Magnetic Monopoles

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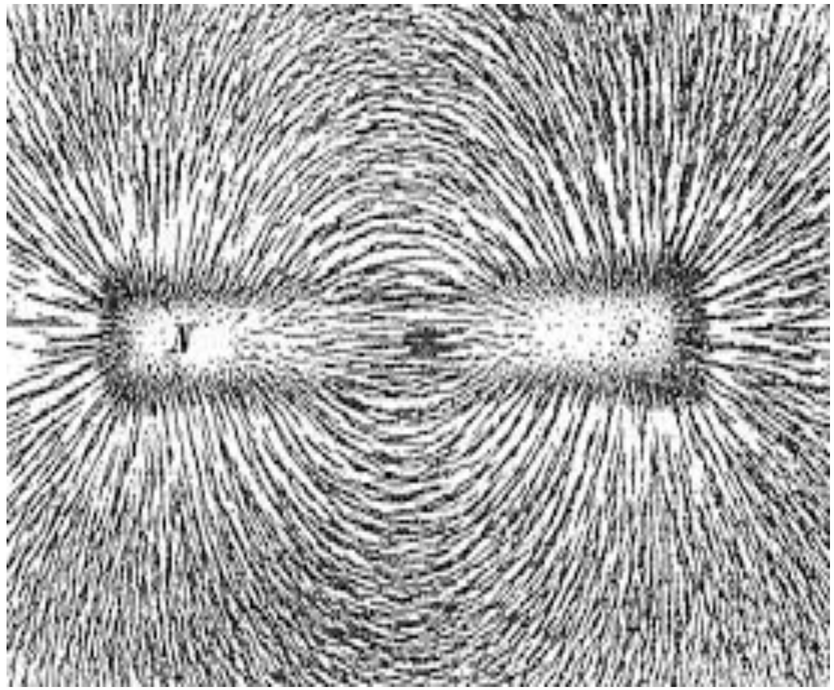
Trinity Mathematical Society, February 2014

North and South Poles

It’s the law: $\nabla \cdot \vec{B} = 0$



Magnetic monopoles are not allowed: $\vec{B} = \frac{g}{4\pi r^2}\hat{\vec{r}}$

And the Law is Non-Negotiable

$$\vec{B} = \nabla \times \vec{A} \quad \Rightarrow \quad \nabla \cdot \vec{B} = 0$$

Quantum Mechanics Doesn't Seem to Help

The vector potential is necessary in quantum mechanics.

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$$

This has physical consequences….

Gauge Symmetry

In general the electric and magnetic fields can be written as

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}$$

These are left invariant under the *gauge symmetry*

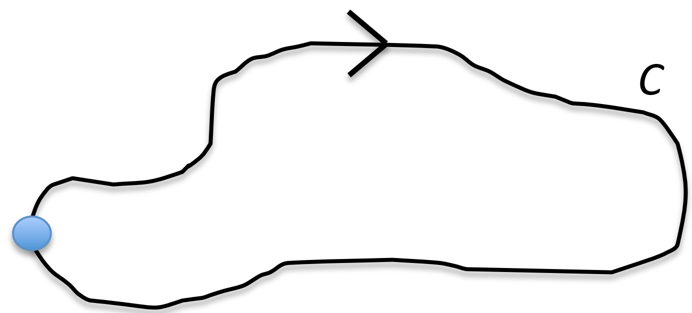
$$\phi \to \phi + \frac{\partial \chi}{\partial t} \qquad \vec{A} \to \vec{A} + \nabla \chi$$

The physical quantities are anything that you can build from the gauge fields that are invariant under this gauge symmetry.

This includes *E* and *B*. But there is more as well…

Aharonov-Bohm Effect

Take a quantum particle. Move it along some closed trajectory *C* in space.



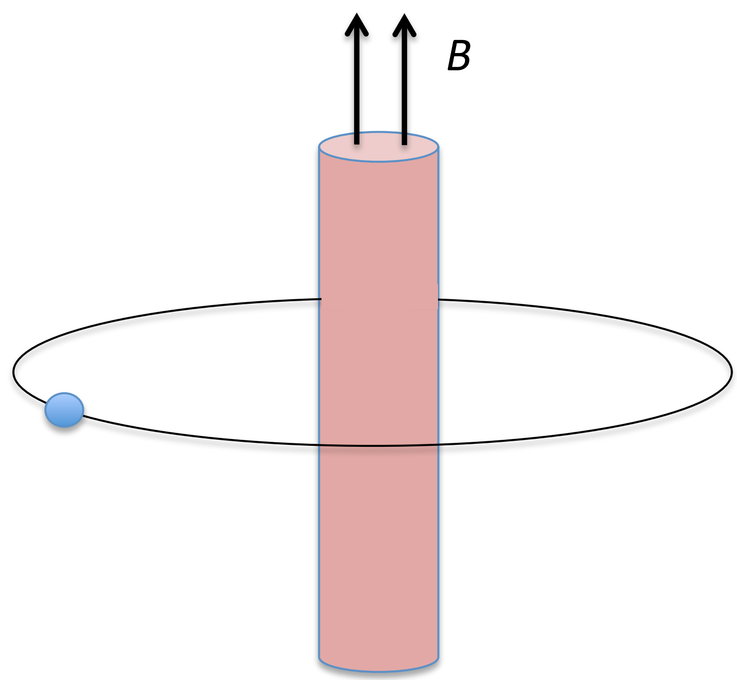
The wavefunction picks up a phase.

$$\psi(\vec{x}) \rightarrow e^{ie\alpha}\psi(\vec{x}) \qquad \text{with} \qquad \alpha = \oint_C \vec{A} \cdot d\vec{x}$$

$\hbar = 1$

Aharonov-Bohm Effect

The phase shift is measurable even if the particle never goes near a magnetic field



$$\psi(\vec{x}) \to e^{ie\alpha}\psi(\vec{x}) \qquad \alpha = \oint_C \vec{A}\cdot d\vec{x} = \int \vec{B}\cdot d\vec{S}$$

Looking for a loophole

None of this bodes well for magnetic monopoles

- Quantum mechanics needs the gauge potential *A.*
- The existence of the gauge potential means no monopoles.

What goes wrong if we just try to write down a gauge potential for a monopole?

$$A_\varphi = \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta} \quad \Longrightarrow \quad \vec{B} = \frac{g}{4\pi r^2}\hat{\vec{r}}$$

singular at $\theta = \pi$

But should we be working with a singular *A?*

[Dirac, 1931]

Gauge Symmetry as the Loophole

Gauge potentials related by $\vec{A} \to \vec{A} + \nabla\chi$ are the same. But no one says that a gauge potential has to be defined everywhere.

valid everywhere but $\theta = \pi$ →

$$A_\varphi^N = \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}$$

valid everywhere but $\theta = 0$ →

$$A_\varphi^S = \frac{g}{4\pi r}\frac{1+\cos\theta}{\sin\theta}$$

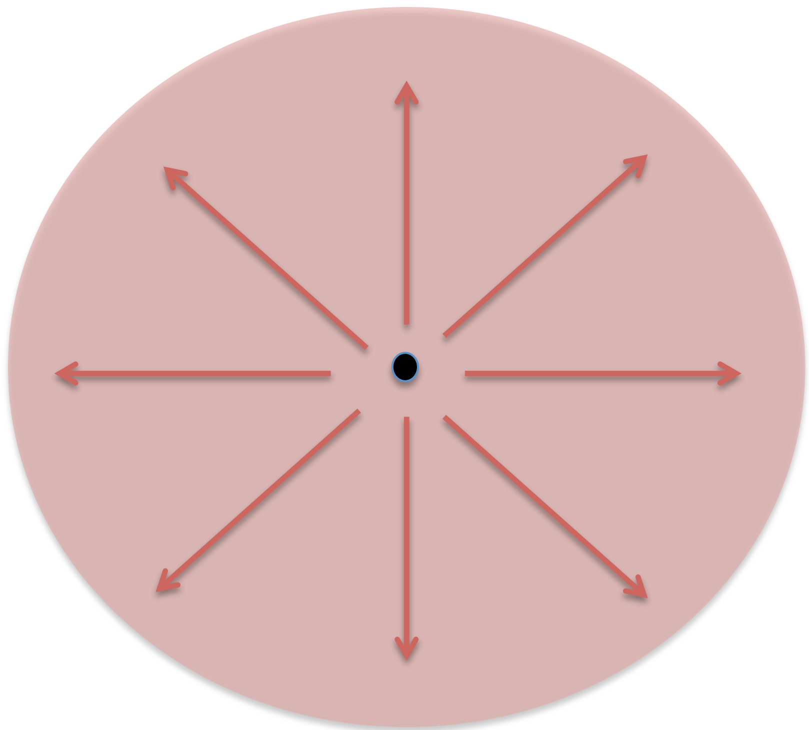
$$\Rightarrow \quad \vec{B} = \frac{g}{4\pi r^2}\hat{\vec{r}}$$

Where they overlap, these two potentials differ by a gauge transformation.

[argument due to Wu and Yang]

Dirac Quantisation

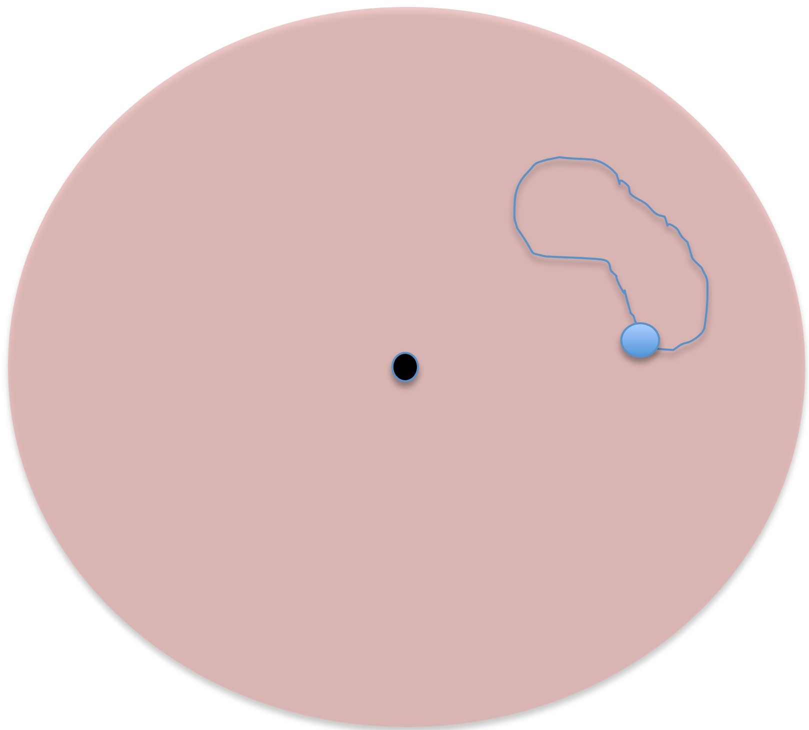
Suppose that we have a magnetic monopole



$$\vec{B} = \frac{g}{4\pi r^2}\hat{\vec{r}} \quad \Longrightarrow \quad \int \vec{B} \cdot d\vec{S} = g$$

Dirac Quantisation

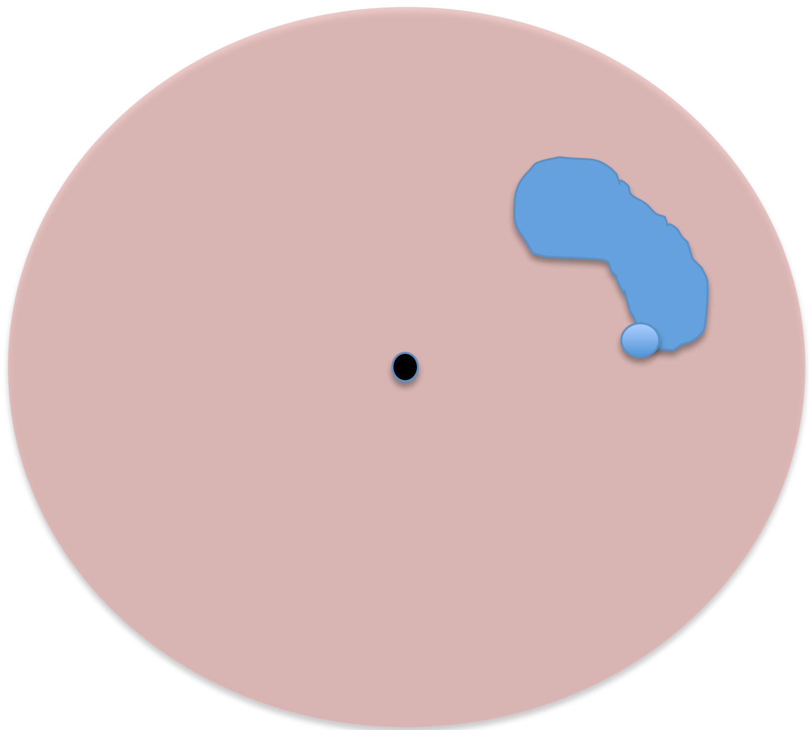
Move an electron around in its vicinity



$$\psi(\vec{x}) \to e^{ie\alpha}\psi(\vec{x}) \qquad \alpha = \oint_C \vec{A} \cdot d\vec{x}$$

Dirac Quantisation

Move an electron around in its vicinity



$$\psi(\vec{x}) \to e^{ie\alpha}\psi(\vec{x}) \qquad \alpha = \oint_C \vec{A} \cdot d\vec{x} = \int_S \vec{B} \cdot d\vec{S}$$

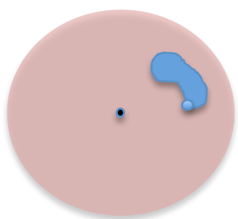
Dirac Quantisation

But there's an ambiguity in how we choose the surface *S*

$$\psi(\vec{x}) \to e^{ie\alpha'}\psi(\vec{x}) \qquad \alpha' = \oint_C \vec{A}\cdot d\vec{x} = -\int_{S'} \vec{B}\cdot d\vec{S}$$

Dirac Quantisation

These must give the same answer



$$\psi(\vec{x}) \to e^{ie\alpha}\psi(\vec{x})$$

$$\alpha = \int_S \vec{B} \cdot d\vec{S}$$

$$\psi(\vec{x}) \to e^{ie\alpha'}\psi(\vec{x})$$

$$\alpha' = -\int_S \vec{B} \cdot d\vec{S}$$

$$e^{ie\alpha} = e^{ie\alpha'} = e^{-ie(g-\alpha)}$$

Dirac Quantisation

$$e^{-ieg} = 1 \quad \Longrightarrow \quad \boxed{eg = 2\pi n} \qquad n \in \mathbf{Z}$$

A relationship between electric and magnetic charge

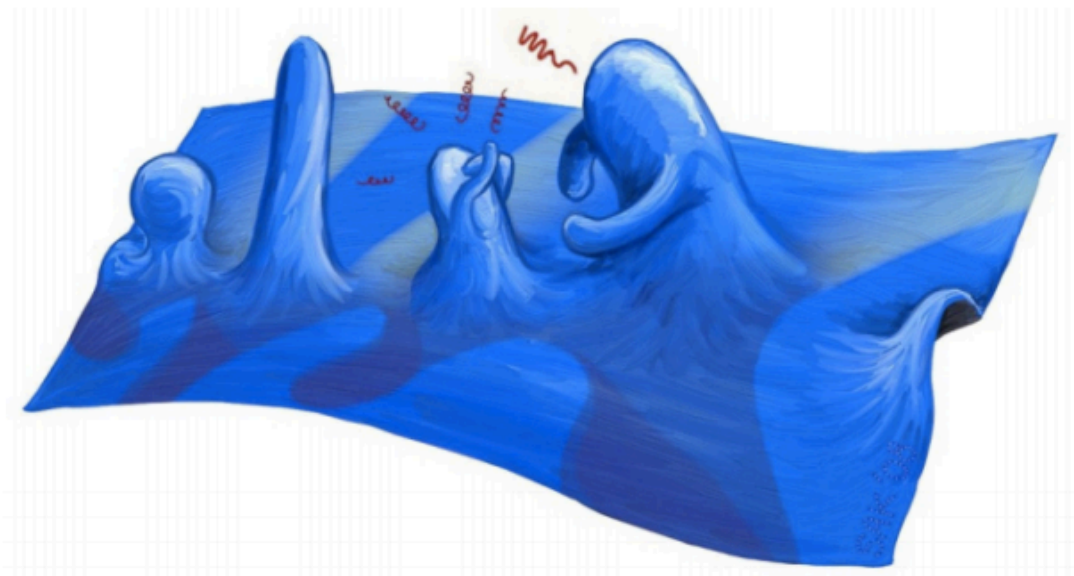
[Dirac, 1931]

The Next Step

Why monopoles have to appear…

[‘t Hooft; Polyakov 1974]

Particles Arise From Fields



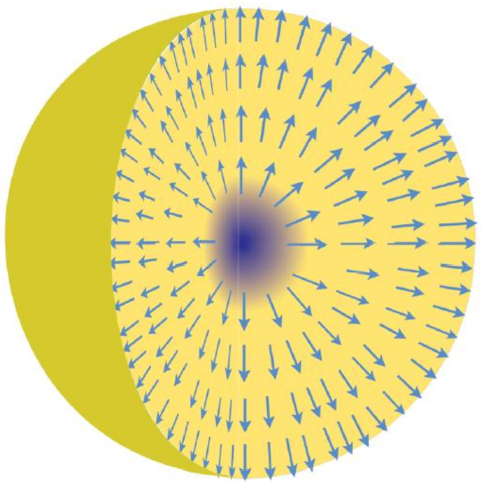
Usually particles appear as quantum fluctuations of fields

Particles Arise From Fields

But particles could appear in another way…..as solitons

't Hooft-Polakov Monopole

Magnetic monopoles appear as solitons in *certain* field theories.



These are solutions to classical partial differential equations.
(Non-linear versions of Maxwell's equations)

Monopoles in Our Universe?

Our best theory of the Universe is the *Standard Model*

$$SU(3) \times SU(2) \times U(1)$$

The Standard Model does not contain monopoles.

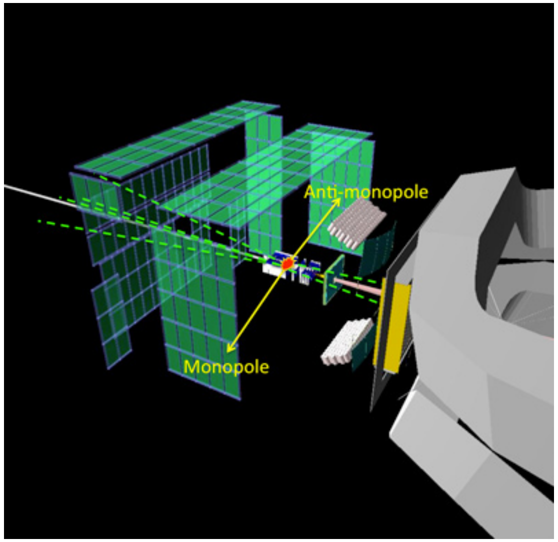
But *anything* that goes beyond the Standard Model does!

e.g. $$SU(3) \times SU(2) \times U(1) \subset SU(5)$$

Where are They?

We expect them to be *very* heavy: way beyond what the LHC can produce.

But…



MoEDAL detector…part of LHCb

Monopoles and PDEs

Monopoles are solutions to partial differential equations. Here are the simplest

$$B_i = \mathcal{D}_i \phi$$

$$B_1 = \partial_2 A_3 - \partial_3 A_2 - i[A_2, A_3]$$

$$B_2 = \partial_3 A_1 - \partial_1 A_3 - i[A_3, A_1]$$

$$B_3 = \partial_1 A_2 - \partial_2 A_1 - i[A_1, A_2]$$

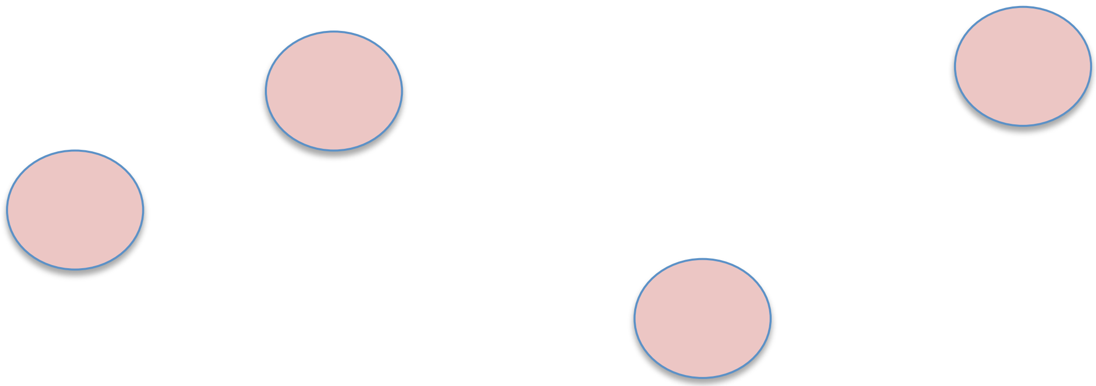
$$\mathcal{D}_i \phi = \partial_i \phi - i[A_i, \phi]$$

A_i and ϕ are Hermitian 2x2 matrices

Monopoles and PDEs

These equations have a rather special property. They admit many many solutions.

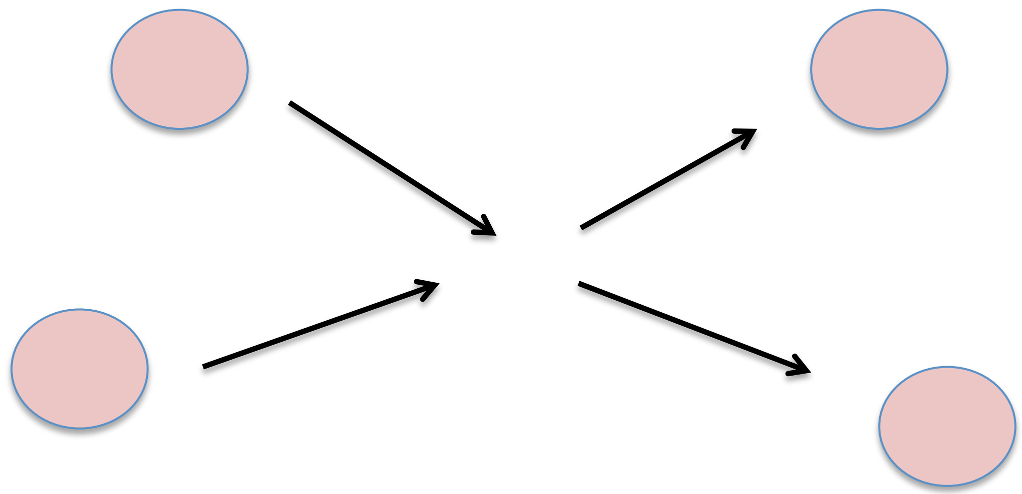
Fix the magnetic charge *n*. Then the most general solution has *4n* parameters.



This should be thought of as *n* monopoles.

- Each can sit anywhere in $\mathbf{R}^3$.
- Each has an extra degree of freedom showing how it's embedded in the 2x2 matrix

Monopole Scattering



Looks like we have to solve nasty, time dependent partial differential equations

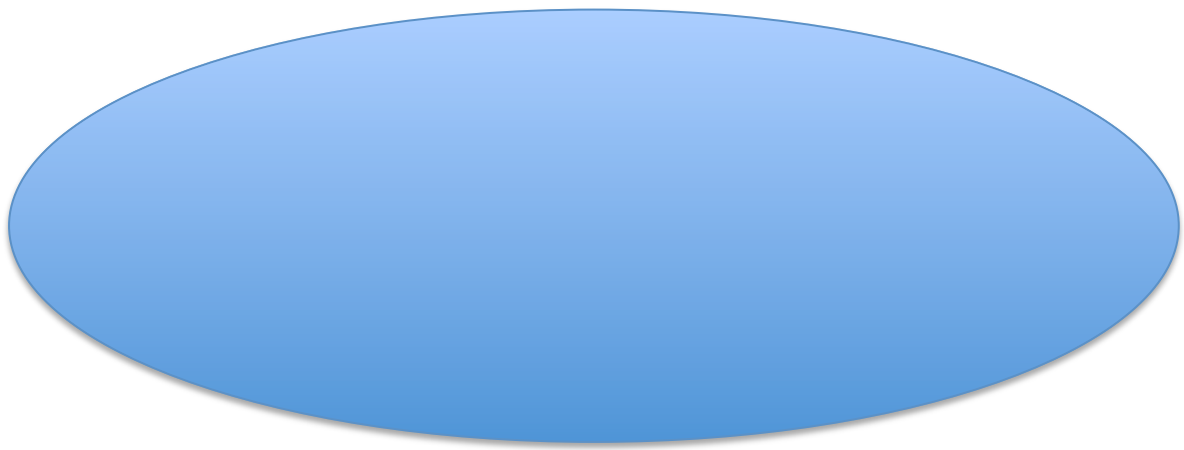
Monopoles and Geometry

Idea: If the scattering is very slow, a photograph at any time will look like a static monopole configuration

[Manton]

The Moduli Space

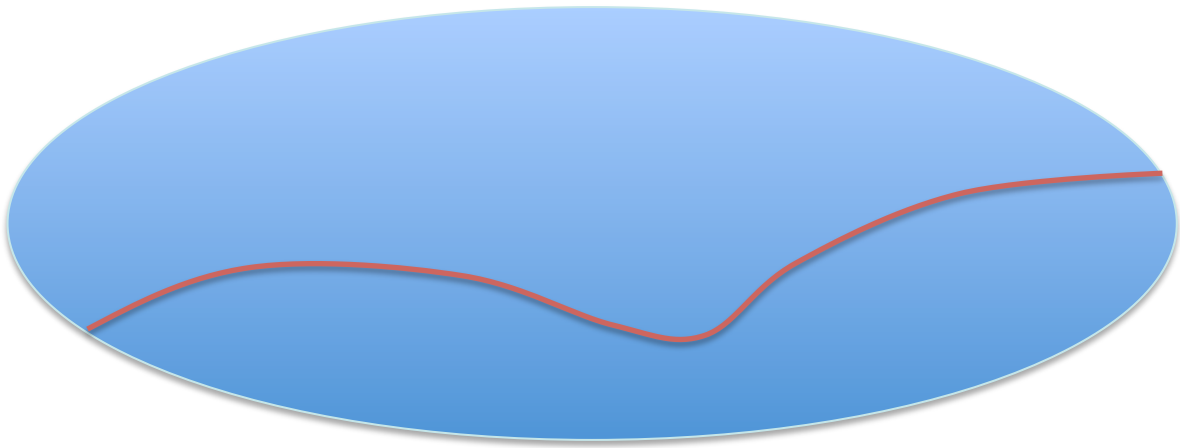
This is the space of solutions to the monopole equations of fixed charge, *n*



It is a manifold (space) of dimension *4n*

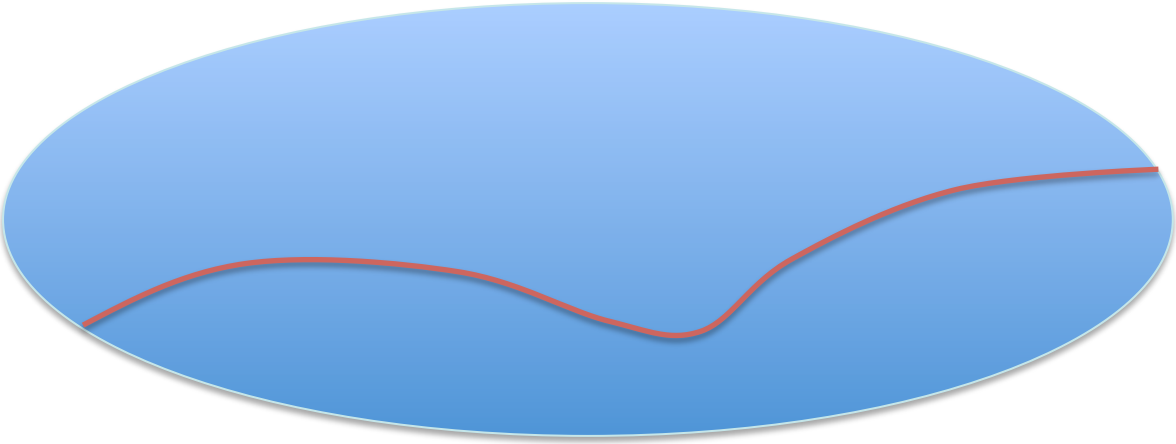
Scattering on the Moduli Space

Low-energy scattering of monopoles traces out a path on the moduli space



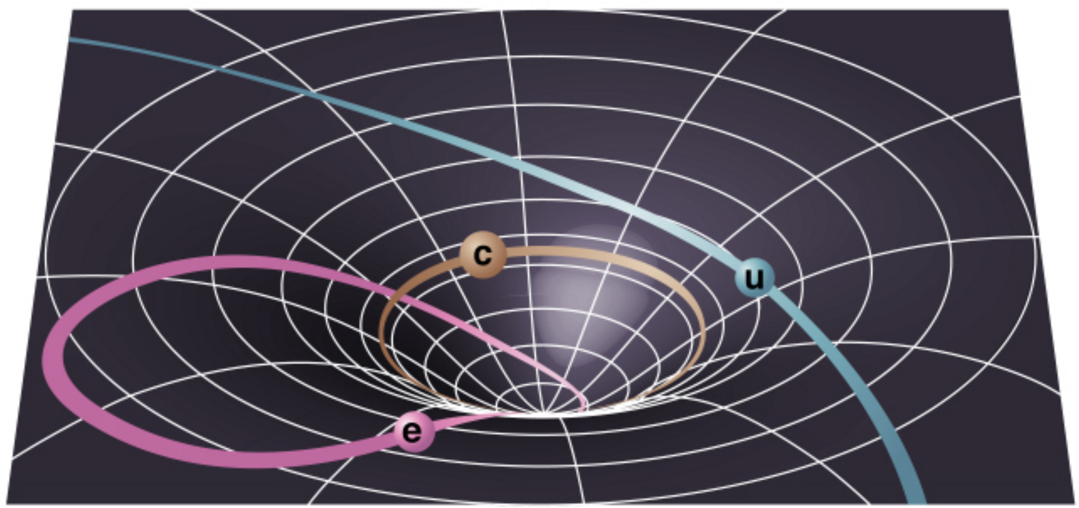
Scattering on the Moduli Space

<u>Manton's Idea</u>: Make the moduli space curved so that this is the natural path



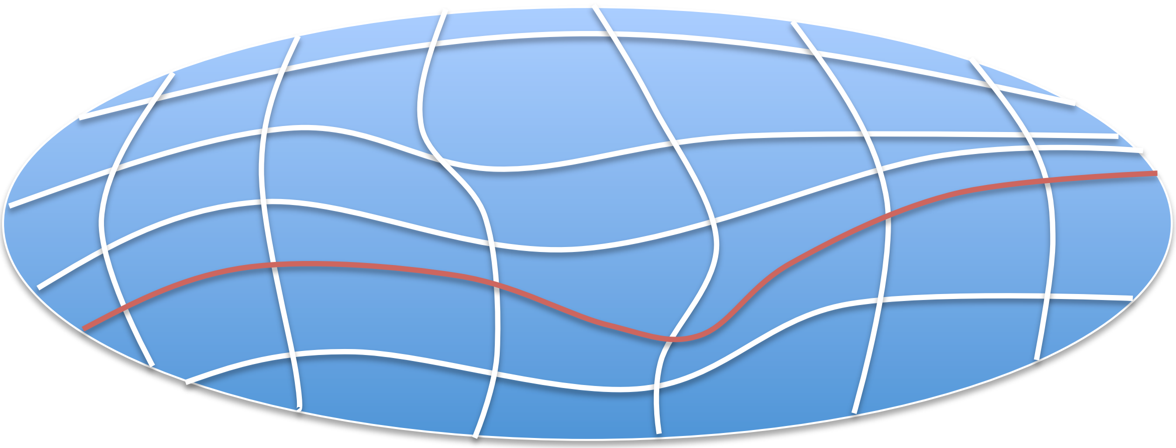
It's Like General Relativity

…but in higher dimensions



Manton Metric

Make the moduli space curved so that this is the natural path



There is a *metric* on the moduli space that does this.

Properties of the Metric

These curved manifolds turn out to have lots of interesting properties

Related to “quaternionic” manifolds (hyperKahler)

[Atiyah and Hitchin]

Monopoles and Geometry

This is the beginning of a long story connecting monopoles and geometry

The story is likely not yet finished

[Sen, Seiberg and Witten, Donaldson, Taubes…]

Thank you for your attention