Quantum Dynamics of Supergravity

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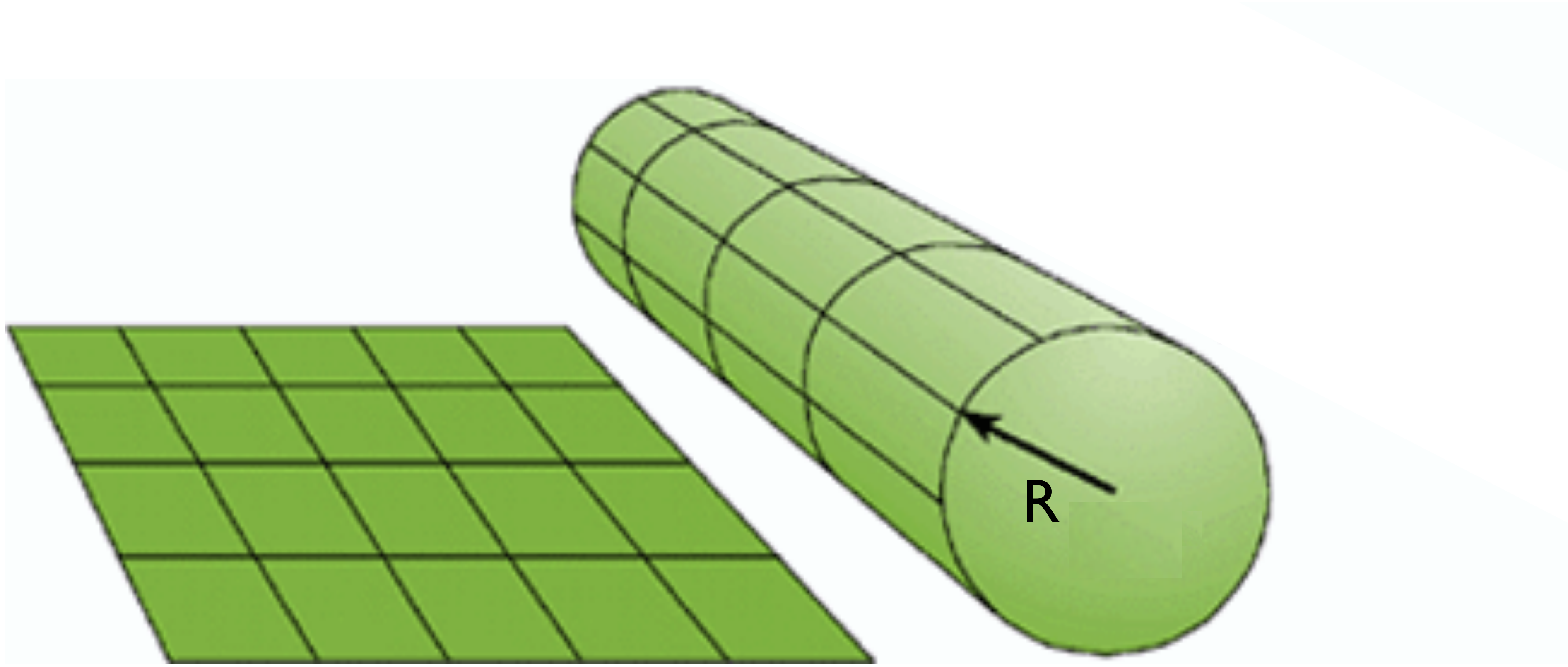
An Old Idea: Euclidean Quantum Gravity

$$\boxed{\mathcal{Z} = \sum_{\text{topology}} \int \mathcal{D}g \; \exp\left(- \int d^4x \, \sqrt{g} \, \mathcal{R} \right)}$$

A Preview of the Main Results

Kaluza-Klein Theory: $\mathcal{M} = \mathbb{R}^{1,d-1} \times \mathbf{S}^1$



There is a long history of quantum instabilities of these backgrounds

- Casimir Forces
- Tunneling to “Nothing”

Appelquist and Chodos ‘83
Witten ‘82

The Main Result

Kaluza-Klein compactification of *N=1* Supergravity is unstable.

$$\boxed{\mathcal{W} \sim \exp\left(-\frac{\pi R^2}{4G_N} - i\sigma\right)}$$

Kaluza-Klein dual photon: $\partial_\mu \sigma \sim \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}$

Grimm and Savelli 2011

And Something Interesting Along the Way…

Quantum Gravity has a hidden infra-red scale!

$$\Lambda_{\rm grav} \ll M_{\rm pl}$$

This is the scale at which gravitational instantons contribute

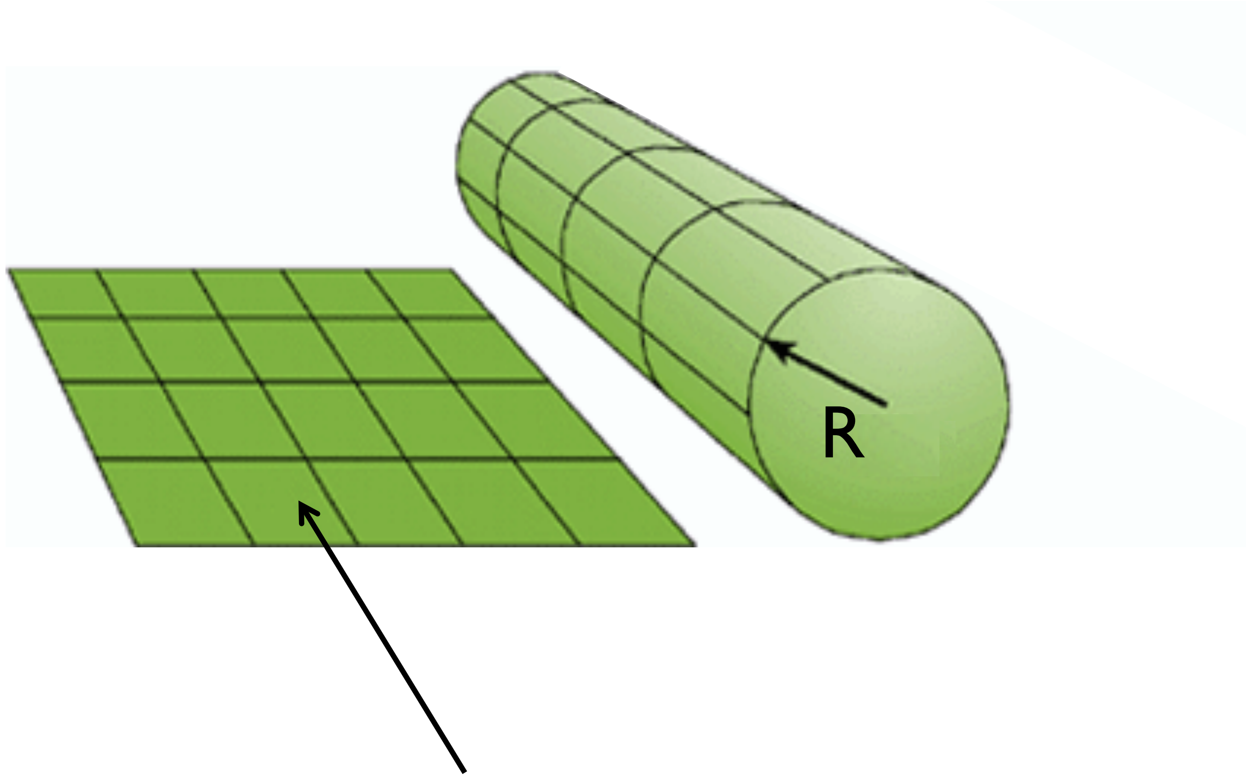
The Theory: *N=1* Supergravity

$$S = \frac{M_{\rm pl}^2}{2} \int d^4x \, \sqrt{-g} \left(\mathcal{R}_{(4)} + \bar{\psi}_\mu \gamma^{\mu\nu\rho} \mathcal{D}_\nu \psi_\rho \right)$$

Compactify on a Circle

$$ds^2_{(4)} = \frac{L^2}{R^2} ds^2_{(3)} + \frac{R^2}{L^2} \left(dz^2 + A_i dx^i\right)^2 \qquad z \in [0, 2\pi L)$$

$$\mathcal{M} = \mathbb{R}^{1,2} \times \mathbf{S}^1$$



Fields $R(x^i)$ and $A_i(x^i)$ live here

L is fiducial scale

Classical Low-Energy Physics

$$
\begin{aligned}
S_{\text{eff}} &= \frac{M_{\text{pl}}^2}{2}\int d^4x\,\sqrt{-g}\,\mathcal{R}_{(4)} \\
&= \frac{M_3}{2}\int d^3x\,\sqrt{-g_{(3)}}\left[\mathcal{R}_{(3)} - 2\left(\frac{\partial R}{R}\right)^2 - \frac{1}{4}\frac{R^4}{L^4}F_{ij}F^{ij}\right]
\end{aligned}
$$

$$M_3 = 2\pi L M_{\text{pl}}^2$$

Or, if we work with the dual photon $\partial_\mu \sigma \sim \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}$

$$S_{\text{eff}} = \int d^3x\,\sqrt{-g_{(3)}}\,\left[\frac{M_3}{2}\,\mathcal{R}_{(3)} - M_3\left(\frac{\partial R}{R}\right)^2 - \frac{1}{M_3}\frac{L^2}{R^4}\left(\frac{\partial\sigma}{2\pi}\right)^2\right]$$

Goal: Understand quantum corrections to this action.

Perturbative Quantum Corrections

Perturbative Quantum Corrections

One-Loop Effects:

- UV divergences
  - Running Gauss-Bonnet term
  - The anomaly
- Finite corrections ~ $1/M_{\rm pl}^2 R^2$
  - Casimir forces
  - Other corrections

One-Loop Divergences

't Hooft and Veltman '74

At one-loop in pure gravity, there are three logarithmic divergences

$$\mathcal{R}^2 \quad , \quad \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} \quad , \quad \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$$

These two can be absorbed by a field redefinition of the metric

The Riemann^2 term can be massaged into Gauss-Bonnet.

$$\chi = \frac{1}{8\pi^2}\int d^4x \; \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2$$

This is purely topological. It doesn't affect perturbative physics around flat space.

The Gauss-Bonnet Term

$$S_\alpha = \frac{\alpha}{8\pi^2} \int d^4x \; \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2 = \alpha\chi$$

The coupling runs logarithmically

$$\alpha(\mu) = \alpha_0 - \alpha_1 \log\left(\frac{M_{UV}^2}{\mu^2}\right)$$

where the *beta function* is given by

Christensen and Duff '78
Perry '78; Yoneya '78

$$\alpha_1 = \frac{1}{48 \cdot 15}\left(848N_2 - 233N_{3/2} - 52N_1 + 7N_{1/2} + 4N_0\right)$$

For us…

$$\boxed{\alpha_1 = 41/48}$$

A New RG-Invariant Scale

$$\alpha(\mu) = \alpha_0 - \alpha_1 \log\left(\frac{M_{UV}^2}{\mu^2}\right)$$

As in Yang-Mills, we can replace the log running with an RG invariant scale

$$\boxed{\Lambda_{\text{grav}} = \mu \exp\left(-\frac{\alpha(\mu)}{2\alpha_1}\right)}$$

This scale will be associated with non-trivial spacetime topologies.

(We will see an example)

Another Divergence: The Anomaly

The classical action is invariant under rotations of the phase of the fermion.

This *U(1)*$_R$ symmetry does not survive in the quantum theory.

$$\nabla_\mu J_5^\mu = \frac{1}{24 \cdot 16\pi^2} \left(21 N_{3/2} - N_{1/2}\right) {}^\star\mathcal{R}_{\mu\nu\rho\sigma}\,\mathcal{R}^{\mu\nu\rho\sigma}$$

The phase of the fermion can be absorbed by shifting the theta term

$$S_\theta = \frac{\theta}{16\pi^2}\int d^4x\sqrt{-g}\;{}^\star\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$$

Topological Terms

One-loop effects tell us that we should consider two topological terms

$$S_\alpha = \frac{\alpha}{32\pi^2} \int d^4x \sqrt{g}\ {}^\star\mathcal{R}^\star_{\mu\nu\rho\sigma}\,\mathcal{R}^{\mu\nu\rho\sigma}$$

$$S_\theta = \frac{\theta}{16\pi^2} \int d^4x \sqrt{-g}\ {}^\star\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$$

In supergravity, these two coupling constants sit in a chiral multiplet

$$\tau_{\text{grav}} = \alpha + 2i\theta$$

Finite Quantum Corrections

Casimir Energy:

$$V_{\text{eff}} = -\frac{N_B - N_F}{720\pi}\frac{L^3}{R^6}$$

Appelquist and Chodos ’83

Supersymmetry means that N_B=N_F and this Casimir energy vanishes.

But there are other effects….

Finite Quantum Corrections

One loop corrections to the kinetic terms give

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\left(M_3 + \frac{5}{16\pi}\frac{L}{R^2}\right)\mathcal{R}_{(3)} - \left(M_3 - \frac{1}{6\pi}\frac{L}{R^2}\right)\left(\frac{\partial R}{R}\right)^2 - \left(M_3 + \frac{11}{24\pi}\frac{L}{R^2}\right)^{-1}\frac{L^2}{R^4}\left(\frac{\partial\sigma}{2\pi}\right)^2$$

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Something important: these two numbers are different!

The Complex Structure

The two fields *R* and σ must combine in a complex number

Classically:

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \left(\frac{\partial R}{R}\right)^2 + \frac{1}{M_3^2}\frac{L^2}{R^4}\left(\frac{\partial\sigma}{2\pi}\right)^2 \\ &= \frac{1}{(\mathcal{S}+\mathcal{S}^\dagger)^2}\partial\mathcal{S}\partial\mathcal{S}^\dagger\end{aligned}$$

$$\boxed{\mathcal{S} = 2\pi^2 M_{\text{pl}}^2 R^2 + i\sigma}$$

The Complex Structure

The two fields *R* and σ must combine in a complex number

At one-loop

$$\mathcal{L}_{\rm eff} = \left(1 - \frac{1}{6\pi}\frac{L}{M_3 R^2}\right)\left(\frac{\partial R}{R}\right)^2 + \left(1 + \frac{11}{24\pi}\frac{L}{M_3 R^2}\right)^{-1}\frac{L^2}{R^4}\left(\frac{\partial \sigma}{2\pi}\right)^2$$

We want to write this in the form

$$\mathcal{L}_{\rm eff} = K(\mathcal{S}, \mathcal{S}^\dagger)\,\partial\mathcal{S}\partial\mathcal{S}^\dagger$$

$$\Longrightarrow \quad \boxed{\mathcal{S} = 2\pi^2 M_{\rm pl}^2 R^2 + \frac{7}{48}\log(M_{\rm pl}^2 R^2) + i\sigma}$$

Non-Perturbative Quantum Corrections

Gravitational Instantons

Look for other saddle points of the action

We want these to contribute to the (super)potential. They must obey

$$\mathcal{R}_{\mu\nu\rho\sigma} = \pm {}^{\star}\mathcal{R}_{\mu\nu\rho\sigma}$$

Taub-NUT Instantons

The appropriate metrics are given by the multi-Taub-NUT solutions

Gibbons and Hawking ‘78

$$ds^2 = U(\mathbf{x})d\mathbf{x}\cdot d\mathbf{x} + U(\mathbf{x})^{-1}\left(dz + \mathbf{A}\cdot d\mathbf{x}\right)^2$$

with $$U(\mathbf{x}) = 1 + \frac{L}{2}\sum_{a=1}^{k}\frac{1}{|\mathbf{x}-\mathbf{X}_a|} \quad \text{and} \quad \nabla\times\mathbf{A} = \pm\nabla U$$

From the low-energy 3d perspective, these look like Dirac monopoles.

This is the gravitational verson of Polyakov’s famous calculation. Polyakov ‘77

Gross ’84

Hartnoll and Ramirez ‘13

The Boundary of the Space

The boundary of Taub-NUT is not the same as the boundary of flat space.

$$\partial(\mathbb{R}^3 \times \mathbf{S}^1) = \mathbf{S}^2 \times \mathbf{S}^1 \quad \text{but} \quad \partial(\mathrm{TN}_k) = \mathbf{S}^3/\mathbf{Z}_k$$

Should we include such geometries in the path integral?

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Should we include such geometries in the path integral?

Yes!

c.f. Atiyah-Hitchin with boundary a circle fibre over ***RP***^{2} for which the answer is probably no!

Zero Modes of Taub-NUT

$$ds^2 = U(\mathbf{x})d\mathbf{x}\cdot d\mathbf{x} + U(\mathbf{x})^{-1}\left(dz + \mathbf{A}\cdot d\mathbf{x}\right)^2$$

$$U(\mathbf{x}) = 1 + \frac{L}{2}\sum_{a=1}^{k}\frac{1}{|\mathbf{x}-\mathbf{X}_a|} \quad \text{and} \quad \nabla\times\mathbf{A} = \pm\nabla U$$

$3k$ bosonic zero modes

$2k$ fermionic zero modes

⟹ Only $k=1$ solution contributes to the superpotential

Doing the Computation

Action, Zero Modes, Jacobians, Determinants, Propagators….

The Determinants

$$\text{dets} = \frac{\det(\text{Fermions})}{\det(\text{Bosons})}$$

Supersymmetry $\Longrightarrow$ $\text{dets} = 1 \; ?$ Hawking and Pope ‘78

The Determinants

In a self-dual background, you can write the determinants as

$$\mathrm{dets} = \left. \frac{\det' \not{D}^\dagger \not{D}}{\det \not{D} \not{D}^\dagger} \right|_{\mathrm{spin}-3/2}^{1/4} \left. \frac{\det' \not{D}^\dagger \not{D}}{\det \not{D} \not{D}^\dagger} \right|_{\mathrm{spin}-1/2}^{-1/2}$$

A somewhat detailed calculation gives

$$\mathrm{dets} = A \left(\mu^2\right)^{41/48} \left(\frac{1}{R^2}\right)^{7/48}$$

An ugly number

UV cut-off scale

We’ve seen these fractions before!

The Superpotential

The calculation gives

$$\mathcal{W} = C\left(\frac{\mu^2}{M_{\rm pl}^2}\right)^{41/48}\left(\frac{1}{M_{\rm pl}^2R^2}\right)^{7/48} e^{-S_{\rm TN}-i\sigma}e^{-\tau_{\rm grav}}$$

$$C = \frac{\left(4e^{24\zeta'(-1)-1}\right)^{7/48}}{2(4\pi)^{3/2}}$$

$$S_{\rm TN} = 2\pi^2 M_{\rm pl}^2 R^2$$

Topological terms

All the pieces now fit together

$$\boxed{\mathcal{W} = C\left(\frac{\Lambda_{\rm grav}^2}{M_{\rm pl}^2}\right)^{41/48} e^{-\mathcal{S}}}$$

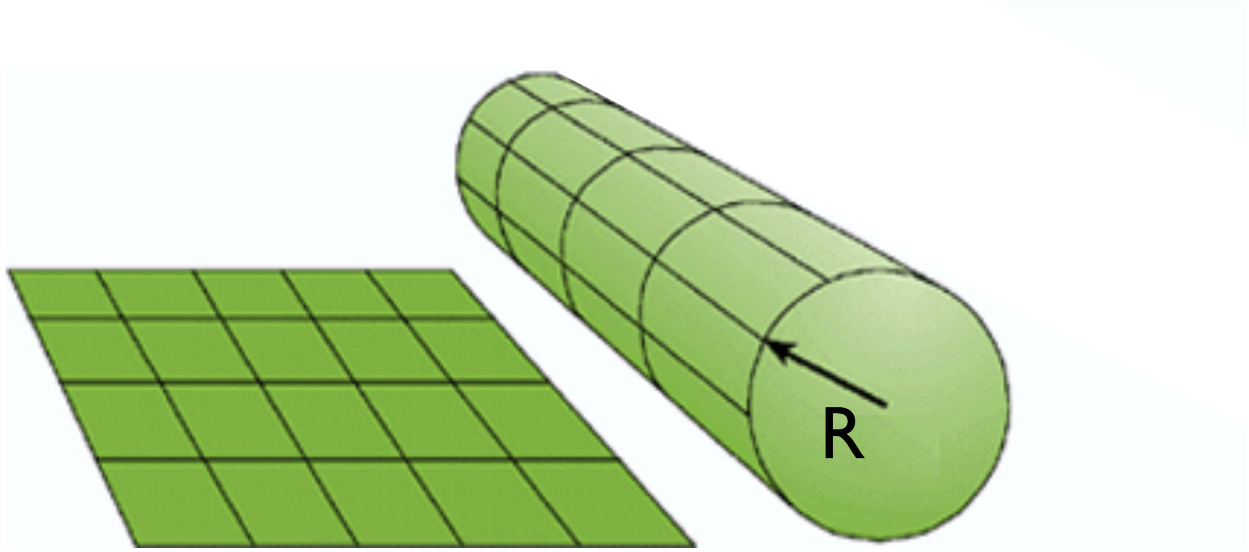
with $\mathcal{S} = 2\pi^2 M_{\rm pl}^2 R^2 + \dfrac{7}{48}\log(M_{\rm pl}^2 R^2) + i\sigma$

The Potential

Kaluza-Klein compactification of *N=1* supergravity is unstable

$$\boxed{V \sim M_3^3 (R\Lambda_{\text{grav}})^{41/24} \exp\left(-4\pi^2 M_{\text{pl}}^2 R^2\right)}$$

The ground state has $R \longrightarrow \infty$



Open Questions

$$\Lambda_{\text{grav}}$$

What is this good for?

What is it in our Universe?

Thank you for your attention