The Project

- Study non-Abelian Berry phase in string theory and other supersymmetric systems.

- Various results:
  - Exact results on Berry phase in strongly coupled systems
  - D0-branes: anyons, Hopf maps
  - Gravitational Precession and AdS/CFT

- Based on work with Julian Sonner
  - arXiv:0809.3783 and 0810.1280
- Also earlier work with Julian and Chris Pedder
Review of Non-Abelian Berry Phase

- Berry, Wilczek and Zee
- Prepare system in an energy eigenstate.
- Change parameters slowly. Adiabatic theorem ensures that the system clings to the energy eigenstate (as long as we avoid degeneracies).
Berry Phase

- **Question:** Perform a loop in parameter space. The original state will come back to itself *up to a phase*. What is this phase?

- **Answer:** There is the usual dynamical phase $e^{-iEt}$. But there is another contribution that is independent of time, but depends on the path taken in parameter space. This is Berry’s phase.

- There is also a non-Abelian generalization. Suppose that the energy eigenstate is n-fold degenerate for all values of the parameters. The state now comes back to itself up to a U(n) rotation. (Wilczek and Zee).
Non-Abelian Berry Phase

Space of all parameters, $\vec{\lambda}$

Space of ground states. (Normalize $E=0$)

For each $\vec{\lambda}$, introduce an arbitrary set of bases for the ground states:

$$|n_{a}(\vec{\lambda})\rangle \quad a = 1, \ldots, n$$
Non-Abelian Berry Phase

We want to know the evolution of the state $|\psi(t)\rangle$ under

$$i\partial_t |\psi\rangle = H(\vec{\lambda}(t)) |\psi\rangle = 0$$

We write $|\psi_a(t)\rangle = U_{ab}(t) |n_b(\vec{\lambda}(t))\rangle$ (which assumes the adiabatic theorem)

$$\implies |\dot{\psi}_a\rangle = \dot{U}_{ab} |n_b\rangle + U_{ab} |\dot{n}_b\rangle = 0$$

$$\implies U^\dagger_{ac} \dot{U}_{ab} = -\langle n_a | \dot{n}_b \rangle$$

$$= -\langle n_a | \partial_{\vec{\lambda}} |n_b\rangle \cdot \dot{\vec{\lambda}}$$

$$\equiv -i \vec{A}_{ba} \cdot \dot{\vec{\lambda}}$$
Non-Abelian Berry Connection

The rotation of the state after a closed path is given by

$$U = P \exp \left( -i \oint \vec{A} \cdot d\vec{\lambda} \right)$$

where $\vec{A}_{ba}$, a Hermitian $u(n)$ connection over the space of parameters, is given by

$$\vec{A}_{ba} = -i\langle n_a | \partial_{\vec{\lambda}} | n_b \rangle$$

Note: this is really a connection. Changing the basis at each point, changes the connection by

$$|n'_a(\vec{\lambda})\rangle = \Omega_{ab}(\vec{\lambda}) |n_b(\lambda)\rangle \quad \implies \quad \vec{A}' = \Omega \vec{A} \Omega + i(\partial_{\vec{\lambda}} \Omega) \Omega^\dagger$$
An Example of Abelian Berry Phase

The canonical example of Berry phase is a spin $\frac{1}{2}$ particle in a magnetic field

$$H = -\vec{B} \cdot \vec{\sigma}$$

It’s easy to write down the ground states and compute the Berry connection

$$A_\psi = \frac{1 - \cos \theta}{2B \sin \theta}$$

$$\epsilon_{ijk} F_{jk} = \frac{B_i}{2B^3}$$

This is the Dirac monopole!

$$B_1 = B \sin \theta \sin \psi$$
$$B_2 = B \sin \theta \cos \psi$$
$$B_3 = B \cos \theta$$
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The Magnetic Monopole

- This is a gauge connection over the space of magnetic fields…confusing!

- The singularity at $B=0$ reflects the fact that the two states are degenerate at this point. (Our “Wilsonian effective action” breaks down).

- Moving on a path that avoids the singularity gives rise to a phase $\exp\left(-i \int dS \cdot F\right)$. The physics is dictated by the singularity, even though we steer clear of it!

- The magnetic monopole first appeared in the context of Berry’s phase. (Two years before Dirac’s paper!)
Berry Phase in Susy Quantum Mechanics

- Exact Results for Berry Phase
- Based mostly 0809.3783 and 0810.1280
Part 1: Susy Quantum Mechanics

- \( N=(2,2) \) Susy QM is the dimensional reduction of \( N=1 \) theories in four dimensions. (Sometimes called \( N=4 \) or \( N=4a \))

Why Berry Phase?

- Witten index implies multiple ground states
  - Naturally get non-Abelian connections

- Berry phase is the BPS quantity!

- We will show that the Berry connection must satisfy certain equations.
**Supersymmetric Parameters**

- **Key point:** Parameters can be thought of as the lowest (bosonic) components of supermultiplets
  - complex parameters of the superpotential live in chiral multiplets
  - triplets of parameters live in vector multiplet

Coordinates on parameter space are bosonic components of supermultiplets.
Supersymmetric Holonomy

\[ U = T \exp \left( -i \oint \vec{A} \cdot \dot{\lambda} \, dt \right) \]

- The holonomy should itself be supersymmetric. Generalise to...

\[ U = T \exp \left( -i \oint A \, dt \right) \]

with \[ A = \vec{A} \cdot \dot{\lambda} + \text{susy completion} \]

- \( A \) is a U(N) valued object. To be invariant under susy, we require

\[ \delta A = \frac{d\Theta}{dt} + i[A, \Theta] \]
Chiral Multiplets: \((\phi, \psi_\alpha, F)\)

\[
\mathcal{A} = A(\phi, \phi^\dagger)\dot{\phi} + G(\phi, \phi^\dagger)F + B(\phi, \phi^\dagger)\psi\psi + C(\phi, \phi^\dagger)\bar{\psi}\psi
\]

- A, G, B and C are all NxN matrices. Susy requires

\[
C = [D, D^\dagger] = -[G, G^\dagger]
\]

\[
B = DG \quad \text{and} \quad D^\dagger G = 0 \quad \text{where} \quad D = \frac{\partial}{\partial\phi} + i[A, \cdot]
\]

- These are the Hitchin equations. G has the interpretation of a particular correlation function in the QM

- Multiple chiral multiplets give the tt* equations
  - c.f. Cecotti and Vafa
Vector Multiplets: \((A_0, \vec{X}, \chi_\alpha, D)\)

\[
A = \vec{A}(X) \cdot \dot{\vec{X}} - H(X)D + \vec{C}(X) \cdot \bar{\chi} \vec{\sigma} \chi
\]

- A, H and C are all NxN matrices. Susy requires

\[
C_i = D_i H \equiv \frac{\partial H}{\partial X_i} + i[A_i, H]
\]

\[
F_{ij} = \epsilon_{ijk} D_k H
\]

- These are the Bogomolnyi equations. Their solutions describe BPS monopoles. Again, H is a correlation function in the QM

- Multiple vector multiplets give a generalization of these equations.
An Example: Spin $\frac{1}{2}$ Particle on a Sphere

$$H = -\frac{\hbar^2}{2m} \nabla^2 1_2 - \hbar \vec{B} \cdot \vec{\sigma} \cos \theta + \frac{1}{2} B^2 \sin^2 \theta 1_2$$

- This is a truncation of the N=(2,2) CP$^1$ sigma-model.
- The triplet of parameters B sit in a vector multiplet

- The theory has two ground states for all values of B
  - This means that we get a U(2) Berry connection $\mathbb{R}^3$
The Berry Connection

- The Berry connection must satisfy the Bogomolnyi equation
  \[ F_{ij} = \epsilon_{ijk} \mathcal{D}_k H \]
- For this particular model, $H$ is the correlation function
  \[ H^a_b = \langle a | \cos \theta | b \rangle \]
- This means that we can just write down the answer
  \[ A_i = \epsilon_{ijk} \frac{B_j \sigma^k}{2B^2} \left( 1 - \frac{2Bm/\hbar}{\sinh(2Bm/\hbar)} \right) \]
- Important point: the Berry connection is smooth at the origin
The 't Hooft-Polyakov connection has an expansion

\[ A_i = \epsilon_{ijk} \frac{B_j \sigma^k}{2B^2} \left( 1 - \frac{4Bm}{\hbar} e^{-2Bm/\hbar} + \ldots \right) \]

1-instanton effect

1-loop determinants around the background of the instanton are non-trivial. (c.f. 3 dimensional field theories)

Higher effects are instanton-anti-instanton pairs
  - i-bar pairs can contribute to BPS correlation functions
Summary of Part 1

Summary:
- Non-Abelian Berry connection is BPS quantity.
- Exact results are possible, exhibiting interesting and novel behaviour.

Applications and Future work
- Mathematical: equivariant cohomology and curvature of bundles of harmonic forms
- Quantum Computing: connections for non-Abelian anyons in FQHE states
- Black Holes:
  - Relationship to attractor flows (c.f. de Boer et al.)
  - Entanglement of black holes in Denef’s quantum mechanics
Part 2: Berry Phase of D0-Branes

- Based on arXiv:0801.1813
The Berry Phase of D0-Branes

- Consider SU(2) Susy quantum mechanics with N=2,4,8 and 16 supercharges.
- This describes relative motion of two D0-branes in d=2,3,5 and 9 spatial dimensions.

$$L = \frac{1}{2g^2} \text{Tr} \left( (D_0 X_i)^2 + \sum_{i<j} [X_i, X_j]^2 + i \bar{\psi} D_0 \psi + \bar{\psi} \Gamma^i [X_i, \psi] \right)$$

$$\{ \Gamma^i, \Gamma^j \} = 2 \delta^{ij} \quad i, j = 1, \ldots, d$$

- Work in the Born-Oppenheimer approximation
- Separate D0-branes so that SU(2) breaks to U(1)
- Construct the Hilbert space for excited strings.

- Question: How does Hilbert space evolve as D0-branes orbit?
With N=2 supercharges, the D0-branes move on the plane.
- They are fractional D0-branes, trapped at a singularity of a G2-holonomy manifold.

The Berry phase of the ground state is a minus sign picked up after a single orbit.

This means that after the exchange of particles, the wavefunction changes by

\[ |\Omega\rangle \rightarrow \pm i|\Omega\rangle \]

The D0-branes are anyons!

U(N) matrix model is a description of a gas of N anyons.
D0-Branes in Three Dimensions

- With N=4 supercharges, the D0-branes move in d=3 spatial dimensions
  - They are trapped at a CY singularity
- The ground state does not feel a Berry phase
- Excited states do: the Berry phase is the Dirac monopole. The effective motion of the D0-branes is governed by
  \[ L \sim \frac{1}{g^2} \dot{X}_i^2 + A_i^{\text{Dirac}}(X) \dot{X}_i - |X| \]
- The states follow non-relativistic Regge trajectories
  \[ E^3 \sim g^2 J(J + q) \]

Angular Momentum, J  
Dirac monopole charge, q=1
The \( N=8 \) and \( N=16 \) Theories

- The D0-branes move in \( d=5 \) and \( d=9 \) respectively.
- The excited states are degenerate, multiplets of R-symmetry.
- The Berry phase is non-Abelian.

\[
|\psi_a\rangle \longrightarrow P \exp \left( -i \oint \vec{A} \cdot d\vec{X} \right)_{ab} |\psi_b\rangle
\]

- \( SU(2) \) Yang monopole for \( N=8 \)
  \[
  \frac{1}{8\pi^2} \int_{S^4} \text{Tr} F \wedge F = +1
  \]
- \( SO(8) \) Octonionic Monopole for \( N=16 \)
  \[
  \frac{1}{4!(2\pi)^4} \int_{S^8} \text{Tr} F \wedge F \wedge F \wedge F = +1
  \]
Berry, Hopf and Supersymmetry

There is a nice relationship appearing here between supersymmetry, the four division algebras, and the Hopf maps

\[
\begin{align*}
S^1 &\cong \mathbb{R}P^1 \\
S^2 &\cong \mathbb{C}P^1 \\
S^4 &\cong \mathbb{H}P^1 \\
S^8 &\cong \mathbb{O}P^1
\end{align*}
\]

These non-Abelian Berry phases have appeared in the condensed matter literature: Zhang et al, Bernevig et al.
Summary of Part 2

- N=2 Susy: Sign Flip
  - anyons

- N=4 Susy: Dirac Monopole
  - deformed Regge trajectories

- N=8 Susy: SU(2) Yang Monopole

- N=16 Susy: SO(8) Octonionic Monopole
  - hint at octonionic structure?
Part 3: Berry Phase and AdS/CFT

- Based on arXiv:0709.2136
$N=8$ Susy Quantum Mechanics

The D0-D4-Brane System in IIA String Theory

$$\mathcal{L}_{D0} = \frac{1}{g^2} (\ddot{\vec{X}}^2 + \bar{\lambda} \dot{\lambda}) + \sum_{i=1}^{N} |D_t \phi_i|^2 + |D_t \bar{\phi}_i|^2 + \bar{\psi}_{\alpha i} D_t \psi_{\alpha i}$$

$$- \sum_{i=1}^{N} \vec{X} \cdot (|\phi_i|^2 + |\bar{\phi}_i|^2) + \bar{\psi}_{\alpha i} (\vec{X} \cdot \vec{\Gamma}_{\alpha \beta}) \psi_{\alpha i}$$

+ Yukawa + Potential

with $\Gamma_a$ five $4 \times 4$ matrices such that $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$
Born-Oppenheimer Approximation

- Make D0-branes heavy: $g^2 \rightarrow 0$
- Treat $X$ as a fixed parameter, and quantize the D0-D4 strings
- Integrate out the D0-D4 strings to write an effective action for $X$
Quantizing the Fermions

\[ \{ \psi_\alpha, \bar{\psi}_\beta \} = \delta_{\alpha\beta} \]  

Creation and Annihilation Operators

\[ \alpha = 1, \ldots, 4 \]

Define \( |0\rangle \) such that \( \psi_\alpha |0\rangle \). Then the fermionic sector of D0-D4 strings gives

<table>
<thead>
<tr>
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**Ground State**
Quantizing the Fermions

\[ \{ \psi_\alpha, \bar{\psi}_\beta \} = \delta_{\alpha\beta} \]

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**Berry Phase**

**Question:** Sit in one of the two excited states in the sector $\bar{\psi}_\alpha |0\rangle$. Adiabatically move the D0-brane around the D4. What is the Berry phase?

**Answer:** It is the *SU(2) Yang Monopole*. This is a rotationally symmetric connection over $\mathbb{R}^5$

$$\int_{S^4} \text{Tr} F \wedge * F = 8\pi^2$$

This has appeared before as Berry’s phase in the condensed matter literature.
Yin and Yang

- In the $\bar{\psi}_\alpha |0\rangle$ sector, the Berry connection is the Yang monopole.

- In the $\bar{\psi}_\alpha \bar{\psi}_\beta \bar{\psi}_\gamma |0\rangle$ sector, the Berry connection is the anti-Yang, or Yin, monopole.

\[
A^\text{Berry}_\mu = \begin{pmatrix}
A^\text{Yang}_\mu & 0 \\
0 & A^\text{Yin}_\mu
\end{pmatrix} \xrightarrow{\text{g.t.}} \frac{X_\nu \Gamma_{\nu\mu}}{X^2}
\]

\[
\Gamma_{\mu\nu} = \frac{1}{4i} [\Gamma_\mu, \Gamma_\nu]
\]
Supergravity and Strong Coupling

- The previous analysis is valid when $g^2 N \ll X^3$
- When $g^2 N \gg X^3$ we can instead replace the D4-branes by their supergravity background

\[ ds^2 = \frac{1}{H} \left( -dt^2 + d\vec{y}^2 \right) + \frac{H}{1 + \frac{g^2 N}{X^3}} \, d\vec{X}^2 \]
\[ e^{2\phi} = H^{-1/2} \]

We now place a probe D0-brane in this background and read off the low-energy dynamics.

* Caveat: This isn’t quite the usual AdS/CFT: there is a non-renormalization theorem at play here.
The D0-Brane Probe

The low-energy dynamics of the D0-brane in the D4-brane background is given by

\[ \mathcal{L}_{D0} = \frac{1}{2} H(X) \dot{X}^2 + H(X) \bar{\lambda}_\alpha D_t \lambda_\alpha + \ldots \]

The covariant derivative for the spinor is

\[ D_t \lambda_\alpha = \dot{\lambda}_\alpha + \dot{X} \cdot \bar{\omega}_\alpha^\beta \lambda_\beta \]

which ensures parallel transport of free spinors.
Gravitational Precession

- The excited states that we studied at weak coupling carry the same quantum numbers as a spinning particle at strong coupling. The quantum Berry connection maps into *classical gravitational precession* of the spin.

- In the near horizon limit of the D4-branes,
  \[
  H = 1 + \frac{g^2 N}{X^3} \rightarrow \frac{g^2 N}{X^3}
  \]

- The spin connection of the metric in the near horizon limit is
  \[
  \omega_\mu = \frac{3}{2} \frac{X_\nu \Gamma_{\nu\mu}}{X^2}
  \]

- which differs by 3/2 from weak coupling result. (Smooth, or level crossing?)
Summary of Part 3

- Summary
  - Berry’s phase is associated to Yang Monopole.
  - Berry’s Phase in strongly coupled system is gravitational precession.

- Questions
  - Relation to six-dimensional (2,0) theory and Wess-Zumino terms?