Quantum Hall Matrix Models

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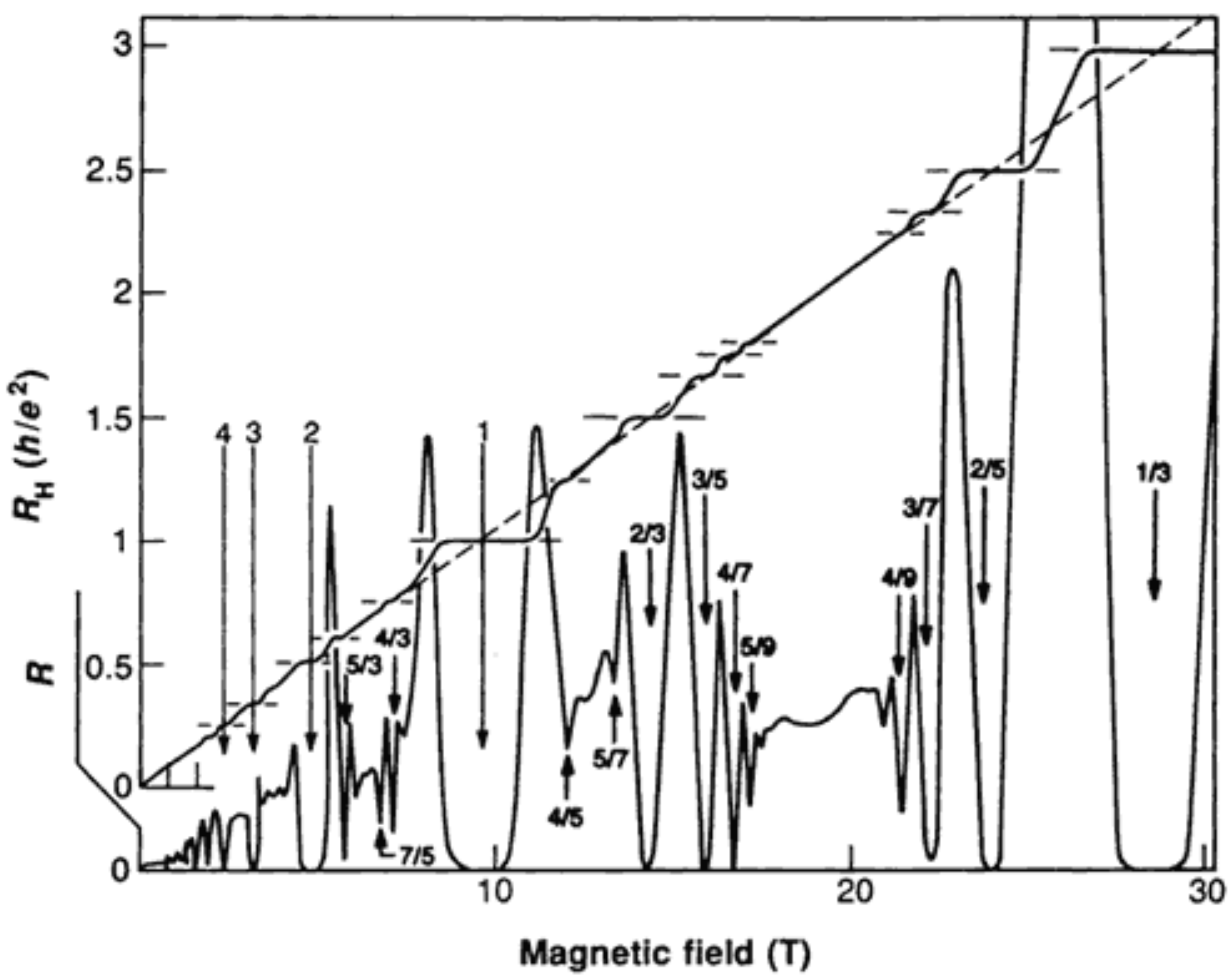
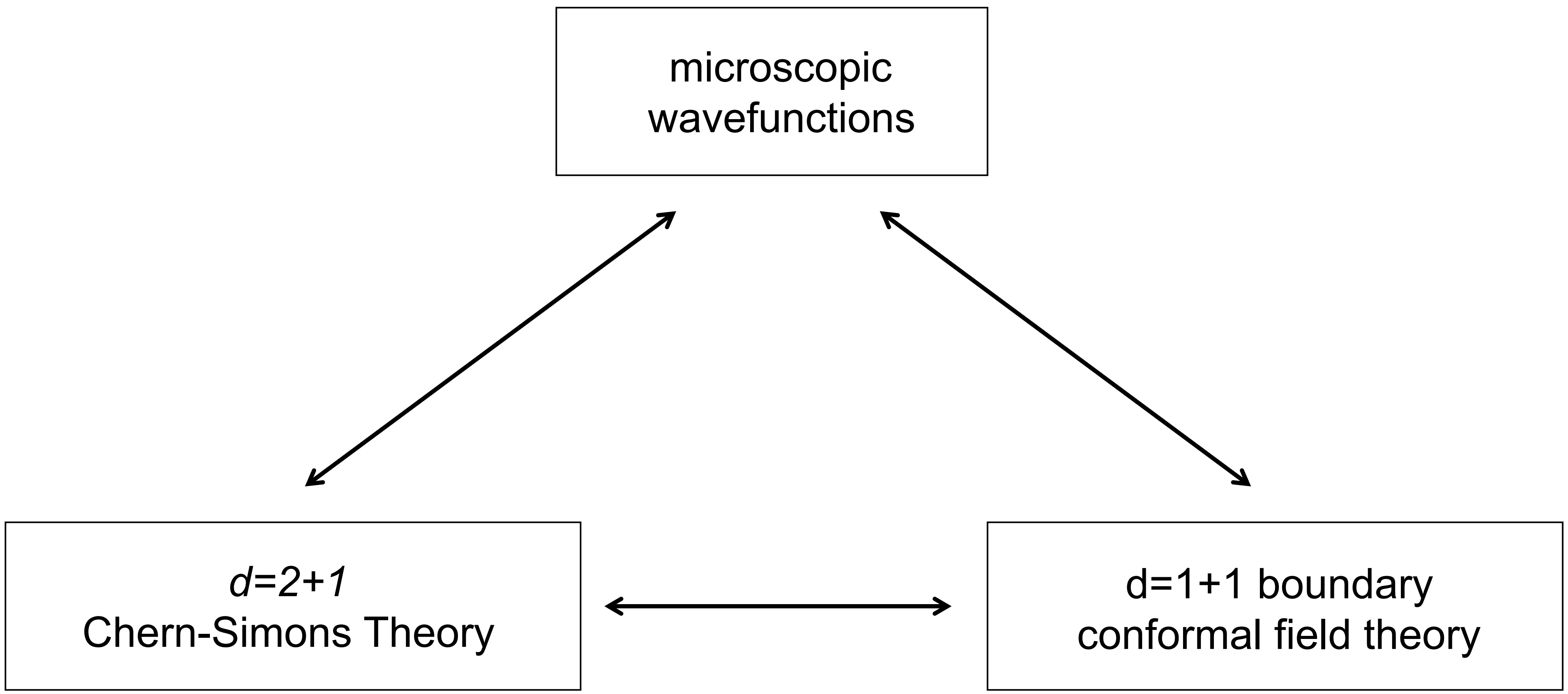
Based on work with Nick Dorey and Carl Turner

arXiv:1508.00580 (Phys. Rev B) and arXiv:1603.09688, 1604.05711

The Quantum Hall Effect



Theory of the Quantum Hall Effect



The Matrix Model

The dynamics of N electrons is described by a $U(N)$ matrix model

$$S = \int dt\ \frac{iB}{2}\,\mathrm{Tr}\,\left(Z^\dagger \mathcal{D}_t Z\right) + i\sum_{i=1}^{p} \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p)\,\mathrm{Tr}\,\alpha - \omega\,\mathrm{Tr}\,Z^\dagger Z$$

The fields are:

- $U(N)$ gauge field α
- adjoint valued, complex matrix Z
- p fundamental vector φ_i

The symmetries are:

- $U(N)$ gauge symmetry
- $SU(p)$ global symmetry

When $p=1$, this was previously written down by Polychronakos (2001), inspired by Susskind.

Getting a Feel for the Matrix Model

$$\boxed{S = \int dt\ \frac{iB}{2}\,\mathrm{Tr}\,\left(Z^\dagger \mathcal{D}_t Z\right) + i\sum_{i=1}^{p} \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p)\,\mathrm{Tr}\,\alpha - \omega\,\mathrm{Tr}\,Z^\dagger Z}$$

Restrict to a single electron, *N=1*

$$S_{N=1} = \int dt\ \frac{iB}{2} Z^\dagger \dot{Z} + \sum_{i=1}^{p} i\varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p)\alpha - \omega\, Z^\dagger Z$$

An electron in a strong magnetic field (lowest Landau level)

A finite dimensional Hilbert space. The electron carries *SU(p)* spin

A harmonic trap

The Matrix Model

The dynamics of *N* electrons is described by a *U(N)* matrix model

$$S = \int dt \; \frac{iB}{2} \, \mathrm{Tr} \, \left(Z^\dagger \mathcal{D}_t Z \right) + i \sum_{i=1}^{p} \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p) \, \mathrm{Tr} \, \alpha - \omega \, \mathrm{Tr} \, Z^\dagger Z$$

Results:

- This describes the dynamics of N vortices in a *U(p)* Chern-Simons theory
- The ground states are quantum Hall wavefunctions
- In the large *N* limit, this is a WZW CFT!

Where did the Matrix Model Come From?

The Origin of the Matrix Model

$$\boxed{S = \int dt\ \frac{iB}{2}\,\mathrm{Tr}\,\left(Z^\dagger \mathcal{D}_t Z\right) + i\sum_{i=1}^{p} \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p)\,\mathrm{Tr}\,\alpha - \omega\,\mathrm{Tr}\,Z^\dagger Z}$$

Claim: This describes the dynamics of vortices in a *U(p)* Chern-Simons theory

Tong, '03; Tong and Turner '15

The Origin of the Matrix Model

$$\boxed{S = \int dt \; \frac{iB}{2} \,\mathrm{Tr}\, \left(Z^\dagger \mathcal{D}_t Z \right) + i \sum_{i=1}^{p} \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p)\, \mathrm{Tr}\, \alpha - \omega \,\mathrm{Tr}\, Z^\dagger Z}$$

First hint: equation of motion for α is the Gauss' law for the matrix model

$$\frac{B}{2}[Z, Z^\dagger] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger = (k+p)\mathbf{1}_N$$

We divide out by gauge transformations $Z \to UZU^\dagger \quad \text{and} \quad \varphi \to U\varphi \quad \text{with } U \in U(N)$

These equations also describe the moduli space of vortices $\mathcal{M}_N$ in *U(p)* gauge theories

Hanany and Tong '03

Effective Description of Quantum Hall Physics

Start with U(1) theory.

$$S = \int d^3x \quad - \frac{k}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho$$

emergent *U(1)* gauge field

background electromagnetism

- This is the effective theory for the Laughlin state with $\nu = \frac{1}{k}$
- Hall conductivity is $\sigma_H = \frac{1}{2\pi k}$
- charged particles = anyonic quasi-holes
- electron current $J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho$

Effective Description of Quantum Hall Physics

Step 1: Add dynamical matter fields

$$S = \int d^3x \quad - \frac{k}{4\pi}\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \frac{1}{2\pi}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho + i\phi^\dagger \mathcal{D}_0 \phi - \frac{1}{2m}\mathcal{D}_\alpha \phi^\dagger\, \mathcal{D}_\alpha \phi - \frac{\pi}{mk}|\phi|^4$$

dynamical quasi-hole

Note
- non-relativistic matter
- contact interaction fixed point of RG

Effective Description of Quantum Hall Physics

Step 2: Turn on a background magnetic field, *B*

$$S = \int d^3x \quad -\frac{k}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho - \frac{B}{2\pi}a_0 + i\phi^\dagger\mathcal{D}_0\phi - \frac{1}{2m}\mathcal{D}_\alpha\phi^\dagger\,\mathcal{D}_\alpha\phi - \frac{\pi}{mk}|\phi|^4$$

background field $A_i = -\frac{B}{2}\epsilon_{ij}x^j$

- *B* looks like a background charge density for the fundamental excitations ϕ
- Gauss' law for a_0 is

$$f_{12} = \frac{2\pi}{k}\left(|\phi|^2 - \frac{B}{2\pi}\right)$$

- Ground state has $f_{12} = 0$ and $|\phi|^2 = \frac{B}{2\pi}$. This is no longer a quantum Hall state

Electrons as Vortices

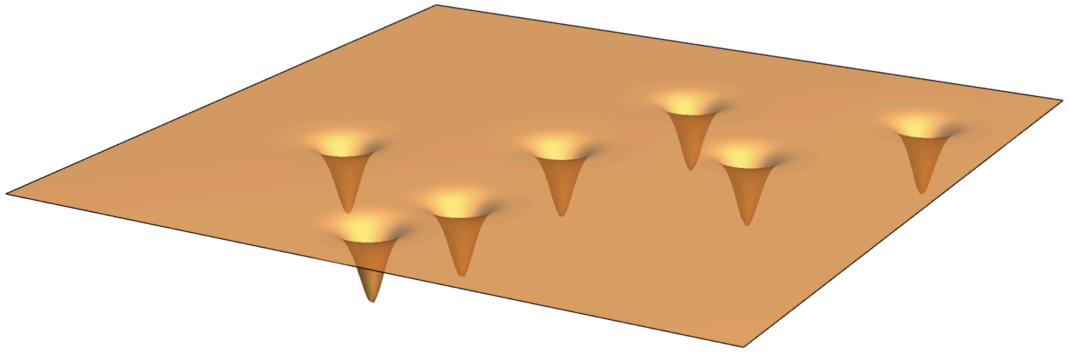
$$|\phi|^2 = \frac{B}{2\pi} \quad \Longrightarrow \quad \text{U(1) broken} \quad \Longrightarrow \quad \text{vortices = electrons}$$

Vortices have a lovely property: they are BPS!

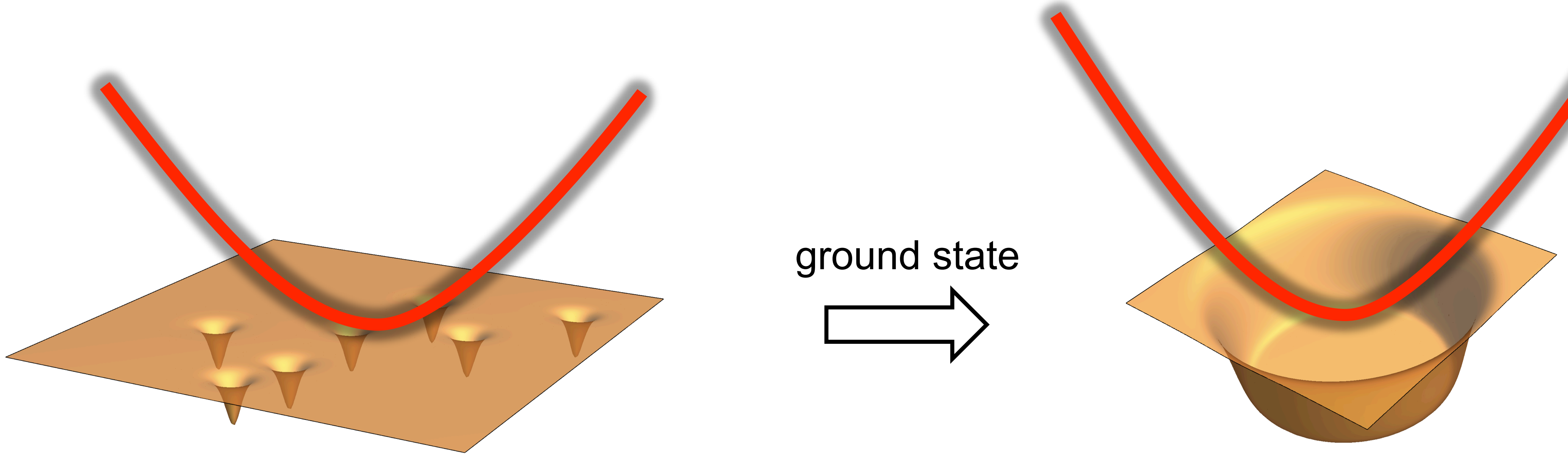
$$\boxed{\begin{gathered} f_{12} = \frac{2\pi}{k}\left(|\phi|^2 - \frac{B}{2\pi}\right) \\ \mathcal{D}_z \phi = 0 \end{gathered}}$$

Electrons as Vortices

Throw in *N* vortices. There is no force between them, so they can sit anywhere on the plane



We add a harmonic trap: $V_{\text{trap}} = -\omega|z|^2 f_{12}$

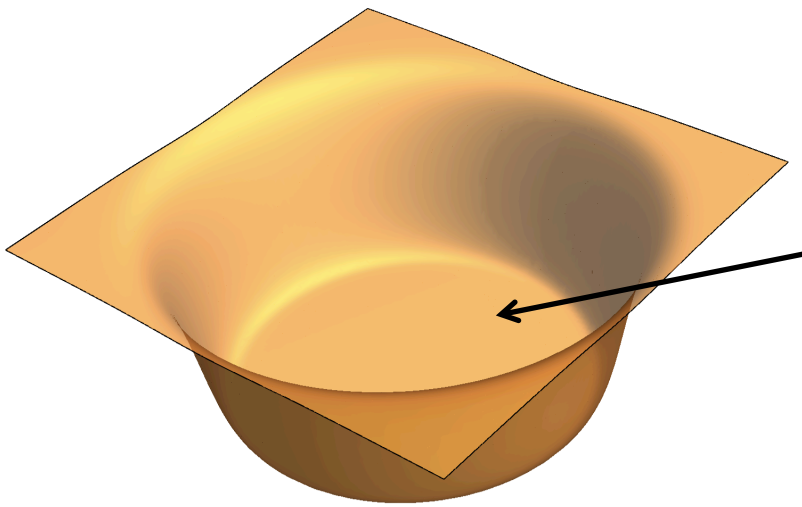


What Did This Buy Us?

We have a large vortex droplet of size $R \approx \sqrt{\frac{2kN}{B}}$

Outside the vortex,

$$|\phi|^2 = \frac{B}{2\pi}$$



Inside the vortex,

$$|\phi|^2 \approx 0$$

- Inside the vortex, we have recovered the quantum Hall ground state.
- But now we have a microscopic picture of this…it is governed by the vortex dynamics.
  - This vortex dynamics is the *p=1* matrix model!
- Repeat for *U(p)* Chern-Simons theory to get general matrix model.

Non-Abelian Hall States

Dorey, Tong and Turner, arXiv:1603.09688

Repeat with U(p) Chern-Simons theories

$$U(p)_{k',k} = \frac{U(1)_{k'} \times SU(p)_k}{\mathbf{Z}_p}$$

We understand the vortex dynamics when $k' = (k+p)p$

Solving the Matrix Model

Solving the Matrix Model

$$\boxed{S = \int dt \; \frac{iB}{2}\,\mathrm{Tr}\,\left(Z^\dagger \mathcal{D}_t Z\right) + i\sum_{i=1}^{p} \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p)\,\mathrm{Tr}\,\alpha - \omega\,\mathrm{Tr}\,Z^\dagger Z}$$

Quantisation: $\frac{B}{2}[Z_{ab}, Z^\dagger_{cd}] = \delta_{ad}\delta_{bc}$ and $[\varphi_{i\,a}, \varphi^\dagger_{j\,b}] = \delta_{ab}\delta_{ij}$

Hilbert Space: Introduce "vacuum" $Z_{ab}|0\rangle = \varphi_i|0\rangle = 0$ and act with creation operators

Gauge Constraint:

$$\boxed{\frac{B}{2}[Z, Z^\dagger] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger = (k+p)\mathbf{1}_N}$$

Hamiltonian:

$$H = \omega \sum_{a,b=1}^{N} Z^\dagger_{ab} Z_{ba}$$

The Quantum Ground State for *p=1*

Let's start with the case *p=1*

$$\frac{B}{2}[Z, Z^\dagger] + \varphi\varphi^\dagger = (k+1)\mathbf{1}_N \quad \Longrightarrow$$

- Physical states are SU(N) gauge invariant
- Physical states have specific U(1) charge

$$\sum_{a=1}^{N} \varphi_a^\dagger \varphi_a = kN$$

$$\boxed{|\mathrm{ground}\rangle_k = \left[\epsilon^{a_1 \dots a_N} \varphi_{a_1}^\dagger (Z\varphi)_{a_2}^\dagger \dots (Z^{N-1}\varphi)_{a_N}^\dagger\right]^k |0\rangle}$$

act with *k* baryon operators

Polychronakos, 01
Hellerman and van Raamsdonk '01

The Quantum Ground State for *p=1*

$$|\mathrm{ground}\rangle_k = \left[\epsilon^{a_1\ldots a_N}\varphi^\dagger_{a_1}(Z\varphi)^\dagger_{a_2}\ldots(Z^{N-1}\varphi)^\dagger_{a_N}\right]^k|0\rangle$$

Compare to the Laughlin states

$$|\mathrm{Laughlin}\rangle_k = \prod_{a<b}(z_a - z_b)^k\, e^{-\frac{B^{\mathrm{ext}}}{4}\sum|z_a|^2} = \left[\epsilon^{a_1\ldots a_n} z^0_{a_1} z_{a_2}\ldots z^{n-1}_{a_n}\right]^k e^{-\frac{B^{\mathrm{ext}}}{4}\sum|z_a|^2}$$

Closely related $\quad |0\rangle = |\mathrm{Laughlin}\rangle_1$

$$|\mathrm{ground}\rangle_k \rightarrow |\mathrm{Laughlin}\rangle_{k+1} \qquad |z^a - z^b| \gg \sqrt{\frac{1}{B}}$$

These are quantum Hall states at filling fraction $\nu = \dfrac{1}{k+1}$

Karabali and Sakita, '01

Quasi-Hole Excitations for *p=1*

The matrix model also has *quasi*-holes. A single quasi-hole at position η is given by

$$|\eta\rangle_k \;\propto\; \det(Z^\dagger - \eta)\; |\text{ground}\rangle_k$$

Alternatively, *m* quasi-holes at positions η_i are given by

$$|\eta_1, \ldots, \eta_m\rangle_k \;\propto\; \prod_{i=1}^{m} \det(Z^\dagger - \eta_i^\dagger)\; |\text{ground}\rangle_k$$

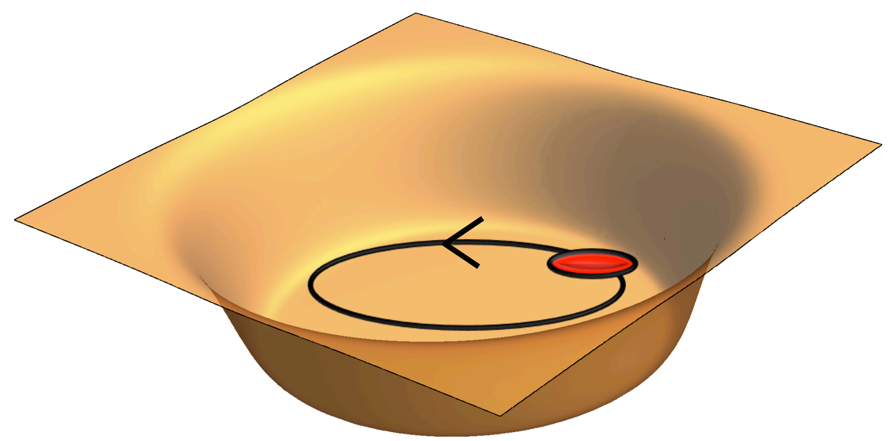
These agree with the appropriate Laughlin wavefunctions at large distance

Can show explicitly that these have fractional charge and fractional statistics…

Tong and Turner ‘15

Quasi-Hole Charge

Consider one quasi-hole $$|\eta\rangle_k \;\propto\; \det(Z^\dagger - \eta)\;|\text{ground}\rangle_k \qquad\qquad \eta = re^{i\theta}$$

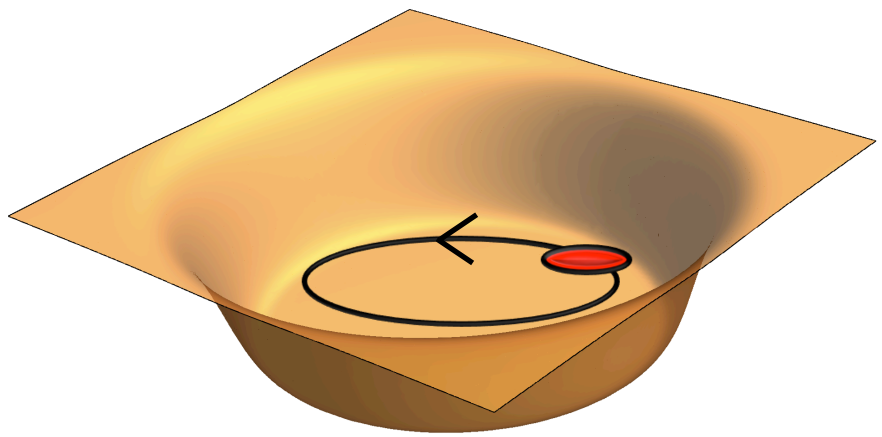


Rotate in circle. Pick up a Berry phase $$\Theta(r) = -i\int_0^{2\pi} d\theta\ {}_k\langle\eta|\,\frac{\partial}{\partial\theta}|\eta\rangle_k$$

Compare to expected phase picked up by a particle of charge q_{QH} $$\Theta(r) = \Phi q_{\mathrm{QH}} = \pi r^2 B^{\mathrm{ext}} q_{\mathrm{QH}} = \frac{2\pi^2\mu r^2}{e} q_{\mathrm{QH}}$$

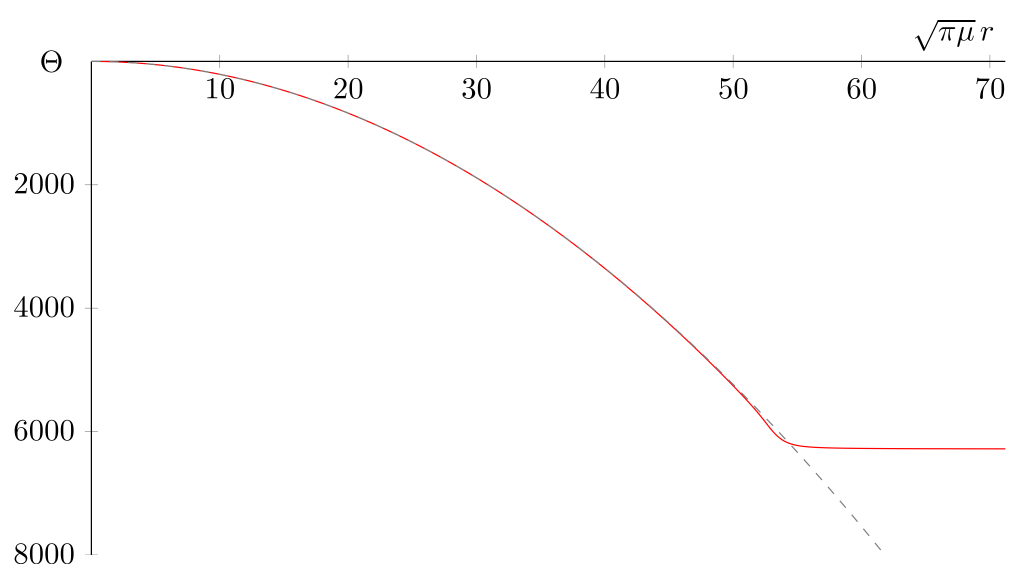
c.f. Laughlin states and the "plasma analogy"
(Arovas, Schrieffer, Wilczek '84)

Quasi-Hole Charge



Analytic calculation possible!

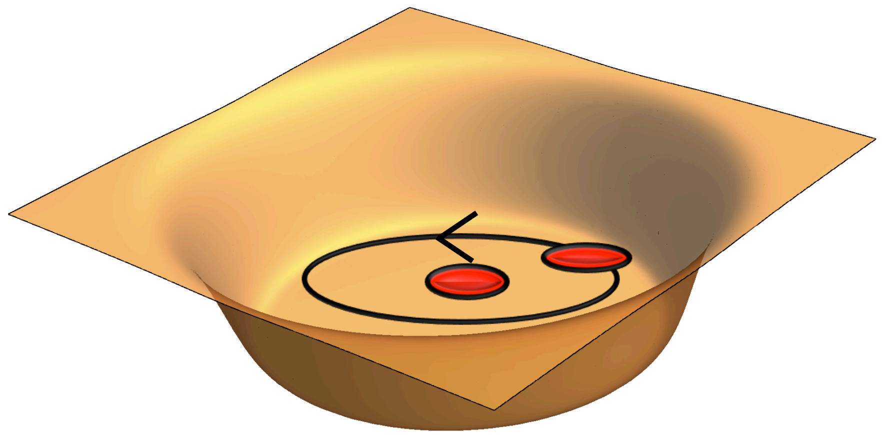
$$\Theta(r) = -2\pi^2\mu r^2 \left(\frac{n}{(n-1)k+1} \frac{{}_1F_1(1-n, 2-n-1/k,\, \pi\mu r^2/k)}{{}_1F_1(\,-n, 1-n-1/k,\, \pi\mu r^2/k)} \right)$$



$$\boxed{q_{\mathrm{QH}} = -\frac{e}{k}}$$

Quasi-Hole Statistics

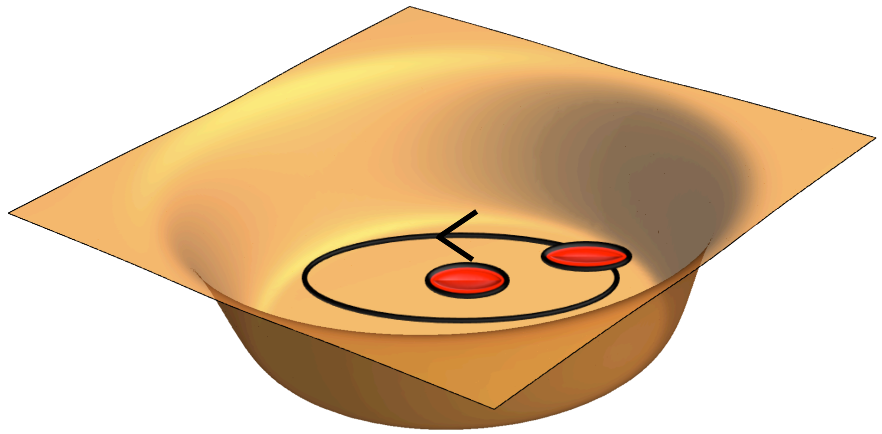
Consider two quasi-hole $|0,\eta\rangle_k \;\propto\; \det(Z^\dagger)\det(Z^\dagger-\eta^\dagger)\;|\mathrm{ground}\rangle_k$ $\qquad \eta = re^{i\theta}$



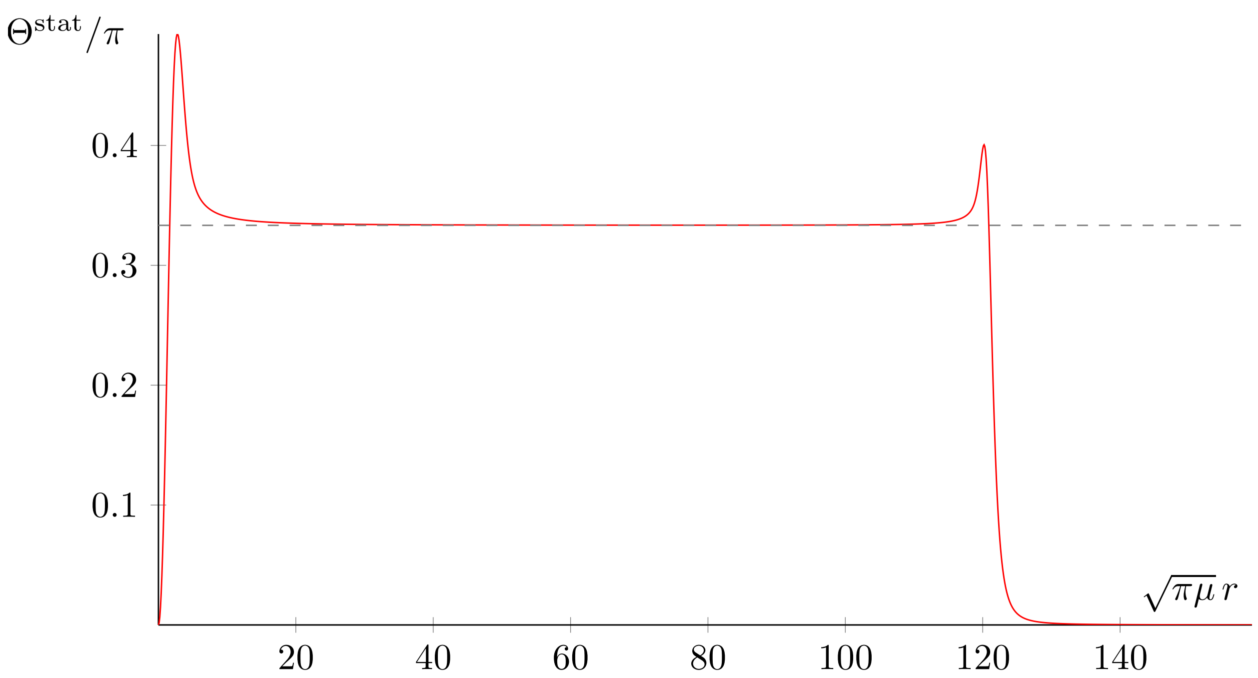
Rotate one quasi-hole around the other

$$2\Theta^{\mathrm{stat}}(r) = -i\int_0^{2\pi} d\theta\;{}_k\langle 0,\eta|\,\frac{\partial}{\partial\theta}|0,\eta\rangle_k \;-\Theta(r)$$

Quasi-Hole Statistics



$$2\Theta^{\rm stat}(r) = \frac{2\pi^2\mu r^2}{k}\left(n\,\frac{{}_2\tilde{F}_2(1+1/k,1-n;1+2/k,2-n-1/k;\,\pi\mu r^2/k)}{{}_2\tilde{F}_2(1/k,-n;\,2/k,\,1-n-1/k;\,\pi\mu r^2/k)}\right) - \Theta_r$$



$$\boxed{\Theta^{\rm stat} = \frac{\pi}{k}}$$

The quasi-holes are anyons

The General Quantum Ground State

$$S = \int dt \; \frac{iB}{2} \,\mathrm{Tr}\, \left(Z^\dagger \mathcal{D}_t Z \right) + i \sum_{i=1}^{p} \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p)\,\mathrm{Tr}\,\alpha - \omega\,\mathrm{Tr}\, Z^\dagger Z$$

- The ground states now describe particles with *SU(p)* spin.
- They are spin singlets only when *N* is divisible by *p*.
  - (Otherwise they transform in some representation of *SU(p)*
- They describe *non-Abelian* quantum Hall states with filling fraction

$$\nu = \frac{p}{k+p}$$

Dorey, Tong and Turner ‘16

The Quantum Ground State

If *N* is divisible by *p* then the ground state is an *SU(p)* singlet

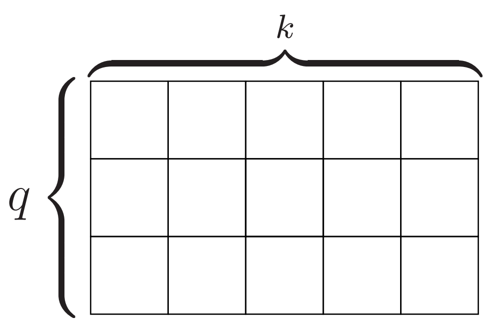
$$|\mathrm{ground}\rangle_k = \left[\epsilon^{a_1\ldots a_N}\left[(\varphi_1)^\dagger_{a_1}\ldots(\varphi_p)^\dagger_{a_p}\right]\left[(Z\varphi_1)^\dagger_{a_{p+1}}\ldots(Z\varphi_p)^\dagger_{a_{2p}}\right]\ldots\left[(Z^{\frac{N}{p}-1}\varphi_1)^\dagger_{a_{N-p+1}}\ldots(Z^{\frac{N}{p}-1}\varphi_p)^\dagger_{a_N}\right]\right]^k|0\rangle$$

These describe states with filling fraction

$$\boxed{\nu = \frac{p}{k+p}}$$

If *N* not divisible by *p* then the ground state transforms in a representation of *SU(p)*

If *N*=*q* mod *p* then the ground state sits in the representation



The General Quantum Hall Ground State

- Now our particles carry spin degrees of freedom under the SU(p).
- Their long distance structure reproduces many famous wavefunctions

<u>Examples</u>

- SU(2) at level k=1: The (2,2,1) Halperin wavefunction for spin ½ particles

$$\psi(z,w) = \prod_{i<j}^{N/2}(z_i^{\uparrow}-z_j^{\uparrow})^2\prod_{k<l}^{N/2}(z_k^{\downarrow}-z_l^{\downarrow})^2\prod_{i,k}(z_i^{\uparrow}-z_k^{\downarrow})$$

- SU(2) at level k=2: The Moore-Read wavefunction at ν=½ (for spin 1 particles)

$$\psi = \mathrm{Pf}\left(\frac{|ij\rangle_1}{z_i-z_j}\right)\prod_{i<j}(z_i-z_j)^2$$

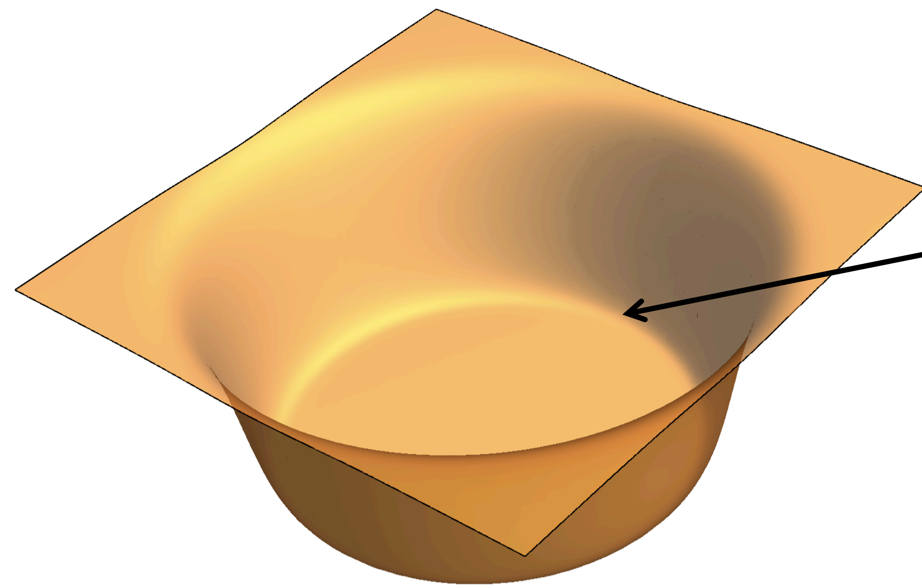
- SU(p) at level k=2: spin-singlet versions of Read-Rezayi states

In general, these are a class of wavefunctions constructed by Blok and Wen '92.

The Matrix Model as a CFT

Boundary Conformal Field Theory

Outside the vortex
$U(p)$ gauge symmetry
broken



Inside the vortex,
$U(p)$ gauge symmetry
unbroken

We have engineered a Chern-Simons theory with boundary…

- There should be chiral edge modes, described by a WZW conformal field theory
- These same edge modes should be excitations of vortices

$\Rightarrow$ In the large N limit, our matrix model should be the $U(p)_k$ WZW CFT!

Quantum Hall States and WZW Model

Our quantum Hall states are those of U(p) Chern-Simons theory…

… so should be related to the boundary U(p) WZW model

First piece of evidence: the wavefunction are correlation function of the *U(p)* WZW model

They obey the Knizhnik-Zamalodchikov equations

$$\left(\frac{\partial}{\partial z_a} - \frac{1}{k+p}\sum_{b\neq a}^{N}\frac{T_a^\alpha \otimes T_b^\alpha}{z_a - z_b}\right)\langle \mathcal{O}_R(z_1)\ldots\mathcal{O}_R(z_N)\rangle = 0$$

ground state wavefunction

Current Algebra

The smoking gun of a WZW model is the existence of a *current algebra*

Construct *SU(p)* currents from the matrix model

$$\mathcal{J}_{ij}^{m} = \varphi_i^\dagger Z^m \varphi_j \qquad m \geq 0$$

$$\mathcal{J}_{ij}^{m} = \mathcal{J}_{ji}^{|m|\,\dagger} \qquad m < 0$$

Can show that*, in the large *N* limit, these obey a Kac-Moody algebra!

$$[\mathcal{J}_{ij}^{m}, \mathcal{J}_{kl}^{n}] \sim i(\delta_{il}\mathcal{J}_{kj}^{m+n} - \delta_{kj}\mathcal{J}_{il}^{m+n}) + km\ \delta_{m+n,0}\ \left(\delta_{jk}\delta_{il} - \frac{1}{p}\delta_{ij}\delta_{kl}\right)$$

central charge (including quantum shift)

* up to two conjectured identities

The Partition Function

Remarkably, the exact partition function of the matrix model can be computed exactly

$$\mathcal{Z} = \mathrm{Tr}\, t^E \prod_{i=1}^{p} x_i^{J_i}$$

Cartan elements of *SU(p)*

Fugacities for *SU(p)* Cartan

$$\mathcal{Z} = \prod_{a=1}^{N} \oint \frac{d\omega_a}{\omega_a^{k+1}} \prod_{b \neq c} \left(1 - \frac{\omega_b}{\omega_c}\right) \mathcal{Z}_Z \, \mathcal{Z}_\varphi$$

with

$$\mathcal{Z}_Z = \prod_{a,b=1}^{N} \frac{1}{1 - t\omega_a/\omega_b}$$

$$\mathcal{Z}_\varphi = \prod_{a=1}^{N} \prod_{i=1}^{p} \frac{1}{1 - \omega_a x_i}$$

Imposes Gauss' law

Fugacities for *SU(N)* Cartan

The Partition Function for Laughlin States

For the *p=1* matrix models, the computation is easy

$$\mathcal{Z} = \mathrm{Tr}\, t^E \prod_{i=1}^{p} x_i^{J_i}$$

The answer is

$$\mathcal{Z}_0 = \prod_{j=1}^{N} \frac{1}{1-t^j}$$

This is the partition function for a single chiral boson, as expected.

c.f. Wen 1990 from the Laughlin wavefunctions

The Partition Function

For *p>1,* the result is more complicated

$$\mathcal{Z} = \prod_{a=1}^{N} \oint \frac{d\omega_a}{\omega_a^{k+1}} \prod_{b\neq c} \left(1 - \frac{\omega_b}{\omega_c}\right) \mathcal{Z}_Z \, \mathcal{Z}_\varphi$$

with

$$\mathcal{Z}_Z = \prod_{a,b=1}^{N} \frac{1}{1 - t\omega_a/\omega_b}$$

$$\mathcal{Z}_\varphi = \prod_{a=1}^{N} \prod_{i=1}^{p} \frac{1}{1 - \omega_a x_i}$$

Basic Idea:

- Expand components in characters of SU(N) and SU(p).
- These are symmetric polynomials known as Schur polynomials

The Partition Function

For *p>1,* the answer is

$$\boxed{\mathcal{Z} = \mathcal{Z}_0(t)\,\sum_{\lambda} K_{\lambda,\mu}(t)\,S_\lambda(x_1,\ldots,x_p)}$$

Chiral boson

Sum over partitions λ related
to representations of of *SU(p)*

Kostka polynomials.

Schur polynomials
$$\chi_\lambda = S_\lambda(x_1,\ldots,x_p)$$

These relate Schur polynomials with Hall-Littlewood polynomials.

- The information about the number of vortices *N* lies in the partition μ:
-
- When *N* is divisible by *p*, it is $(k,k,k,\ldots,k) = k^N$

Dorey, Tong and Turner, ‘16

The Relationship to the Boundary CFT

In the large *N* limit, these partition functions become something nice!

N divisible by *p*

$$\boxed{\lim_{N\to\infty} \quad \mathcal{Z}_{\mathrm{matrix}}(t, x_i) \longrightarrow \mathcal{Z}_{WZW}(t, x_i)}$$

The partition function of the matrix model and WZW model coincide!

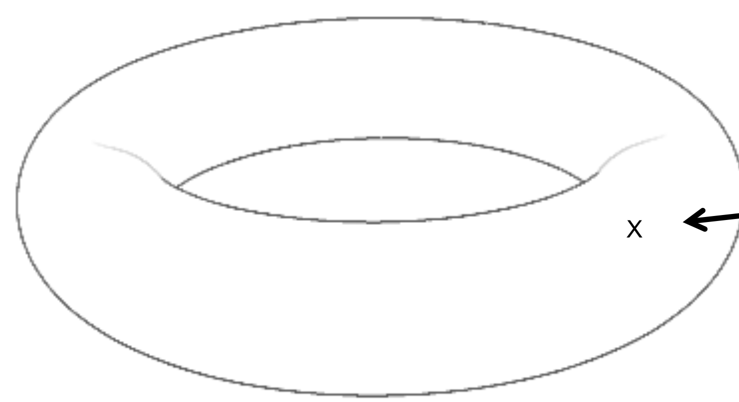
(Kirillov, "*Dilogarithm identities*", 94; Nakayashiki and Yamada, '95)

The Relationship to the Boundary CFT

But the large *N* limit is subtle

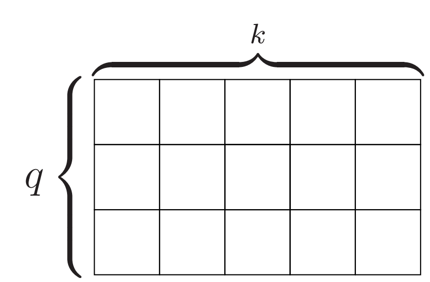
N=*q* mod *p*

$$\boxed{\lim_{N\to\infty} \quad \mathcal{Z}_{\text{matrix}}(t, x_i) \longrightarrow \ \chi^{(q)}_{WZW}(t, x_i)}$$



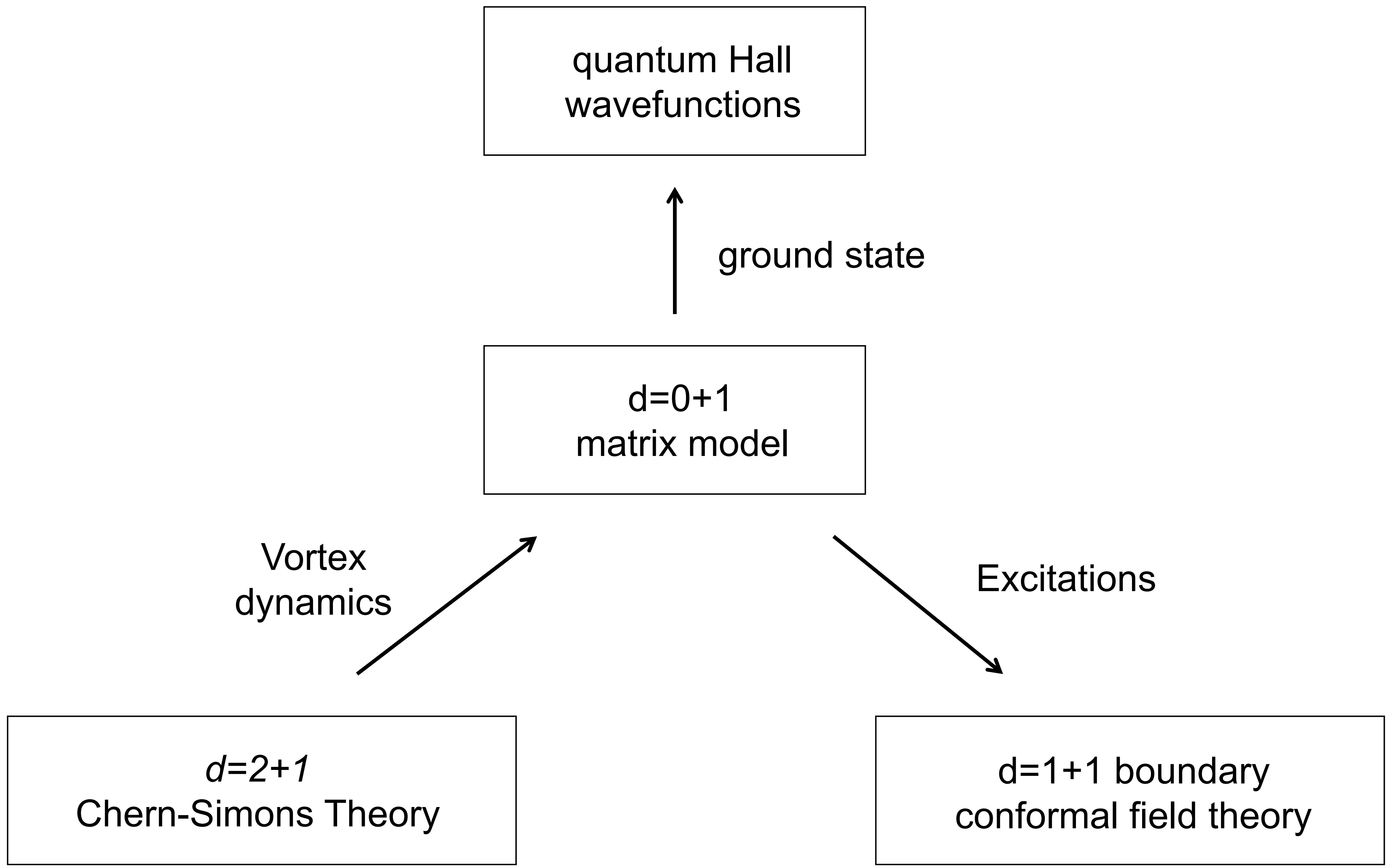
insertion of primary operator
in appropriate representation

This is the character of the WZW model
associated to the representation



This is the representation that the ground state of the matrix model sits in.

Summary


quantum Hall
wavefunctions
ground state
d=0+1
matrix model
Vortex
dynamics
Excitations
d=2+1
Chern-Simons Theory
d=1+1 boundary
conformal field theory

Thank you for your attention

Supersymmetry

A 3d Theory with Supersymmetry

$$S = \int dtd^2x \;\; \left\{ i\phi^\dagger \mathcal{D}_0 \phi + i\psi^\dagger \mathcal{D}_0 \psi - \frac{1}{2m}\mathcal{D}_\alpha\phi^\dagger\, \mathcal{D}_\alpha\phi - \frac{1}{2m}\mathcal{D}_\alpha\psi^\dagger\, \mathcal{D}_\alpha\psi - \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \right.$$
$$\left. -\mu A_0 + \frac{1}{2m}\psi^\dagger B\psi - \frac{\pi}{mk}\left(|\phi|^4 - \mu|\phi|^2 + 3|\phi|^2|\psi|^2\right) \right\}$$

Two supersymmetries:

$$\delta_1\phi = \epsilon_1^\dagger\psi \qquad\qquad \delta_2\phi = \epsilon_2^\dagger\mathcal{D}_{\bar z}\psi$$

$$\delta_1\psi = -\epsilon_1\phi \qquad\qquad \delta_2\psi = \epsilon_2\mathcal{D}_z\phi$$

$$\delta_1 A_z = 0 \qquad\qquad \delta_2 A_z = -\frac{i\pi}{k}\epsilon_2^\dagger\psi\phi^\dagger$$

$$\delta_1 A_0 = \frac{\pi}{mk}\left(\epsilon_1\phi\psi^\dagger - \epsilon_1^\dagger\psi\phi^\dagger\right) \qquad\qquad \delta_2 A_0 = \frac{i\pi}{mk}\left(\epsilon_2^\dagger\phi^\dagger\mathcal{D}_{\bar z}\psi - \epsilon_2\phi\mathcal{D}_z\psi^\dagger\right)$$

A 3d Theory with Supersymmetry

$$\boxed{S=\int dtd^2x\ \left\{i\phi^\dagger\mathcal{D}_0\phi+i\psi^\dagger\mathcal{D}_0\psi-\frac{1}{2m}\mathcal{D}_\alpha\phi^\dagger\,\mathcal{D}_\alpha\phi-\frac{1}{2m}\mathcal{D}_\alpha\psi^\dagger\,\mathcal{D}_\alpha\psi-\frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \right. \\ \left. -\mu A_0+\frac{1}{2m}\psi^\dagger B\psi-\frac{\pi}{mk}\left(|\phi|^4-\mu|\phi|^2+3|\phi|^2|\psi|^2\right)\right\}}$$

particle number

Hamiltonian

momentum

Algebra:

$$\{Q_1,Q_1^\dagger\}=\mathcal{N}\qquad \{Q_2,Q_2^\dagger\}=\frac{m}{2}H\qquad \{Q_1,Q_2^\dagger\}=\hat{P}$$

$$[P,P^\dagger]=-\frac{\pi\mu}{k}\mathcal{N}\qquad [H,Q_1]=-\frac{2\pi\mu}{mk}\,Q_1$$

$$[\mathcal{J},Q_1]=-\frac{1}{2}Q_1\qquad [\mathcal{J},Q_2]=\frac{1}{2}Q_2$$

angular momentum

Gauss' Law

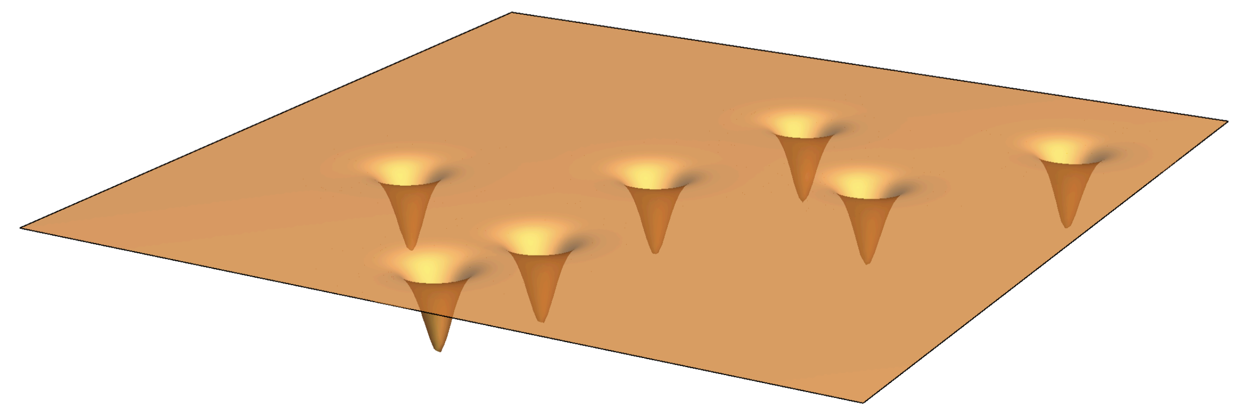
$$S = \int dt d^2x \ \Big\{ i\phi^\dagger \mathcal{D}_0 \phi + i\psi^\dagger \mathcal{D}_0 \psi - \frac{1}{2m} \mathcal{D}_\alpha \phi^\dagger \, \mathcal{D}_\alpha \phi - \frac{1}{2m} \mathcal{D}_\alpha \psi^\dagger \, \mathcal{D}_\alpha \psi - \frac{k}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$
$$- \mu A_0 + \frac{1}{2m} \psi^\dagger B \psi - \frac{\pi}{mk} \left(|\phi|^4 - \mu |\phi|^2 + 3|\phi|^2 |\psi|^2 \right) - V_{\mathrm{trap}} \Big\}$$

$$B = \frac{2\pi}{k} \left(|\phi|^2 + |\psi|^2 - \mu \right)$$

Microscopic Dynamics of Relativistic Vortices

$$B = \frac{2\pi}{k}(|\phi|^2 - \mu)$$

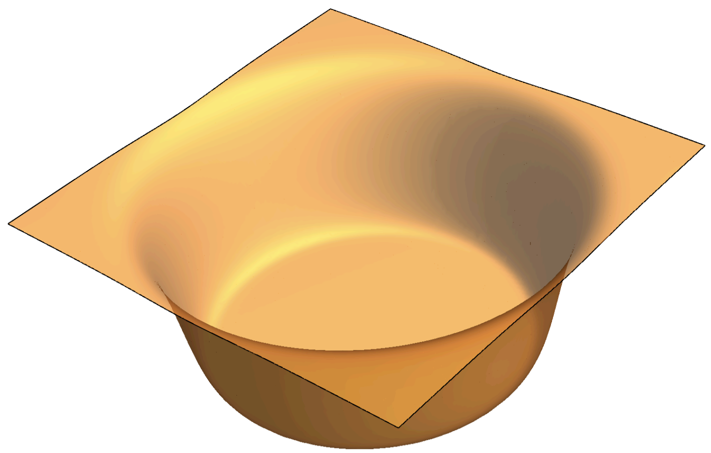
$$\mathcal{D}_z \phi = 0$$



$\mathcal{M}_N$ is *configuration* space

$$S_{\rm vortex} = \int dt \; g_{ab}(X)\dot{X}^a \dot{X}^b$$

Microscopic view of Vortices



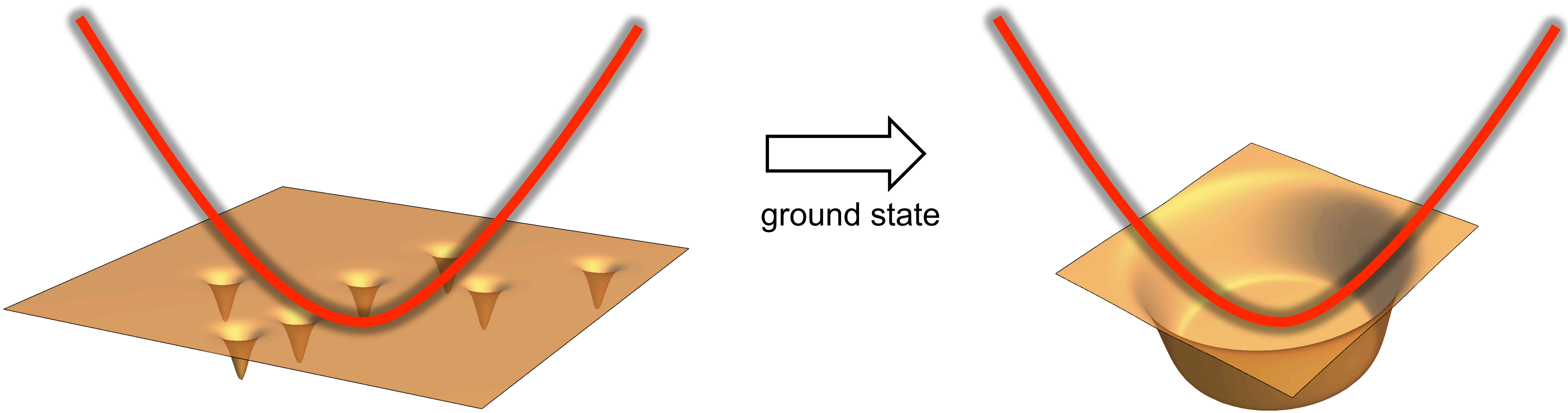
The expected number of available states per unit area is $\dfrac{eB^{\mathrm{ext}}}{2\pi} = \mu$

The area is $\pi R^2 = \dfrac{Nk}{\mu}$. The total number of available states is *Nk.*

$$\Rightarrow \quad \boxed{\nu = \frac{1}{k}}$$

Adding the Harmonic Trap

Evaluted on vortices, the trapping potential is simply the angular momentum.

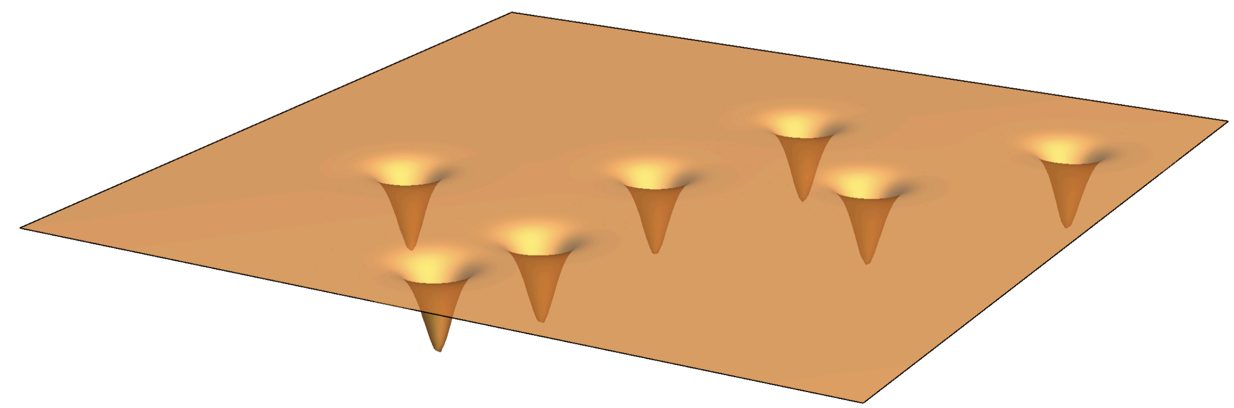


$$S_{\text{vortex}} = \int dt \, \left(\mathcal{F}_a(X)\dot{X}^a - \omega \mathcal{J}(X) \right)$$

Microscopic Dynamics of Non-Relativistic Vortices

$$B = \frac{2\pi}{k}(|\phi|^2 - \mu)$$

$$\mathcal{D}_z \phi = 0$$



$\mathcal{M}_N$ is *phase* space

$$S_{\text{vortex}} = \int dt \, \Big(\mathcal{F}_a(X)\dot{X}^a - \omega \mathcal{J}(X)\Big)$$

with $d\mathcal{F} = \Omega$ the Kahler form

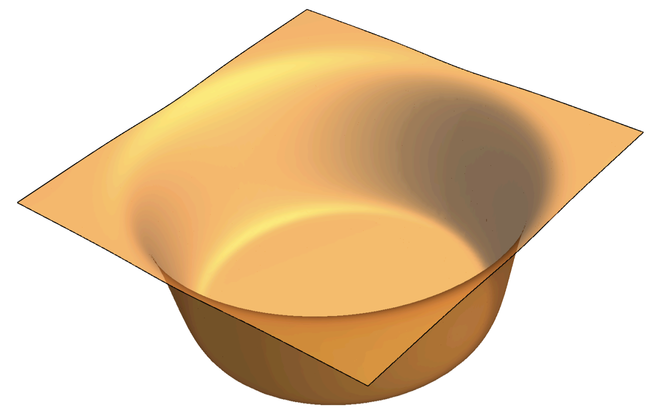
The potential is proportional to angular momentum

$$\mathcal{J} = -\frac{\mu\omega}{2}\int d^2x \; |z|^2 B$$

Manton, '97, Romao and Speight, 04

Edge Excitations

Ground State:



This is a classically incompressible fluid. Perturbative excitations about the ground state are described by

$$S = \sum_{l=1}^{n} \int dt \;\; (ic_l^\star \, \dot{c}_l - \omega l c_l^\star c_l)$$

This is the action for the (first *n* Fourier modes) of a chiral boson, describing ripples along the edge. The continuum limit is the Floreanini-Jackiw action

$$S = -\int dt d\sigma \;\, \partial_t c \, \partial_\sigma c + (\omega R) \partial_\sigma c \, \partial_\sigma c$$

Plan for the Talk