# 2d Dualities Revisited

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Based on 1902.05550 with Andreas Karch and Carl Turner

# The Two Phases of a 2d Majorana Fermion

Consider a 2d Majorana fermion on a Riemann surface *X* with spin structure ρ

$$S_{\mathrm{Maj}} = \int_X i\bar{\chi}\not{D}_\rho\chi + im\bar{\chi}\gamma^3\chi$$

There are two phases:
- $m > 0$
- $m < 0$

Most simply detected by looking at manifold with boundary $\Longrightarrow$ SPT phase

# The Two Phases of a 2d Majorana Fermion

Consider a 2d Majorana fermion on a Riemann surface *X* with spin structure ρ

$$Z_{\mathrm{Maj}}[\rho; m] = \mathrm{Pf}(\not{D}_\rho + m\gamma^3)$$

What happens as the mass changes sign?

If there are zero modes, the partition function changes sign:

$$\mathrm{Pf}(\not{D}_\rho + m\gamma^3)\Big|_{\mathrm{zero\ mode}} = \mathrm{Pf}\begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix} = m$$

# The Two Phases of a 2d Majorana Fermion

Consider a 2d Majorana fermion on a Riemann surface *X* with spin structure ρ

$$Z_{\mathrm{Maj}}[\rho; m] = \mathrm{Pf}(\not{D}_\rho + m\gamma^3)$$

What happens as the mass changes sign?

$$Z_{\mathrm{Maj}}[\rho; -m] = (-1)^{\mathcal{I}[\rho]}\ Z_{\mathrm{Maj}}[\rho; m]$$

with $\mathcal{I}[\rho] = \mathrm{Index}(\not{D}_\rho) \ \in\ \mathbf{Z}_2$

e.g. on the torus, $\mathcal{I}[\rho] = \begin{cases} 1 & \rho = PP \\ 0 & \rho = AP, PA, AA \end{cases}$

# A ‘t Hooft Anomaly in Chiral Symmetry

Consider a 2d Majorana fermion on a Riemann surface *X* with spin structure $\rho$

We can use chiral symmetry to induce a sign of the mass

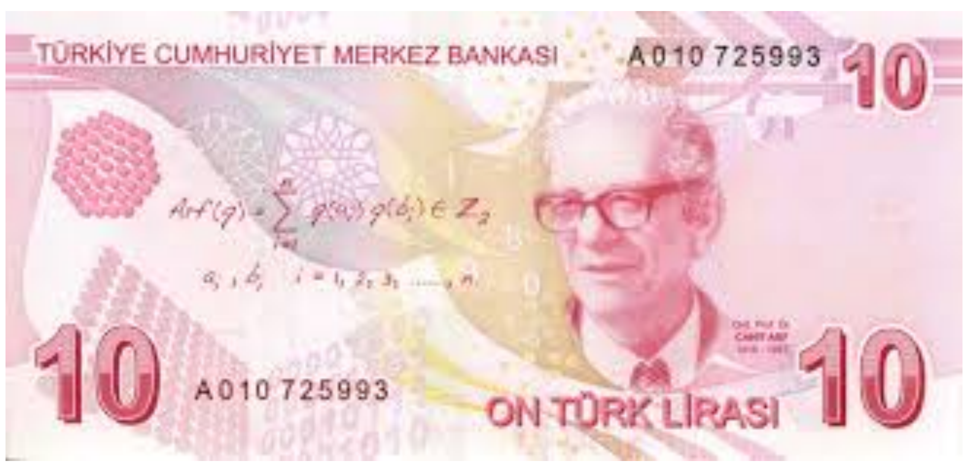
$$\mathbf{Z}_2^{\mathrm{chiral}} : \chi \ \mapsto \gamma^3 \chi$$

This acts as

$$\mathbf{Z}_2^{\mathrm{chiral}} : Z_{\mathrm{Maj}}[\rho; m] \ \mapsto \ Z_{\mathrm{Maj}}[\rho; -m] = (-1)^{\mathcal{I}[\rho]} \ Z_{\mathrm{Maj}}[\rho; m]$$

# Introducing the Arf Invariant

$$\mathcal{I}[\rho] = \mathrm{Arf}[\rho]$$




[Proof of equivalence by Atiyah]

Integrate out a massive Majorana fermion $\Longrightarrow$

- nothing if m > 0
- $i\pi\,\mathrm{Arf}[\rho]$ if m < 0

Kapustin, Thorngren, Turzillo and Wang, ’14

# Attempting a Duality

Ising = Majorana

except...

# A New Ingredient: A $Z_2$ Gauge Field

Consider a scalar field, $\sigma$. We take our theory to have a $Z_2$ symmetry

$$\mathbf{Z}_2 : \sigma \ \mapsto \ -\sigma$$

We can gauge this symmetry. We introduce a $Z_2$-valued gauge field *s.*

$$\int_\gamma s \in \{0, 1\}$$

# Duality 1: Majorana = Ising/$Z_2$

The correct statement is a *fermionic* duality:

$$\int i\bar{\chi}\,\not{D}_\rho\,\chi \quad\longleftrightarrow\quad \int (\mathcal{D}_s\sigma)^2 + \sigma^4 + i\pi\Big[\mathrm{Arf}[s\cdot\rho] + \mathrm{Arf}[\rho]\Big]$$

dynamical $Z_2$ gauge field

a new spin structure

The gauge symmetry acts as $\mathbf{Z}_2^{\mathrm{gauge}} : \sigma \;\mapsto\; -\sigma$

There is also a global symmetry: $\mathbf{Z}_2^{\mathrm{global}} : \chi \;\mapsto\; -\chi$

[This is also known as $(-1)^F$.]

Kapustin, Thorngren, Turzillo and Wang, '14; Senthil, Son, Wang and Xu '18; Tachikawa (unpublished)

# Duality 1: Majorana = Ising/$Z_2$

We can keep track of $Z_2$ global quantum numbers by introducing a background field

$$\int \; i\bar{\chi}\not{D}_{S\cdot\rho}\,\chi \quad \longleftrightarrow \quad \int \; (\mathcal{D}_s\sigma)^2 + \sigma^4 + i\pi\Big[\mathrm{Arf}\,[s\cdot\rho] + \mathrm{Arf}\,[\rho] + \int s\cup S\Big]$$

background $Z_2$ gauge field

dynamical $Z_2$ gauge field

$$\mathbf{Z}_2^{\mathrm{global}} : \chi \;\mapsto\; -\chi \qquad\qquad \mathbf{Z}_2^{\mathrm{gauge}} : \sigma \;\mapsto\; -\sigma$$

Kapustin, Thorngren, Turzillo and Wang, ’14; Senthil, Son, Wang and Xu ‘18; Tachikawa (unpublished)

# Duality 1: Matching Phases

$$\int \; i\bar{\chi}\not{D}_{S\cdot\rho}\,\chi \quad \longleftrightarrow \quad \int \; (\mathcal{D}_s\sigma)^2+\sigma^4+i\pi\Big[\mathrm{Arf}\,[s\cdot\rho]+\mathrm{Arf}\,[\rho]+\int s\cup S\Big]$$

Add mass *m* for fermion ⟹

- *m > 0* gives the trivial phase
- *m < 0* gives *i*π Arf [S.ρ]

Add mass $M^2$ for scalar ⟹

- $M^2 < 0$ then $Z_2$ gauge symmetry is Higgsed
  - Theory sits in trivial phase
- $M^2 > 0$ then we have $Z_2$ gauge theory

$$Z_{\text{scalar}}=\sum_s \exp\left(i\pi\left[\mathrm{Arf}\,[s\cdot\rho]+\mathrm{Arf}\,[\rho]+\int s\cup S\right]\right)\sim\exp\Big(i\pi\,\mathrm{Arf}[S\cdot\rho]\Big)$$

$$\Longrightarrow \quad m \quad \longleftrightarrow \quad -M^2$$

# Building a Duality Web

# Duality 2: Majorana/$Z_2$ = Ising

Start with the first duality:

$$\int \; i\bar{\chi}\not{D}_{S\cdot\rho}\,\chi \quad \longleftrightarrow \quad \int \; (\mathcal{D}_s\sigma)^2 + \sigma^4 + i\pi\Big[\mathrm{Arf}\,[s\cdot\rho] + \mathrm{Arf}\,[\rho] + \int s \cup S\Big]$$

add $i\pi \int S \cup T$ to both sides, and then promote S to a dynamical gauge field.

$\Longrightarrow$ a new *bosonic* duality

$$\int \; i\bar{\chi}\not{D}_{s\cdot\rho}\,\chi + i\pi\left[\mathrm{Arf}\,[S\cdot\rho] + \mathrm{Arf}\,[\rho] + \int s \cup S\right] \quad \longleftrightarrow \quad \int \; (\mathcal{D}_S\sigma)^2 + \sigma^4$$

dynamical $Z_2$ gauge field

background $Z_2$ gauge field

# Duality 3: Kramers-Wannier Duality

Start with the second duality:

$$\int \; i\bar{\chi}\not{D}_{s\cdot\rho}\,\chi + i\pi\left[\mathrm{Arf}\left[S\cdot\rho\right] + \mathrm{Arf}\left[\rho\right] + \int s\cup S\right] \quad \longleftrightarrow \quad \int \; (\mathcal{D}_S\sigma)^2 + \sigma^4$$

Once again promote the background $Z_2$ to a dynamical field. You will find..

$$\int i\bar{\chi}\not{D}_{s\cdot\rho}\,\chi + i\pi\left[\mathrm{Arf}[s\cdot\rho] + \mathrm{Arf}\left[T\cdot\rho\right] + \mathrm{Arf}\left[\rho\right] + \int s\cup T\right]$$

$$\longleftrightarrow \int (\mathcal{D}_t\sigma)^2 - \sigma^4 + i\pi\int t\cup T$$

The left-hand side can be viewed as the chiral transform of our original theory!

# Duality 3: Kramers-Wannier Duality

We conclude:

$$\int (\mathcal{D}_S\sigma)^2 + \sigma^4 \quad \longleftrightarrow \quad \int (\mathcal{D}_t\tilde{\sigma})^2 + \tilde{\sigma}^4 + i\pi \int t \cup S$$

Or Ising = Ising/$Z_2$

If we follow through the matching of masses, we find

$$M^2 \longleftrightarrow -\tilde{M}^2$$

Schroer and Truong '79

# Further Dualities

# Coleman’s Bosonization

Dirac Fermion = Compact Boson

Coleman, ‘75

# Coleman's Bosonization

A more accurate phrasing: Dirac/$Z_2$ = Compact Boson

$$\int i\bar{\chi}_1 \, \not{D}_{(s+S)\cdot\rho}\chi_1 + i\bar{\chi}_2 \, \not{D}_{(s+S+C)\cdot\rho}\chi_2 + i\pi \mathrm{Arf}[s\cdot\rho] \quad \longleftrightarrow \quad \int \frac{1}{2\pi}(\mathcal{D}_{S;C}\theta)^2$$

with $\quad \mathbf{Z}_2^S : \theta \;\mapsto\; \theta + \pi$

$$\mathbf{Z}_2^C : \theta \;\mapsto\; -\theta$$

# Coleman's Bosonization

Alternatively: Dirac = Compact Boson/$Z_2$

$$\int \; i\bar{\chi}_1 \, \not{D}_{S\cdot\rho} \, \chi_1 + i\bar{\chi}_2 \, \not{D}_{(S+C)\cdot\rho} \, \chi_2 \quad \longleftrightarrow \quad \frac{1}{2\pi}(D_{s+S;C}\theta)^2 + i\pi \mathrm{Arf}[s\cdot\rho]$$

This is a fermionic CFT $\Longrightarrow$ It is not modular invariant

$\Longrightarrow$ It lies in a topological phase when gapped

[Aside: can use these methods to map out space of c=1 non-modular invariant CFTs]

# RNS = GS

An important duality in string theory:

8 Majorana fermions = 8 Majorana fermions = 8 Majorana fermions

$\mathbf{8}_V$ $\mathbf{8}_S$ $\mathbf{8}_C$

This also plays a key role in condensed matter physics (Fidkowski and Kitaev)

# RNS = GS

work in progress with Carl Turner

A better phrasing:

8 Majorana fermion coupled to $Z_2$ x $Z_2$ gauge field = 8 Majorana fermions

$\mathbf{8}_V$ $\mathbf{8}_S$

$$\sum_{i=1}^{8} i\bar{\chi}_i \, \not{D}_{s;h}\chi_i + i\pi\left[\int s\cup S + h\cup H\right] \quad \longleftrightarrow \quad \sum_{i=1}^{8} i\bar{\tilde{\chi}}_i \, \not{D}_{H+S;S}\tilde{\chi}_i$$

$$\mathbf{Z}_2^1 : \ \chi_i \ \mapsto \ -\chi_i \qquad \text{and} \qquad \mathbf{Z}_2^2 : \ \chi_i \ \mapsto \gamma^3\chi_i$$

[We also need a further duality, involving Arf, to generate full triality]

# Summary

There are some subtle $Z_2$ gauge symmetries in 2d dualities

But once you fix one of them, you can get them all.

Thank you for your attention