The Flop and the Glop

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Summary

- Work with Kentaro Hori
  - "Aspects of Non-Abelian Gauge Dynamics in Two Dimensional N=(2,2) Theories", hep-th/0609032
  - "Summing the Non-Abelian Instantons", to appear

- What we do:
  - Study basic questions about supersymmetric U(N) and SU(N) gauge theories in two dimensions.
    - Number of Ground States (Witten Index)
    - Does the gauge theory flow to a superconformal theory?
    - Is the conformal theory singular? (Is the ground state normalizable?)

- Why we do it:
  - Relationship to Calabi-Yau manifolds in Grassmannians

The Plan

- Review of Topology Change in String Theory
  - The Flop Transition
  - The Conifold Transition
  - Time Dependence in String Theory
- The Glop Transition
  - Gauged Linear Sigma Models and Calabi-Yau Manifolds
  - The Vacuum Structure of Non-Abelian Gauge Theories
  - A New Topology Changing Transition

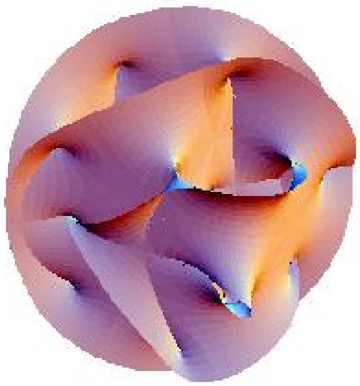
Why Topology Change?

- Why are we interested in topology change?
  - Engineering.
  - Early Universe: (e.g. in string theory the dynamics of moduli, the connectivity of the landscape)

  - More generally, it is an example of a process for which classical general relativity is not sufficient and we require a quantum theory of gravity.

- String theory is supposed to be such a theory…

The Calabi-Yau Moduli Space



A Calabi-Yau manifold X has moduli

$h^{1,1}(X)$ Kahler class (size) moduli

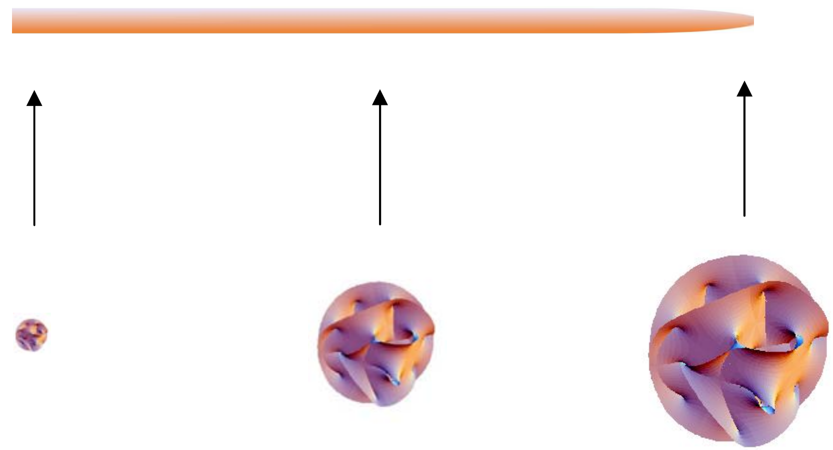
$h^{2,1}(X)$ Complex Struture (shape) moduli

- To change the topology, we *adiabatically* change the moduli until the manifold is singular
- We then study how the string reacts to this singularity, including
  - α' effects
  - g_s effects

The Classical Moduli Space $h^{1,1}(X) = 1$

$$r \in \mathbf{R}^+$$

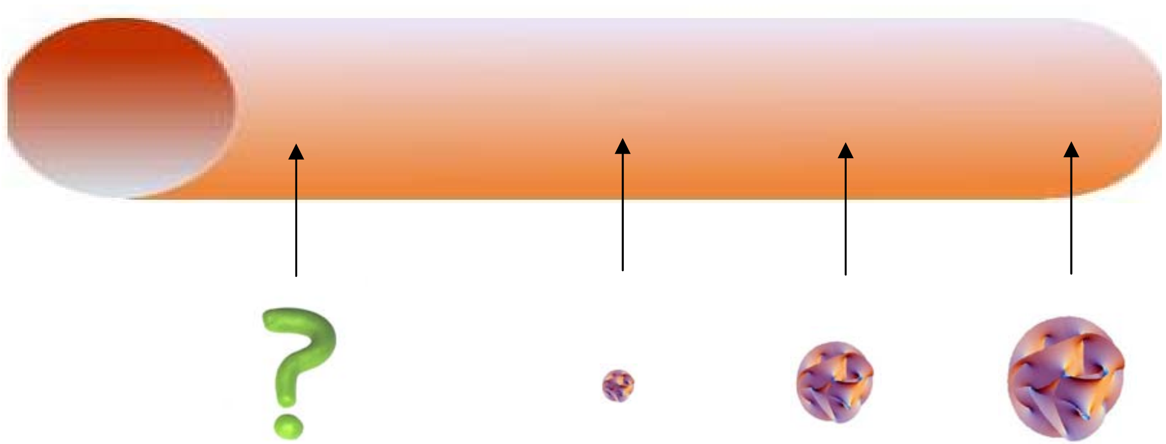
Kahler moduli



- The Kahler modulus is real and positive.
- The moduli space is the half-line
- The manifold becomes vanishingly small at r=0

The Quantum Moduli Space

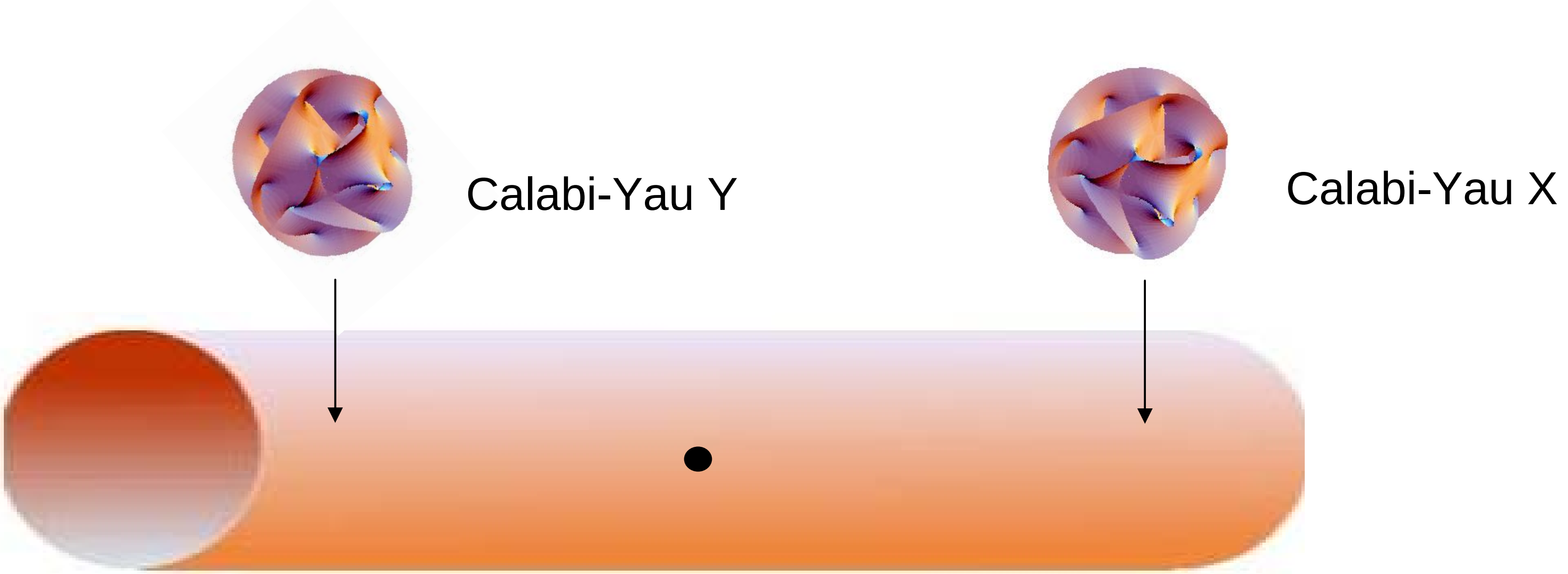
- The quantum geometry is defined by a d=1+1 sigma model on the Calabi-Yau space
- The Kahler class is naturally complexified $t = r + i \int_{\mathbf{S}^2} B_2$



- In this framework, we can ask what happens when we continue r to negative values

The Flop Transition

Witten, ’93
Greene, Morrison and Plesser, ‘93



- The Flop is smooth in perturbative string theory --- i.e. it relies upon α' effects, but not g_s effects.
- A 2-cycle shrinks, and another 2-cycle grows. X and Y have the same Hodge numbers; they differ only by more subtle topological invariants.
- X and Y are “birationally equivalent”

An Example

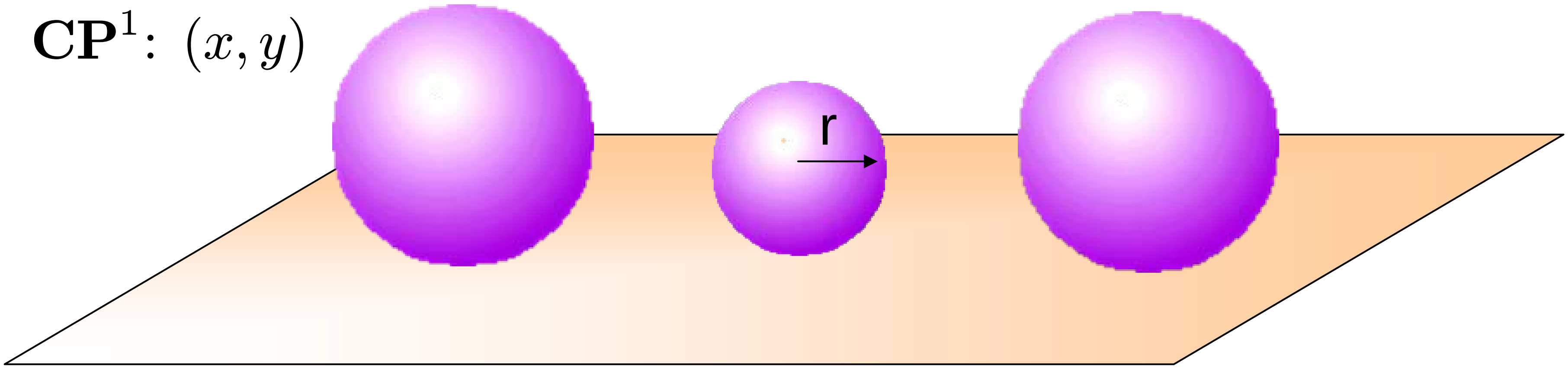
$$|x|^2 + |y|^2 - |z|^2 - |w|^2 = r$$

$$(x, y) \to e^{i\alpha}(x, y)$$

$$(z, w) \to e^{-i\alpha}(z, w)$$

$$r > 0$$

$\mathbf{CP}^1$: (x, y)



$\mathbf{C}^2$ plane: (z, w)

$$X \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$$ or "$X \cong \mathbf{C^2} \times \mathbf{S}^2$"

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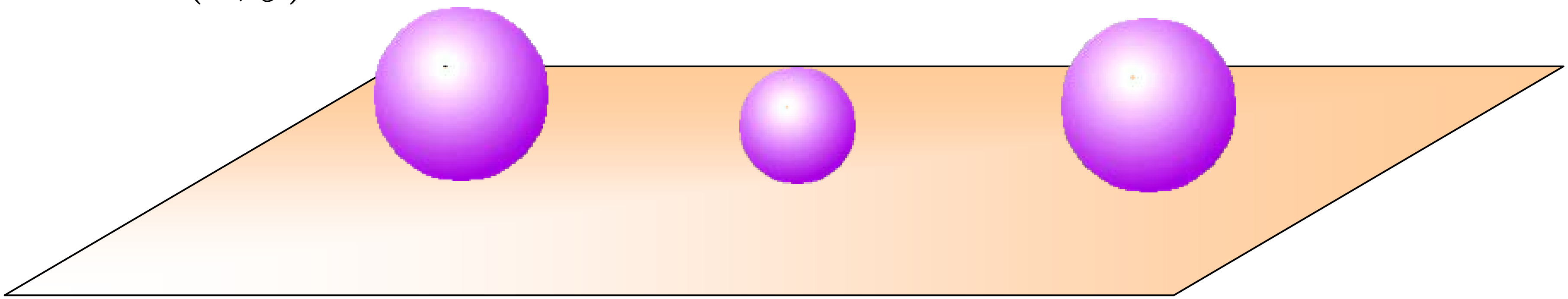

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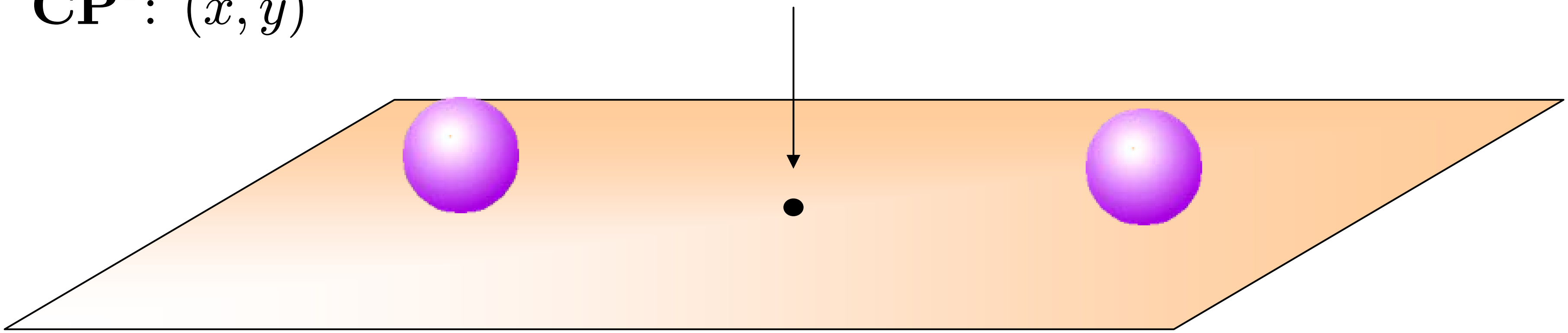
$$\boxed{r = 0}$$

Conifold Singularity

$\mathbf{CP}^1$: (x, y)



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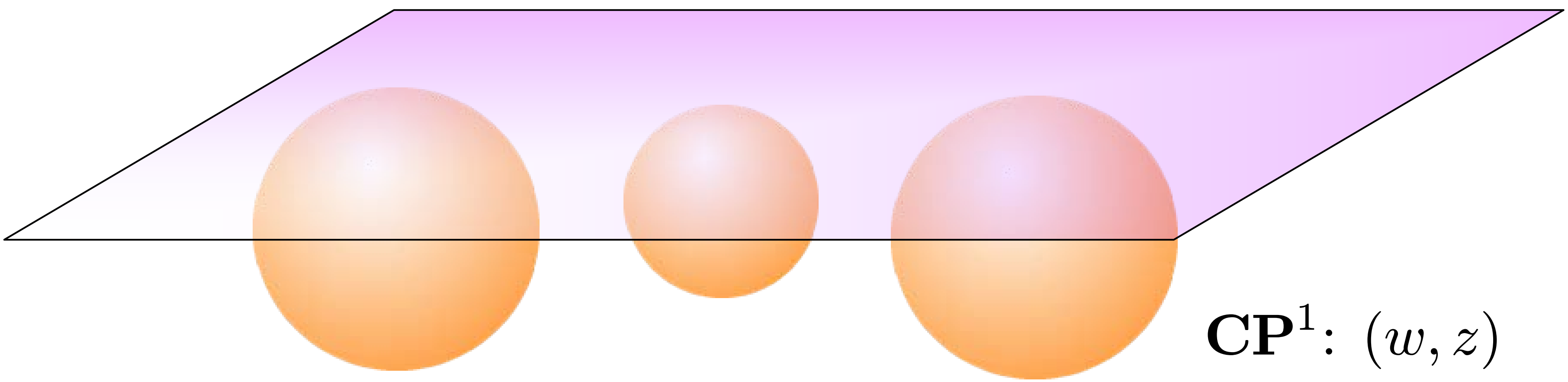
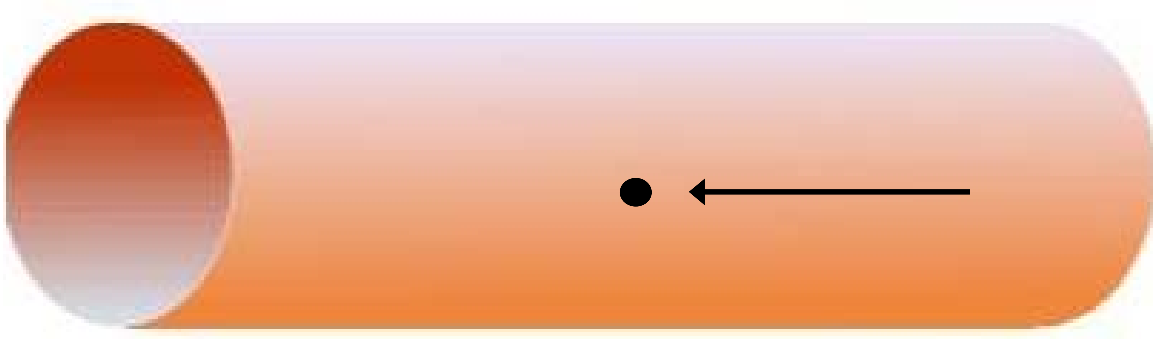


$\mathbf{CP}^1$: (w, z)

$X \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$ or "$X \cong \mathbf{C^2} \times \mathbf{S}^2$"

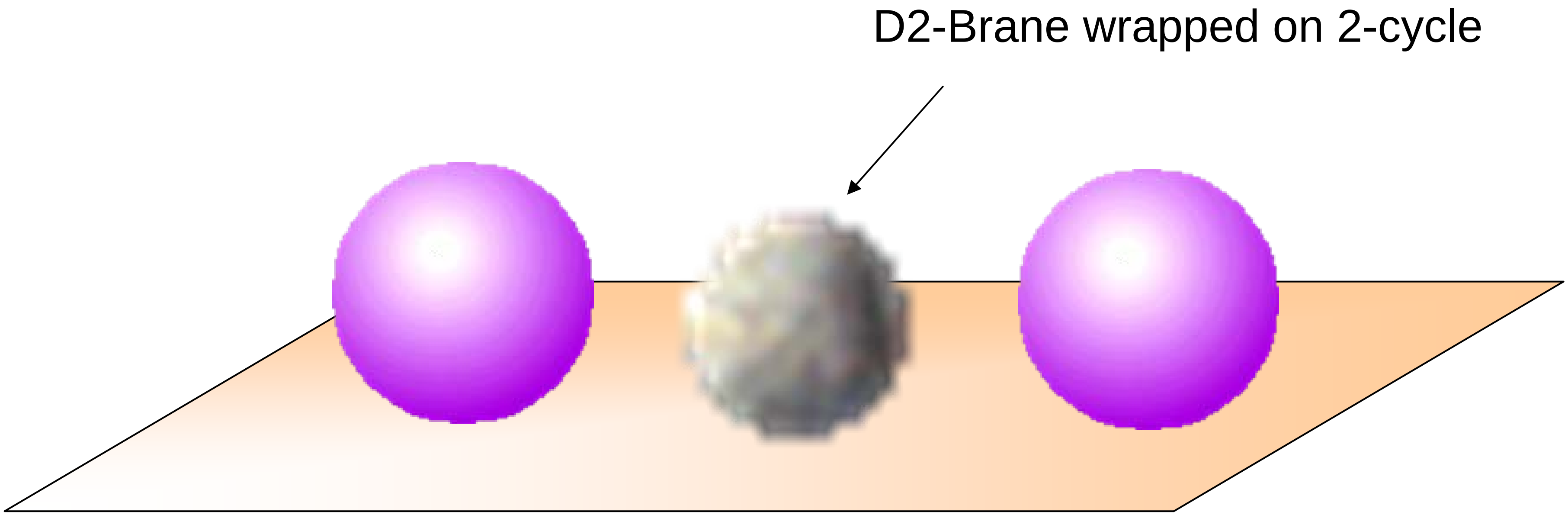
The Conifold Transition



- The flop proceeds by avoiding the singularity in moduli space.
- What happens if we instead choose to hit the singularity?
- At this point correlation functions in the CFT diverge: perturbative string theory is not sufficient --- we need to include g_s effects.

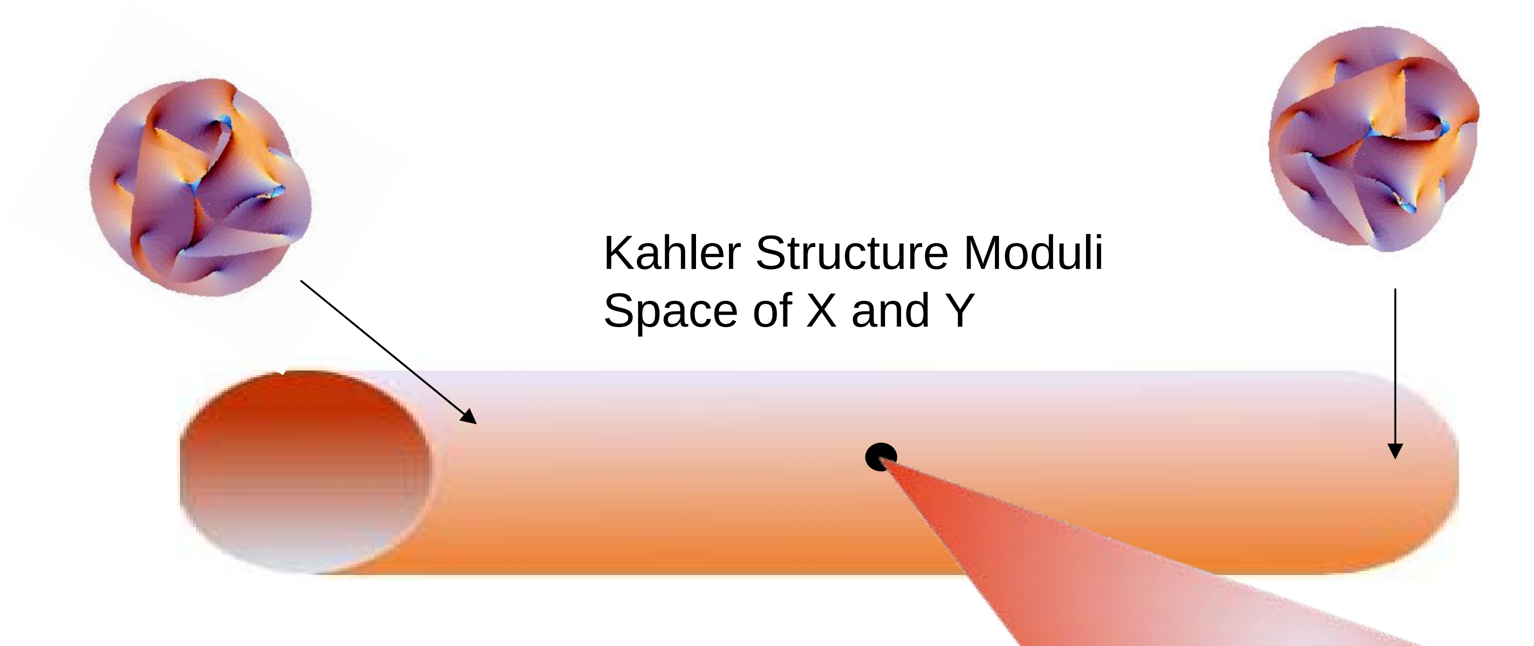
The Conifold Transition

Strominger 1995

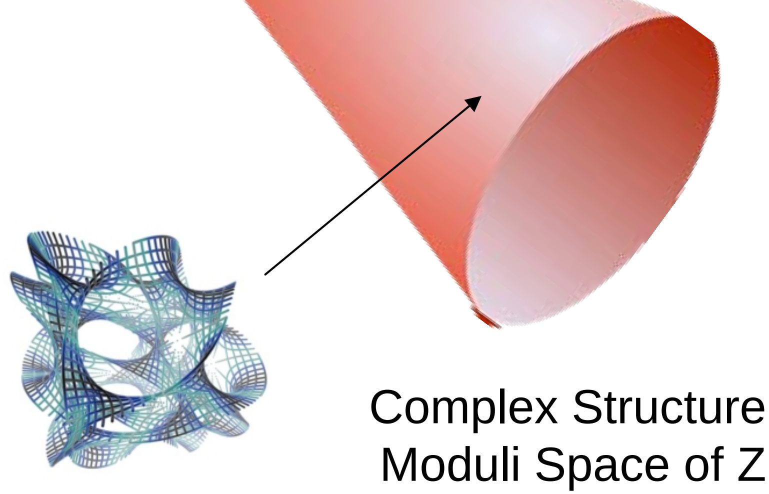


- Mass of D2-brane $\sim r/g_s$. These new, light states are responsible for the breakdown of classical string theory.
- As the D2-branes become massless, they may condense. This quantum phase transition takes us to a new geometry.

The Conifold Transition



- X and Z have different Hodge numbers.
- A 2-cycle of X shrinks
- A 3-cycle of Z grows



Greene, Strominger and Morrison 1995

Time Dependence in String Theory

- The discussion of topology change in string theory is always *adiabatic*. Does it actually occur dynamically?
- Two Problems
  - Moduli Trapping: light particles formed on-shell.
  - Cosmic Censorship and horizon formation
- Both can apparently be overcome in this context.

Other Examples of Topology Change

- M-Theory on G2 Manifolds: Flop transitions

  Atiyah, Maldacena, Vafa
  Atiyah and Witten

- M-Theory on Spin(7) Manifolds: Conifold Transitions

  Gukov, Sparks and Tong

- String theory on Riemann surfaces and tachyon condensation

  Barbon and Rabinovici
  Adams, Liu, McGreevy, Saltman, Silverstein
  Horowitz

The Glop Transition

Work with K. Hori.

- Like the flop, it is smooth in perturbative string theory
- X and Y still have the same Hodge numbers
- X and Y are NOT birationally equivalent. (But are derived equivalent)
- It is the Grassmannian flop --- or “glop” --- transition.
- There is a quantum splitting of the conifold singularity

The Strategy

E. Witten, “Phases of N=2 Theories”, 1993

Gauge Theory → (RG flow) → Conformal Field Theory

Non-Abelian Gauge Theory → Hypersurface in Grassmannians

The vacuum moduli space of the gauge theory is the target space of the sigma-model. The goal is to construct a gauge theory whose vacuum manifold is your favorite CY.

Building a CY from Gauge Theory
1: The Grassmannian

- N=(2,2) supersymmetry in d=1+1
- U(k) Gauge Theory with N fundamental chiral multiplets

$$V = \frac{e^2}{2} \mathrm{Tr} \left(\sum_{i=1}^{N} \phi_i^a \phi^{\dagger}{}_b^i - r\delta_b^a \right)^2 + \ldots$$

mod U(k) gauge action $\phi_i \rightarrow U\phi_i$

- This defines the space of k planes in C^N
- This is the Grassmannian G(k,N). Its size is given by r
- Note: For abelian theories, G(1,N) = CP^{N-1}, the projective space

Building a CY from Gauge Theory

2: The Line Bundle

- BUT: The Grassmannian is not a Calabi-Yau
- This is reflected in the gauge theory, which does not flow to a CFT

$$r \to r(\mu) = r_0 - \frac{N}{2\pi} \log\left(\frac{\Lambda_{UV}}{\mu}\right)$$

- To cancel the running of the Kahler class, we add extra matter
- We choose: S chiral multiplets P_α with charge $-q_\alpha$ under the central U(1) in U(k)
- The criterion for conformal invariance in the infra-red is

$$\sum_{\alpha=1}^{S} q_\alpha = N$$

Building a CY from Gauge Theory
2: The Line Bundle

- The vacuum space of the gauge theory is now Calabi-Yau
- BUT: it is non-compact

$$V = \frac{e^2}{2}\,\mathrm{Tr}\,(\sum_{i=1}^{N} \phi_i^a {\phi^\dagger}_b^i - \sum_{\alpha=1}^{S} q_\alpha |p_\alpha|^2 \delta_b^a - r\delta_b^a)^2 + \ldots$$

modulo U(k) gauge transformations

The CY manifold X is the sum of line bundles over the Grassmannian

$$X \cong \oplus_{\alpha=1}^{S} \mathcal{O}(-q_\alpha) \to G(k,N)$$

Building a CY from Gauge Theory

3: The Hypersurface

- The final step: introduce a potential to restrict to a hypersurface
  - Require holomorphy and gauge invariance

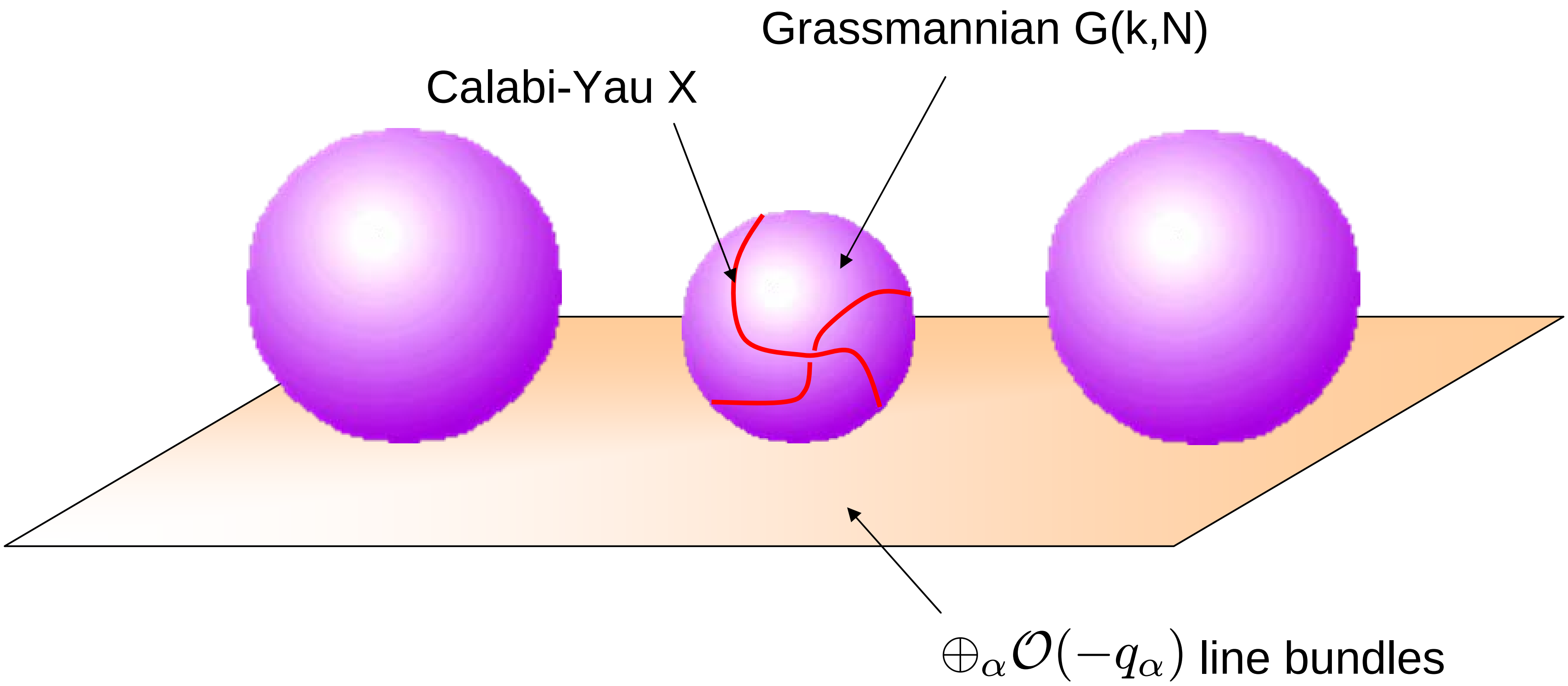
$$\mathcal{W} = \sum_{\alpha=1}^{S} P_\alpha \, G_\alpha(B)$$

  - G_α is a homogeneous polynomial of degree q_α
  - B is the baryon coordinate $B_{i_1 \ldots i_k} = \epsilon_{a_1 \ldots a_k} \Phi^{a_1}_{i_1} \ldots \Phi^{a_k}_{i_k}$

$$V = \sum_{\alpha=1}^{S} |G_\alpha|^2 + \sum_{i=1}^{N} \left| \sum_{\alpha=1}^{S} p_\alpha \frac{\partial G_\alpha}{\partial \phi_i} \right|^2$$

- For suitably generic G, this potential requires p=0. It then restricts to a compact CY X, defined as the hypersurface G=0 in G(k,N).

Building a CY from Gauge Theory



- r determines the Kahler class
- W, the superpotential, determines the complex structure

Building CY 3-folds

- Restricting to 3-folds, this construction yields only a handful of compact Calabi-Yau's as hypersurfaces in Grassmannians
  - c.f. an infinite number of 3-folds as hypersurfaces in projective spaces

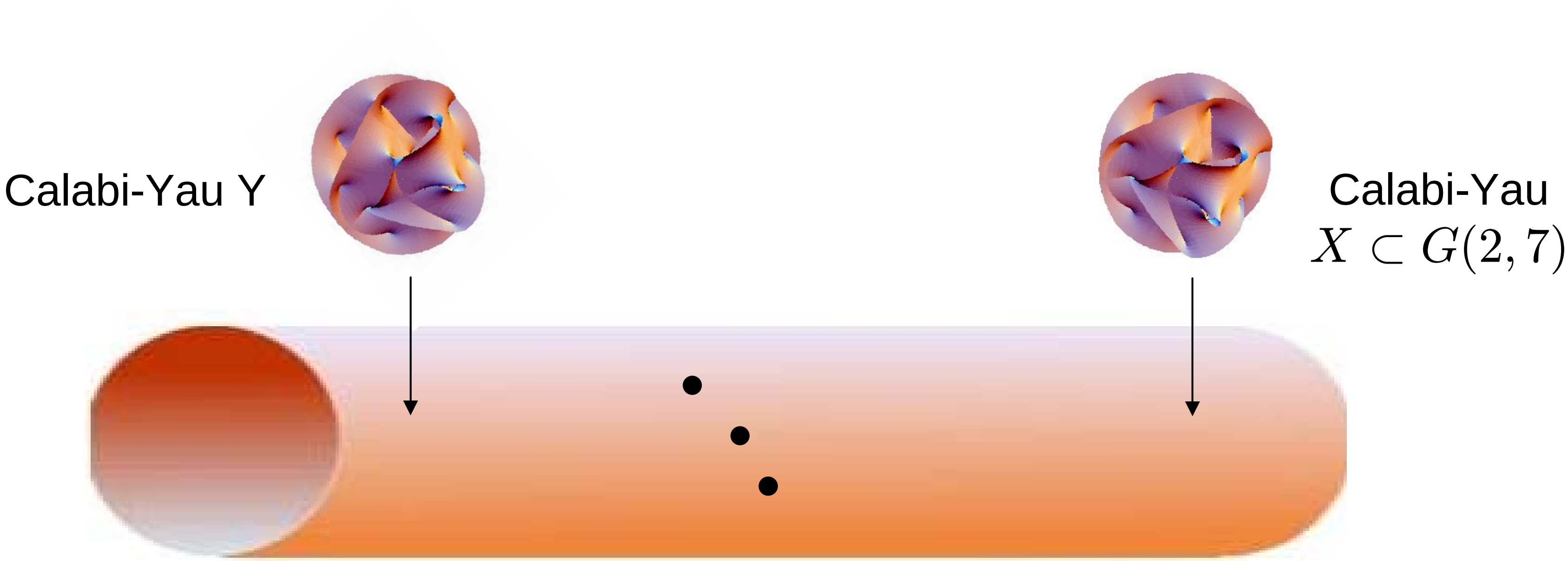
This one is special! →

$G(k,N)$	q_α	$h^{1,1}$	$h^{2,1}$
$G(2,4)$	4	1	89
$G(2,5)$	1,1,3	1	76
$G(2,5)$	1,2,2	1	61
$G(2,6)$	1,1,..,2	1	59
$G(2,7)$	1,1,..,1	1	50
$G(3,6)$	1,1,..,1	1	49

Batyrev et al. 1998

Rodland's Conjecture

1998



- Y is the "Pfaffian Calabi-Yau". Let A be an antisymmetric 7x7 matrix. Consider the locus of matrices which have rank(A)<6. Y is defined by the intersection of this locus with $\mathbf{CP}^6$.
- Rodland showed that X and Y have the same Hodge numbers. By studying mirror manifolds, he conjectured that they lie on the same moduli space.

The Gauge Theory Proof

- U(2) with 7 fundamental chiral multiplets
  + 7 chiral multiplets, with charge -1 under central U(1)

D-term: $$\sum_{i=1}^{N} \phi_i^a \phi^{\dagger}{}_b^i - |p_i|^2 \delta_b^a = r\delta_b^1$$

F-term: $$\mathcal{W} = \mu \; A_k^{ij} \; P^k \epsilon^{ab} \Phi_i^a \Phi_j^b$$

- r >> 0: This is the CY 3-fold in G(2,7)
- r << 0: What do we get?

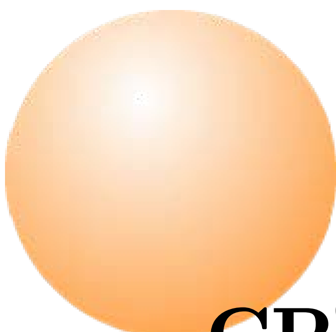
The Other End of Moduli Space

$$\sum_{i=1}^{N} \phi_i^a \phi^{\dagger}{}_b^i - |p^i|^2 \delta_b^a = r\delta_b^1$$

If $r \ll 0$ then $|p^i| \neq 0$

Dividing by U(1) gauge transformations, the p^i parameterize $\mathbf{CP}^6$

Low-Energy Dynamics is:



$\mathbf{CP}^6$

+

SU(2) Gauge Theory
7 fundamental chiral multiplets with masses:

$$\mathcal{W} = \mu \; A_k^{ij}(p^k) \; \epsilon^{ab} \Phi_i^a \Phi_j^b$$

$\mathbf{CP}^6$ coords

Non-Abelian Gauge Dynamics

- To understand the moduli space of the CY, we need to understand the dynamics of SU(2) non-abelian gauge theories, specifically

SU(2) Gauge Theory 7 fundamental chiral multiplets with masses

  - What is the vacuum structure?
  - Does it flow to a conformal theory?
  - What happens as we vary the masses?

The Witten Index

- One of our main results: Consider SU(k) gauge theory with N massive fundamental chiral multiplets.
  - # vacua = combinatoric problem. Let $\omega = \exp(2\pi i/N)$. Choose k distinct integers n_a from {0,1,….,N-1} such that

$$\sum_{a=1}^{k} \omega^{n_a} \neq 0$$

  - # vacua = # possible choices, modulo shifts $n_a \to n_a + 1$

Counting the Vacua

(A cheap proof with loopholes)

- We count the vacua by looking on the *Coulomb branch*
  - Let $\sigma \neq 0$ such that SU(k) breaks to the Cartan subalgebra
  - Integrate out the N chiral multiplets with twisted masses m_i

$$\mathcal{W} = -\sum_{i=1}^{N}\sum_{a=1}^{k}(\Sigma_a - m_i)\left(\log(\Sigma_a - m_i) - 1\right)$$

  - Find solutions to $\partial\mathcal{W}/\partial\Sigma_a = 0$ subject to $\Sigma_1 + \Sigma_2 + \ldots + \Sigma_k = 0$
  - When all masses are equal, the solution is easy

$$\Sigma_a = m - km\frac{\omega^{n_a}}{\sum_{a=1}^{k}\omega^{n_a}}$$

Examples:

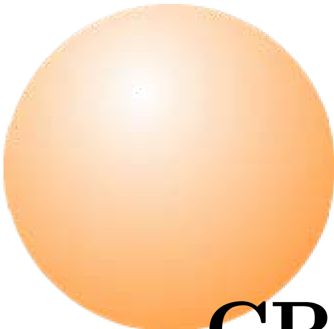
- SU(k) gauge theory with $N \leq k$ fundamental chiral multiplets have no susy vacua.

- SU(k) gauge theory with $N = k + 1$ chiral multiplets has a unique vacuum.

- As the masses $m \to \infty$, the ground states become non-normalizable

- When the masses $m \to 0$, the theory flows to a CFT with normalizable ground state only if a related combinatoric criterion is satisfied

The Other End Revisited



$\mathbf{CP}^6$

+

SU(2) Gauge Theory
7 fundamental chiral
multiplets with masses:

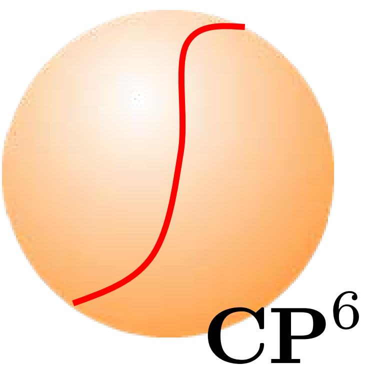
$$\mathcal{W} = \mu \; A^k_{ij}(p^k) \; \epsilon^{ab}\Phi^a_i\Phi^b_j$$

- This SU(2) gauge theory has 3 vacua. But some become non-normalizable as the mass $\mu \to \infty$

- $A(p)_{ij} \equiv A^k_{ij}p^k$ is a 7x7 antisymmetric matrix.
  - When rank(A) = 6, we have 6 massive chirals, 1 massless chiral. There is no susy ground state as $\mu \to \infty$
  - When rank(A) = 4, we have 4 massive chirals, 3 massless chirals. There is a unique susy ground state as $\mu \to \infty$

The Other End Revisited

Supersymmetric ground state exists only on the locus $X \subset \mathbf{CP}^6$ such that A(p) degenerates to rank 4



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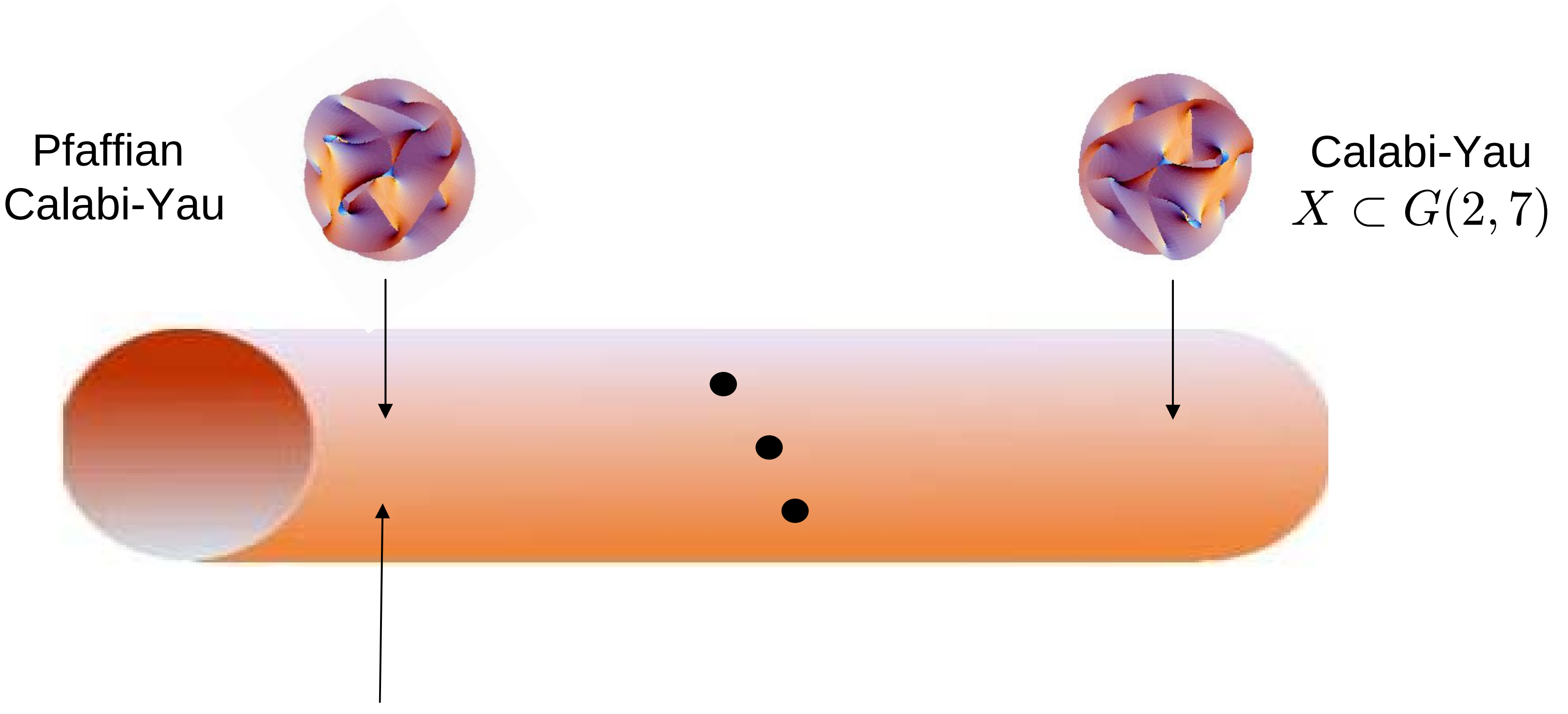
SU(2) Gauge Theory
7 fundamental chiral multiplets with masses:

$$\mathcal{W} = \mu \; A^k_{ij}(p^k) \; \epsilon^{ab} \Phi^a_i \Phi^b_j$$

This is the definition of the Pfaffian Calabi-Yau. It gives a gauge theoretic proof of Rodland's conjecture.

Summary of a Glop



- Here, a *weakly* coupled CFT (large volume Calabi-Yau) arises from the dynamics of a *strongly* coupled gauge theory.
- This novelty is ultimately responsible for the differences between the flop and the glop.

Summary

- New Results in 2d Non-Abelian Gauge Theories
  - Witten Index of SU(k) QCD with N flavors
  - Quantum Splitting of Conifold Singularities
  - Seiberg Dualities: SU(k) and SU(N-k) with baryonic superpotentials flow to the same CFT

- New “Proofs” of Old Results about CY manifolds
  - Mirrors for CY in Grassmannians
  - Rodlands “Glop Transition” (and a generalization to a CY 5-fold).