Quantum Geometry:
What the String Saw

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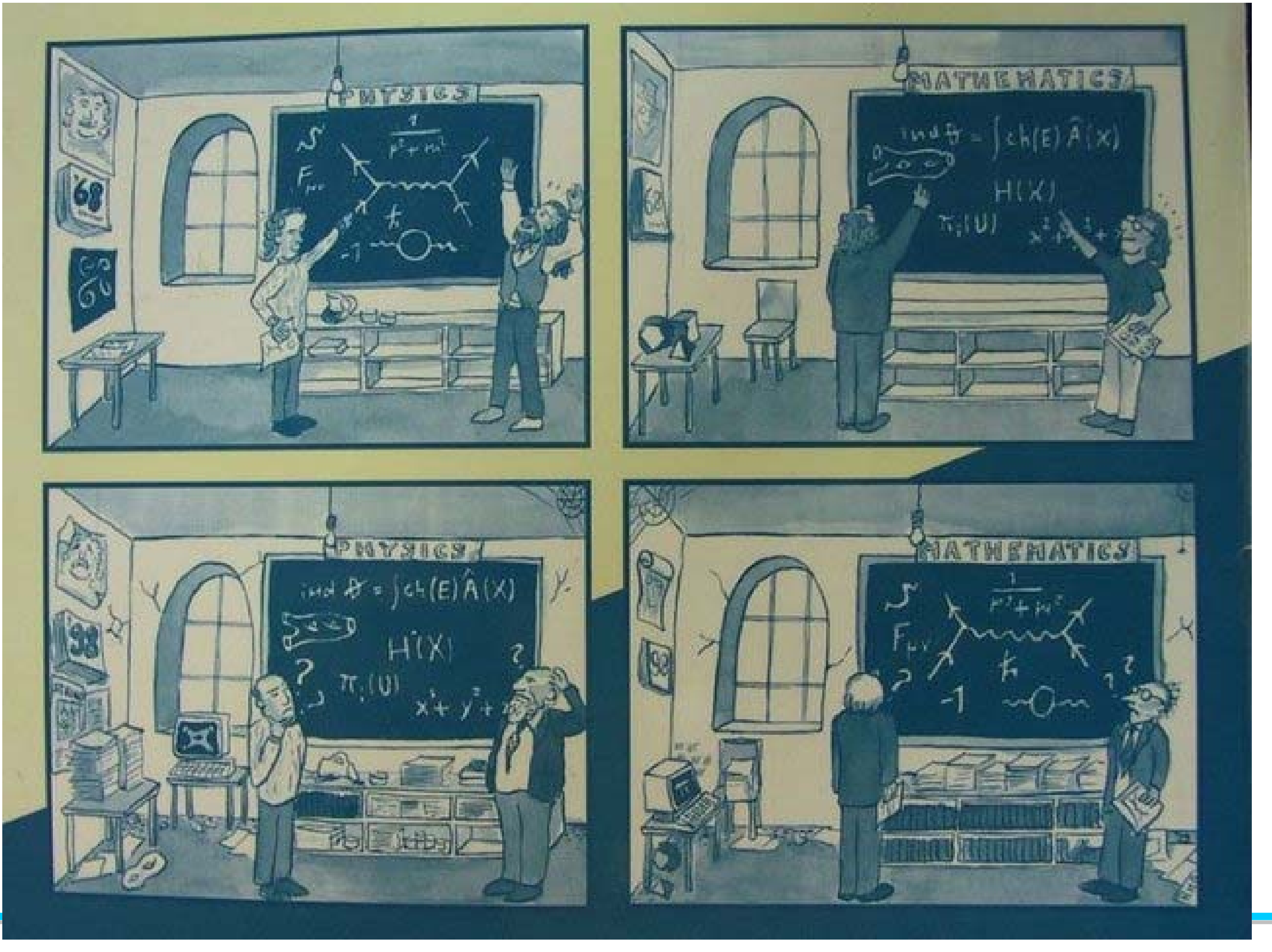
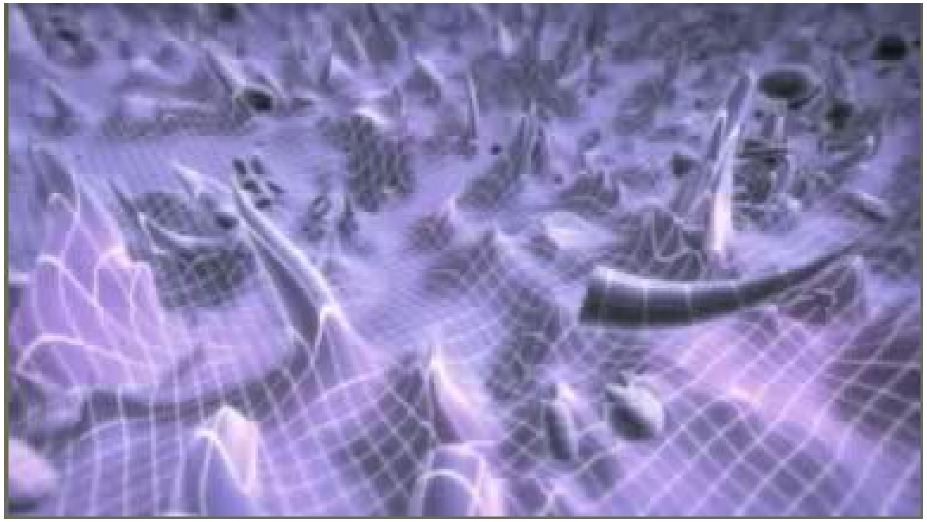
The Unreasonable Effectiveness of Physics in Mathematics



by Robbert Dijkgraaf

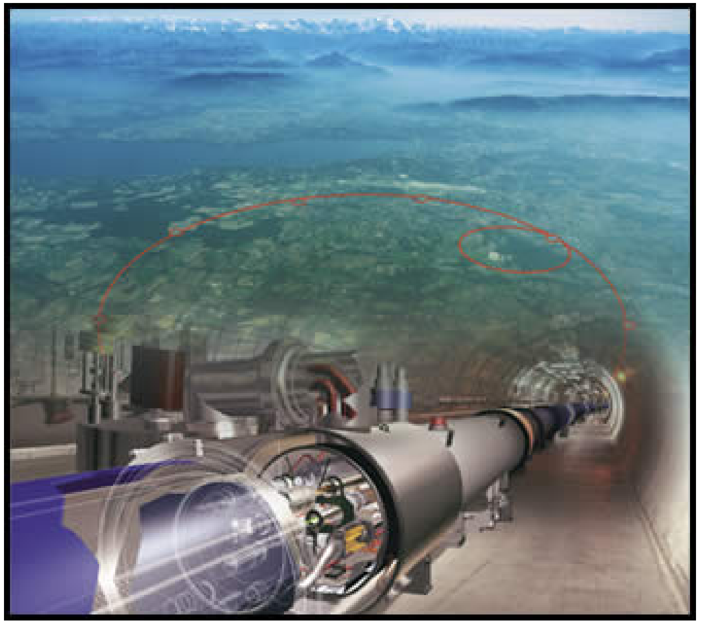
What is Quantum Geometry?



- It depends who you ask!
  - Many different ways to generalize geometry
  - (At least) one of them should be relevant for Nature, describing space and time at distance scales of 10^{-33} cm

The Most Useful Method in Physics

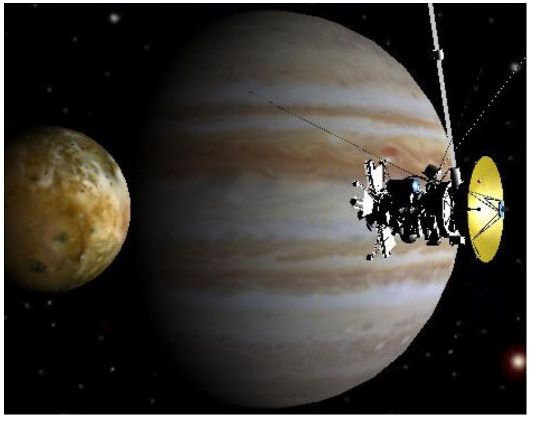
- Take the object you want to understand



- And throw something at it.

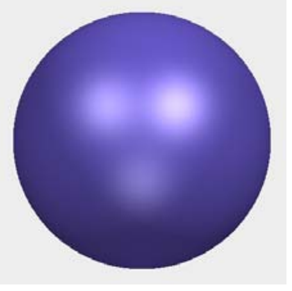
Space Probes

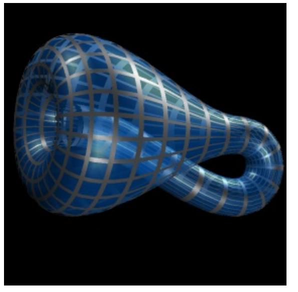
- We can use this trick for mathematics as well as physics
- Take a mathematical object --- in this case, a geometrical space, or *“manifold”.*
- See how objects that we understand well react to this space.



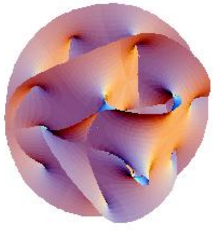
What are we Probing?

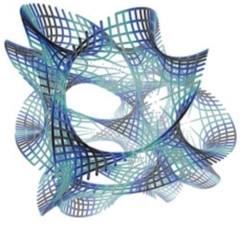
- A geometrical space, or manifold
  - Some simple 2-dimensional examples





  - Or more complicated, higher dimensional examples

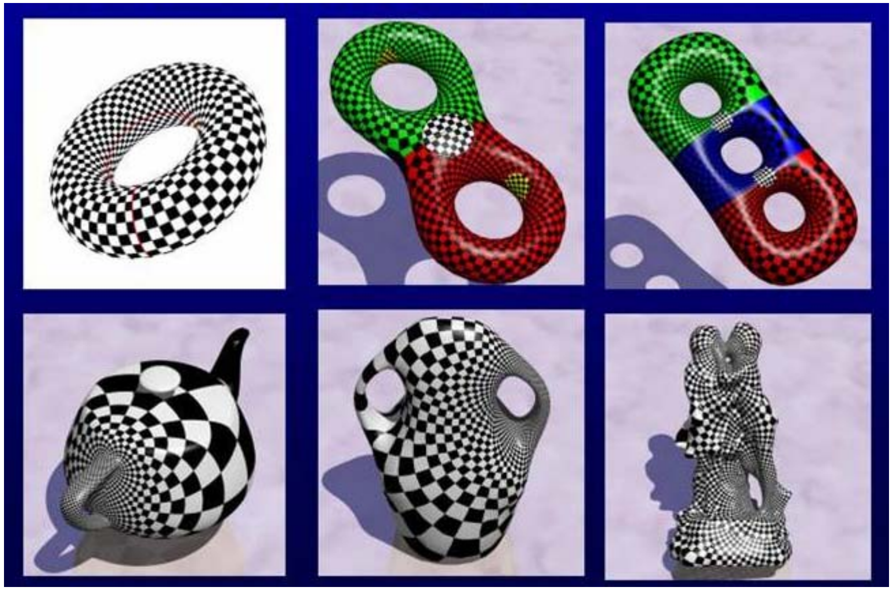




Sections of Calabi-Yau manifolds by A. Hanson

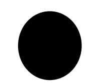
Why are we Probing?

- To understand questions of geometry and topology
  - To classify different types of manifolds and their properties

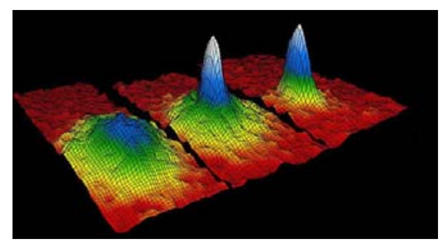


What Will we Probe With?

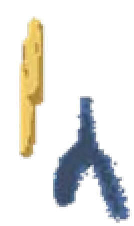
- We will use three different objects as our probe of geometry
  - classical particle



  - quantum particle



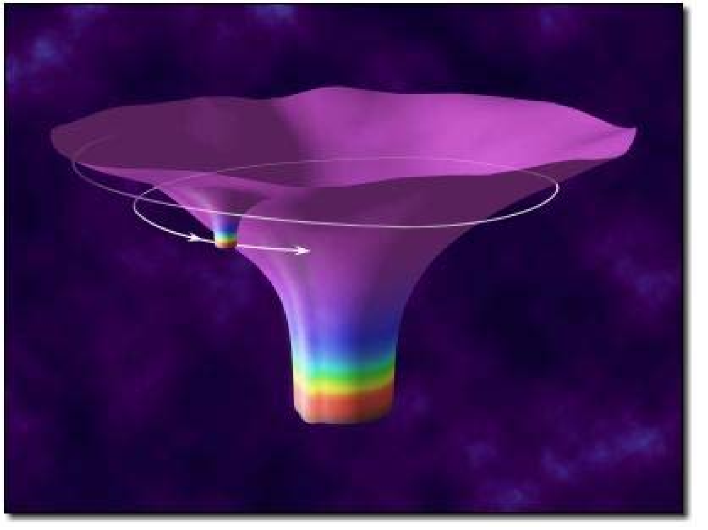
  - quantum string



What the Classical Particle Saw

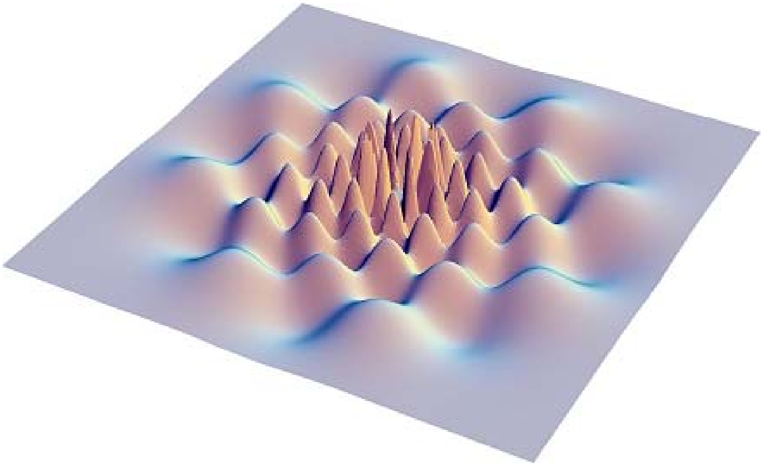
- What does a particle know about a space?
  - Answer: it’s orbits, or trajectories. (Geodesic motion).

- But knowing all the orbits is not a very useful way of characterizing the space
  - orbit of mercury
  - chaotic motion
  - LISA



What the Quantum Particle Saw

- The quantum world is very different. A particle is no longer described by it's trajectories, but now by a wavefunction.



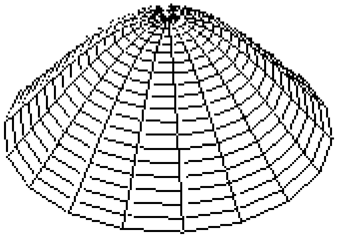
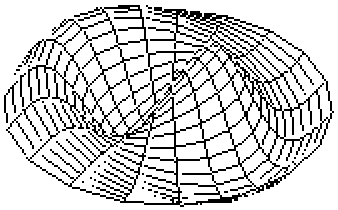
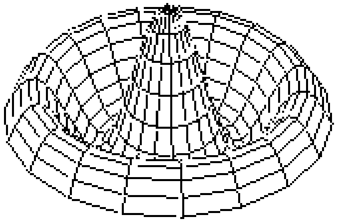
- We may characterize a manifold by the possible energies of a quantum particle living on it. This is the *spectrum* of the manifold

Can you hear the shape of a manifold?

- The spectrum of a manifold is the eigenvalues of the Laplacian

$$\nabla^2 \psi(x) = E^2 \psi(x)$$

- For example, if the manifold is a drum, the spectrum is the notes it will play

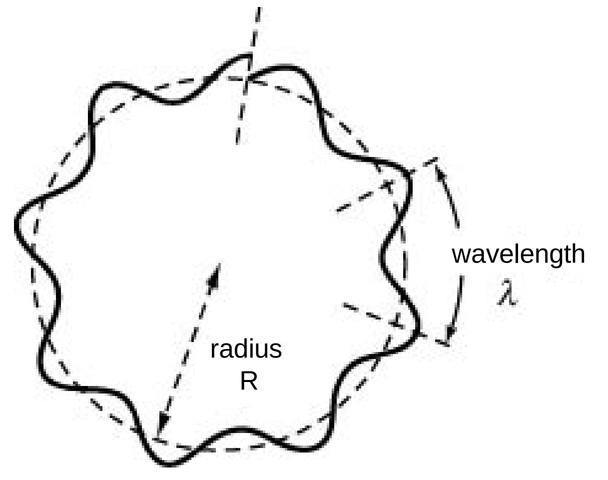




Animations by D. Russell

A Simple Example: A Circle

- The different wavefunctions are Fourier modes
  - Acceptable wavefunctions have an integer number of de Broglie wavelengths going around the circle.
  - A circle of radius R has spectrum

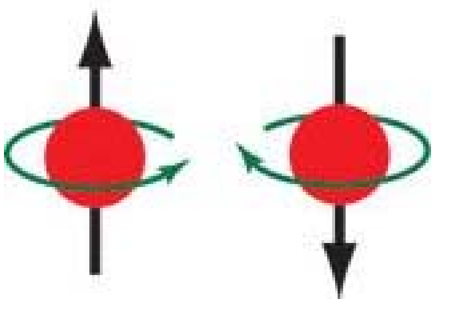
$$E_n = \frac{n}{R} \qquad n = 0, 1, 2, \ldots$$



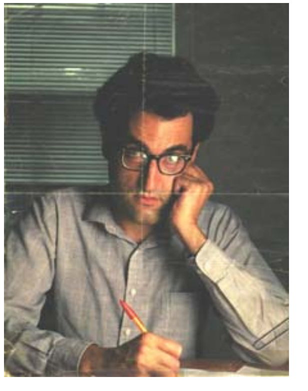
  - If there are extra-dimensions in the universe at distance scale 10^{-16} cm then we will see a characteristic energy spectrum like this at LHC.

What the Spinning Quantum Particle Saw

- The quantum particle sees the spectrum of the manifold. This is a very natural concept mathematically.



- But a particle with spin sees much more.
  - The spinning particle gets trapped by the holes and handles of the manifold. It knows about *topology*.
  - Understanding the ground states of the spinning quantum particle roaming on a manifold is equivalent to one of the most famous problems in mathematics!
  - Exact Statement: Ground States of supersymmetric particle = de Rahm cohomology



Witten, 1982

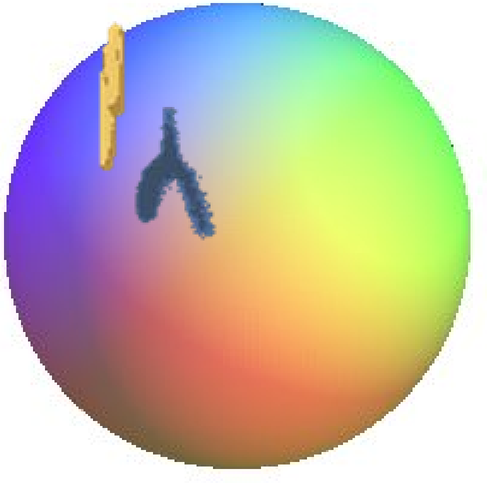
Quantum Particle = Classical Geometry

- Summary:
  - The problem of a quantum mechanical particle moving on a curved space is a way of reformulating large swathes of modern geometry.

- But what about *quantum* geometry?

What the String Saw

- Let's now look at what happens to a quantum string moving on a space



- We can again start by looking at the energy spectrum of the string

A Simple Example: The Circle Again

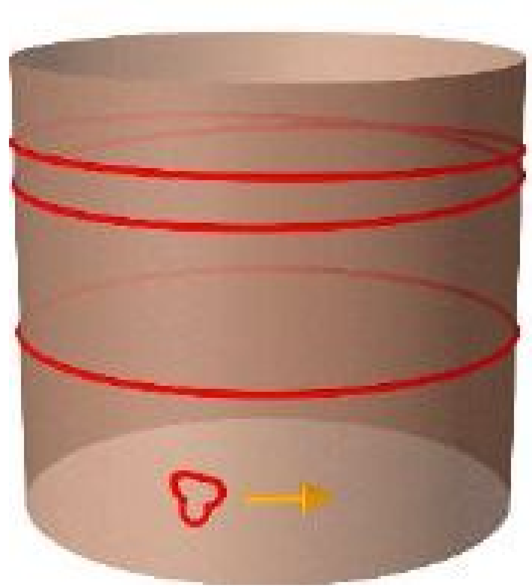
- Small loops of string can move around the circle. These look the same as quantum particles. Their allowed energy is

$$E_n = \frac{n}{R} \qquad n = 0, 1, 2, \ldots$$

- But strings can also wind around the circle. Their energy is quantized as

$$E_m = mR \qquad m = 0, 1, 2, \ldots$$

(for a string of unit mass per length)



Graphics by S. Jensen

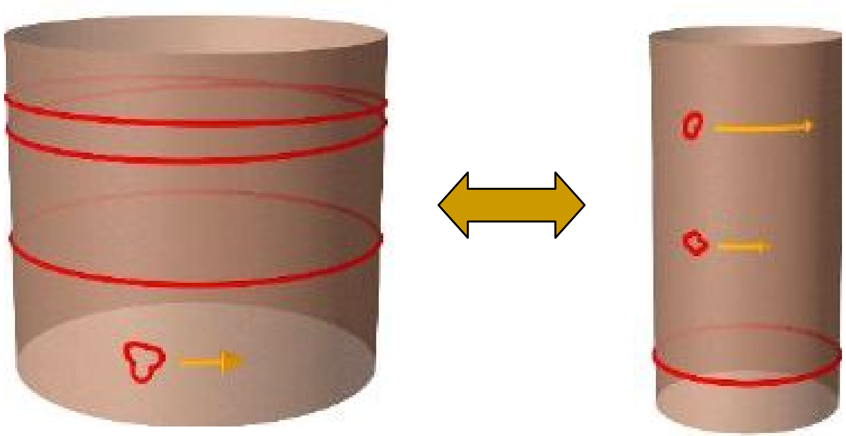
T-Duality: (a boring name for a great idea)

- The energy spectrum of a string moving on a circle of radius R

$$E_{m,n} = mR + n/R \qquad m, n = 0, 1, 2, \ldots$$

- This spectrum is invariant under $R \leftrightarrow 1/R$

- If everything is made out of strings, it is impossible to tell the difference between a circle of length R and a circle of length 1/R!



What Else Did the String See?

- The string doesn't always see what you give it! It will change the shape of the space if it's not happy! The equations which describe this are "Ricci flow".

$$\frac{dg_{ij}}{d\mu} = -R_{ij}$$

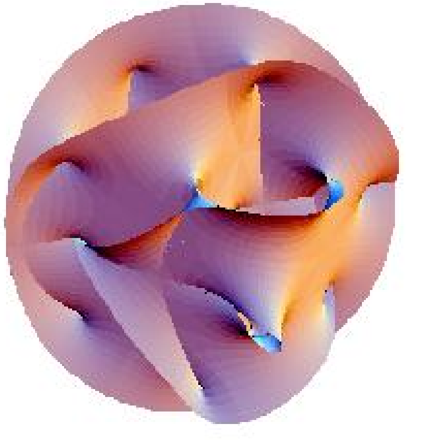
- The types of spaces that strings live happily on are "Calabi-Yau" manifolds. They are Ricci flat

$$R_{ij} = 0$$

The equations of string theory are the same as those Perelman used to prove the Poincare conjecture. No one knows if there's a deep meaning to this!

Calabi-Yau Manifolds

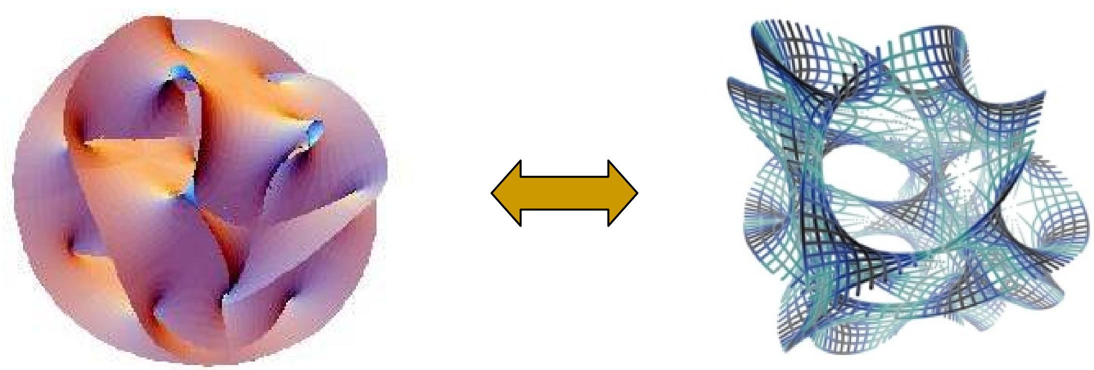
- Calabi-Yau manifolds are one of the key ideas of string theory.
- They are 6-dimensional spaces.
  (Actually any even dimension, but we'll stick to 6)
- They are postulated to be extra dimensions of the universe



- They are complicated!! No explicit metric known.

Mirror Symmetry

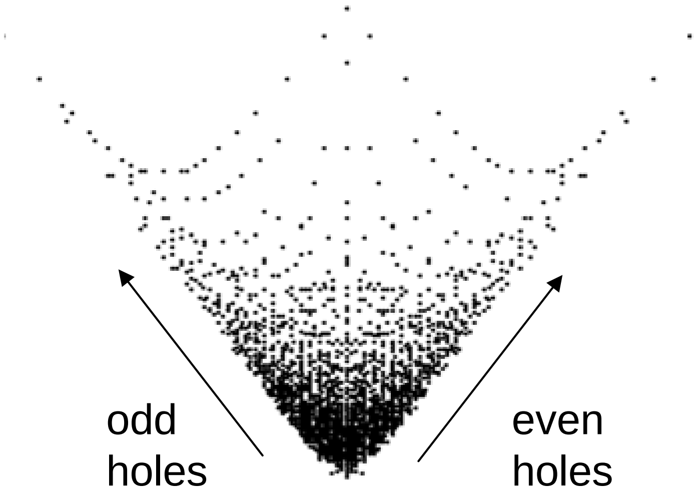
- Is there an $R \leftrightarrow 1/R$ equivalence for these more complicated spaces?
  - Yes – but much richer and more interesting.



- Calabi-Yau manifolds come in pairs. Strings see the two manifolds as the same. Mathematicians see them very differently!

More on Mirror Symmetry

- The two Calabi-Yau spaces are very different.
  - They have different topology. Mirror symmetry swaps even and odd dimensional holes
  - Calabi-Yau manifolds can deform in two different ways: they can change their shape, or change their size. Mirror symmetry swaps these.
  - Mirror symmetry exchanges easy questions for hard questions!! Questions of complex geometry (easy) are equivalent to questions of symplectic geometry (hard). This makes it very very useful.



More Geometry of String Theory

- Using strings as a probe to understand geometry gives us many more insights
  - The meaning of spaces with negative, and complex, volume
  - Understanding the correct way to tear space and change its topology

  - Yet more ideas are revealed by higher dimensional probes of the geometry: membranes and “D-branes”.
  - No doubt many more surprises to come….