What Would Newton Do?

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Adams Society, November 2012

UNIVERSITY OF CAMBRIDGE

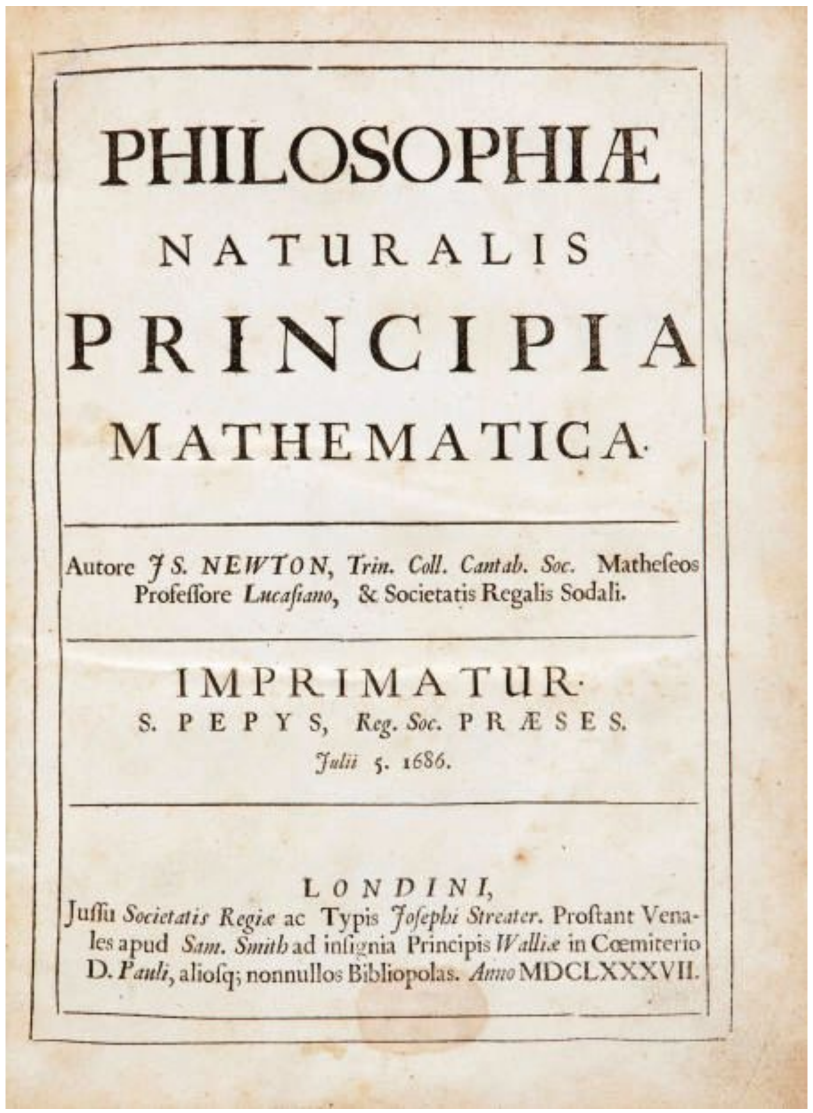
This talk is partly about physics, but partly about the history of physics. While the physics in the talk speaks for itself, the history needs a little further explanation. For this reason, I've added some annotations at the beginning of the talk. Anything I said out loud I've placed in a box. Like this.

I should also stress that I'm no historian. What follows is my cartoon version of history in the 17th century.

This is really a talk about the Principia. This is Newton's masterpiece. First published in 1687, it describes the laws of motion, his law of gravity and then goes into great detail deriving various consequences from these laws. Not just the obvious stuff like the laws of planetary motion and friction, but hugely complicated detail of how the Earth gets squashed as it rotates and the precession of the equinoxes and a study of the tides and the first look at the three body problem of the Sun-Earth-Moon using perturbation theory and all sorts of things. It is rightly viewed as the beginning of theoretical physics.

Now Newton didn't just come up with this stuff over night. It’s the culmination of more than 20 years work. Most of the big breakthroughs were made in 1665 when Cambridge closed due to the plague and Newton returned to his home in Grantham.

Newton's Principia (1687)

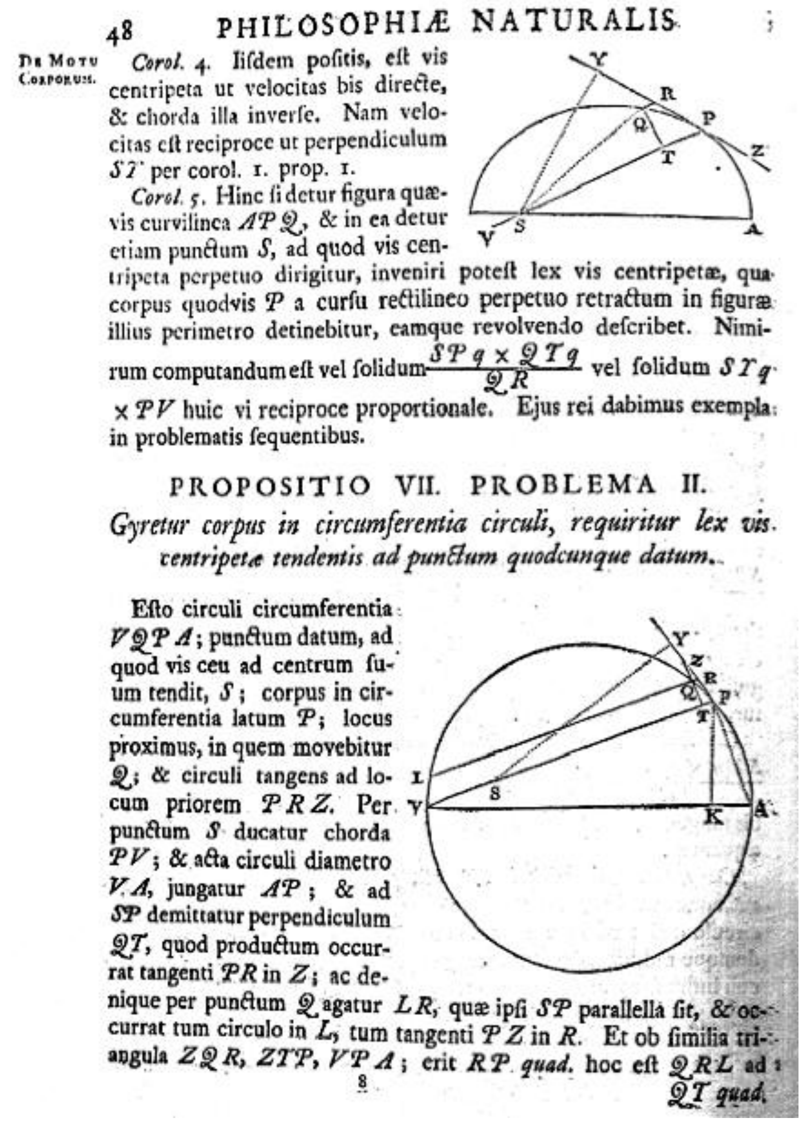


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA

Autore *JS. NEWTON*, *Trin. Coll. Cantab. Soc.* Mathefeos Profeffore *Lucafiano*, & Societatis Regalis Sodali.

IMPRIMATUR
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,
Juffu *Societatis Regiæ* ac Typis *Jofephi Streater*. Proftant Venales apud *Sam. Smith* ad infignia Principis *Walliæ* in Cœmiterio D. *Pauli*, aliofq; nonnullos Bibliopolas. *Anno* MDCLXXXVII.



48 PHILOSOPHIÆ NATURALIS

De Motu Corporum.

Corol. 4. Iifdem pofitis, eft vis centripeta ut velocitas bis directe, & chorda illa inverfe. Nam velocitas eft reciproce ut perpendiculum *ST* per corol. 1. prop. 1.

Corol. 5. Hinc fi detur figura quævis curvilinea *APQ*, & in ea detur etiam punctum *S*, ad quod vis centripeta perpetuo dirigitur, inveniri poteft lex vis centripetæ, qua corpus quodvis *P* a curfu rectilineo perpetuo retractum in figuræ illius perimetro detinebitur, eamque revolvendo defcribet. Nimirum computandum eft vel folidum $\frac{SPq \times QTq}{QR}$ vel folidum $STq \times PV$ huic vi reciproce proportionale. Ejus rei dabimus exempla in problematis fequentibus.

PROPOSITIO VII. PROBLEMA II.

Gyretur corpus in circumferentia circuli, requiritur lex vis centripetæ tendentis ad punctum quodcunque datum.

Efto circuli circumferentia *VQPA*; punctum datum, ad quod vis ceu ad centrum fuum tendit, *S*; corpus in circumferentia latum *P*; locus proximus, in quem movebitur *Q*; & circuli tangens ad locum priorem *PRZ*. Per punctum *S* ducatur chorda *PV*; & acta circuli diametro *VA*, jungatur *AP*; & ad *SP* demittatur perpendiculum *QT*, quod productum occurrat tangenti *PR* in *Z*; ac denique per punctum *Q* agatur *LR*, quæ ipfi *SP* parallella fit, & occurrat tum circulo in *L*, tum tangenti *PZ* in *R*. Et ob fimilia triangula *ZQR*, *ZTP*, *VPA*; erit *RP quad.* hoc eft *QRL* ad *QT quad.*

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But Newton wasn't the only one thinking about gravity. In London, three extraordinarily smart people were also working on these ideas. This group was Edmund Halley (of comet fame), Christopher Wren (of Christopher Wren fame) and, most importantly, Robert Hooke. You probably know Hooke for that stupid spring law, but he was one of the great intellectual forces of the 1600s.

By the late 1670s, it seems likely that all three of these guys knew that there should be an inverse square law --- that is a force that drops off as 1-over-distance-squared --- that is responsible for gravity.

In fact, as I'll show you shortly, it's actually quite easy to see that gravity must follow an inverse-square-law. This is because 50 years earlier, Kepler had written down three laws that the describe the motion of the planets. The first of these states that all planets move on an ellipse. Laws 2 and 3 tell you exactly how fast the planets move on an ellipse. And, it turns out that if you think about circular orbits, Kepler's third law is equivalent to the inverse-square law for gravity. All of this will be covered later.

But what's hard is to show Kepler's first law: to show that an inverse-square law of gravity means that the planets necessarily move on ellipses. That's the tough thing to show.

In 1684, Hooke tells his two friends that he’s figured this out. He claims that he can show how the planets move on an ellipse. They, understandably, say "show us". Hooke kind of mumbles something and says he’s lost the bit of paper and can't do it or won't do it and they leave thinking that he’s full of shit.

But this prompted Halley to come up to Cambridge to meet with Newton. Halley tells Newton about Hooke’s claims and Newton says "oh, I showed that decades ok". Halley says "show me". Newton kind of mumbles and says he can’t find the piece of paper and so on and Halley leaves, probably thinking that Newton is full of shit too.

Except the next month, Newton sends Halley the proof. It's a short 9 page paper called "*De Motu Corporum in Gyrum*", or "*The Motions of Bodies in Orbit*". Halley says "amazing, let's publish". Newton says "well, it's not quite ready. Let me add a few things". Those few things took 2 more years and the result is the 3 books that make up the Principia.

(There's actually an interesting extra story here. When it's finally done, Newton sends it to Halley who had arranged for the Royal Society to publish it. Except, with the delay, they've already spent their publication budget on another book called, wonderfully, "The History of Fishes". And there's no more money to publish Newton. So Halley pays for publication out of his own pocket)

Anyway, there's one key discovery missing from Newton's Principia: that's the discovery of calculus. It appears that he developed this back in 1665 as well. But, for some reason, he decided to keep it secret. Instead, all the proofs in the Principia are written using traditional methods of geometry.

Meanwhile, around the same time that the Principia is published, Leibniz in Germany publishes calculus. And, unlike Newton, Leibniz is really excited about this and telling everyone and, of course, everyone else gets really excited as well because, after all, it's brilliant. And one of the obvious challenges is to take all the proofs in the Principia and rewrite them using the new methods of calculus. This challenge is taken up eagerly by a number of mathematicians and it was very successful; so much so that by the time the second edition of the Principia was published in 1713, no one is thinking about this in terms of geometry. It's all about calculus.

Of course, Newton is kind of pissed about this. So he's jumping up and down shouting that he did it first --- which he probably did, but he should have just told someone. And, in 1715, Newton writes that, of course, he had originally proved everything in the Principia using calculus and later translated it into geometry so that people could understand it better. As far as I can tell, most historians think this is just a lie.

Anyway, this leaves the Principia as something of an enigma. One of the most important scientific books ever written, yet using machinery that was out of date before the second edition was published. There's a quote from William Whewell, an old master of trinity, which sums this up nicely

"The ponderous instrument of synthesis, so effective in Newton's hands, has never since been grasped by anyone who could use it for such purpose; and we gaze at it with admiring curiosity, as some gigantic implement of war, which stands idle among the memorials of ancient days, and makes us wonder what manner of man he was who could wield as a weapon what we can hardly lift as a burden"

William Whewell on Newton’s geometric proofs, 1847

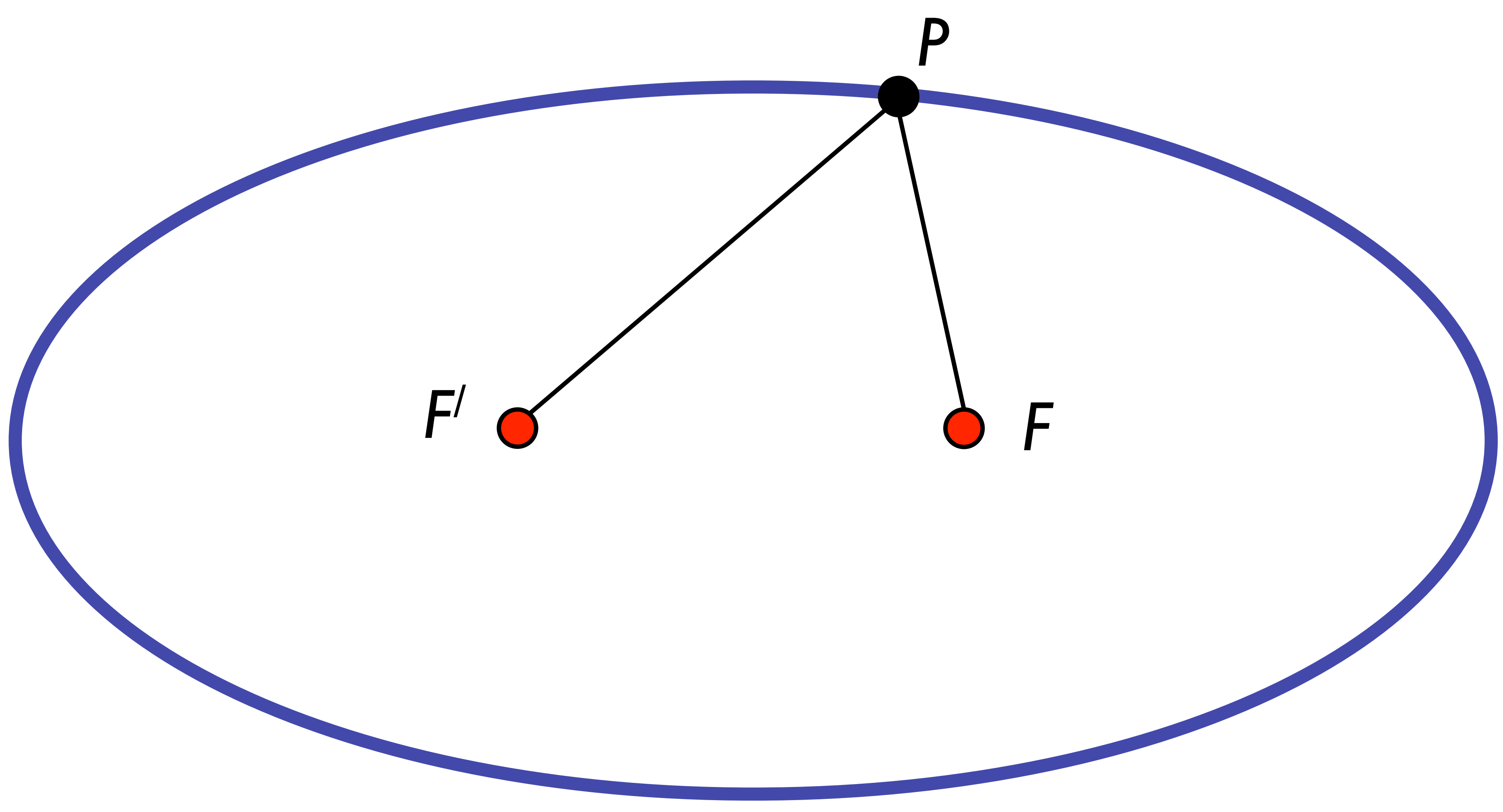
So what I want to do here is show you a geometrical proof of Kepler's laws in the Newtonian style. This proof is not quite the same as the one presented in the Principia. But it's fairly close. And the proof itself has an impressive pedigree. It was first published in a book by Maxwell, who gave credit to Hamilton. It was later independently rediscovered by Feynman. He gave a lecture on this which was lost apart from an audio recording and some scribbled notes. But in the 1990s, the lecture was reconstructed by David and Judith Goodstein and published in a book called "Feynman's lost lecture". If you want to learn more about this, then looks at this book.

(If you really want to learn more about this, then the great astrophysicist Chandrasekhar has written an annotated version of the Principia. Be warned: it's not easy going).

First…

<u>Some Facts About Ellipses</u>

Building an ellipse from a piece of string

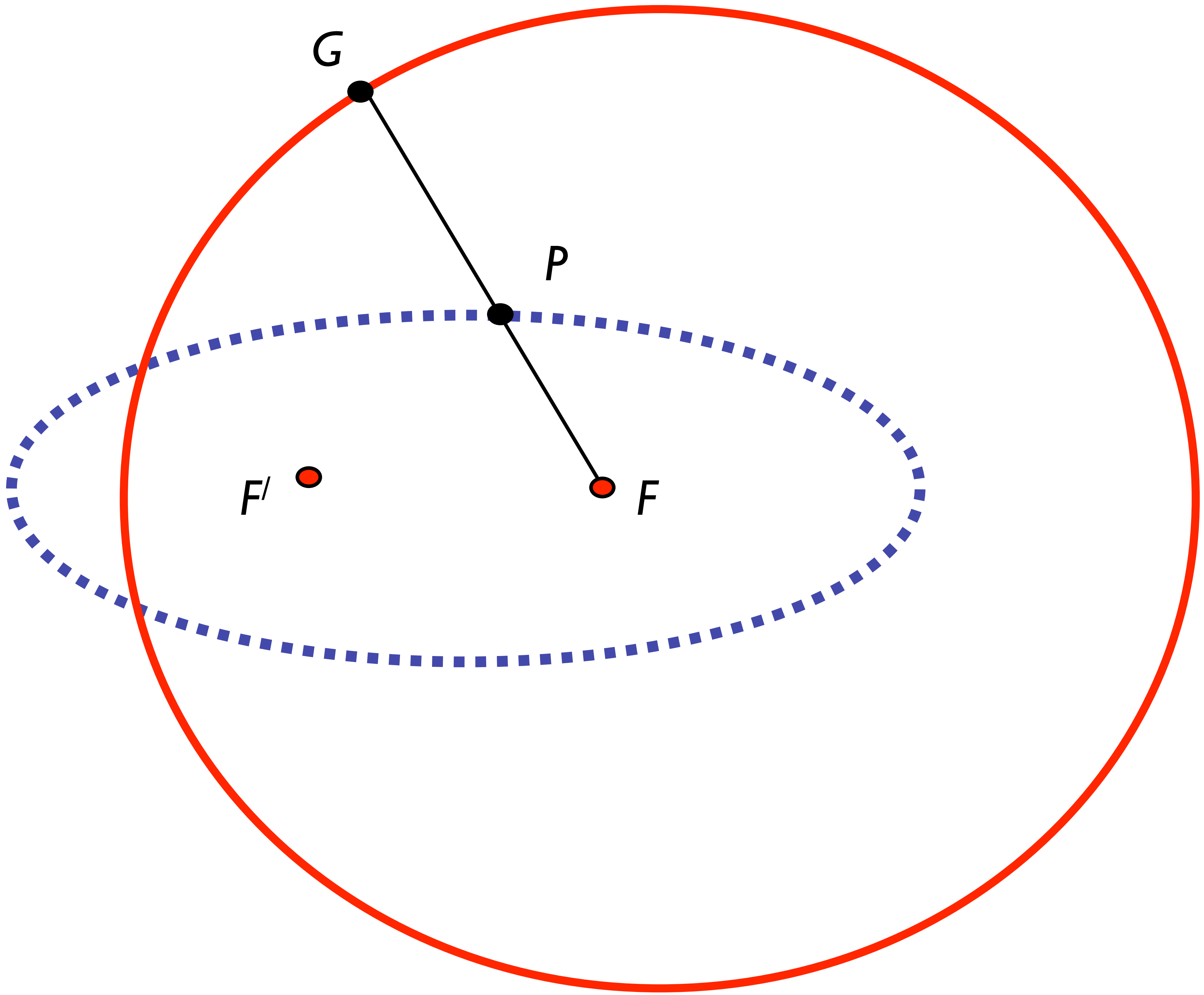


$F'P + FP = \text{constant}$

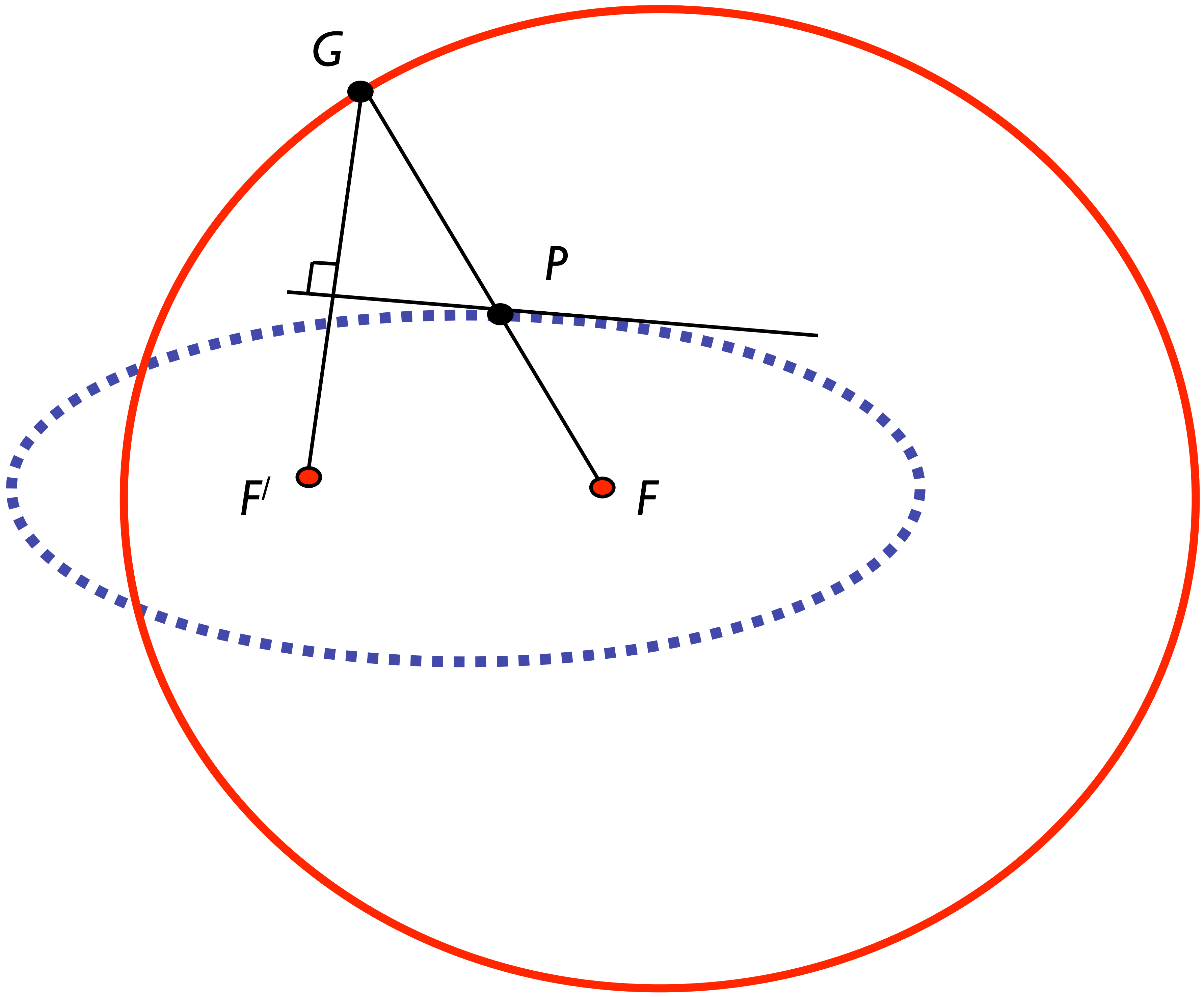
There's an interesting historical fact here. The points F and F' are the foci of the ellipse. Each is called a focus.

As we will see later, the planets move around the Sun on an ellipse with the Sun sitting at one of the focus. And the name *focus* for this point of the ellipse was chosen by Kepler himself. Because it is latin for "fireplace".

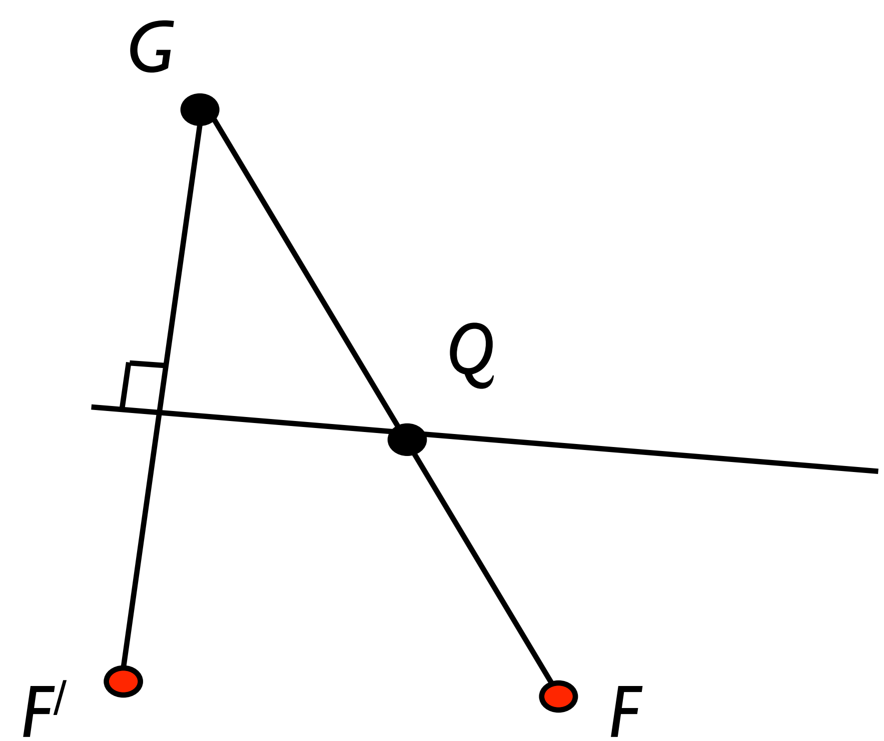
Unpinning the string: draw a circle with centre *F*



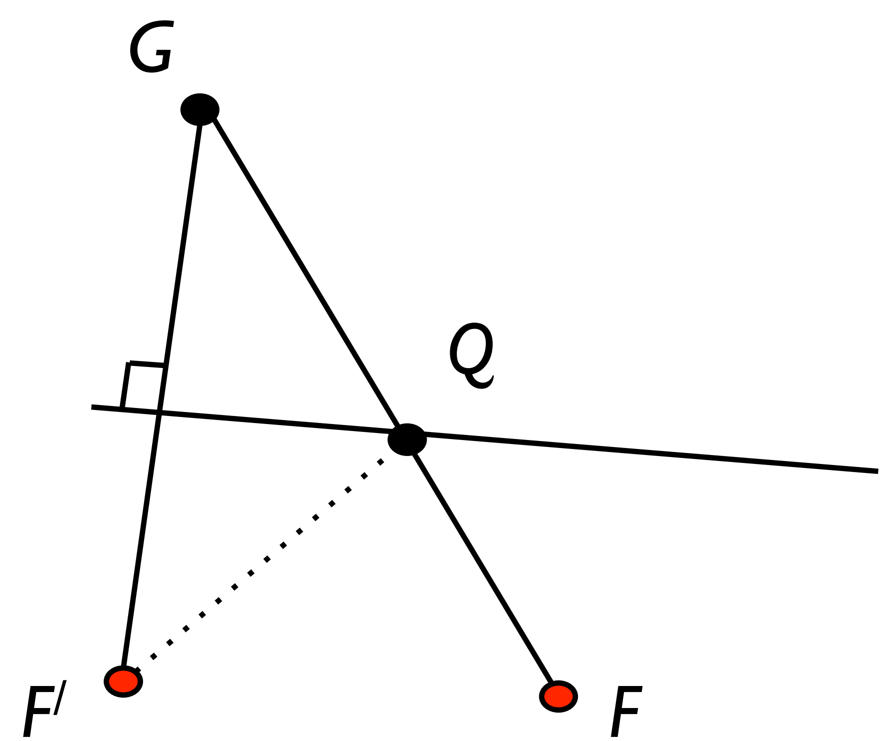
Draw the perpendicular bisector of $F'G$



Claim: The perpendicular bisector intersects the line *FG* at *Q*=*P*

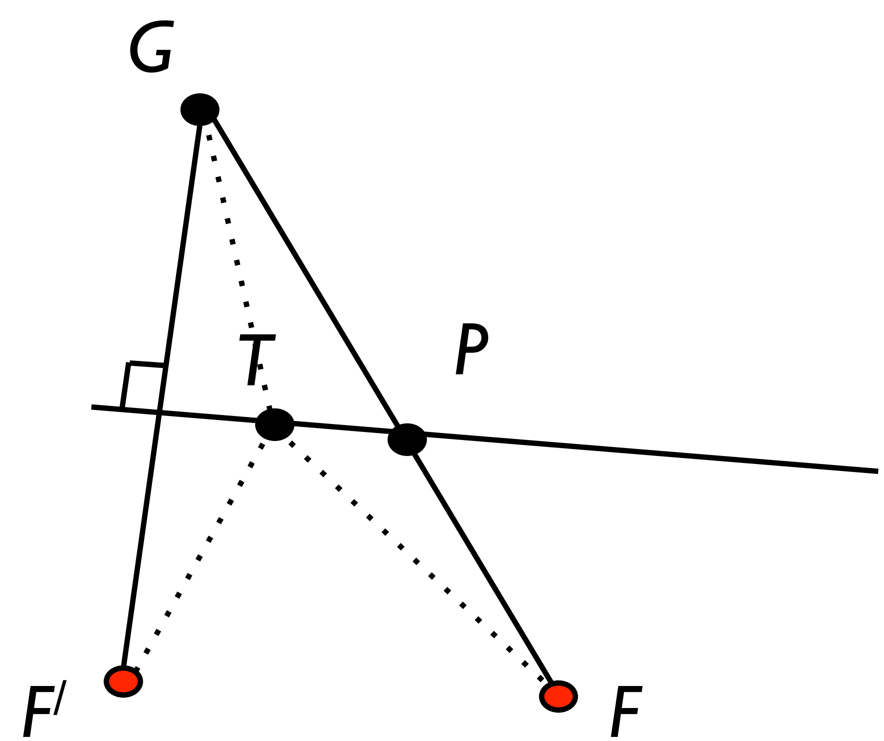


Proof: congruent triangles $\Longrightarrow$ $FQ = QG$



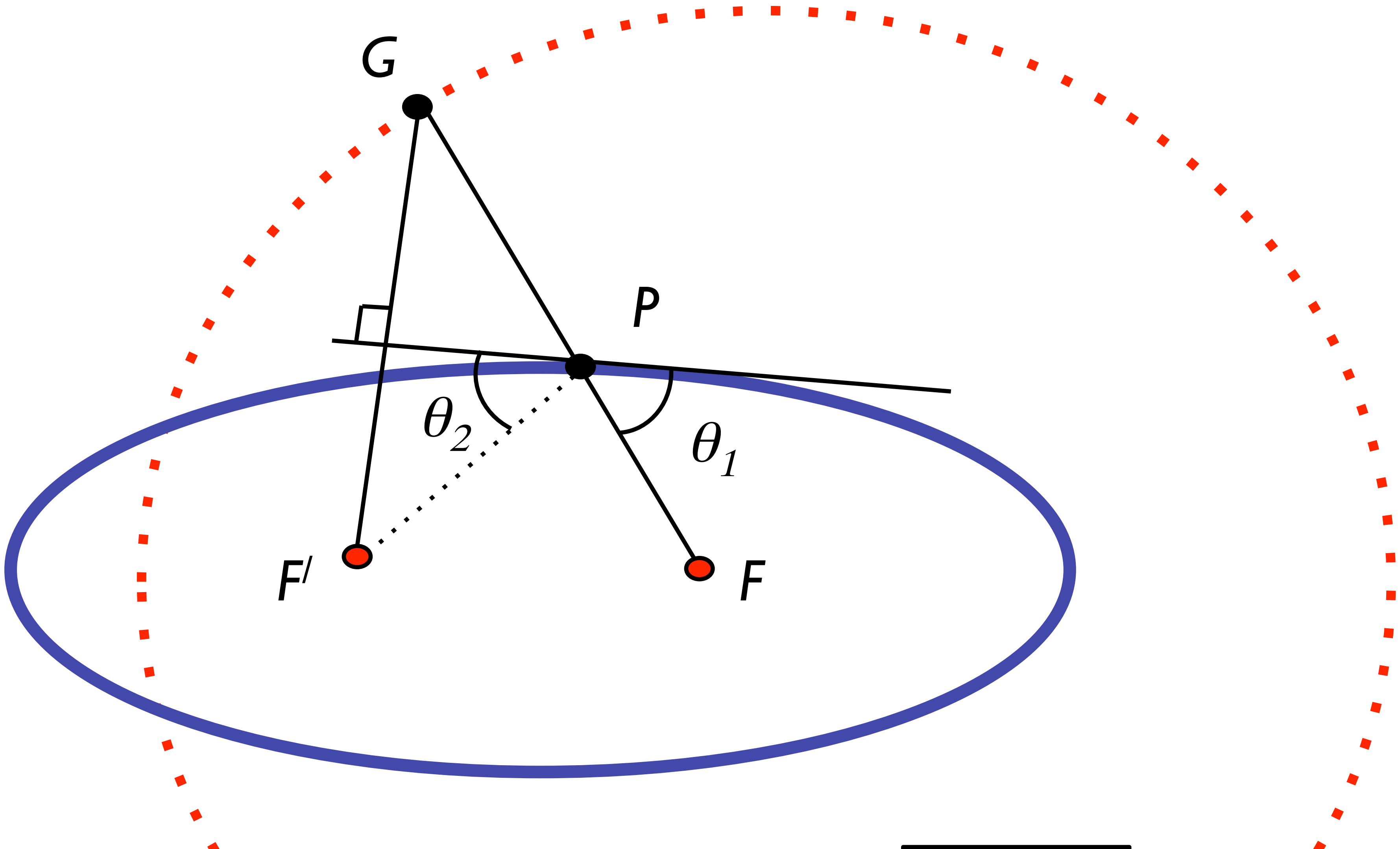
So distance $F'QF$ is the same as $F'G$ which is the length of the original string $\Longrightarrow$ point Q lies on the ellipse

Moreover…No other point on the perpendicular bisector lies on the ellipse



$FTG > FPG$ but $TG = F'G \Longrightarrow FTF' > FPPF'$

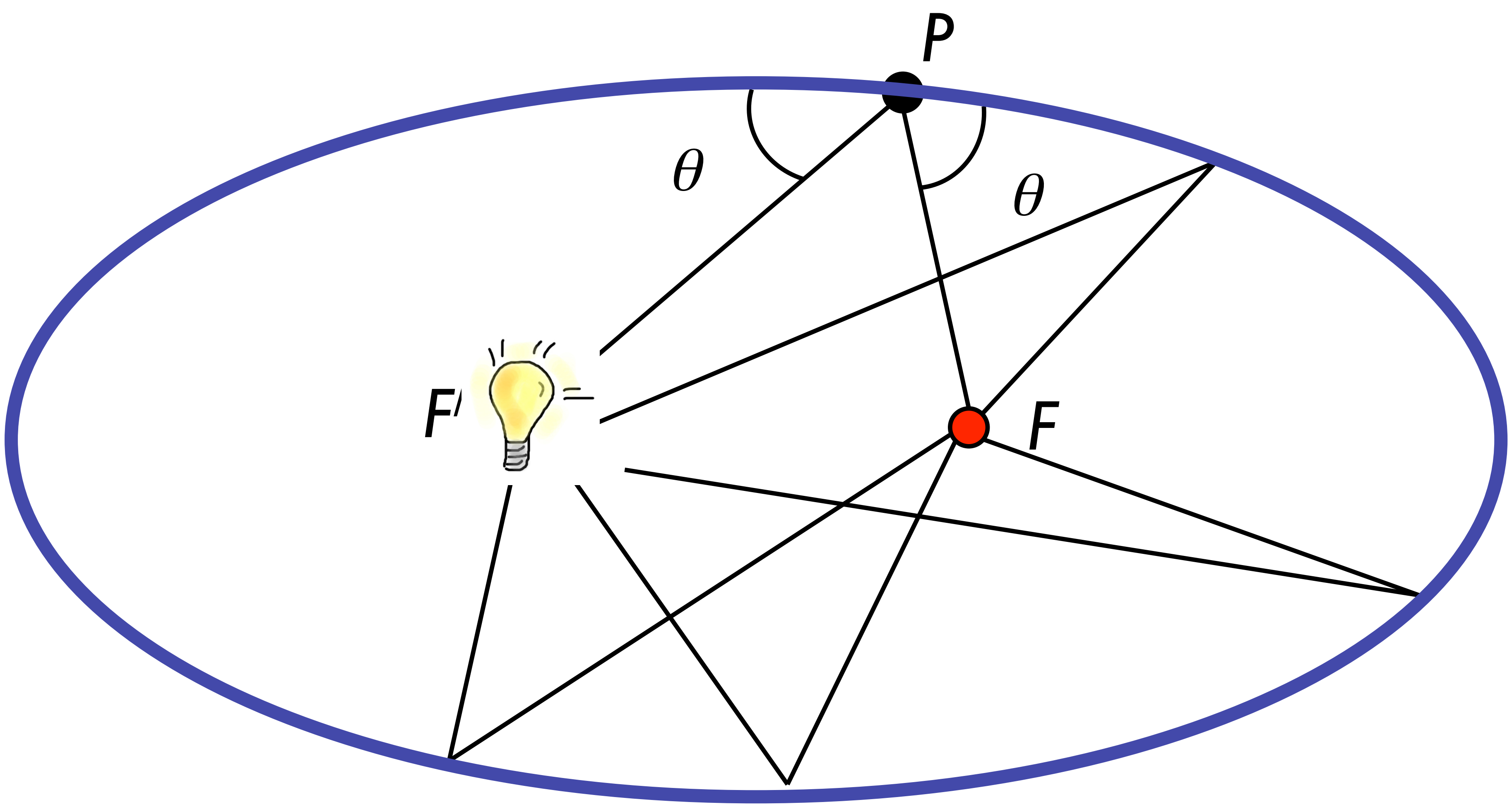
This means the perpendicular bisector is tangent to the ellipse



$\theta_1 = \theta_2$

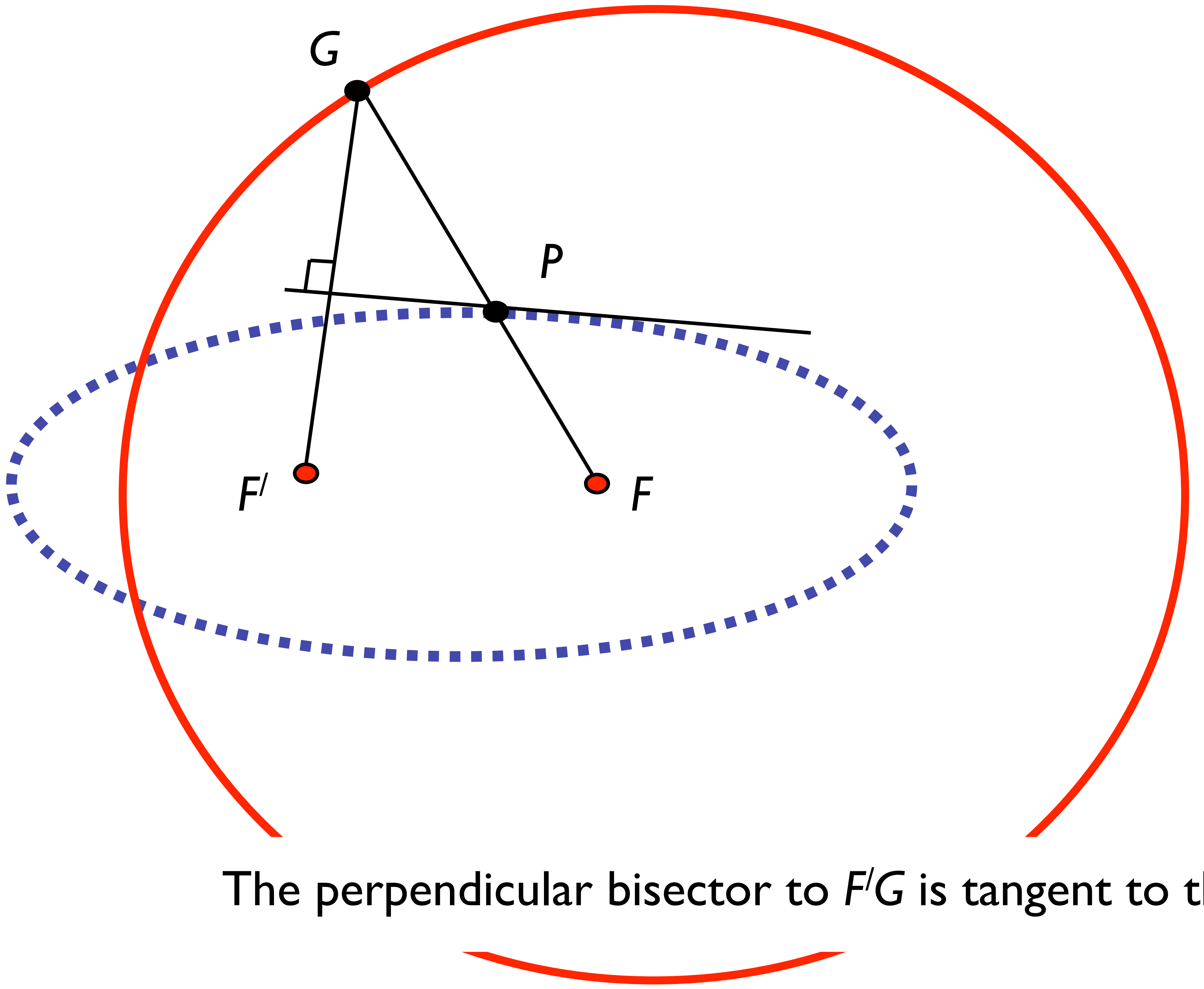
(using the fact that there are congruent triangles in *F/PG*)

A simple physics corollary: an elliptic mirror



All light leaving one focus reaches the other

Punchline to remember!



The perpendicular bisector to *F/G* is tangent to the ellipse at *P*

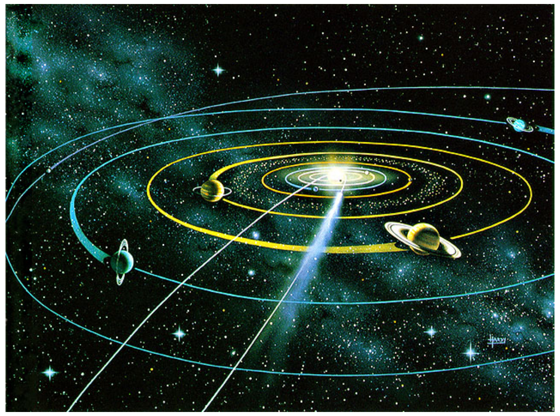
Planetary Orbits

Kepler's laws of planetary motion

1. The orbits of all planets are ellipses, with the Sun at one focus. (1609)
2. A planet sweeps out equal areas in equal times. (1619)
3. The time, *T*, of an orbit is related to the distance, *R*, of the planet by (1627)

$$T \sim R^{3/2}$$

Tycho Brahe
1546-1601



Johannes Kepler
1571-1630

Our modern understanding

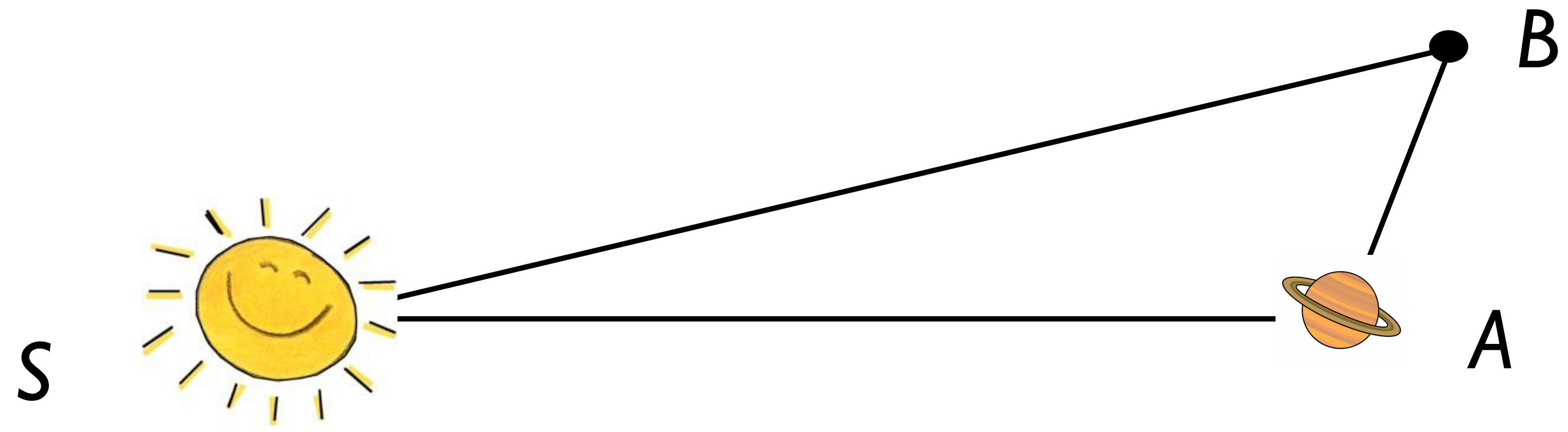
1. The orbits of all planets are ellipses, with the Sun at one focus.
2. A planet sweeps out equal areas in equal times.
3. The time, *T*, of an orbit is related to the distance, *R*, of the planet by

$$T \sim R^{3/2}$$

- Force towards the Sun $\Longrightarrow$ Kepler 2
  - Really just angular momentum conservation.
- Kepler 3 for circular orbits $\Longleftrightarrow$ Inverse square law of gravity.
- Inverse square law $\Longrightarrow$ Kepler 1.

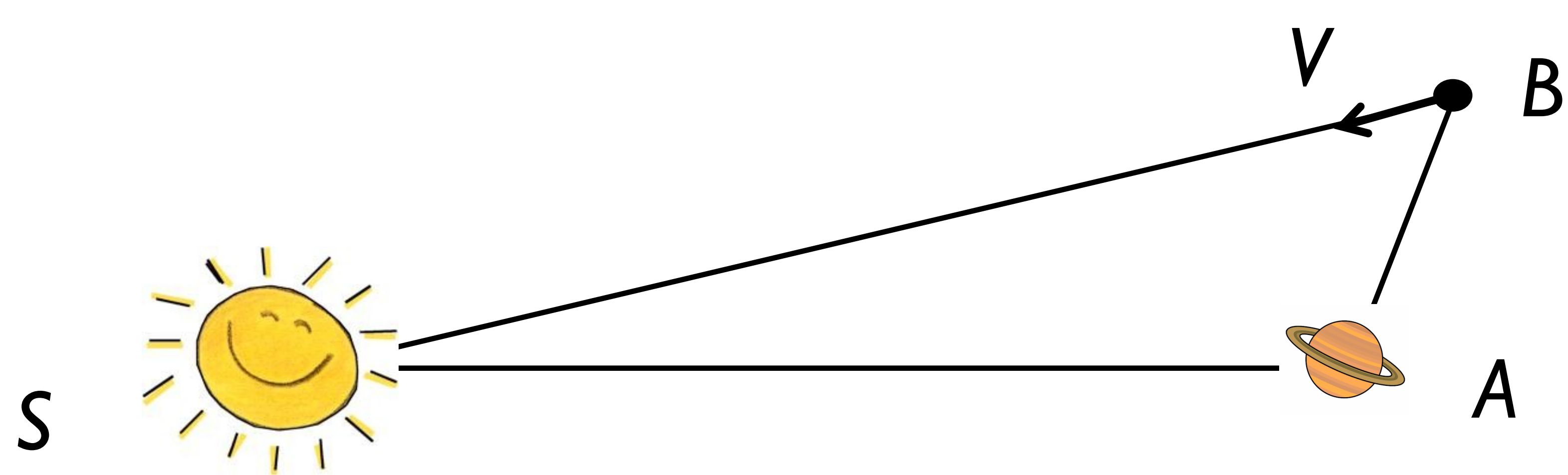
Force towards the Sun

- In time Δt, the planet travels in a straight line AB



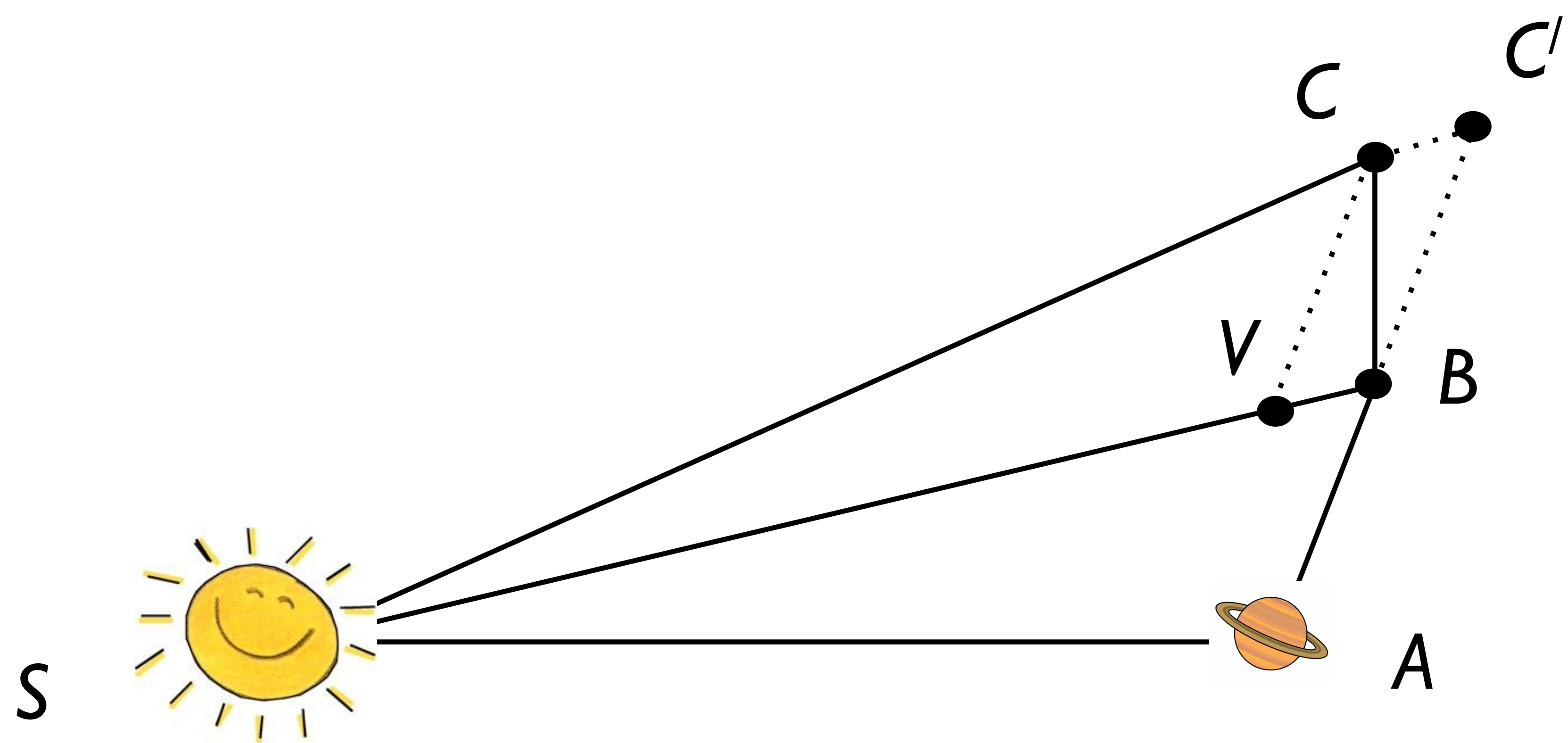
Force towards the Sun

- In time Δt, the planet travels in a straight line *AB*
- It then receives an impulse *BV* towards the Sun



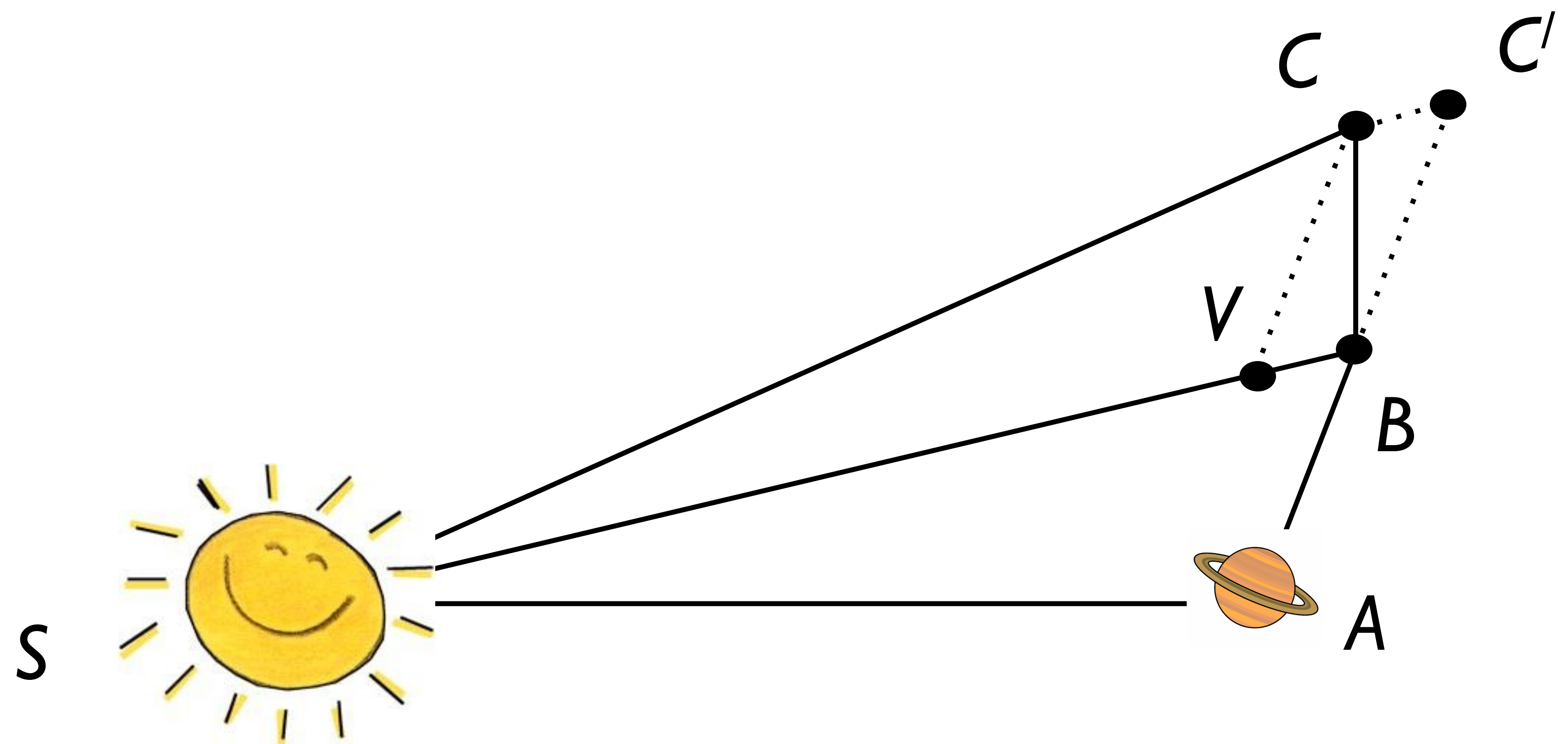
Force towards the Sun

- In time Δt, the planet travels in a straight line *AB*
- It then receives an impulse *BV* towards the Sun
- In the next Δt it travels the vector sum of *AB* + *BV*. This takes it to *C*, instead of $C^{/}$



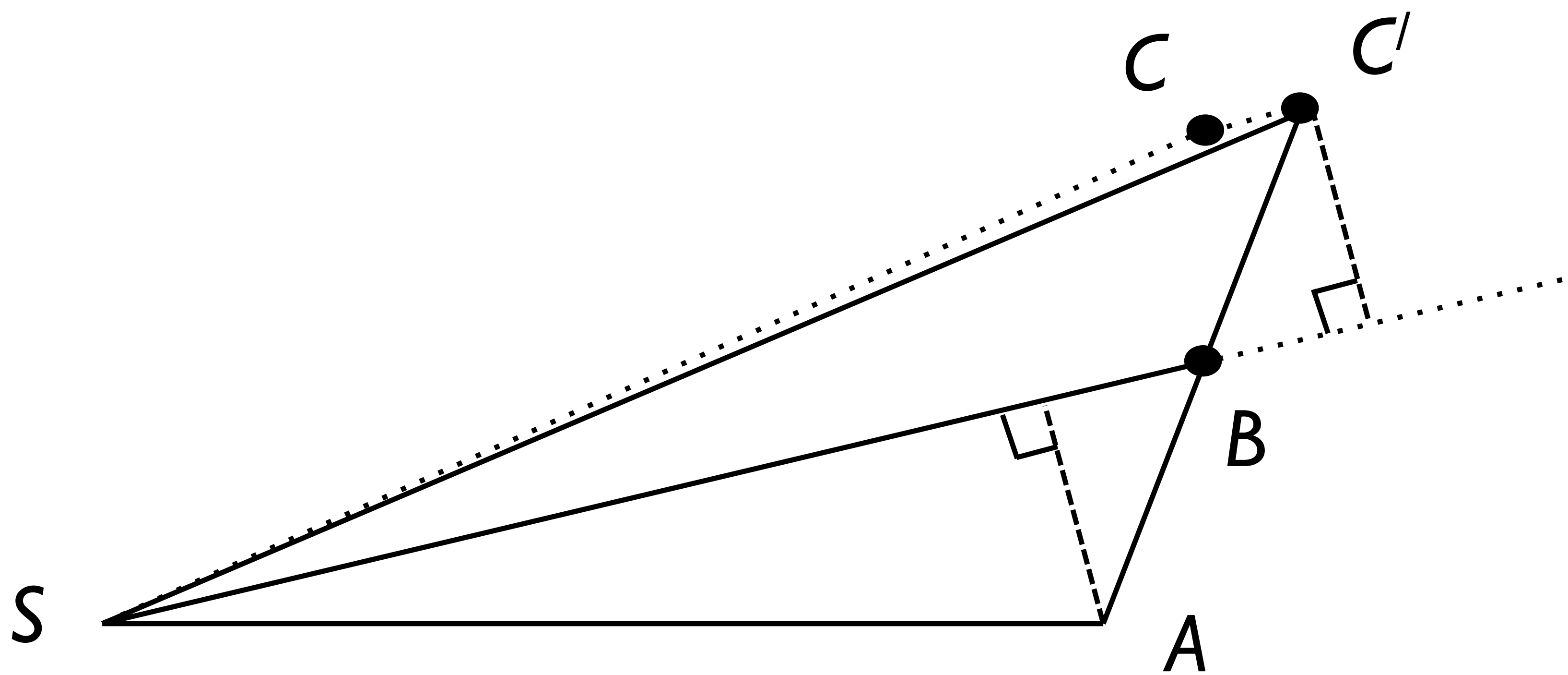
Note: $C^{/}C$ is parallel to *BV*

Kepler's Second Law: Area *SAB* = Area *SBC*



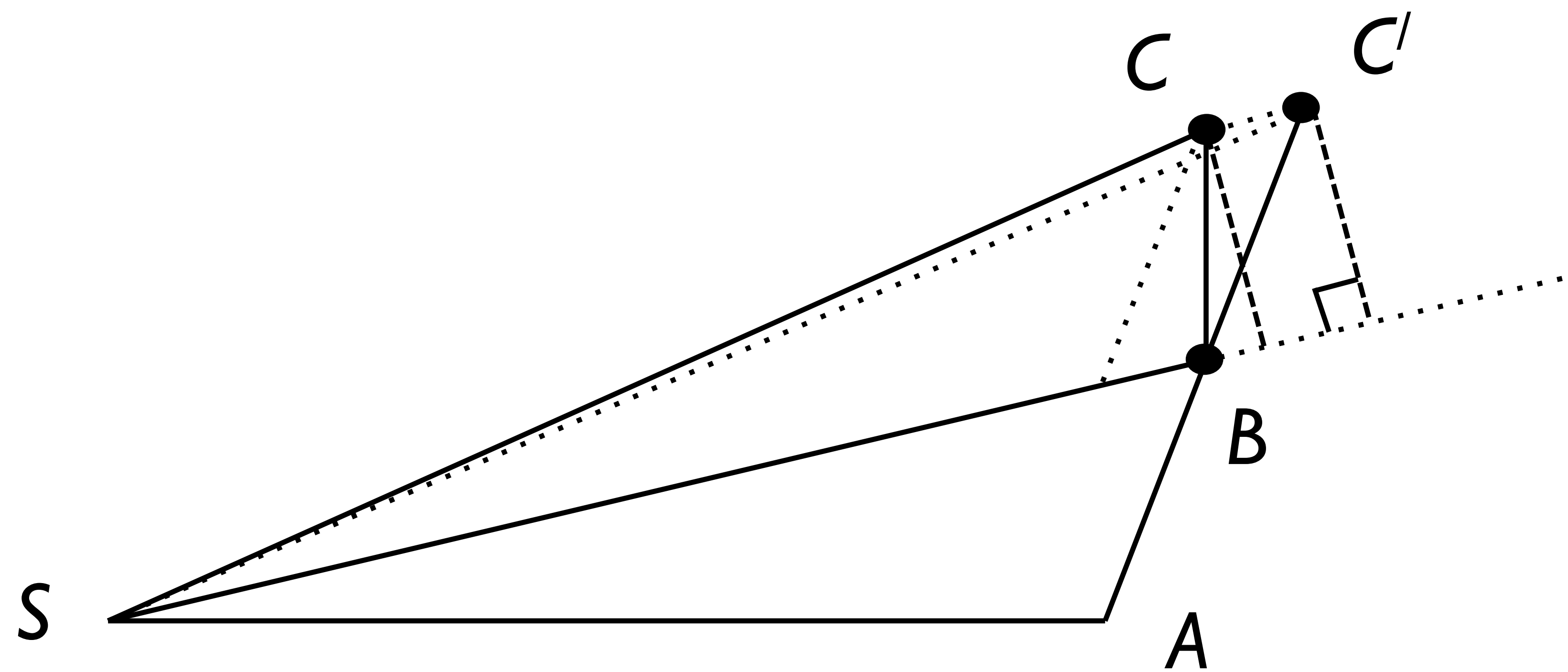
Note: *C*/*C* is parallel to *BV*

First show: Area *SAB* = Area *SBC*′



Area = ½ base *x* height
½ *SB x* same distance

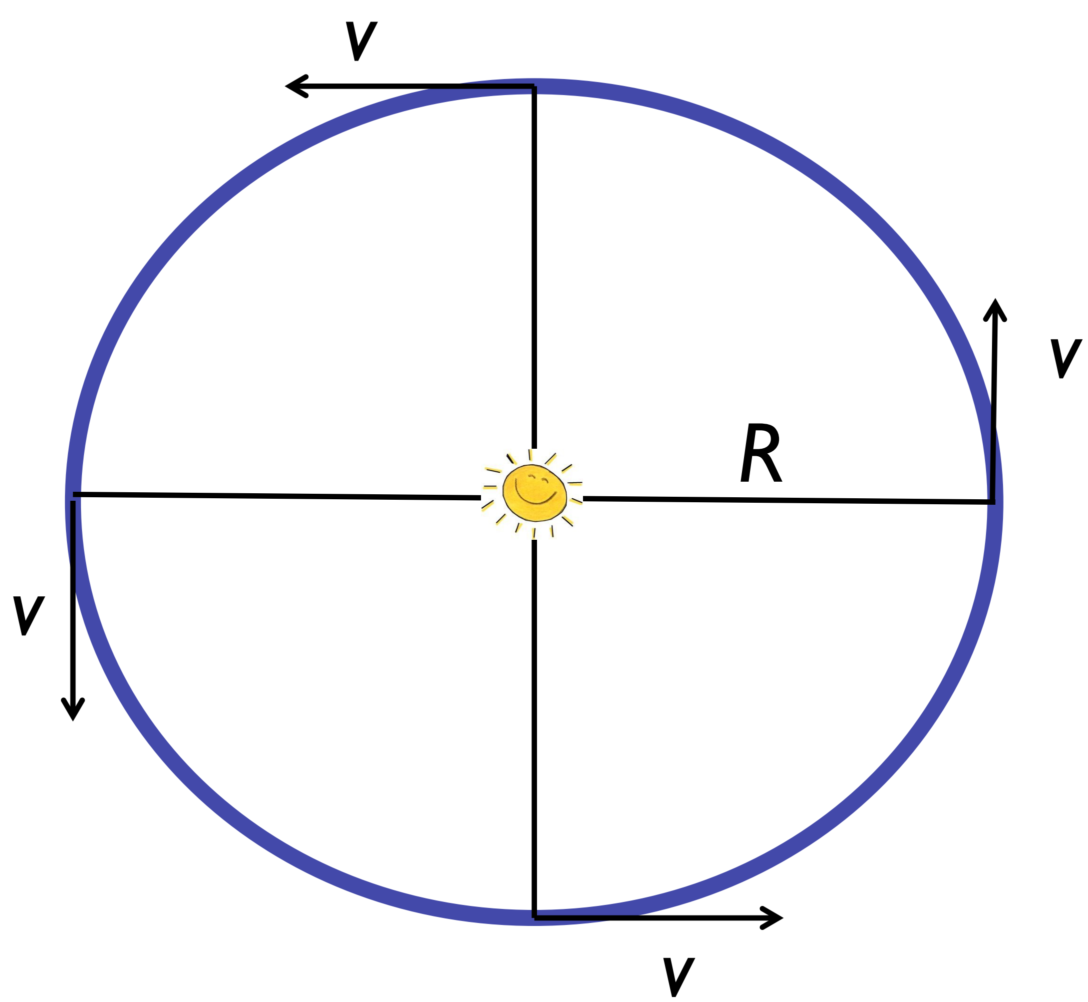
Then show: Area *SAC* = Area *SBC*$^{/}$



Area = ½ base *x* height
½ *SB x* same distance

This demonstrates Kepler's second law

The inverse square law from circular orbits



- From Kepler 2, the speed, *v*, must be constant.
- The time taken to make an orbit is

$$T = \frac{2\pi R}{v}$$

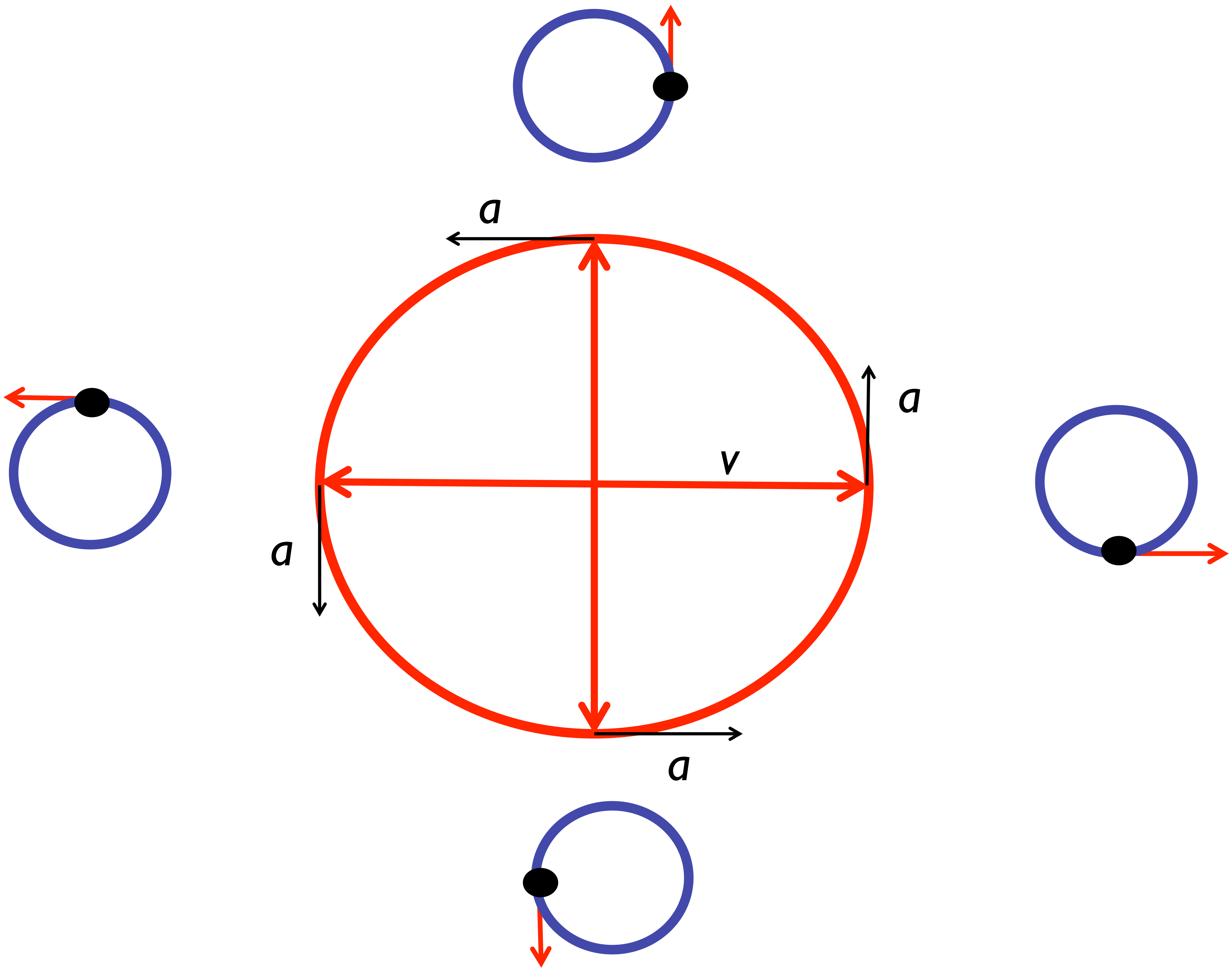
Circular Acceleration

Claim: the acceleration necessary to move in a circle is

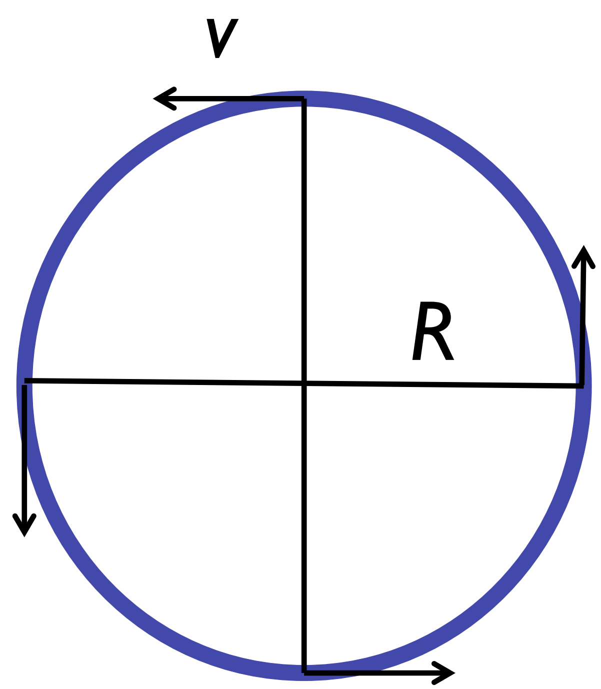
$$a = \frac{v^2}{R}$$

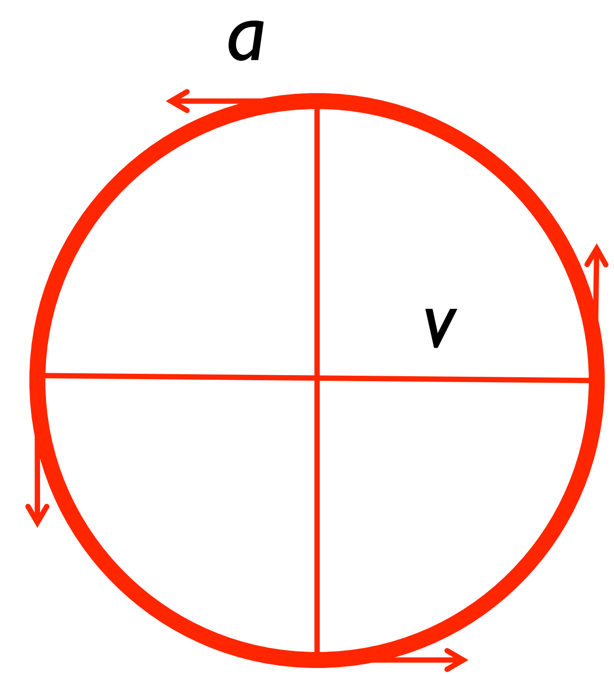
- We will now provide a geometric proof of this well-known fact.
- The main idea used in this proof will also help us later

The velocity vector also rotates around a circle



By analogy, the time taken to move around the circle is





$$T = \frac{2\pi R}{v}$$

$$T = \frac{2\pi v}{a}$$

$$\Longrightarrow \quad a = \frac{v^2}{R}$$

The inverse square law

- We can rewrite this as $a = \frac{R}{T^2}$
- So, using Newton's second law, the force must be

$$F = ma = \frac{mR}{T^2}$$

- Now comparing to Kepler's third law $T \sim R^{3/2}$

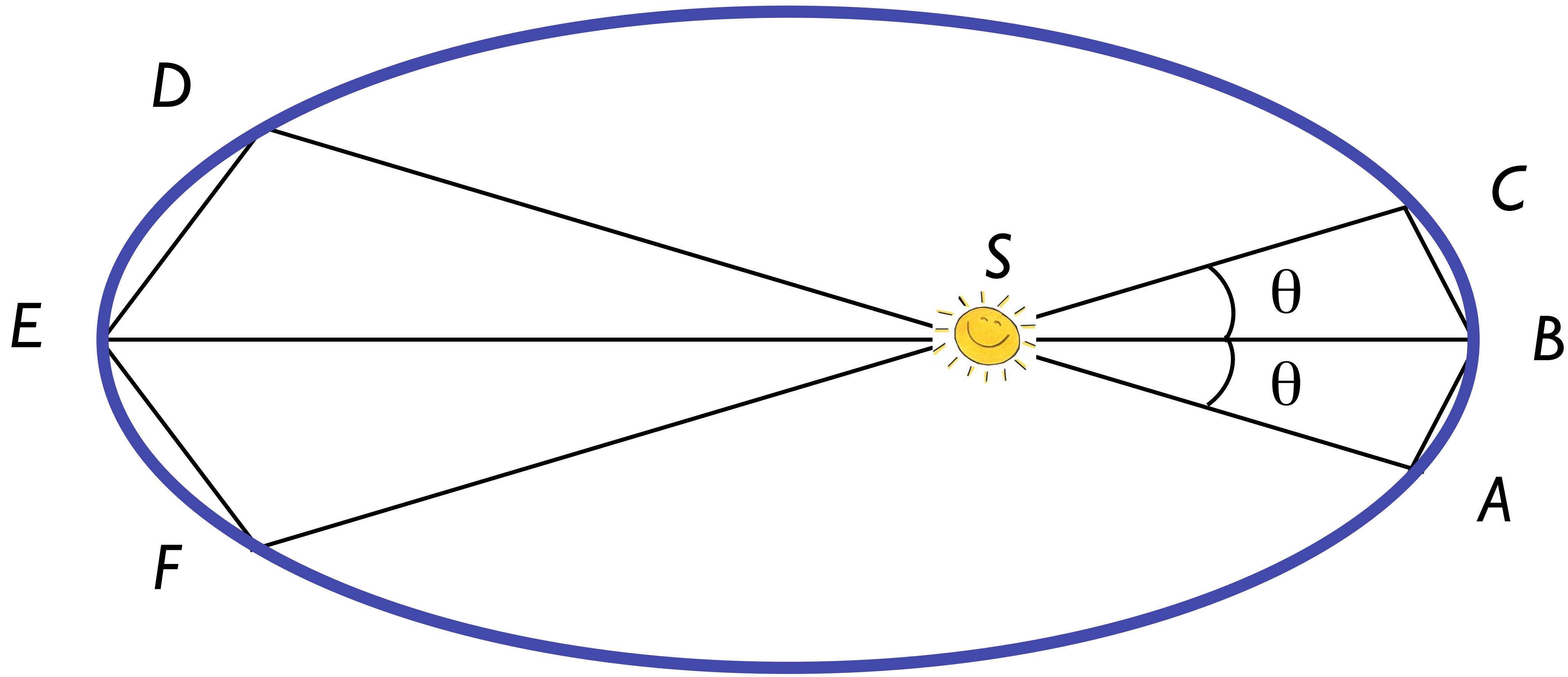
$$\boxed{F \sim \frac{1}{R^2}}$$

This, of course, is Newton's famous law of gravity

Finding ellipses

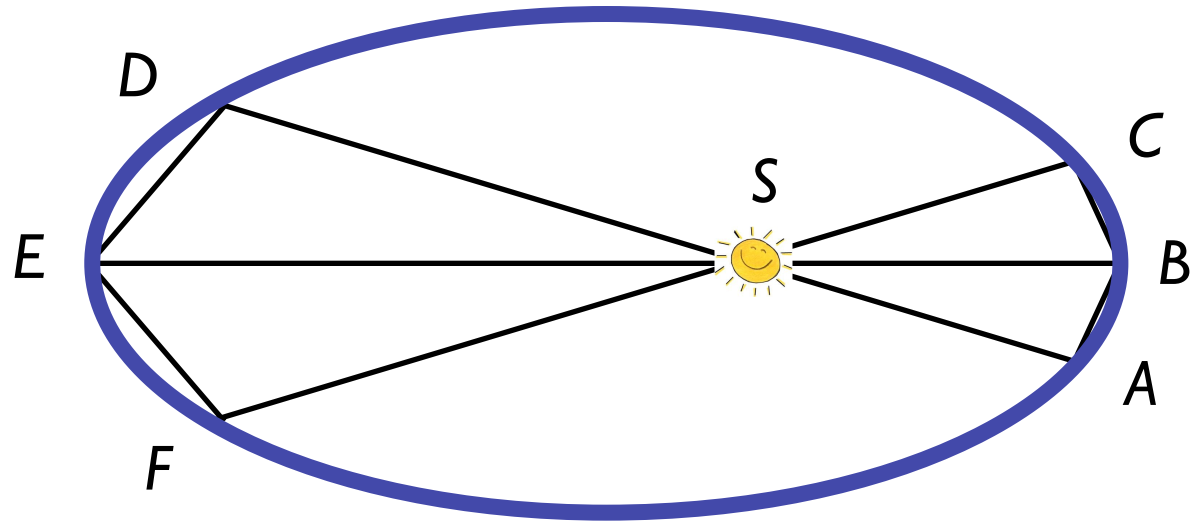
Finally, we need to show that the inverse square law implies that all orbits lie on an ellipse

Take an orbit and divide into equal angles



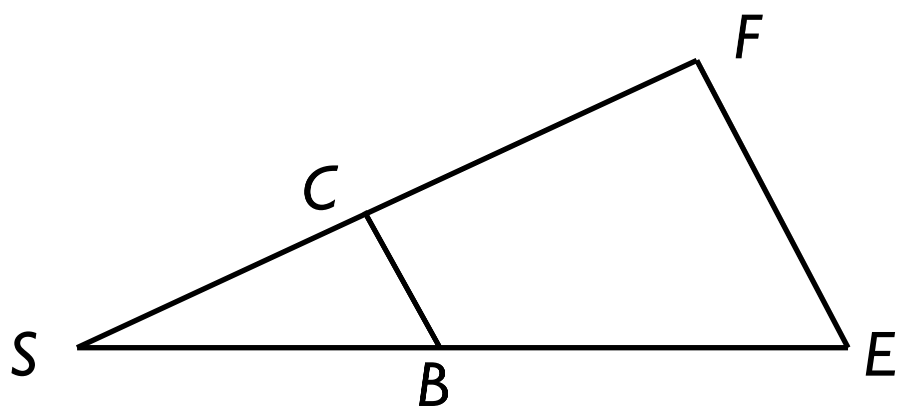
Equal areas in equal times ⟹ Planet goes faster around *ABC* and slower around *DEF*

How much faster does it go?



Claim: Let $\frac{SE}{SB} = x$. Then $\frac{\text{Area}(SEF)}{\text{Area}(SBC)} = x^2$

Proof: Follows because they are similar triangles*. Or, if you prefer, we can flip the triangle *SEF* and overlay it on *SBC*



*This proof is not quite good enough. It assumes that *CB* is parallel to *EF*. It's not very hard to show that this is assumption is not needed.

So…

- The area of a triangle is proportional to R^2
- So the time taken to traverse a segment is proportional to

$$\Delta t \sim R^2$$

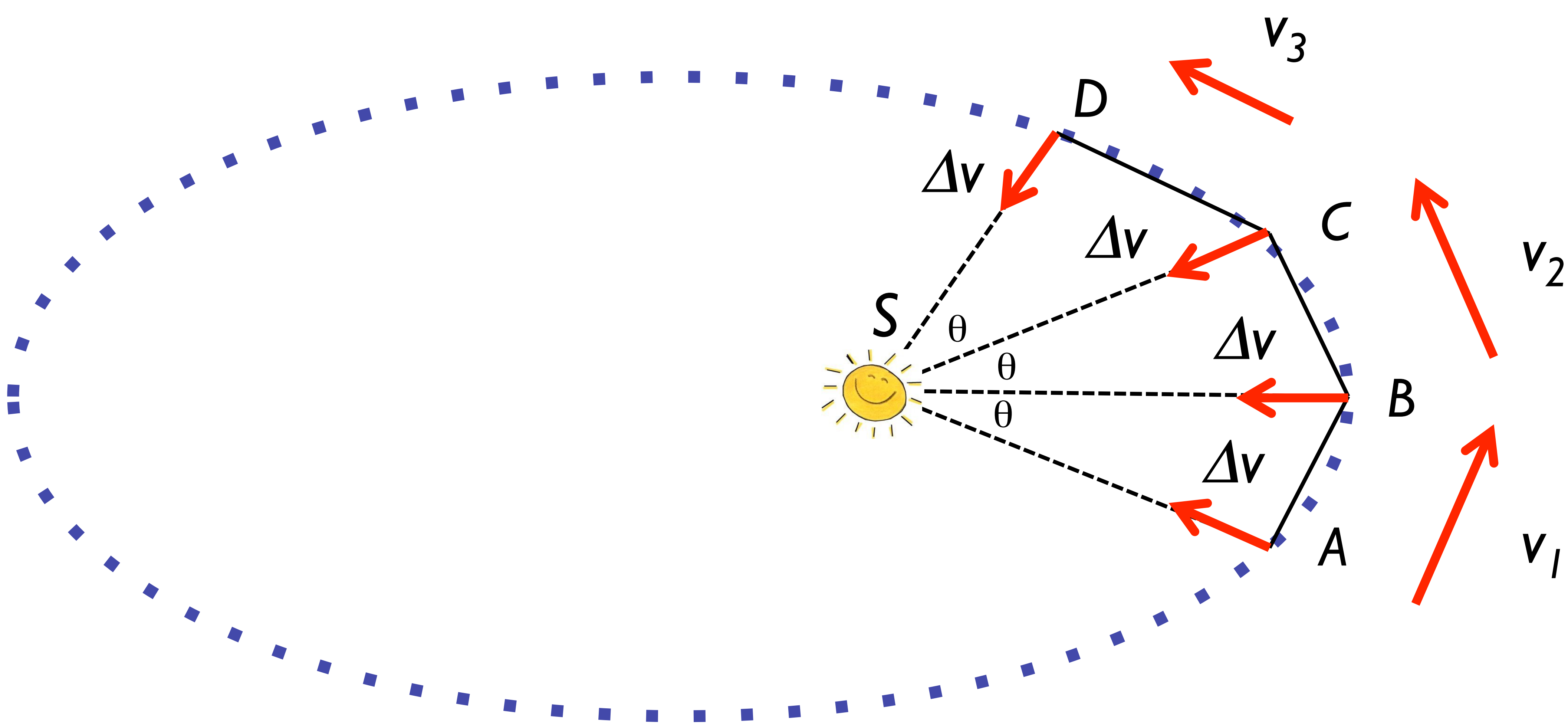
- But the force is proportional to

$$F \sim \frac{1}{R^2}$$

- Which means that the impulse, or change of velocity, felt at each corner is constant

$$\Delta v = F \Delta t = \text{constant}$$

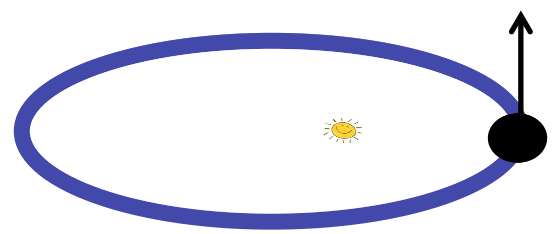
The upshot is that, when divided into equal angles…

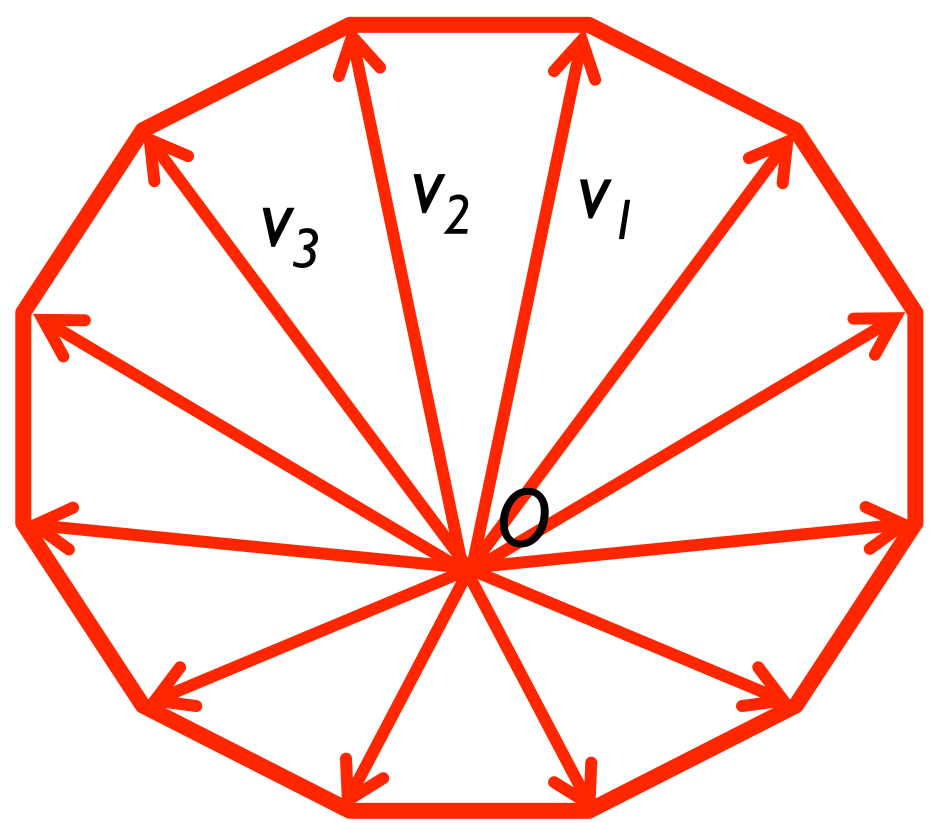


…the size of the change of velocity, Δv, is the same at each kick

This means the associated velocity diagram is a regular polygon

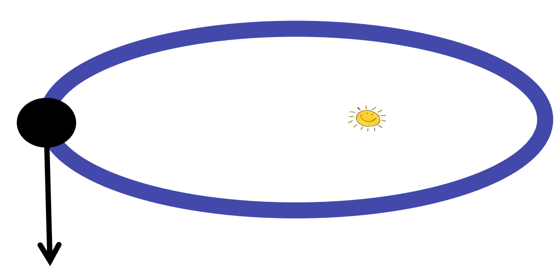
The planet is fastest when closest to the Sun. Here the velocity arrows are larger



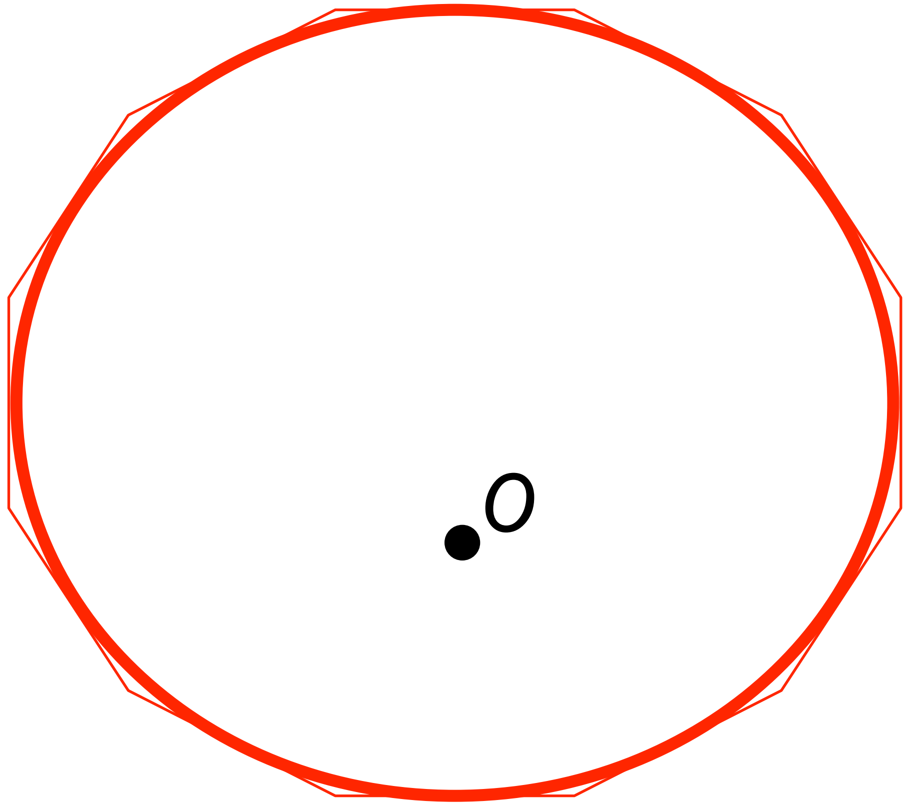


Each side has size Δv

The planet is slower when further from the Sun. Here the velocity arrows are smaller



And, in the limit Δt goes to zero, this becomes a circle

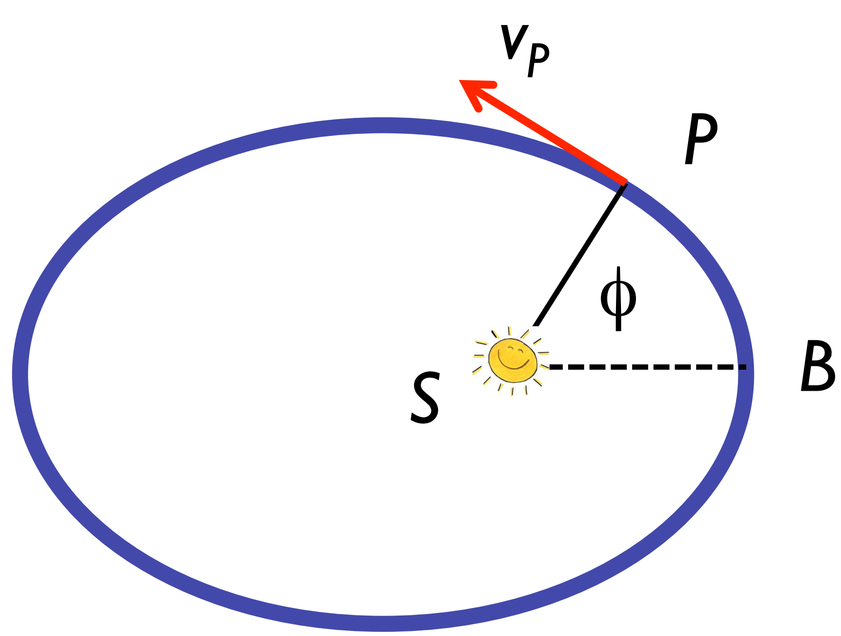


The point *O* is off-centre

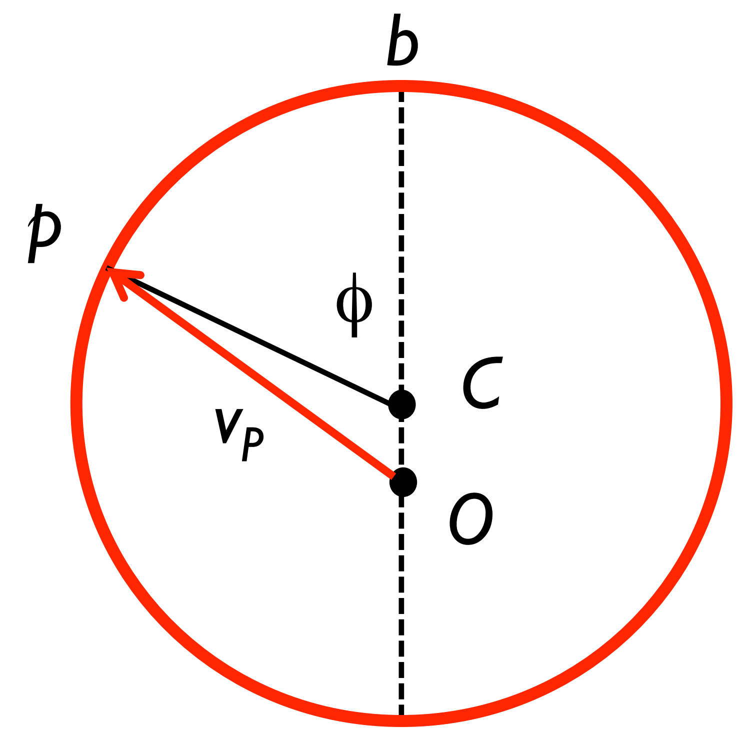
Now we're almost done

Let's look at the real orbit and velocity orbit side by side

<u>Real Space</u>

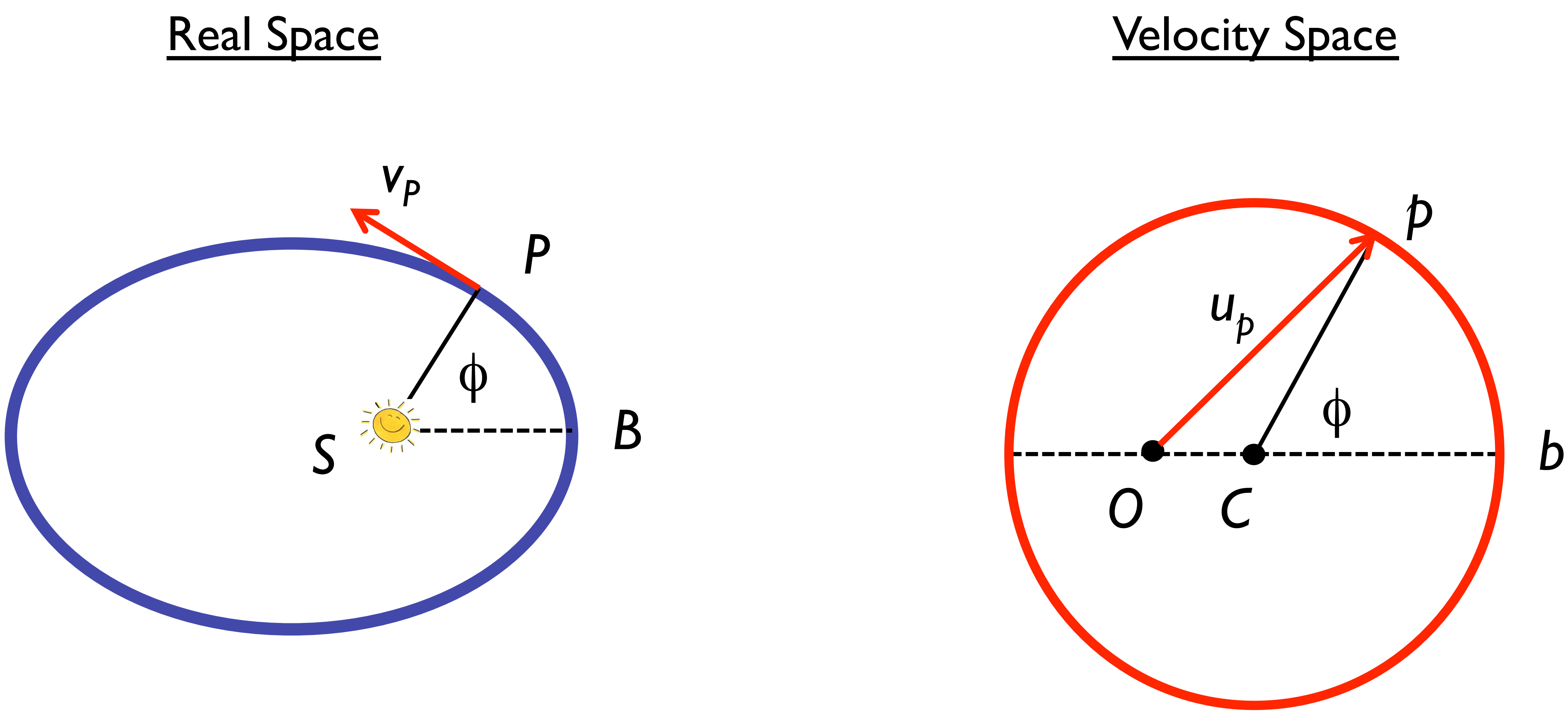


<u>Velocity Space</u>



Note: the angle ϕ is the same in both cases. This is simplest to see by returning to the jerky version of planetary motion and the resulting polygon

Rotate the velocity diagram by 90°

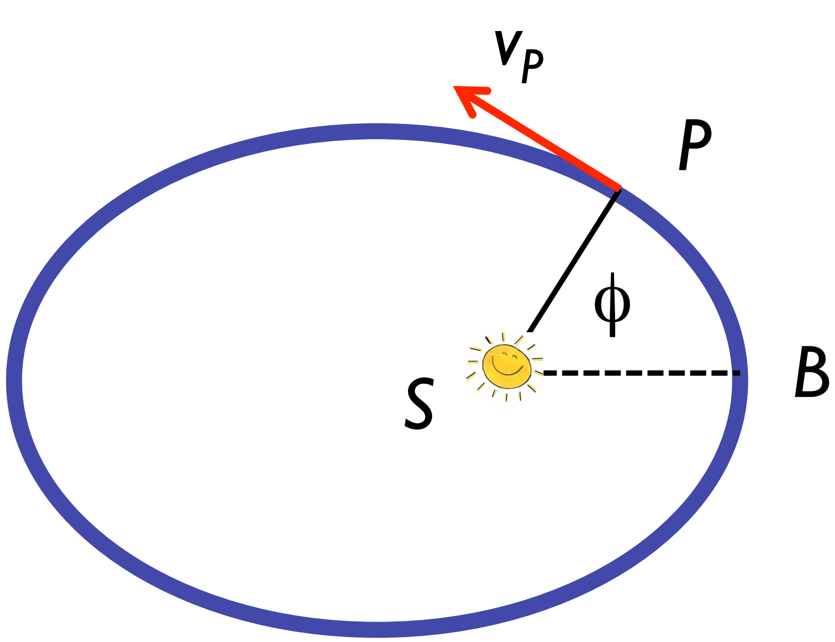


Because we rotated by 90°, we must have that the two vectors are orthogonal

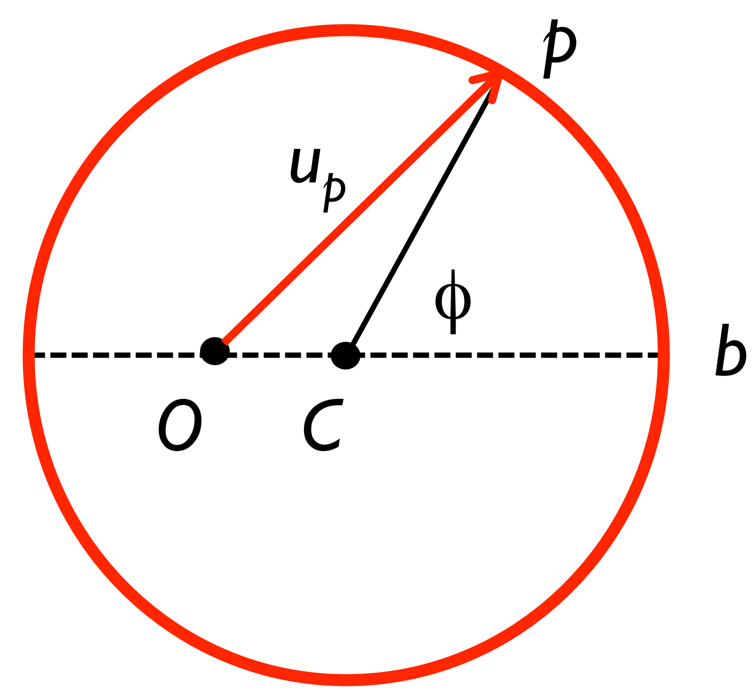
$$\vec{u}_P \cdot \vec{v}_P = 0$$

Rotate the velocity diagram by 90^o

Real Space

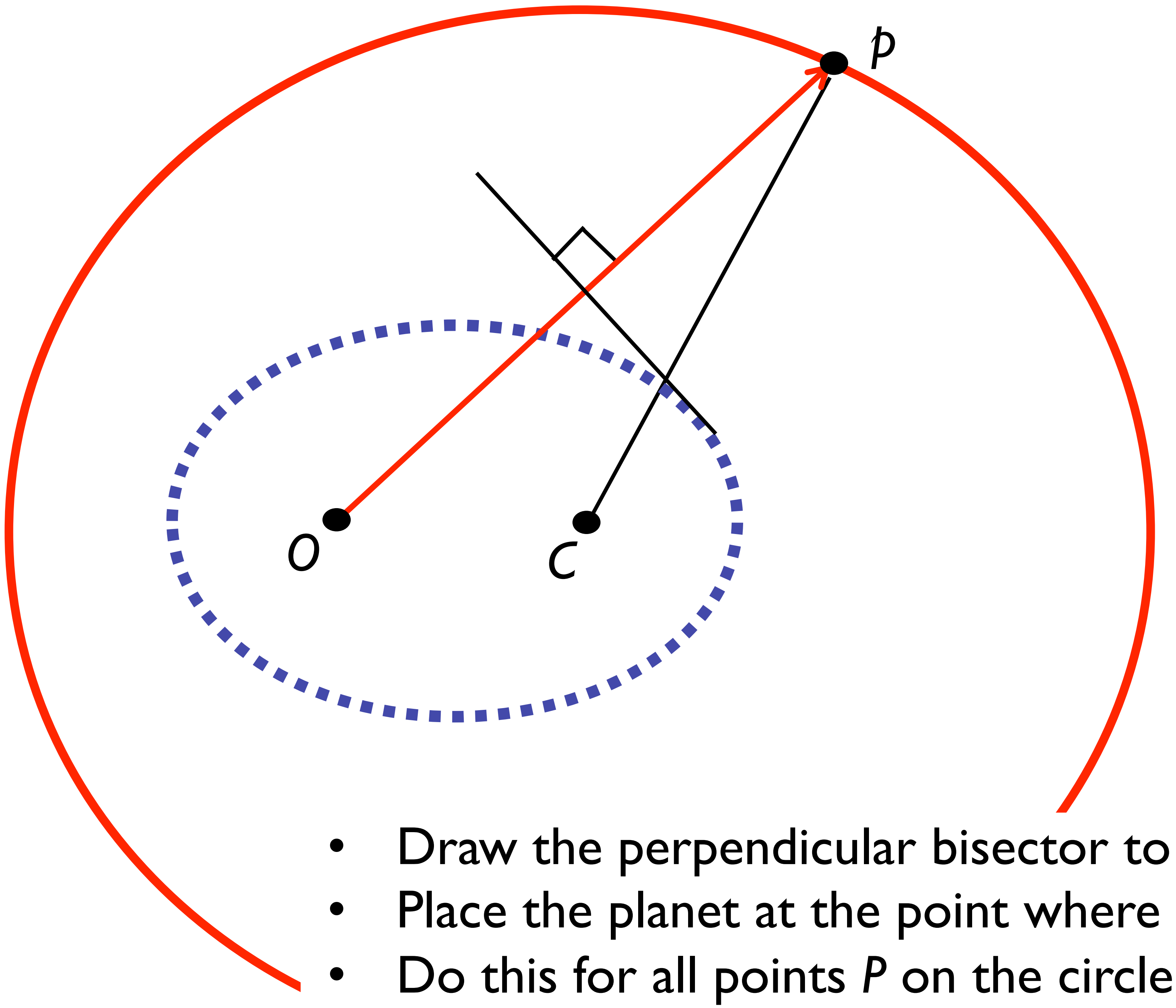


Velocity Space



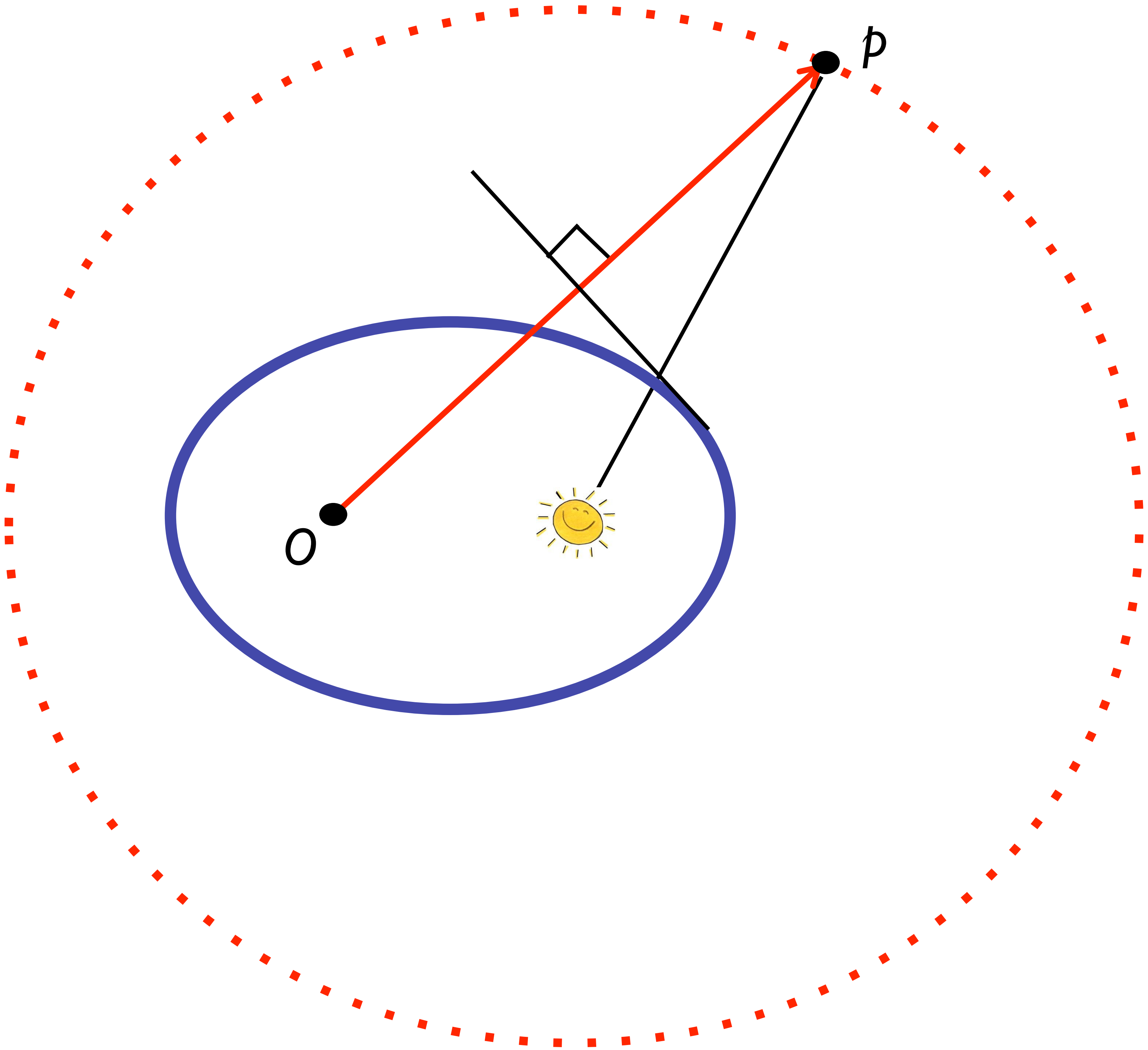
- Goal: Reconstruct the orbit in real space from the motion in velocity space.
- At every point *P*, the tangent to the orbit, v_P, must be perpendicular to u_P.

But we know how to do this…



- Draw the perpendicular bisector to *OP*
- Place the planet at the point where it intersects *CP*
- Do this for all points *P* on the circle and the bisector will always be tanget to the orbit

And the orbit of the planet is an ellipse!



The End