1.32), as

$$h(x,y) = -\mathrm{Re}\, W = -\sum_i \lambda_i \Big( (x^i)^2 - (y^i)^2 \Big) + \ldots$$

We learn that, because of the holomorphy of $W$, for every positive eigenvalue of the Hessian of $h$, there is a corresponding negative eigenvalue. This ensures that every critical point has morse index (1.28) given by $n$ and each contributes exactly the same to the Witten index (1.29) which becomes

$$\mathrm{Tr}\, (-1)^F e^{-\beta H} = (-1)^n \times \text{Number of critical points of } W$$

We learn that in theories with $N = 4$ supersymmetry, every critical point of $W$ is a true $E = 0$ ground state of the quantum theory.

### 1.4.3 Less Supersymmetry and Spinors

It's also possible to consider theories with less supersymmetry than our starting point. In fact, this is easy to achieve. We return to our theory with $N = 2$ supersymmetry and impose a reality condition on the Grassmann variables

$$\psi^{\dagger\, i} = \psi^i$$

Real quantum mechanical Grassmann variables like this are called *Majorana modes* or *Majorana fermions*.

For our current purposes, it will suffice to discuss just the free theory,

$$S = \int dt \, \sum_{i=1}^{n} \left( \frac{1}{2} \dot{x}^i \dot{x}^i + \frac{i}{2} \psi^i \dot{\psi}^i \right) \tag{1.34}$$

This is invariant under a single real supercharge,

$$Q = \sum_i \dot{x}^i \psi^i$$

which obeys $Q^\dagger = Q$ and generates the supersymmetry transformation

$$\delta x^i = \epsilon \psi^i \quad \text{and} \quad \delta \psi^i = -\epsilon \dot{x}^i$$

This is usually referred to as $N = 1$ supersymmetry. (You will sometimes see the terminology $N = \frac{1}{2}$ supersymmetry in the literature, counting complex supercharges rather than real.)

Here our interest lies in a very specific property of these theories: how should we think of the internal degrees of freedom generated by the real fermions $\psi^i$? There is a very pretty answer to this question. To see this, first note that the momentum conjugate to the fermion is

$$\frac{\partial L}{\partial \dot{\psi}^i} = \frac{i}{2} \psi^i$$

The canonical commutation relation for the fermion is then

$$\{\psi^i, \psi^j\} = 2\delta^{ij} \qquad i, j = 1, \ldots, n$$

But this is a very familiar equation: it is the Clifford algebra, usually written in terms of gamma matrices

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \qquad i, j = 1, \ldots, n$$

This means that the fermions in this theory should be viewed as gamma matrices! The Clifford algebra has a unique irreducible representation of dimension $2^{n/2}$ if $n$ is even and $2^{(n-1)/2}$ if $n$ is odd. This strongly suggests that the internal degrees of freedom of the particle described by the action (1.34) have something to do with spinors on $\mathbb{R}^n$.

It is straightforward to construct these internal degrees of freedom. First, let's assume that $n$ is even. (We will discuss the case of $n$ odd below.) We pair up the Majorana modes into complex fermions

$$c^i = \frac{1}{\sqrt{2}}\left(\psi^{2i} + i\psi^{2i-1}\right) \qquad i = 1, \ldots, \frac{n}{2}$$

Then the complex $c^i$ operators form the usual algebra of fermionic creation and annihilation operators that we're used to

$$\{c^{\dagger\, i}, c^j\} = \delta^{ij} \quad \text{and} \quad \{c^{\dagger\, i}, c^{\dagger\, j}\} = \{c^i, c^j\} = 0$$

and we can use them to build the familiar fermionic Fock space starting with $|0\rangle$ that obeys $c^i|0\rangle = 0$ and then acting with $c^{\dagger\, i}$. Following the discussion in Section 1.4.1, we see that the fermions fill out an internal space with

$$\text{Dimension of Internal Space} = 2^{n/2}$$

This is precisely the dimension of a Dirac spinor on $\mathbb{R}^n$.

There is more to say about these spinors. Under a rotation in $\mathbb{R}^n$, the Dirac spinor transforms in the representation generated by $\Sigma^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$. (See the lectures on Quantum Field Theory for more details of this.) However, in even dimension, as we have here, this is not an irreducible representation. It is composed of two smaller representations known as *chiral spinors* or *Weyl spinors*.

These arise because we can always construct an operator $\hat{\gamma}$ that is analogous to $\gamma^5$ in four dimensions. In general, this is given by multiplying all the gamma matrices together, with a suitable factor of $i$ to ensure Hermiticity,

$$\hat{\gamma} = i^{n/2}\gamma^1 \ldots \gamma^n$$

This obeys $\hat{\gamma}^2 = 1$ and $\{\hat{\gamma}, \gamma^i\} = 0$. The existence of the $\hat{\gamma}$ operator means that all the internal states can be decomposed into two different camps: those with eigenvalue $\hat{\gamma} = +1$ and those with eigenvalue $\hat{\gamma} = -1$. In the language of the Dirac equation, these are spinors of different chirality.

In the context of our supersymmetric quantum mechanics, this $\hat{\gamma}$ operator has a very natural meaning. The eigenvalues are simply states with an even or odd number of $c^\dagger$ operators excited. In other words, this plays the role of our fermion number.

$$\hat{\gamma} = (-1)^F$$

This means that $\hat{\gamma}$ determines whether states live in $\mathcal{H}_B$ or $\mathcal{H}_F$.

The punchline of this argument is that quantising real fermions, appropriate for $N = 1$ supersymmetry, gives Dirac spinors on $\mathbb{R}^n$, at least for $n$ even. These have dimension $2^{n/2}$. Meanwhile, while quantising complex fermions, appropriate for $N = 2$ supersymmetry, gives forms on $\mathbb{R}^n$. These have dimension $2^n$. We'll have use for quantum mechanics with $N = 1$ supersymmetry in Section 3.3 where we discuss the Atiyah-Singer index theorem.

As an aside, clearly the construction of spinors and forms on $\mathbb{R}^n$ from Grassmann degrees of freedom is closely related. This also suggests that you can take $2^{n/2}$ different Dirac spinors and bundle them together to look like forms. Such a construction is called *Kähler-Dirac fermions*. It won't play a role in these lectures, but arises in a number of other areas of physics including topological twisting of field theories and lattice gauge theory where it goes by the name of *staggered fermions*.

#### The Case of $n$ Odd: A Subtle Anomaly

We still have to understand the case of $n$ odd. Here there is a surprise. Quantum mechanical theories with an odd number of Majorana modes don't make any sense! They are an example of what is sometimes called an *anomalous* quantum theory: a seemingly sensible classical theory that cannot be quantised.

The argument is straightforward. Consider two, non-interacting Majorana fermions, $\psi^1$ and $\psi^2$. From the discussion above, we can construct a single complex fermion $c = (\psi^1 + i\psi^2)/\sqrt{2}$ and this acts on a two-dimensional Hilbert space spanned by $|0\rangle$ and $c^\dagger|0\rangle$.

But, by the factorisation of Hilbert spaces, that means that a single Majorana fermion, say $\psi^1$, must act on a Hilbert space of dimension $\sqrt{2}$. And that's nonsense! You can reach the same conclusion if you use the path integral to compute $\text{Tr}\, e^{-\beta H}$, which just counts the dimension of the Hilbert space when $H = 0$, Again, after suitable regularisation, you find $\sqrt{2}$.

For us, this means that theories with $N = 1$ supersymmetry are restricted to describe a particle moving in an even dimensional space, like $\mathbb{R}^n$ with $n$ even.