Geometry Through the Eyes of Physics

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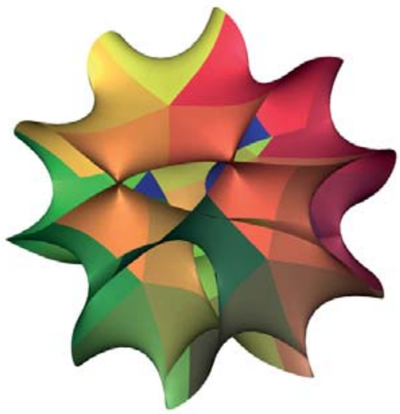
It's no secret that there is a close connection between geometry and physics. Probably the most famous example is the theory of General Relativity, in which the force of gravity is recast in terms of the geometry of space and time. The purpose of this article, however, is not to wax poetic about geometry in Nature. Instead, I'd like to describe how things work the other way round, when Nature gets into geometry. I will try to explain how we can use ideas from physics to give new insight into mathematics.

To tell the story, we'll need two simple ideas: one from maths and one from physics. From maths, the main character is a *manifold*. If you haven't heard of this before, then you should have in the back of your mind a curved, closed surface, like that of a sphere or a torus. A manifold is a generalisation of this shape to higher dimensions. The purpose of geometry is to understand the properties of different manifolds, the relationships between them and the language we need to describe them. Meanwhile, from physics, the only object that we'll need to begin with is the humble particle. Our plan is as follows: we'll place the particle on the manifold and let it roam around. By understanding the behaviour of the particle, we'll try to infer various properties of the underlying space.

To start, we'll think about a particle obeying the laws of classical mechanics. Here there are few surprises and the particle does exactly what you would expect: it rolls around, guided by the contours of the space. The path it takes has some special mathematical properties and is called a *geodesic*. But the particle is too limited to know anything very deep about the underlying manifold. Its perspective is too parochial; it knows only about the small region in its immediate neighbourhood and has little to tell us about the global properties of the manifold.



A Calabi-Yau manifold
Andrew J Hanson, Indiana University

Geometry and Quantum Mechanics

Things get more interesting when we turn to quantum mechanics. In the quantum world, the particle no longer has a definite position. Instead, things are more uncertain and we have to talk in the language of probabilities. The mathematical description of a quantum particle is in terms of a wavefunction, $\psi(\mathbf{x})$. This is a complex valued function, with $\mathbf{x}$ a set of coordinates which label points on the manifold. The probability of finding a particle at the point $\mathbf{x}$ is proportional to $|\psi(\mathbf{x})|^2$.

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The fact that the quantum particle spreads out in a wave of uncertainty gives it more power. It can feel its way all over the manifold. It knows about the global structure of the space. The state of the particle is described by the Schrödinger equation

$$-\nabla^2\psi = E\psi \qquad (1)$$

You've probably seen the symbol ∇^2 before. It's called the Laplacian. Roughly speaking, it means that you should differentiate ψ twice with respect to every coordinate that it depends upon. The first time you see the Laplacian is usually in the context of flat $\mathbb{R}^3$, where $\mathbf{x} = (x,y,z)$ and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

There is an obvious generalisation of this to different dimensions. But, most importantly, there is also a generalisation of the Laplacian to manifolds that are curved. In this case, the Laplacian depends on the *metric* on the manifold which means that the symbol contains within it information about the distances between different points on the manifold.

The E in equation (1) is just a real number. Physicists would identify it with the energy of the particle. The key idea is that the Schrödinger equation doesn't admit solutions $\psi(\mathbf{x})$ for any value of E. Instead, there are only solutions when the energy E takes certain, discrete values. Moreover, because ∇^2 depends on the underlying space, so too does the list of allowed energies. This provides a very different way of thinking about geometry. You give me a manifold and specify its shape and curvature (or, more precisely, its topology and metric). With that information, I solve the Schrödinger equation and hand you back a list of numbers E. That list of numbers is called the spectrum of the Laplacian and it contains, encoded with it, much of the information about the manifold. This way of thinking is called *spectral geometry*.

There is a more down-to-earth version of spectral geometry, made famous by the mathematician Mark Kac in an article called *"Can One Hear the Shape of a Drum?"*. The frequencies at which a drum beats are again governed by the equation (1), now with particular boundary conditions imposed by the shape of the rim of the drum. The question is: if you know all the frequencies, can you figure out the shape? The answer, it turns out, is no, but you can extract a lot of information. Similarly, it is known in geometry that the spectrum is not necessarily sufficient to determine uniquely the underlying manifold. Nonetheless, the study of spectral geometry is a rich subject, with different properties of the manifold encoded in the spectrum in interesting ways.

It will be useful to work through a (very) simple example of spectral geometry: the one-dimensional circle. We will label the position along the circle by the coordinate x. If the circle has radius R, we should identify $x \equiv x + 2\pi R$. The Schrödinger equation now reads

$$-\frac{d^2\psi}{dx^2} = E\psi.$$

The solutions are simply $\psi = e^{inx/R}$. The information that the space is a circle arises through the requirement that ψ is single valued, so that $\psi(x) = \psi(x + 2\pi R)$. This tells us that we must have $n \in \mathbb{Z}$. The spectrum of the circle is therefore just a tower of numbers

$$E = \frac{n^2}{R^2}, \qquad n \in \mathbb{Z}$$

We'll return to this shortly.

Although I introduced spectral geometry by thinking about quantum physics, the subject wasn't discovered by physicists. Nonetheless, it's pleasing that it sits so naturally in the framework of quantum mechanics and there are many further related connections between the two subjects. For example, a more complicated quantum mechanical Hamiltonian which has a property called *supersymmetry* naturally captures the de Rahm or Dolbeault cohomology of the manifold. In this way, many of the great results from differential geometry can be recast in the language of quantum mechanics. However, rather than exploring these directions here, I would instead like to tell you about something novel and surprising that came out of thinking about geometry in the language of physics.

Geometry and String Theory

String theory is currently the best guess that we have for a unified theory of gravity and quantum mechanics. The basic idea is, on the face of it, slightly daft: string theory postulates that, at the fundamental level, if you look deep inside every particle, you will see a tiny vibrating loop of string. At the moment there is no experimental evidence for string theory. Nonetheless, it is a powerful mathematical framework. Here we're going to bring that framework to bear on questions in geometry. We use the same strategy that we've seen above and ask: what is the energy spectrum of a string moving on a manifold?

Let's return to our example of the circle. Now there are two different things that the string can do. First, the string can form a little loop which then moves around the circle. Because, from afar, this loop of string looks like a particle, it shouldn't be too much of a surprise to learn that the energy spectrum is identical to that of a particle: $E = n^2/R^2$ with $n \in \mathbb{Z}$. But the string can also do something that the particle can't: it can stretch itself all around the circle. You can think of the string as an elastic band; stretching it costs energy and a string which winds m times around the circle has energy $E = (2\pi mR)^2$, with $m \in \mathbb{Z}$. This means that the energy spectrum of a string moving on a circle consists of two towers of numbers

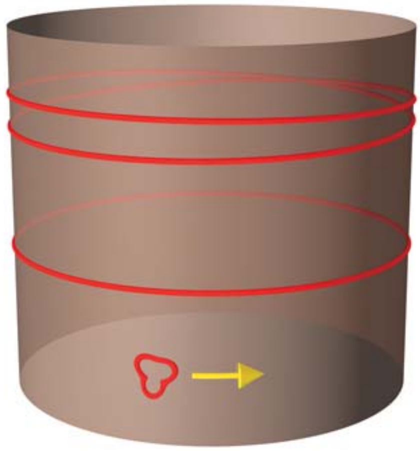
$$E = \frac{n^2}{R^2} + 4\pi^2 m^2 R^2, \quad n, m \in \mathbb{Z}.$$

But there's something interesting here. This set of numbers remains the same if we swap

$$R \longleftrightarrow \frac{1}{2\pi R}. \tag{2}$$

This means that, if all you're given is this list of numbers, then you can't tell the difference between very big circles of size R and very small circles of size $1/2\pi R$. As far as the string is concerned, these circles look exactly the same! Of course, we've only discussed the energy spectrum of the string but it turns out that all properties of the string remain invariant under the interchange (2). Strings really can't tell the difference between big circles and small circles. This beautiful fact has a rubbish name: it is called *T-duality*.

The confusion of strings extends to other manifolds as well. Roughly speaking, manifolds come in pairs. Although particles view these pairs very differently, to a string they look identical. (This is literally true of a special class of manifolds called *Calabi-Yau* and there is a slightly generalised version of the statement for other manifolds). But these two manifolds are not related in a simple way like the big and small circles. Instead, at first sight, the two manifolds seem to have nothing to do with each other. Typically, they don't even have the same topology (i.e. the same number of holes).



How strings can behave
Steuard Jensen, Alma College

This pairing between manifolds is called *mirror symmetry*. The string's inability to distinguish between these two manifolds turns out to be a great strength. For a start, we learn that there's a very surprising and unexpected relationship between manifolds. Moreover, it turns out that mathematicians were often able to say a lot about one of these manifolds, but almost nothing about the other. Yet, according to string theory, the two manifolds should be identical; you just have to look at them in the right way. Any question that you can answer about the first manifold is telling you something interesting about the other. (Technically, questions in *complex geometry* for the first manifold are turned into questions in *symplectic geometry* for the second). Mirror symmetry then becomes a powerful tool which allows you to reinterpret properties of one manifold to provide answers to previously unsolved questions about the other.

Mirror symmetry was discovered almost 25 years ago. In the intervening time, it has become one of the most vibrant areas of research in geometry, with insight coming from both mathematicians and physicists. There is, admittedly, a difference in the style of research. Physicists tend not to be overly consumed with matters of rigour, relying instead on an intuition for how Nature should work to build conjecture upon conjecture. Mathematicians, of course, are not content until each conjecture becomes a proof. Yet this is one of an increasing number of areas in which mathematicians and physicists find themselves exploring the same questions hand in hand. It is a relationship which has enriched both communities.