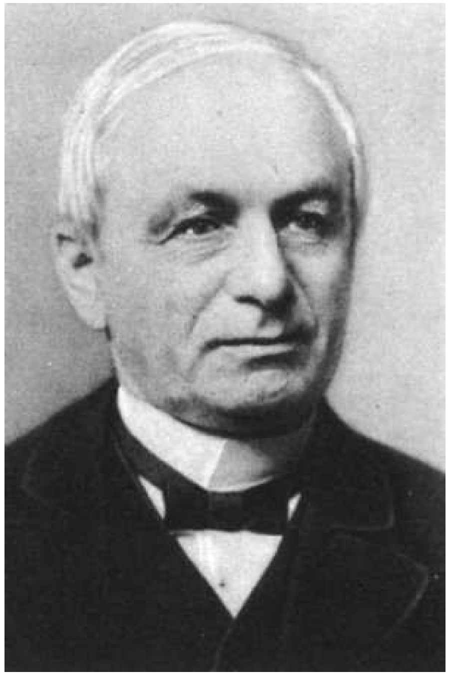
Physics and the Integers

David Tong

UNIVERSITY OF CAMBRIDGE

Trinity Maths Society, 2010



"God made the integers,
the rest is the work of man"

Leopold Kronecker, circa 1880

I always hated this quote. I always thought that the guy was just wrong. It doesn't seem to gel with the way I understand of the laws of physics.

However, recently I learned that *everyone* thought this guy was wrong. In fact, Kronecker's quote is part of his polemic against developments in mathematics in the late 1800s. Things like irrational numbers, Cantor's set theory and the Bolzano-Weierstrass theorem. In other words, all the cool things in maths. Kronecker was an old, distinguished mathematician who thought these new developments were undignified.

More than 100 years later, no mathematician would deny the importance of the developments that Kronecker railed against. His wider viewpoint is largely discredited. Yet I suspect that most mathematicians (and even many physicists) harbour some sympathy for his statement. The integers do hold a special place in the heart of mathematics. Many of the most famous unsolved conjectures relate to the properties of the primes. More importantly, the integers are where we start mathematics. They are its bedrock. They are how we count.

“I don’t believe that atoms exist”

Ernst Mach, 1897

At the same time, a parallel debate was happening in the world of physics, concerned not with the building blocks of numbers but the building blocks of matter.

Although the Greeks discussed the possibility of atoms, they weren't really considered to be a useful concept until the 1800s. Two scientists in particular strove to make atoms respectable – James Clerk Maxwell and Ludwig Boltzmann.

They didn't explain how to observe atoms. No one believed it would ever be possible to see them directly. Instead they showed how from the assumption of the existence of atoms, one could derive many of the known laws of physics --- in particular concerning thermodynamics and gases --- from Newton's laws.

But not everyone found this to be a convincing demonstration of atoms.

There were two main arguments: One, primarily philosophical, was articulated most strongly by Mach. He claimed that one shouldn't talk about objects that one can't directly see.

The other objection came from scientists who were smitten by two of the great advances in 19th century physics: Maxwell's field theory of electromagnetism and Clausius's statement of the conservation of energy. Both seemed to describe continuous substances so (the argument went) perhaps everything should be continuous.

Eventually the scientists agreed that atoms existed. The philosophers under the sway of Mach never did.

There’s actually a tragic end to this story.

In the later years, Boltzmann was prone to serious bouts of depression, often brought on by his arguments about the existence of atoms. In 1906, he killed himself.

But Boltzmann never realised that he had won the argument. To a generation of younger scientists – Planck, Millikan, Einstein – it was obvious that atoms existed. It was only the old guard that couldn’t be convinced. As Planck himself later said:

“A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die”

The trouble with counting

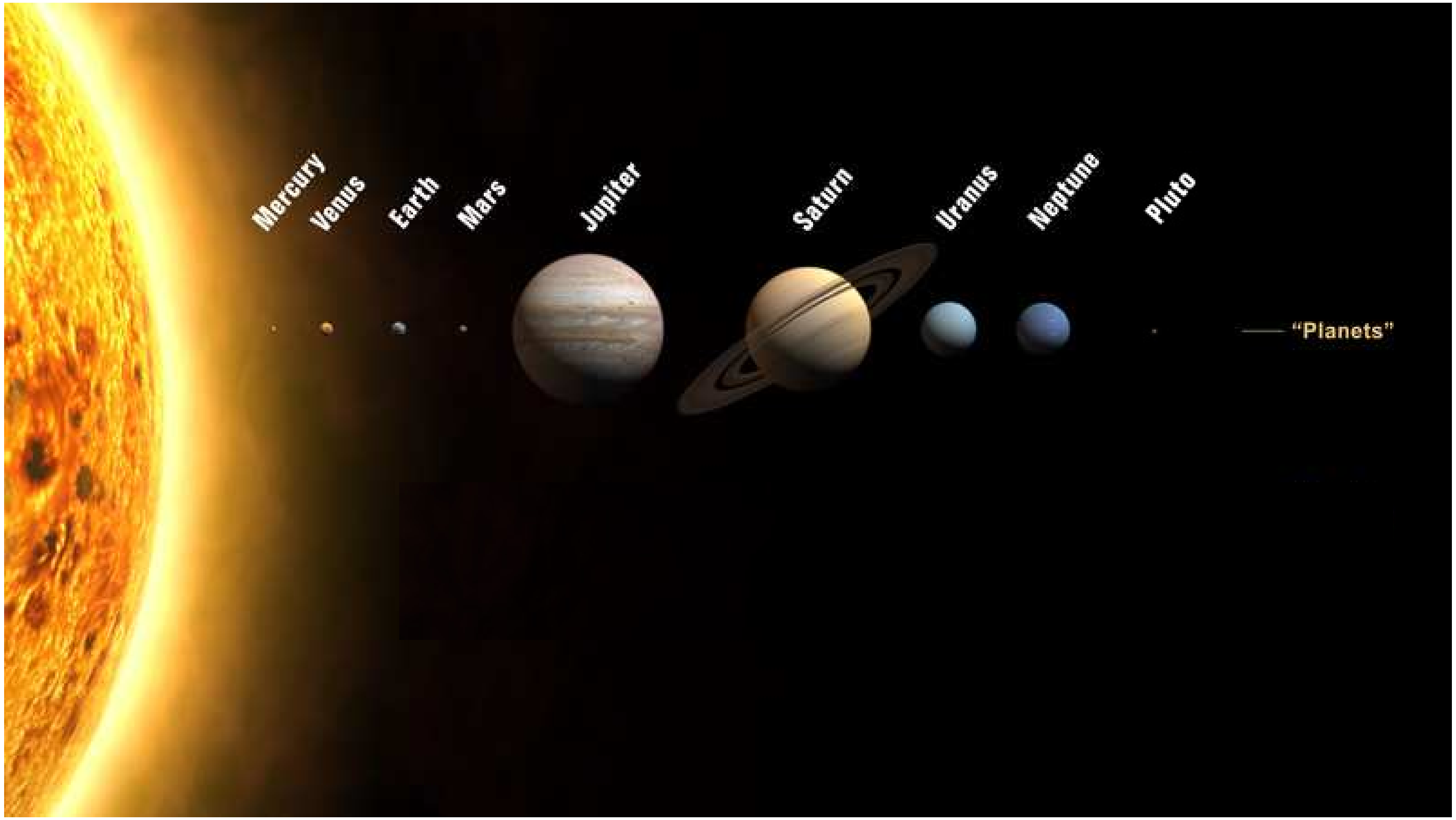
Let’s start our discussion by seeing why it’s not at all obvious that the integers have any place in physics.

How many ripples are there in this picture? Of course, it's not really a sensible question. But it shows that in a world where everything is fluid and continuous, the integers seem unlikely to play a role.

This slide is here simply to honour Benoit Mandelbrot who died last week. But it does illustrate that even when you think the answer to a question should be an integer, it's not always the best answer. What's the dimension of the coastline? Apparently somewhere around 1.3.

Both these are examples were chosen to highlight situations where it's not easy to count. So let's look at an example where it's very clear that you should be able to count things using the integers.

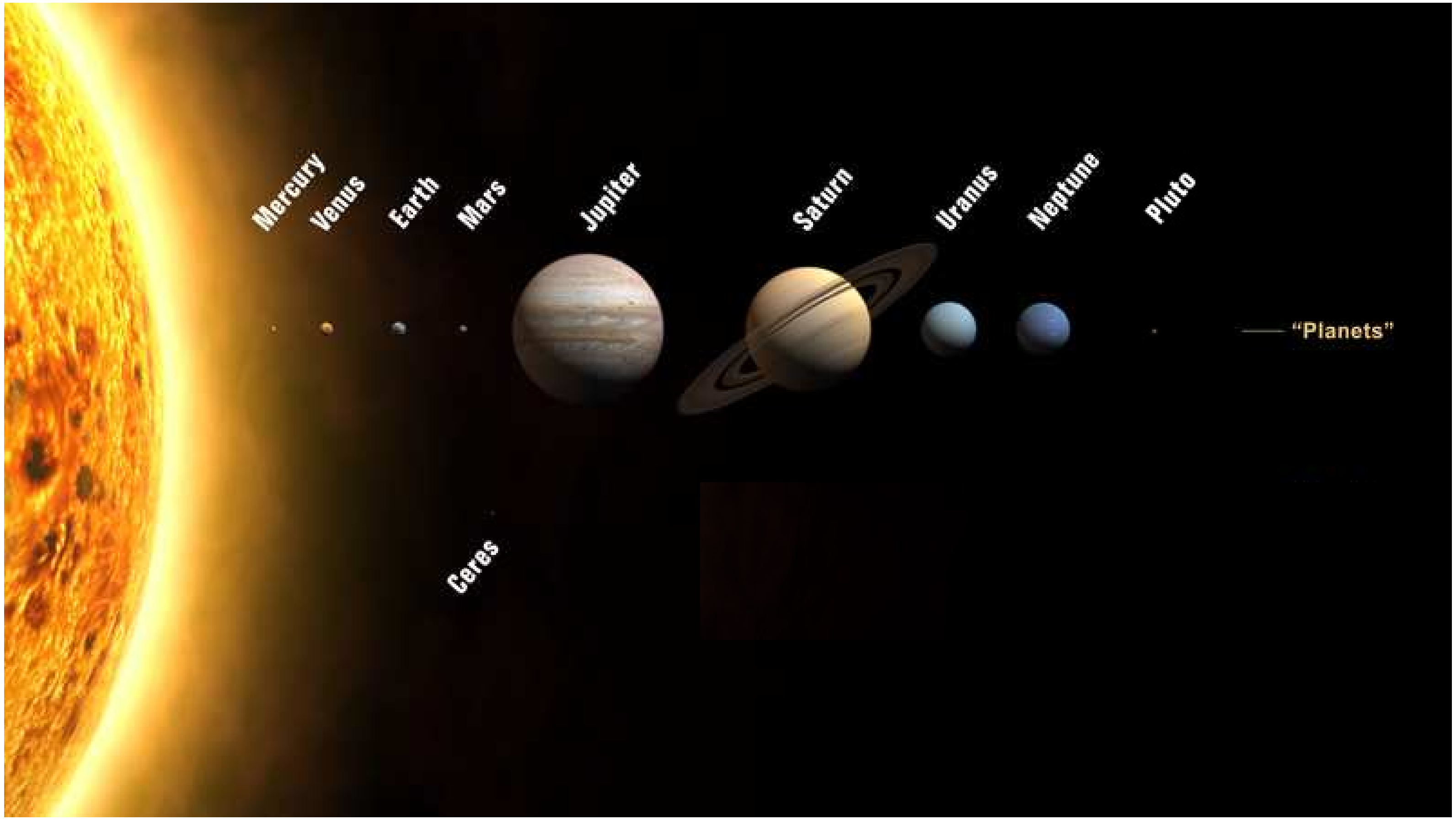
How Many Planets in the Solar System?



I was told at school that there are 9 planets in orbit around the sun. The last to be found was Pluto, discovered in 1930. And for 75 happy years that was the planetary A-list of our solar system.

But nothing lasts forever…

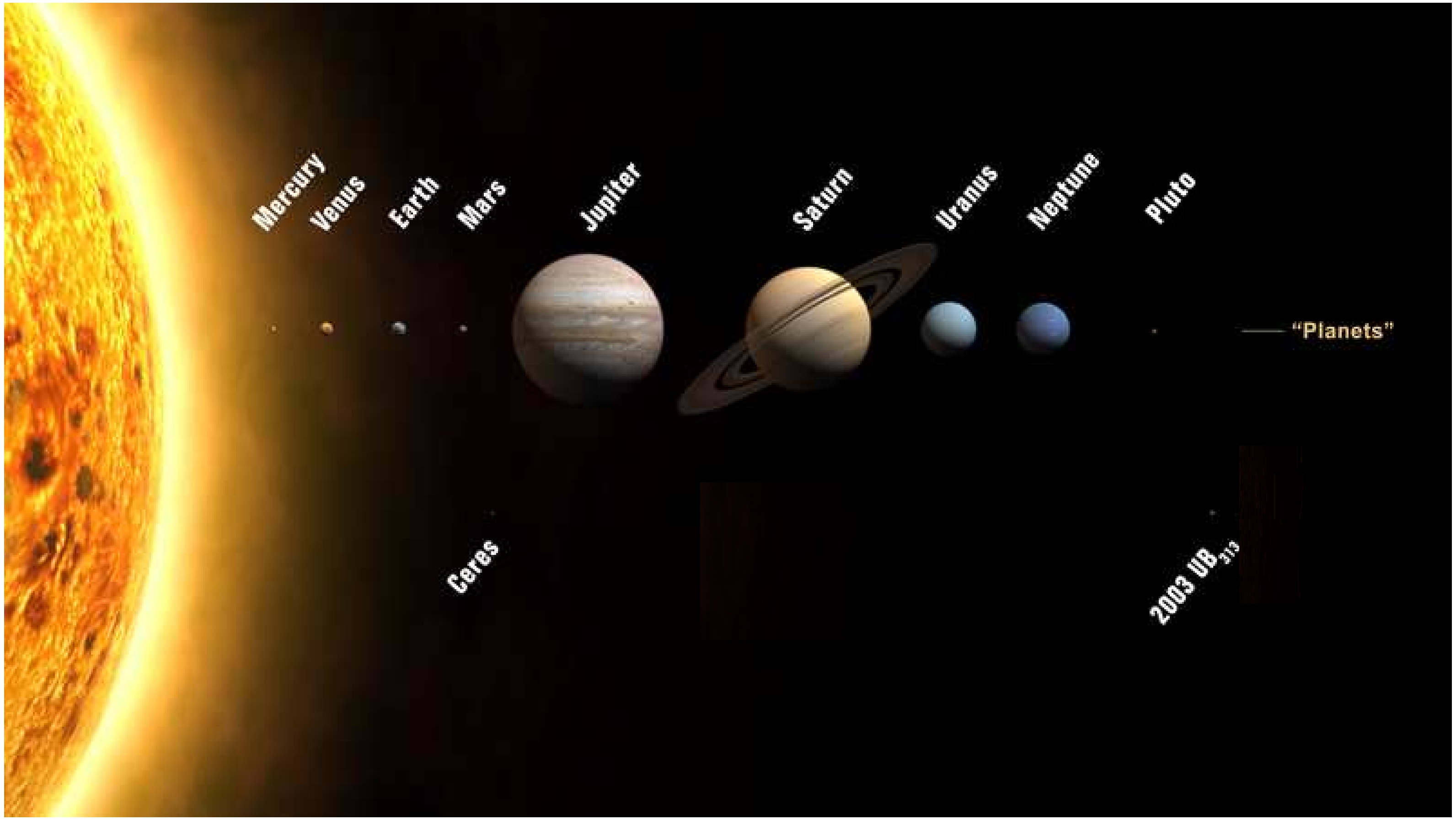
How Many Planets in the Solar System?



Pluto isn’t the first planet to have its status doubted. For the first 50 years of the 19^{th} century, an object named Ceres was classified as a planet. This has a radius of about 500 km – a little less than half the size of Pluto.

In the mid 1800s it was decided that Ceres wasn’t big enough to warrant planetary status and it was put in a new class of objects called “asteroids”.

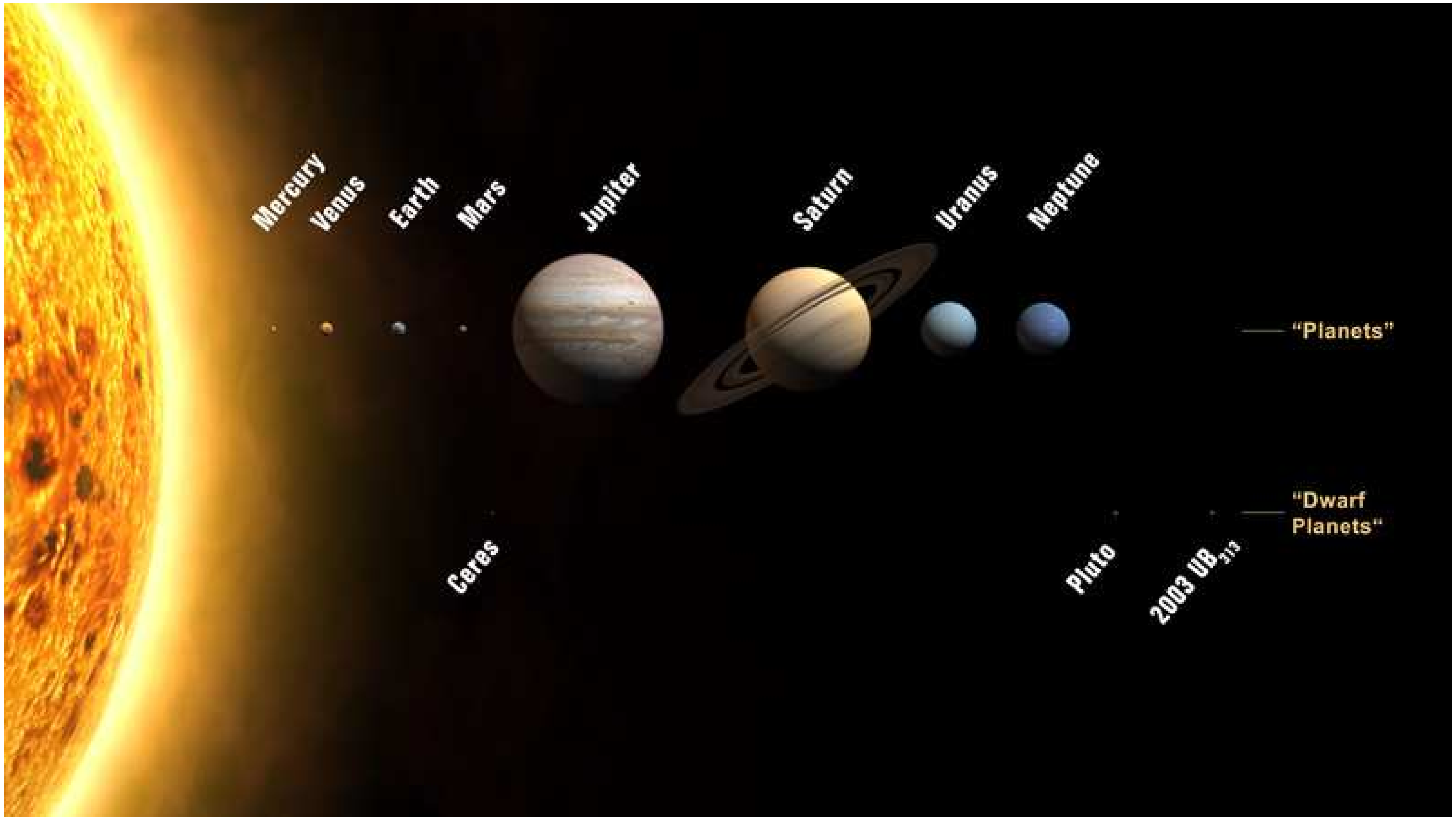
How Many Planets in the Solar System?



The recent trouble started in 2005 when Eris was discovered. It sometimes goes by the catchier name of 2003UB. (It was discovered in 2005 from images taken in 2003).

Eris is more than twice the size of Pluto, with a radius of around 2,500 km, and is about 3 times further from the Sun. This discovery caused all sorts of worries for people that like to count. Did we now have 10 planets or 9?

How Many Planets in the Solar System?



Eventually, as I'm sure you all know, it was decided to reclassify Pluto, Ceres and Eris as dwarf planets. There are two others in this gang – Haumea and Makemake (both of which are larger but lighter than Pluto).

These objects are just a few of the hundreds of thousands that lie in the Kuiper belt. There are almost certainly more dwarf planets to be found. But, more importantly, there are objects that range in size from a few thousand kilometers to a few microns. You can only decide which objects are planets and which are merely lumps of rock if you employ a totally arbitrary definition of what it means to be a planet.

Of course, none of this teaches us anything about planets. The take-home message is that it's difficult to put objects in one-to-one correspondence with the integers. The problem is not with the counting; it's with defining the objects you want to include.

If we want to find the integers in physics, we need Nature to provide us with objects which are naturally discrete.

The integers in Nature

Periodic Table of the Elements

- hydrogen
- alkali metals
- alkali earth metals
- transition metals
- poor metals
- nonmetals
- noble gases
- rare earth metals

1 H																	2 He
3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne
11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
55 Cs	56 Ba	57 La	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
87 Fr	88 Ra	89 Ac	104 Unq	105 Unp	106 Unh	107 Uns	108 Uno	109 Une	110 Unn								

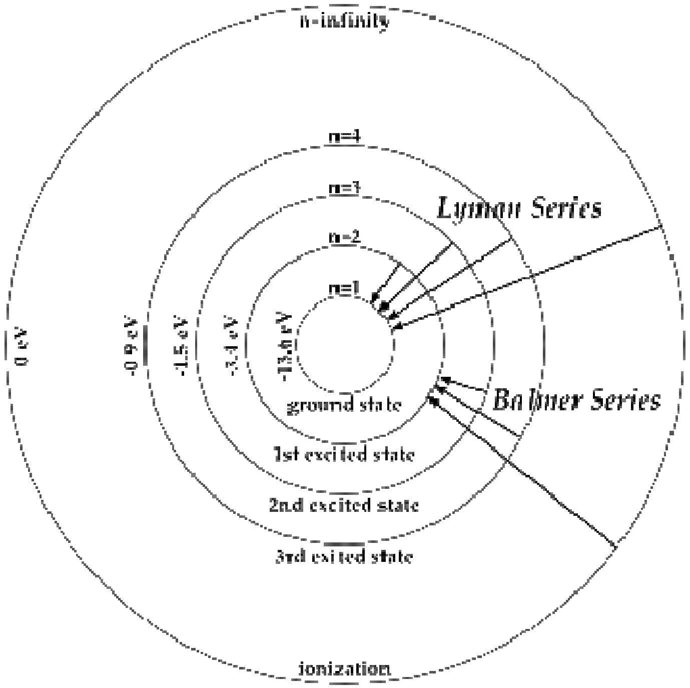
58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu
90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr

Fortunately, discrete objects do exist. While the definition of a planet may be arbitrary, the definition of an atom, or an elementary particle, is not.

Historically the first place that the integers appeared in science was the periodic table of elements. The integers labelling the elements -- which we now know count the number of protons -- are honest. Regardless of what happens in physics, they are here to stay. For example, I will happily take bets that we will never observe a stable element sitting between titanium and vanadium that contains the square-root of 500 protons.

The hydrogen atom

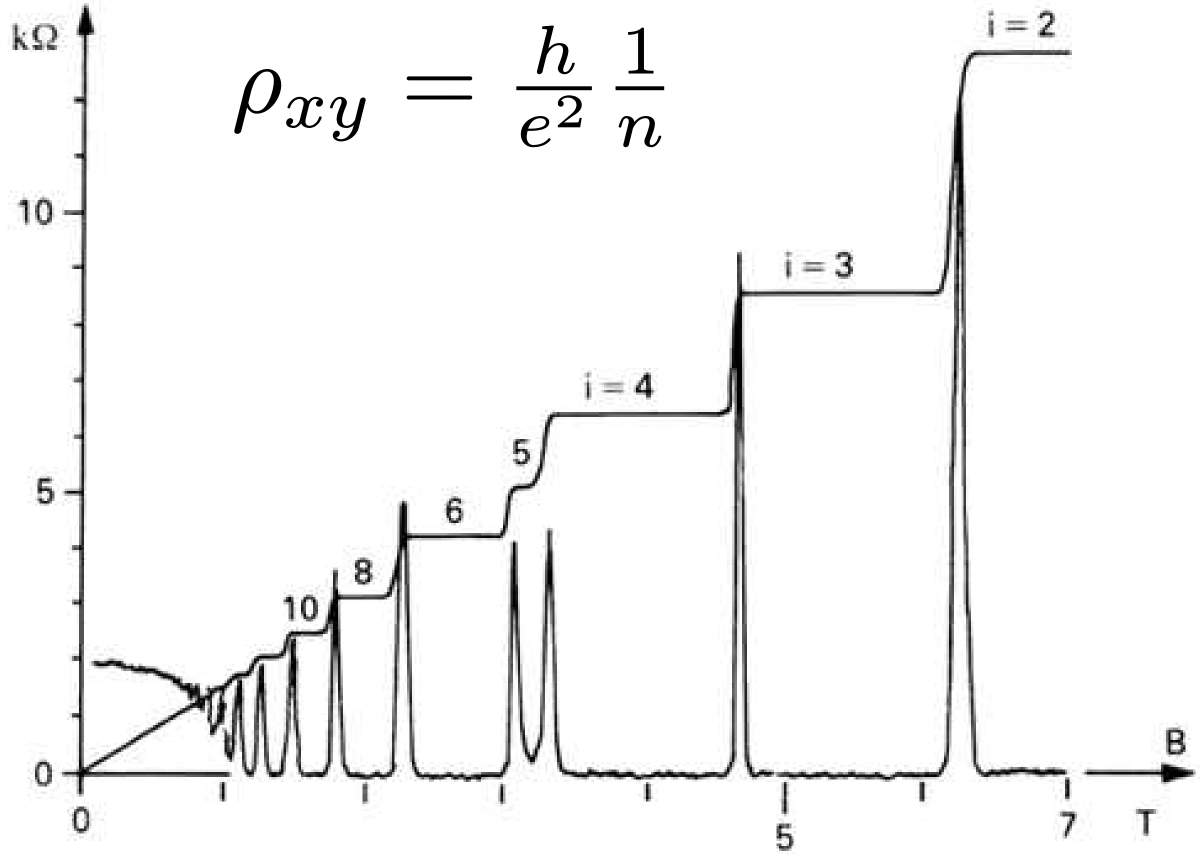
$$E = -\frac{13.6}{n^2}\ ev$$



Once we’re in the atomic world, the integers are everwhere. This is what the “quantum” means in quantum mechanics

In the second year quantum course, you learn that the hydrogen atom has energy levels that go as -1/n^2, which are beautifully illustrated in the observed spectral lines.

The integer quantum Hall effect



You can even coax the integers to appear in macroscopic phenomena. The graph shows the quantum Hall effect that occurs when electrons are placed in a magnetic field.

The Hall resistivity measures the resistance to a current in the y-direction when you drop a voltage across the x-direction. At low temperatures you see these surprising plateaux. This result is even more astonishing when you think that it's happening in a dirty system. You can even drill a hole through the sample and the integers survive. The integer values are measured to more than one part in a billion – one of the most exact results in all of physics. In fact, they're measured more accurately than h/e^2 is known so now it's just assumed that one sees integer values and this measurement is used to *define* h/e^2.

But the integers aren't inputs…

…they are outputs

But in all these cases, the integers aren’t sitting there as the fundamental constituents of the laws of Nature. They are a consequence of solving continuous equations.

Schrödinger Equation

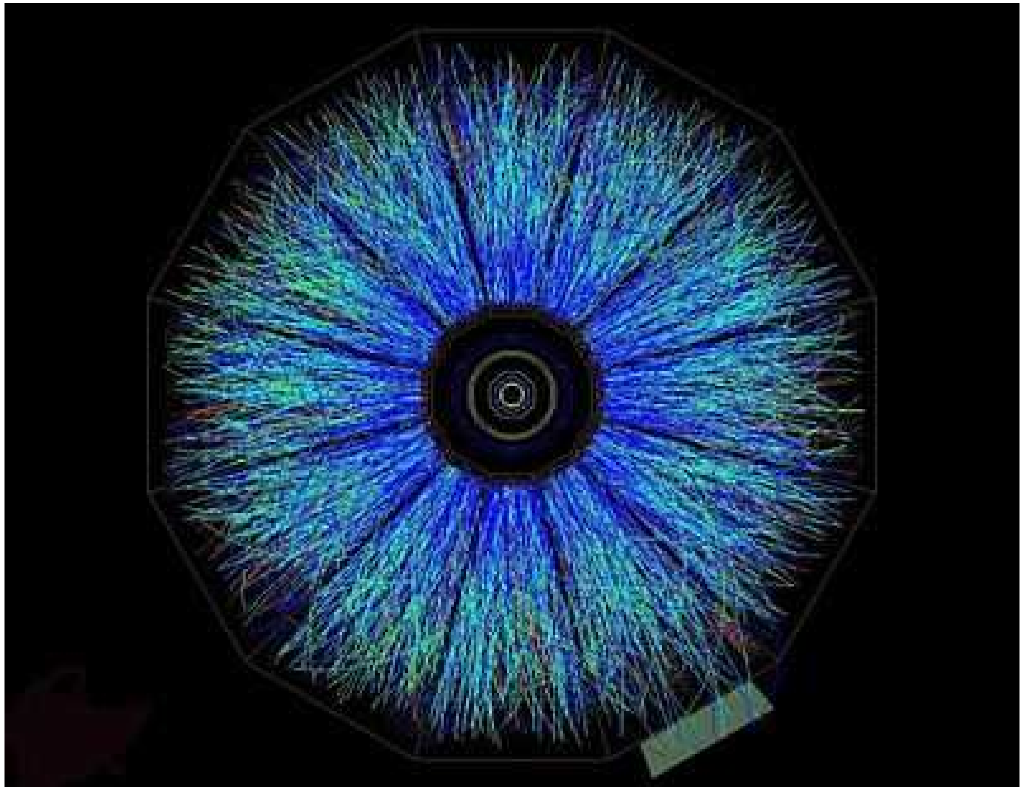
$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) - \frac{e^2}{r}\psi(\vec{r}) = E\psi(\vec{r})$$

$$\Longrightarrow \quad E = -\frac{me^4}{8h^2n^2}$$

There are no integers in the Schrodinger equation for the hydrogen atom. The solutions are labelled by integers because of a normalization condition on the wavefunction so that it has a physical interpretation. The basic equation is continuous. Only its solutions have integer character.

The correct statement should be: God made the complex numbers, the rest is the work of Schrodinger. (Or, if your more mathematically inclined, the work of the eigenvalue problem for continuous Hermitian operators).

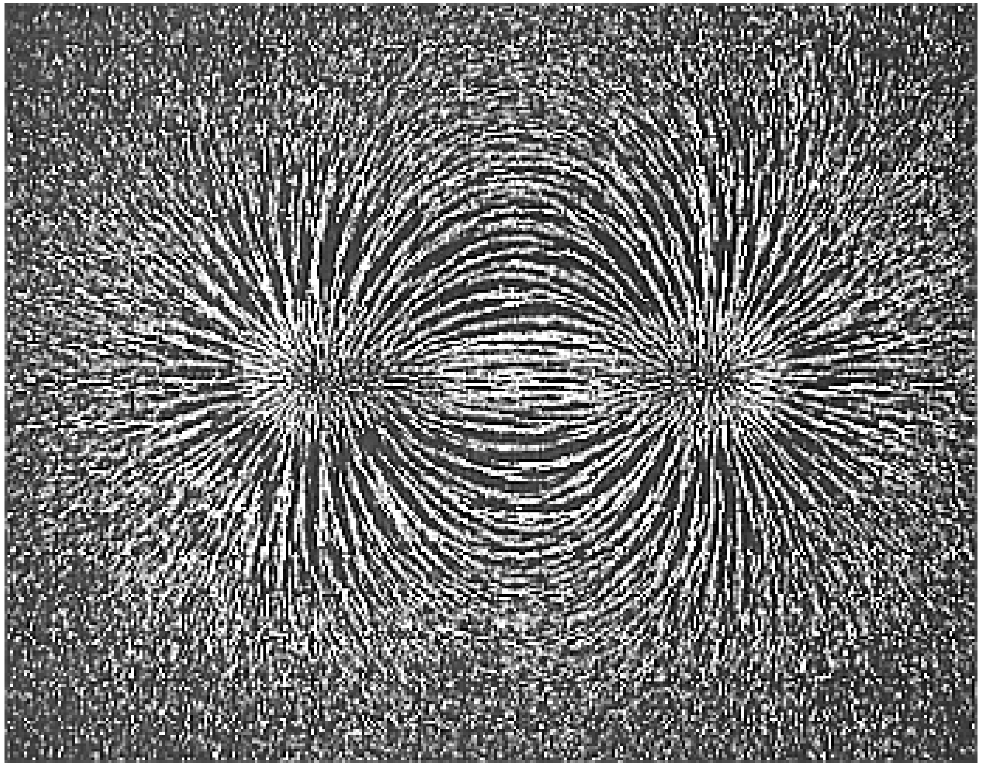
Particles are *not* the building blocks



The same is also true of elementary particles. This is a picture from RHIC, a particle collider just outside of New York. Unlike most colliders, it smashes together gold nuclei at high energies, resulting in an explosion of 10,000 particles. (The LHC does something similar part of the time).

But what I want to tell you about this picture is that the elementary building blocks of Nature are not particles. The particles are emergent, just like the integers in the hydrogen atom. The fundamental constituents of Nature are continuous fluid-like quantities spread throughout space. We call them *fields*.

Fundamental Objects are Quantum Fields



The magnetic and electric fields should be familiar from high school physics. The picture shows the magnetic field of a bar magnetic, painted in iron filings.

Small ripples in the electric and magnetic fields give rise to light. But once these light waves are fed into Schrodinger's equation (i.e. once they are quantized) the light is actually made of many particles called photons.

But this same story holds for all other particles in Nature. The electrons in this room are little ripples of something called the *electron field*. These ripples are forced to have discrete energies by quantum mechanics and these lumps of energy are what we call particles.

The very existence of particles is on exactly the same footing as the appearance of the integers in the hydrogen spectrum. They are not the fundamental objects. They are outputs of the theory.

In some sense, those 19^{th} century scientists who doubted Boltzmann were right: the underlying constituents of Nature are continuous, fluid-like substances.

The integers are *emergent*

The punchline of this part of the talk is twofold:

First, the integers do arise in physics. And this is a non-trivial statement: it didn’t have to be that way.

However, the integers are not the building blocks of the laws of physics. They are emergent quantities, no more fundamental than the concepts of temperature. Or smell. Or the offside rule.

What About Future Laws of Physics?

Let’s leave the known laws of physics behind for the moment and turn to the future. We know that the final laws of physics have yet to be written. It’s a fairly common speculation that, in their final form, the laws of physics will be based on the integers, or some similar discrete mathematics.

Such speculations often come from computer scientists or people thinking of the world through the lens of information and computation. They envision the laws of Nature as something akin to a computer algorithm.

So is this the way the world is? Are we living inside a computer algorithm? Of course, the honest answer is that no one knows.

Here I’m not going to give an answer – or even an opinion – but I would like to describe an issue which I think is pertinent to the discussion, but which seems to get very little airtime. I view it as an important open problem in physics but most people have never heard of it…

An Important Open Problem

No one knows how to discretize
the current laws of physics

Before discussing future laws of physics, it seems sensible to first look at the known laws of physics and ask whether we can make these discrete. For an operational description of *discrete* I'll ask whether we can formulate the laws of physics in a manner which can be simulated on a computer.

The very surprising answer is that no one knows how to simulate the current laws of physics on a computer.

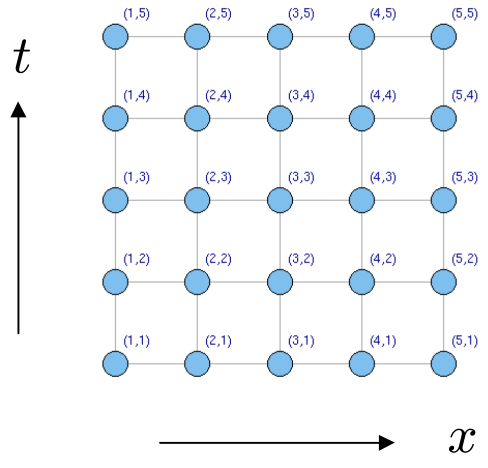
Let me firstly explain what I mean by this. There are many equations that arise in physics that are difficult to simulate – Navier Stokes, Yang-Mills, Einstein. All of them have computational difficulties which makes it difficult to study turbulence, the proton structure or black hole mergers. But in each case I view this as a computational issue rather than a problem of principle. The problem that I have in mind is much deeper.

The technical name for the problem is "chiral fermions on the lattice" where lattice just means a discretization and chiral fermions is the technical bit. In the next few slides, I'll try to give a flavour of this issue.

For classical physics, it is trivial

Differential Equation $\Longrightarrow$ Difference Equation

e.g. $\ddot{x} = \frac{F(x)}{m} \quad \Longrightarrow \quad \frac{x_{n+2} - 2x_{n+1} + x_n}{(\Delta t)^2} = \frac{F(x_n)}{m}$



Let me first describe why it’s surprising that we can’t discretize the laws of Nature.

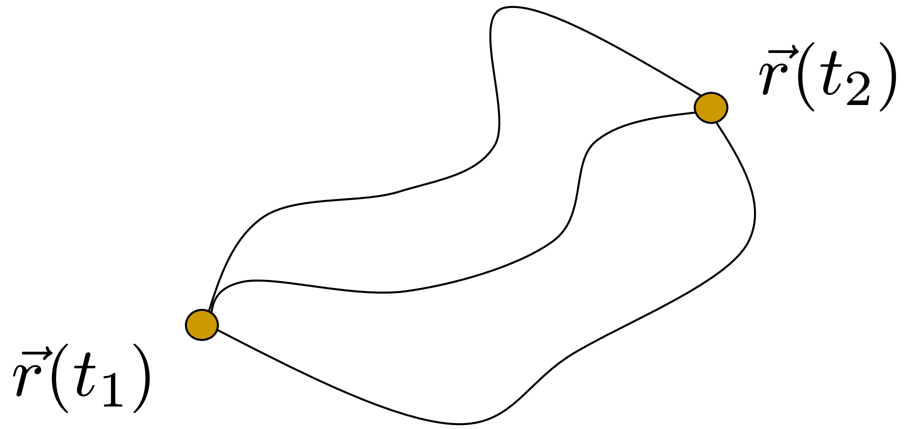
In the classical world, the laws of physics are written in terms of differential equations. Here it’s a simple matter to replace differential equations with difference equations which can then be put on a computer.
In this way, we typically view the underlying space as a lattice, discrete in both space and time.

So there’s no problem with classical physics. What about quantum physics?

Quantum mechanics, not too difficult

Feynman Path Integral

$$Prob = |\mathcal{A}|^2 \qquad \mathcal{A} = \sum_{\text{paths}} \exp(iS/\hbar)$$

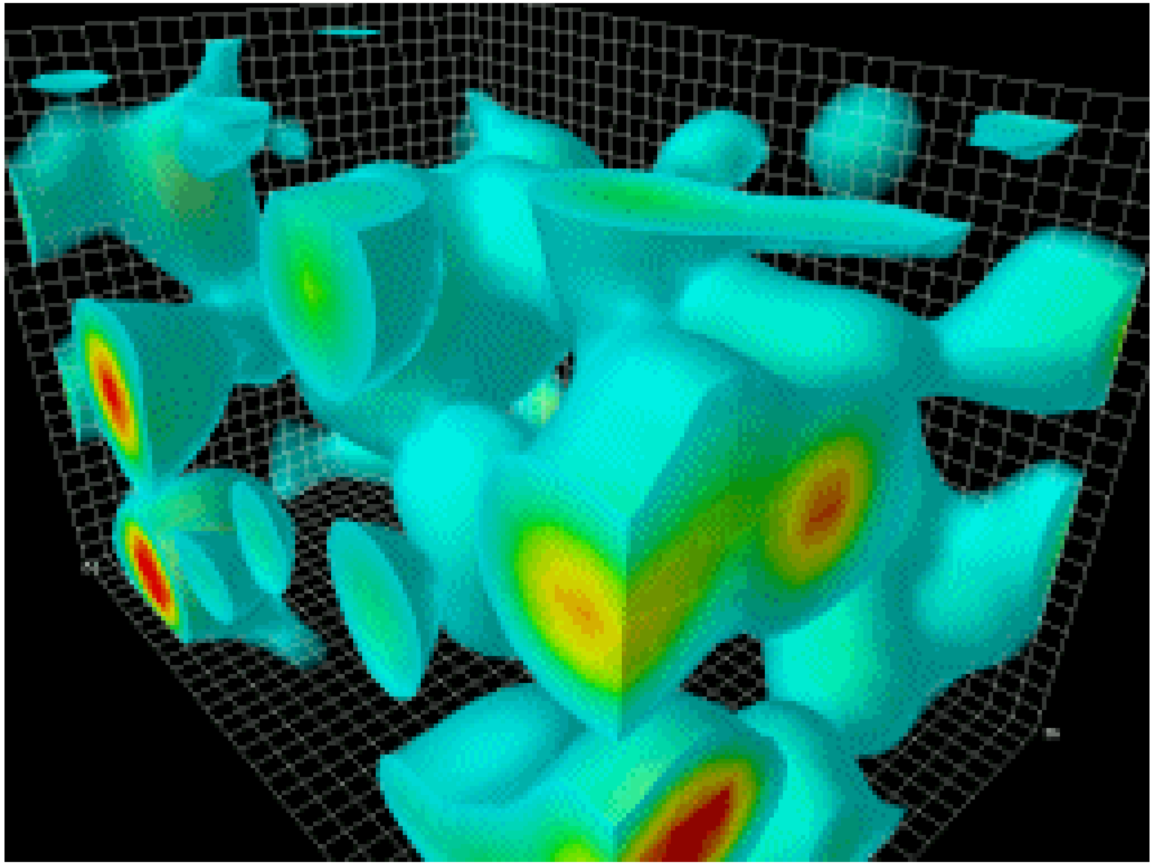


Let's first talk about quantum mechanics, which means the quantum theory of particles. There are different formulations of the theory. The one you learn as an undergraduate is the Schrodinger equation. Since this is a differential equation, we can again discretize it without difficulty.

Another framework is the *Feynman Path Integeral*. This is mathematically equivalent to the Schrodinger equation but provides a novel perspective on quantum mechanics. One thinks of a particle as taking all possible paths between two points. To each path, you associate a number, *S*, called the action. You then sum over all possible paths, weighted with exp(iS/hbar) to get the probability.

Since the Feynman path integeral is equivalent to the Schrodinger equation, you shouldn't be surprised to hear that it's fairly easy to discretize the path integral: you think of the particles as moving on an underlying lattice, just as you did for the classical theory.

Path integral for fields



Lattice QCD

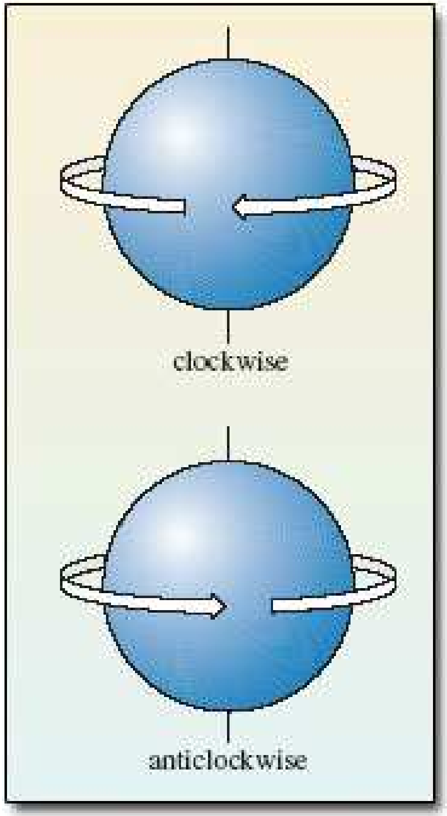
But I’ve already told you that the fundamental constituents of Nature are not particles, but are fields. The quantum theory of fields is unimaginatively called *Quantum Field Theory*.

Here the Schrodinger way of thinking turns out to be very cumbersome. The field is already a function over space. The wavefunction should assign a probability (amplitude) to each configuration of the field. In other words, the wavefunction is a function of a function. Which quickly becomes messy.

However, the Feynman path integral is perfectly adapted for use in quantum field theory. Now one has to sum over all possible field configurations, weighted with some number.

The picture shows a representative example of a field configuration of the gluon (described by the Yang-Mills equation). You can see that it’s wildly fluctuating. This is something which is numerically very intensive to simulate, but there is no problem in principle.

Discrete Dirac fermions getting difficult

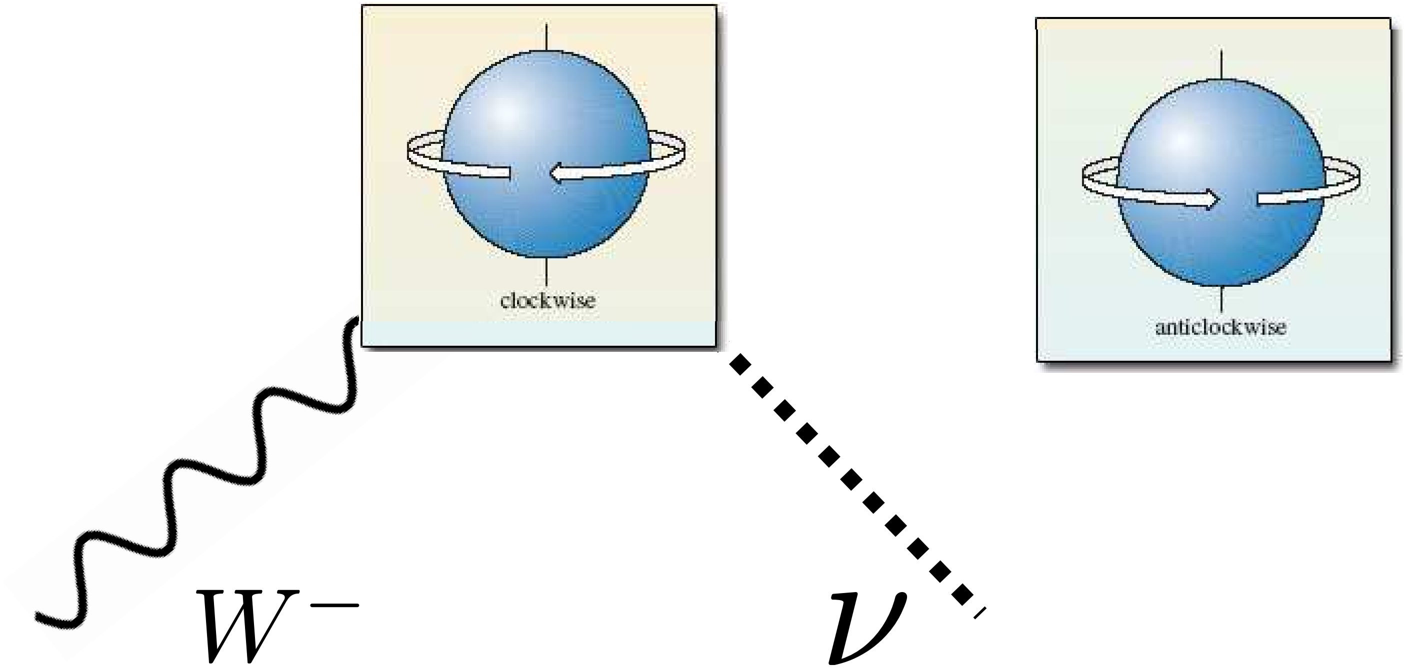


Problems first arise when we think of discretizing fermions. These are particles of matter, like electrons or quarks. They carry a property called *spin*. At a rough level, you can think of the electron as something like a spinning ball. It either spins clockwise or it spins anti-clockwise.

Fermions are strange objects. If you rotate them by 360 degrees, they don’t look the same as they did before. Similarly, if you walk around them once, then they don’t look the same as they did before. The fermion only comes back to itself if you rotate it by 720 degrees, or walk around it twice.

This already causes a bit of a headache when trying to put them on a lattice. But, after several decades of work, it’s now understood how to do it. (The last big breakthrough in this came in the late 1990s when it was understood how to put massless fermions on a lattice).

Discrete chiral fermions are not understood



The problem comes when the interaction of the electron depends on the way it’s spinning. This happens in the Weak force. The force is mediated by the W-boson. And the W-boson interacts differently with electrons that are spinning clockwise and those that are spinning anti-clockwise.

Theories that treat clockwise and anti-clockwise spinning particles differently are said to be *chiral*. No one knows how to formulate these on a computer.

I appreciate that the above discussion is very simplistic. If you want to learn more, then there is a nice review article by David Kaplan: arXiv:0912.2560

Of course, it’s possible that we simply haven’t been smart enough to figure out how to put chiral fermions on a lattice. But maybe something deeper is going on. The obstacles that lie behind attempts to discretize chiral fermions are related to aspects of geometry, topology, index theorems and a physics version of Hilbert's Hotel known as the quantum anomaly. All of these rely on the continuous nature of the field. It may well be that our failure is telling us something important about reality.

What does this mean?

- We haven’t been smart enough yet
- Something deeper?