The Quantum Hall Effect

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(And why these three guys won last week's Nobel prize)

UNIVERSITY OF CAMBRIDGE

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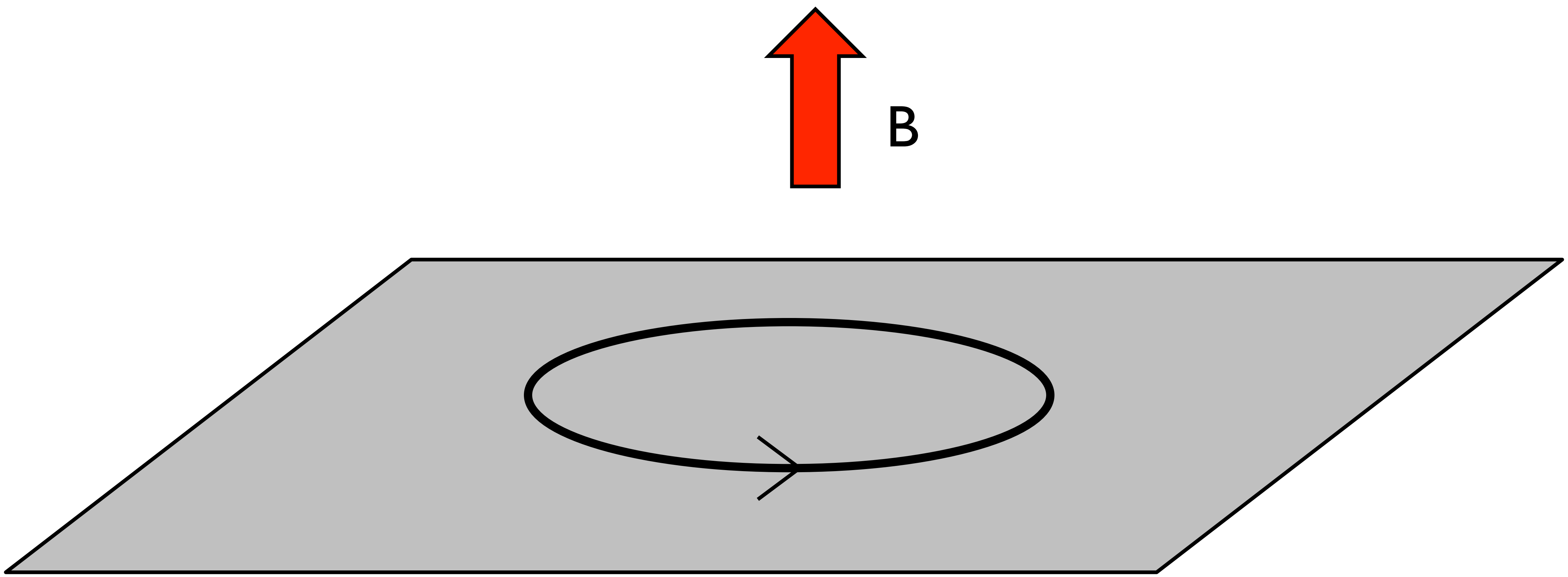
Electron in a Magnetic Field



$$m\ddot{\mathbf{x}} = -e\dot{\mathbf{x}} \times \mathbf{B} \quad \Rightarrow \quad \begin{aligned} x &= \frac{v}{\omega}\cos\omega t \\ y &= -\frac{v}{\omega}\sin\omega t \end{aligned}$$

$$\omega = \frac{eB}{m}$$

The Classical Hall Effect

friction = resistance

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - e\mathbf{v} \times \mathbf{B} - \frac{m\mathbf{v}}{\tau}$$

applied electric field
in x-direction

magnetic field
in z-direction

In equilibrium, we solve for velocity *v*. The solution takes the form

$$\mathbf{v} = \sigma\mathbf{E}$$

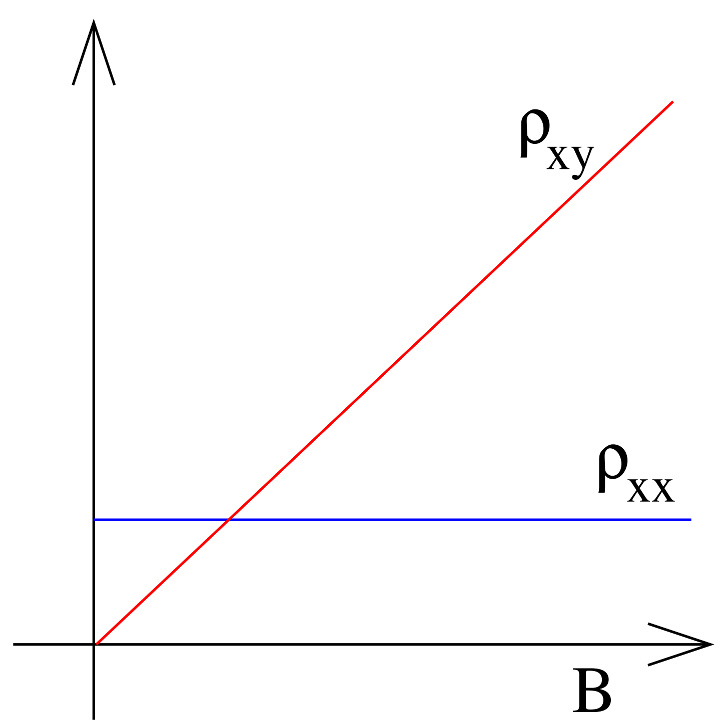
with σ a 2x2 matrix called the *conductivity*.

The Classical Hall Effect

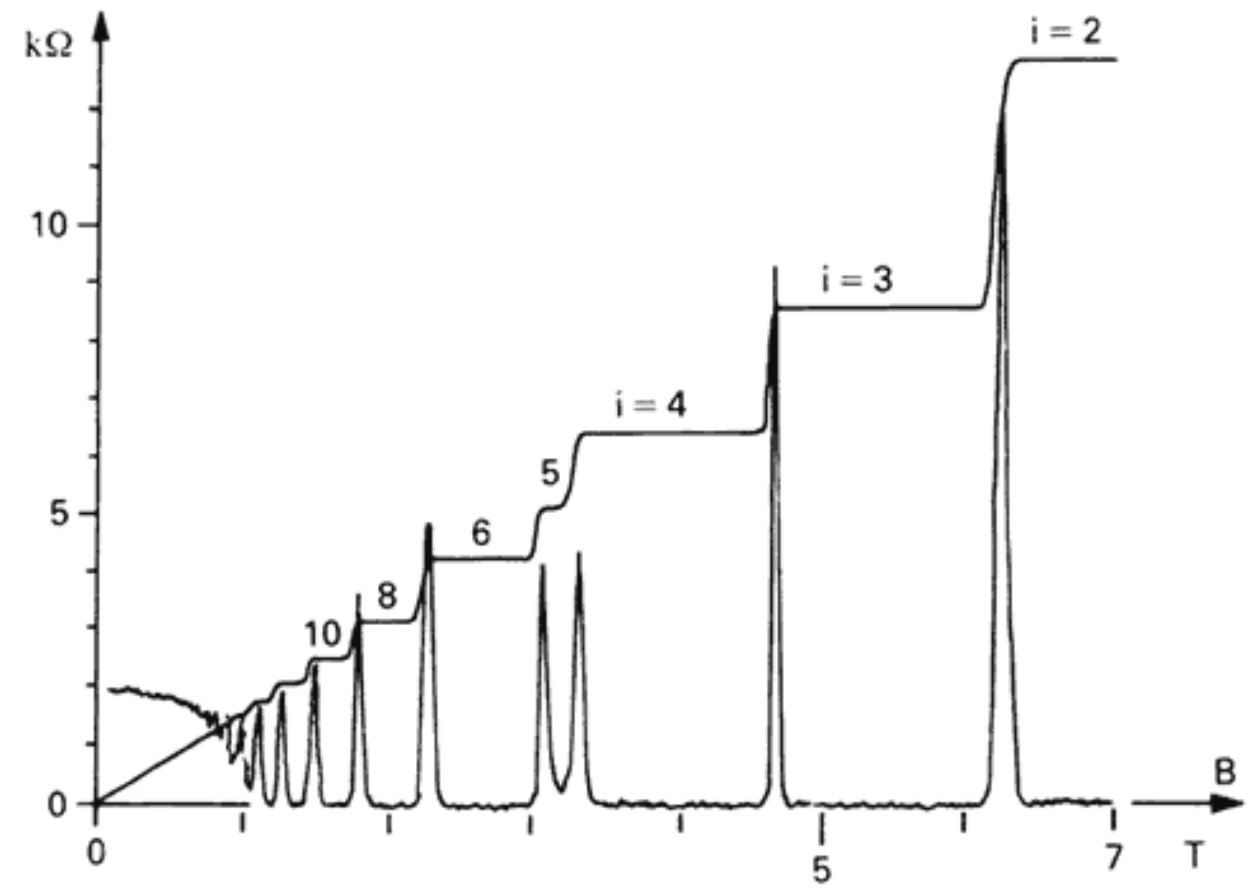
We usually plot the resistivity matrix

$$\rho = \sigma^{-1} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix}$$

The classical calculation above tells us how the resistivity should change with *B*



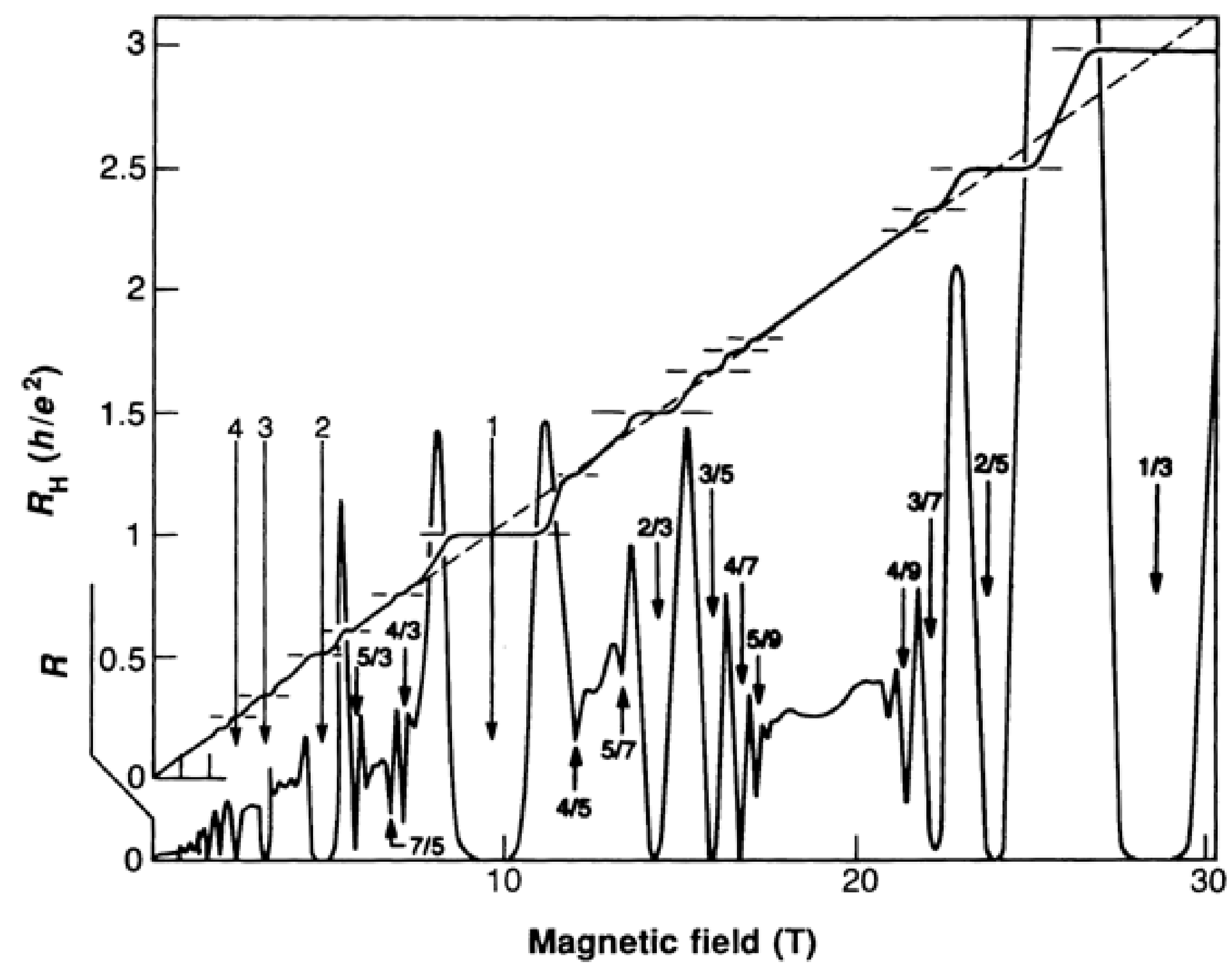
Integer Quantum Hall Effect



$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\,\frac{1}{\nu} \qquad\qquad \nu \in \mathbf{Z}$$

von Klitzing, Dorda and Pepper, 1981. (Nobel prize 1985)

The Fractional Quantum Hall Effect



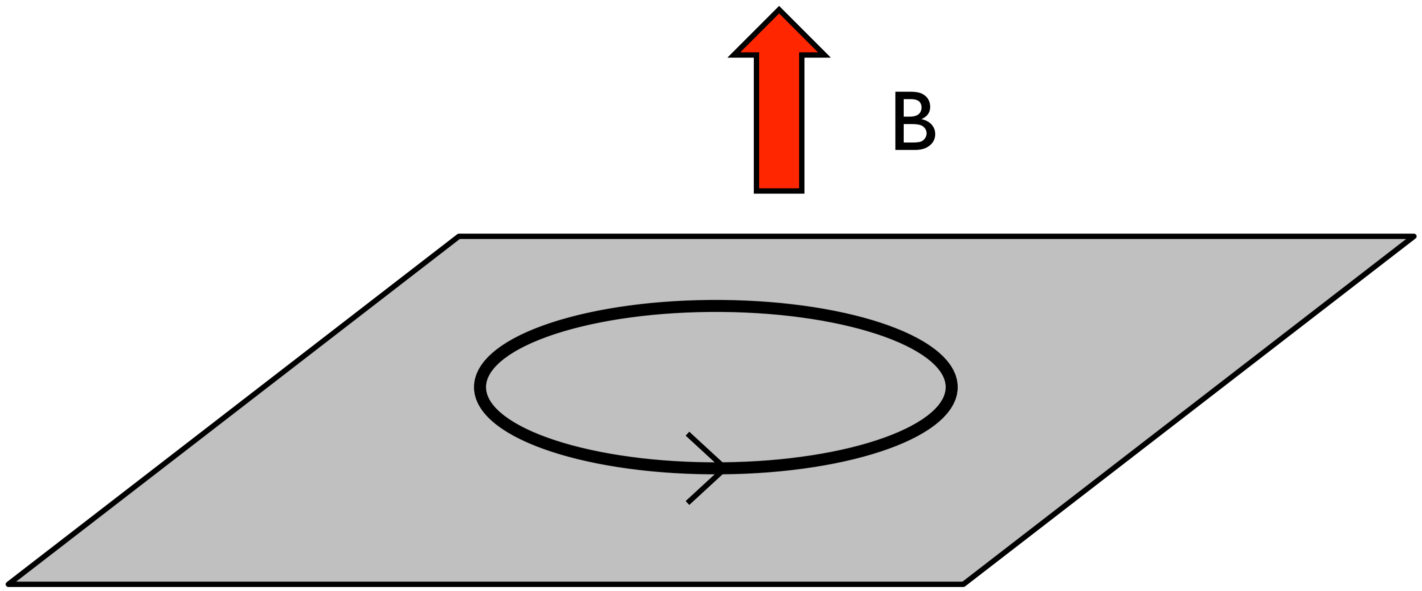
$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu} \qquad \nu \in \mathbf{Q}$$

Tsui, Stormer and Gossard, 1982. (Nobel prize with Laughlin 1998)

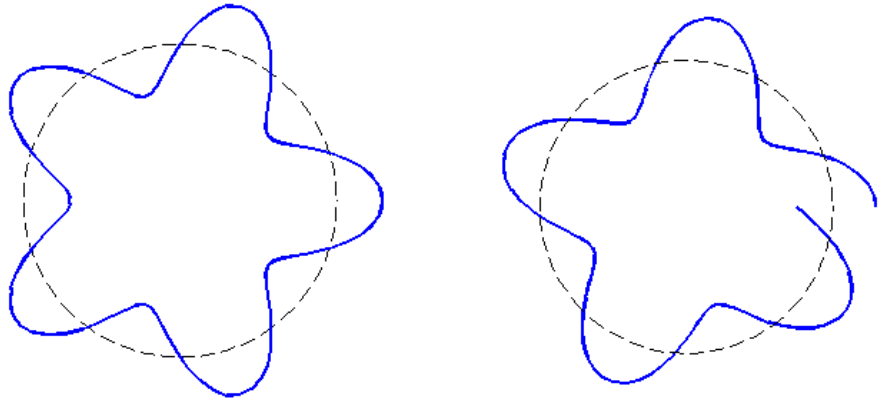
Understanding the Integer Quantum Hall Effect

A rough explanation:

Particles go in circles but circles must be compatible with de Broglie wavelength



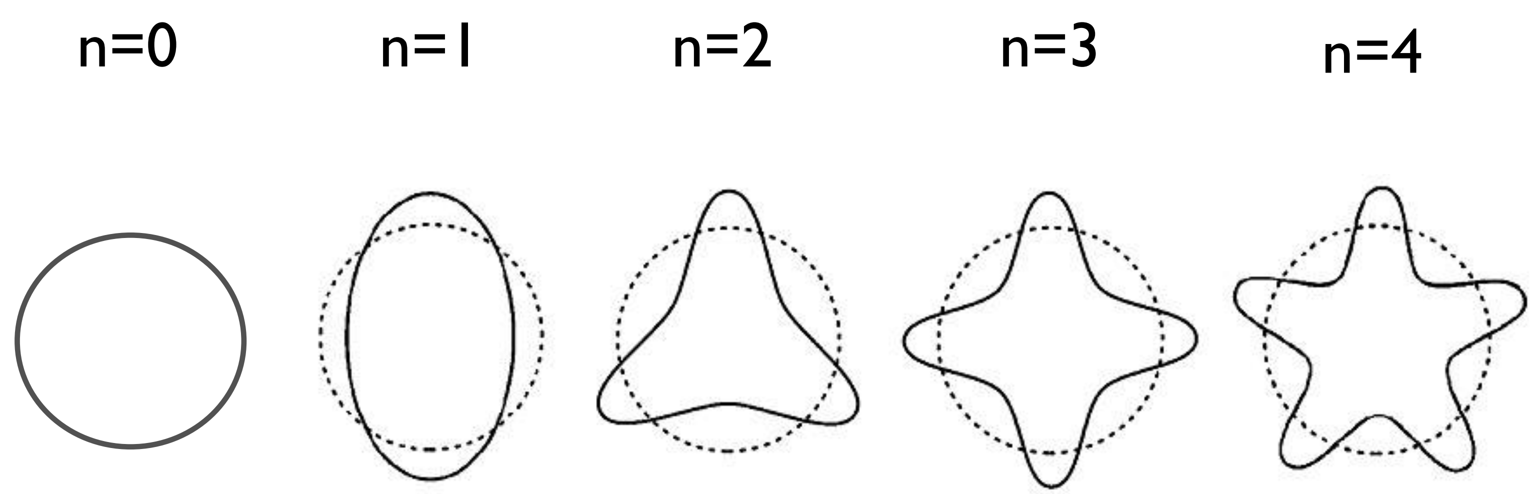
This is allowed… …but this is not



The energy of the particle depends on how many wavelengths sit in its orbit.

Integer Quantum Hall Effect

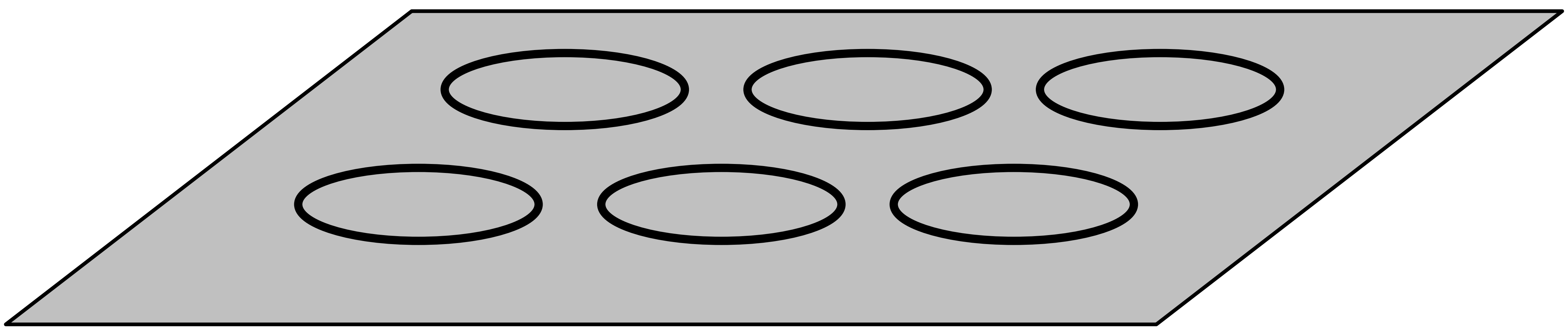
$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \qquad \omega = \frac{eB}{m}$$



But we can place these orbits anywhere on the plane

Understanding the Integer Quantum Hall Effect

The n=0 orbits can sit in many different positions

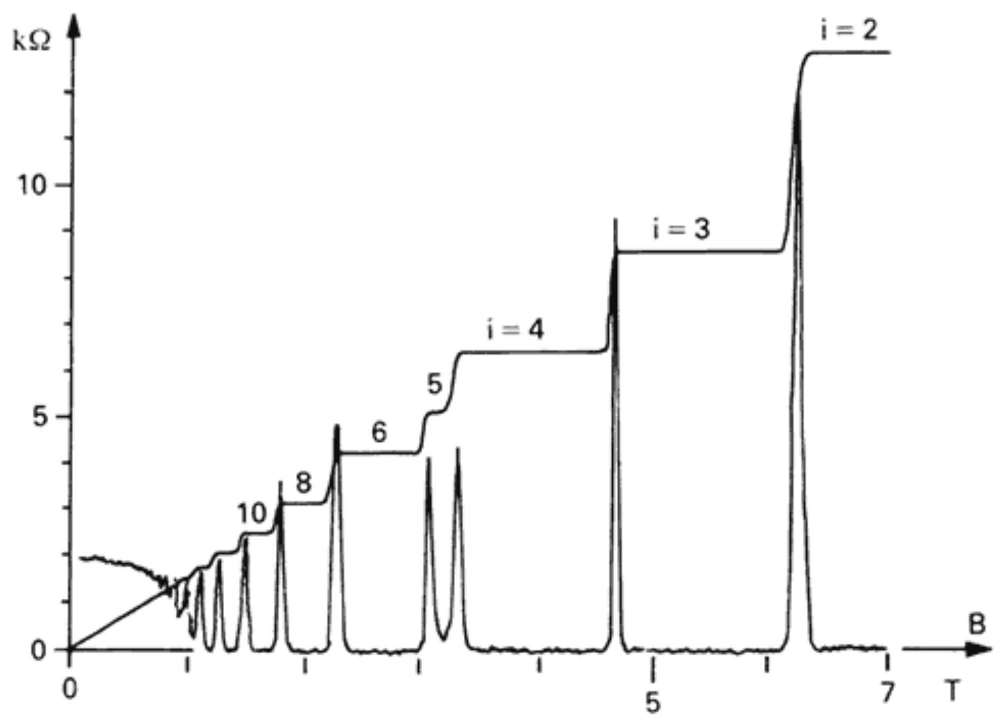


Total number of states with energy E_n is $\mathcal{N} = \frac{AB}{\Phi_0}$ $\quad \Phi_0 = \frac{2\pi\hbar}{e}$

area

The energy levels $E_n = \hbar\omega \left(n + \frac{1}{2} \right)$ with this degeneracy are called *Landau levels*

Integer Quantum Hall Effect



$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}$$

$\nu \in \mathbf{Z}$ is counting the number of fully filled Landau levels

A full explanation of why the plateau exists needs disorder

An Aside: Topology in Physics

The Nobel Prize for Topology in Physics

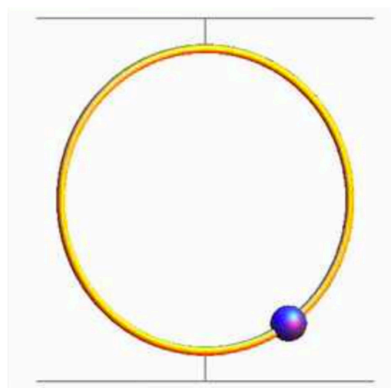
David Thouless

Duncan Haldane

Michael Kosterlitz

An Aside: Topology in Physics

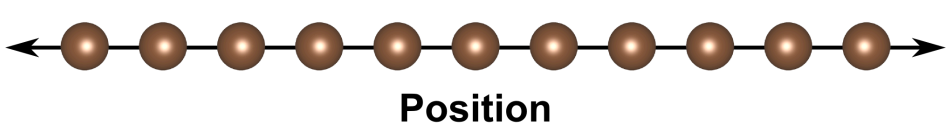
Here's a different way of thinking about the integer quantum Hall effect. First some basic quantum mechanics

Space is periodic: 

Momentum is discrete $p = \frac{\hbar n}{R}$

But the converse is also true

Space is discrete



Momentum is periodic

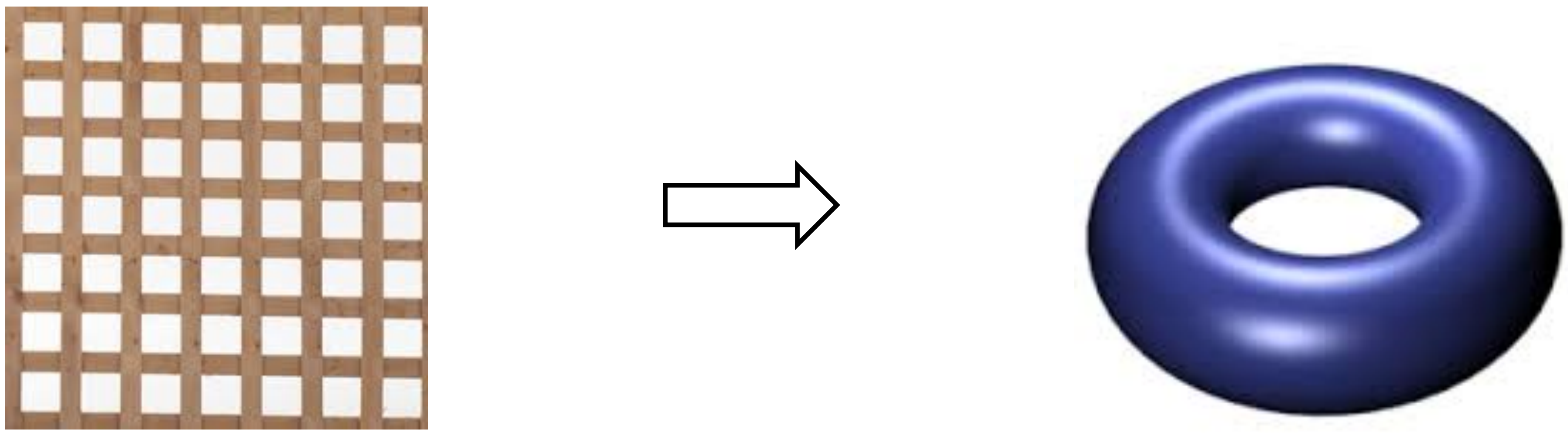
$$p \in \left[-\frac{\hbar\pi}{a}, \frac{\hbar\pi}{a}\right)$$

A periodic momentum space is called the *Brillouin zone*

An Aside: Topology in Physics

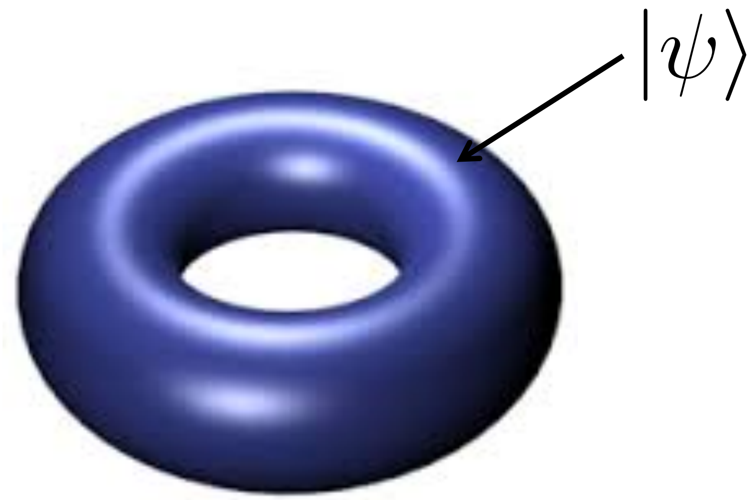
An electron in a material lives on a two-dimensional lattice

Its momentum lives on a two-dimensional torus



- Each point on this torus is a *state* of the electron, described by a wavefunction $\psi(p)$
- This wavefunction has a complex phase
- This phase can "wind" as we go around the torus.

An Aside: Topology and Physics



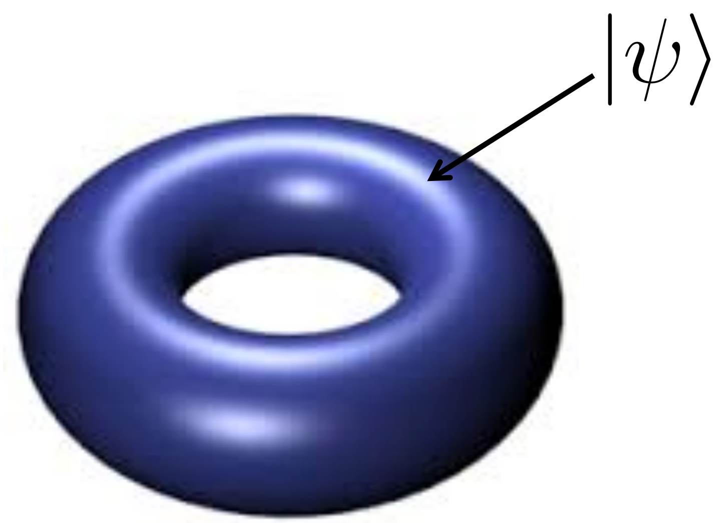
The TKNN formula (where T = Thouless) relates this winding to the Hall conductivity

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} C$$

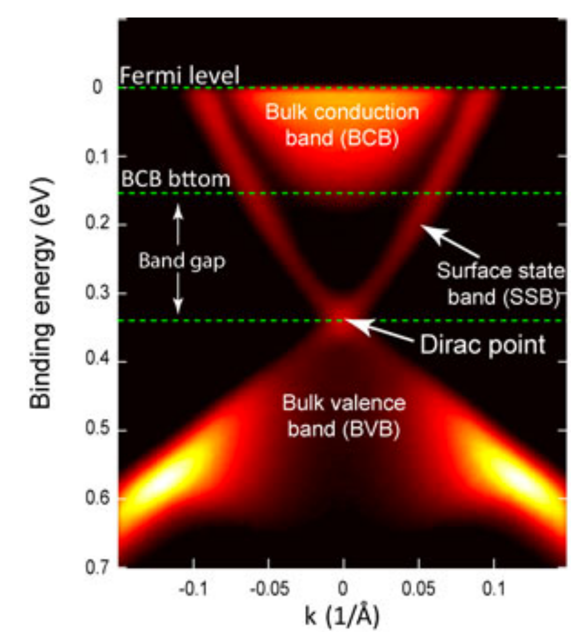
First Chern number

An Aside: Topology and Physics

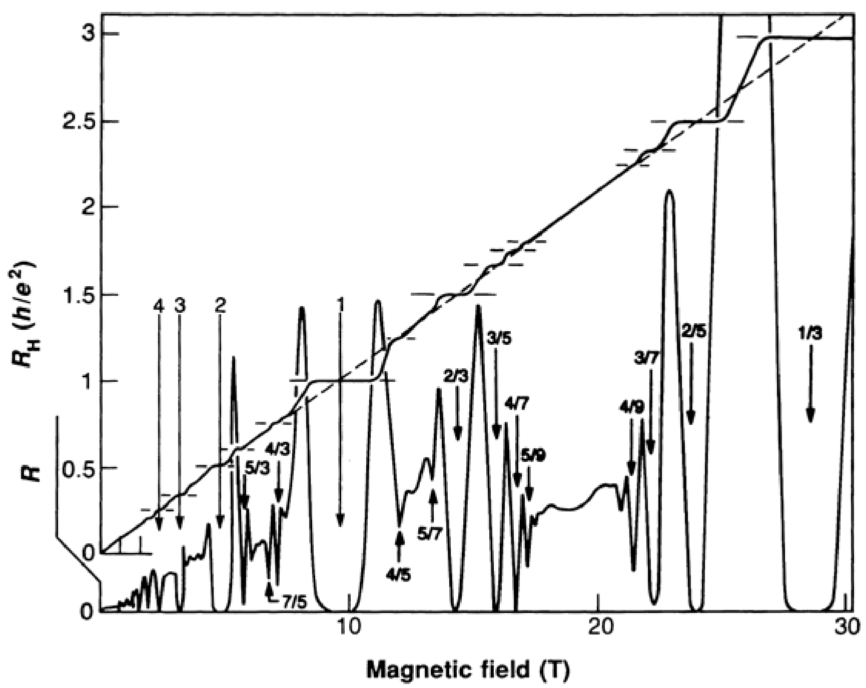
This idea lay almost-dormant for around 30 years!



- Until *topological insulators* were discovered in 2007
- No magnetic fields in sight, but the same idea of winding around the Brillouin zone
- Wonderful things happen on their surface



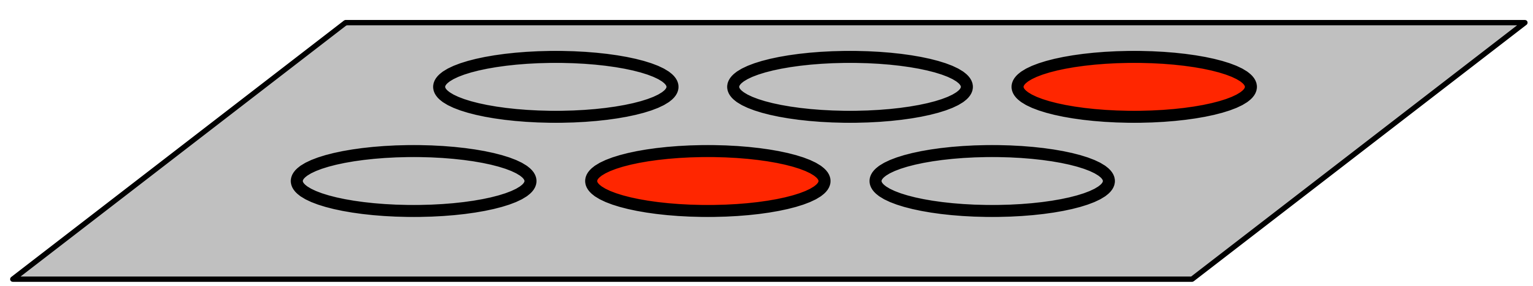
Back to…the Fractional Quantum Effect



$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}$$

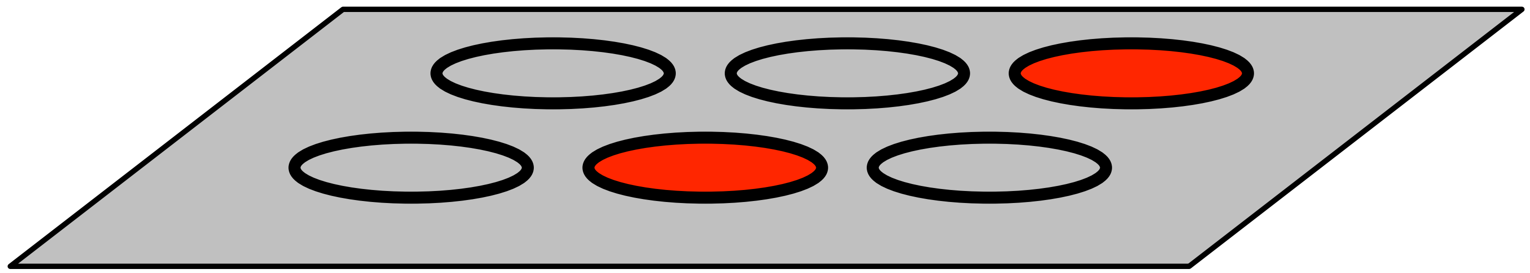
Now ν is the filling fraction of the lowest Landau level

e.g. ν=1/3



What picks the correct choice of filled states?......Interactions!

The Fractional Quantum Effect



Solving problems with many interacting electrons is hard *

Laughlin wavefunction: $$\psi(z_i) = \prod_{i<j} (z_i - z_j)^3 e^{-\sum_{i=1}^{n} |z_i|^2/4l_B^2}$$

$$z = x + iy$$

This describes a liquid of electrons

$$l_B = \sqrt{\frac{\hbar}{eB}}$$

* hard = no one knows how to do it.

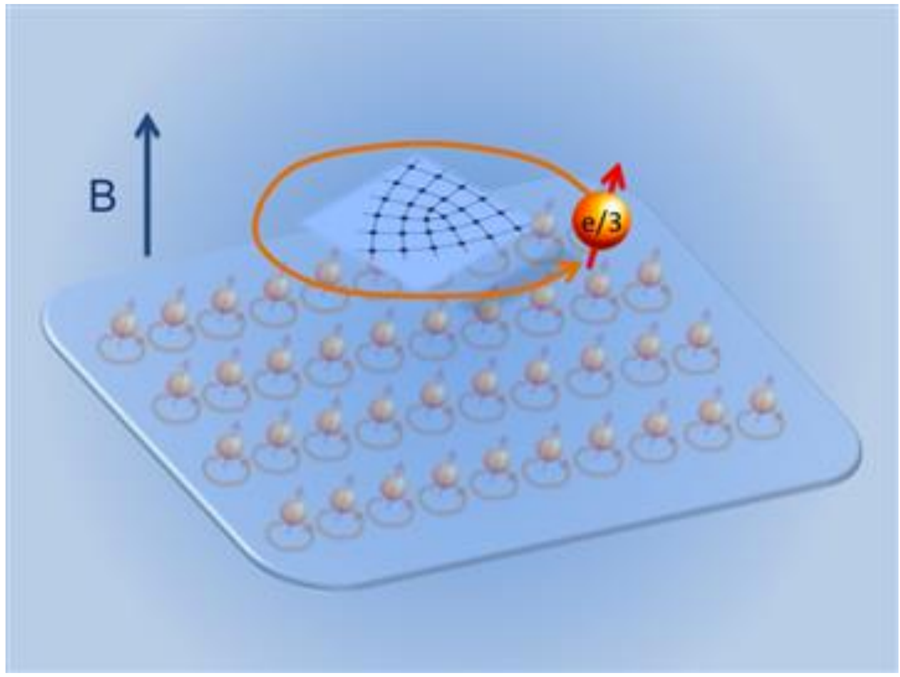
The Fractional Quantum Hall Effect

Laughlin's wavefunction predicts many surprising things. Here is the most startling

The excitations of the $\nu=1/3$ quantum Hall state have charge

$$q = \pm\frac{e}{3}$$

The indivisible electron has split into three pieces!



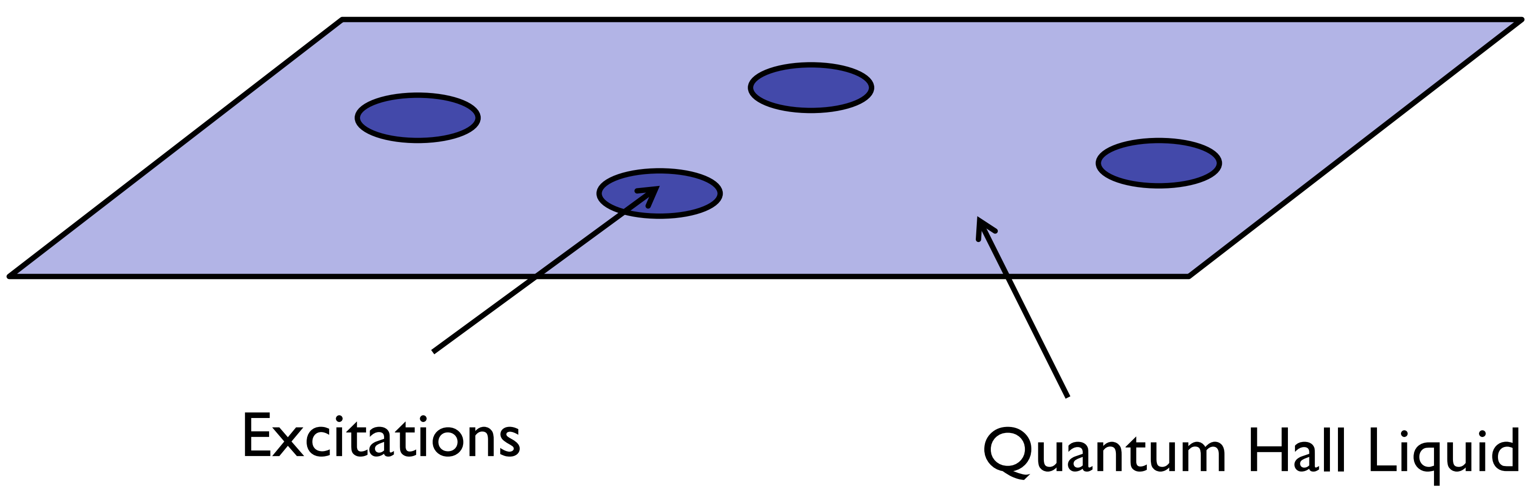
Moreover, this particle is neither boson nor fermion…it is an *anyon.*

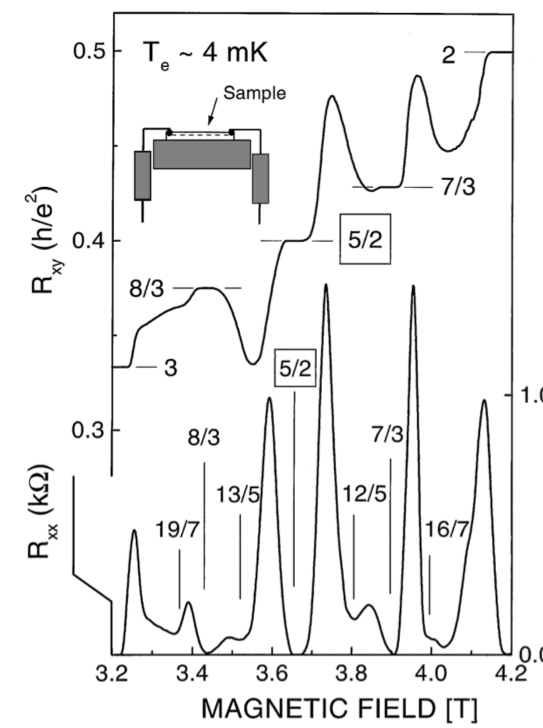
Non-Abelian Quantum Hall States

The is one more stage in the quantum Hall story…

$\nu = 5/2$

The most interesting physics is in the excitations above the ground state

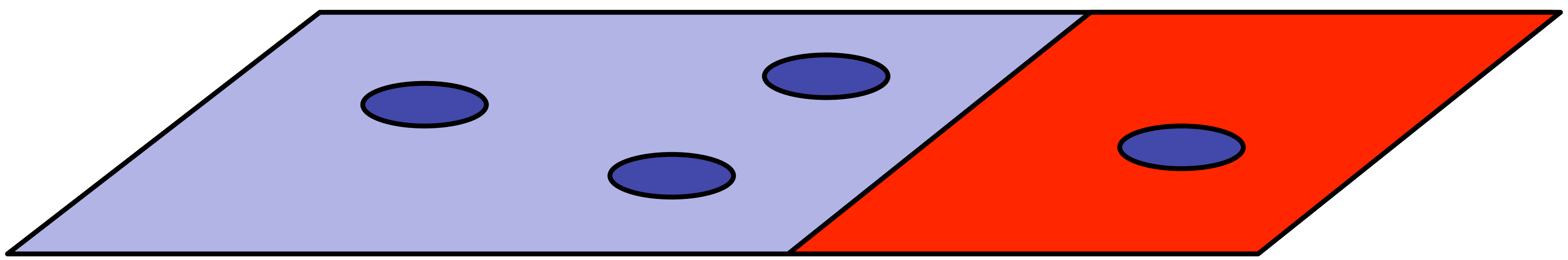




The *N* excitations do not have a unique state. The number of states is $\boxed{2^{N/2}}$

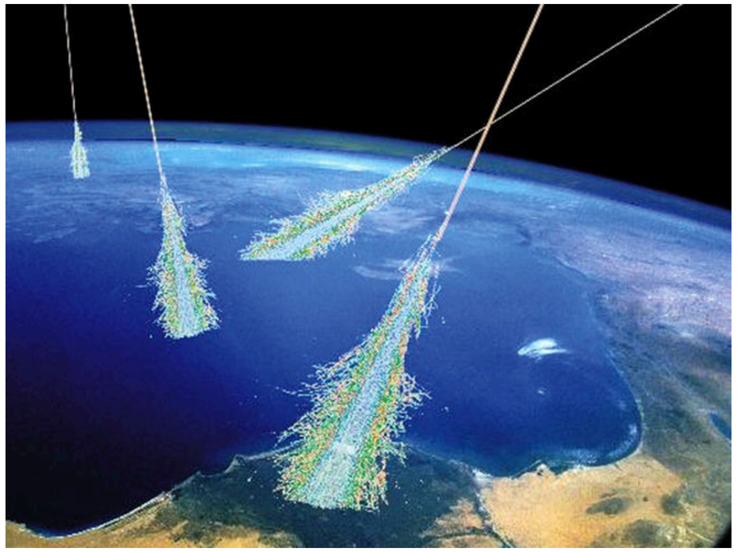
Global Properties of Non-Abelian Anyons

- $2^{N/2}$ states is a strange number
- If each particle had two different states (e.g. spin up/down), we would get 2^N



The state is a global property of the system. If we only have access to a subset of the system, there's no way of telling which state we're in.

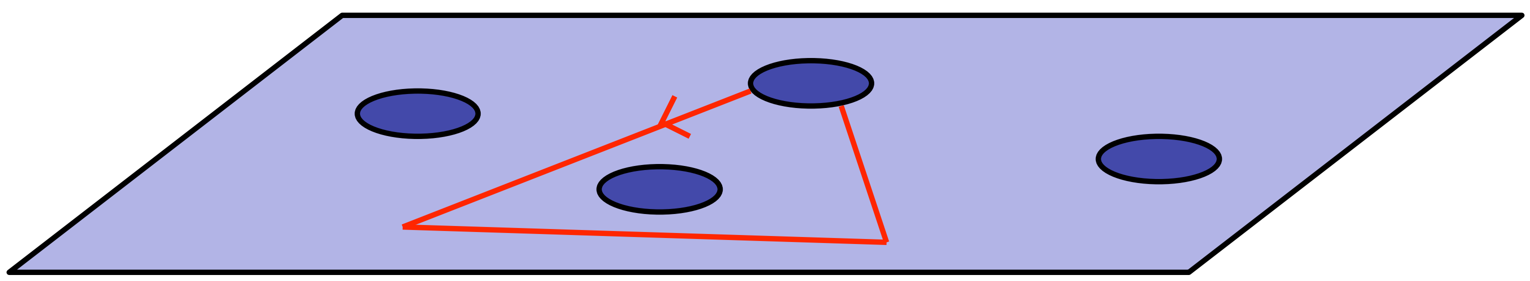
This makes these quantum states robust…



Topological Quantum Computing

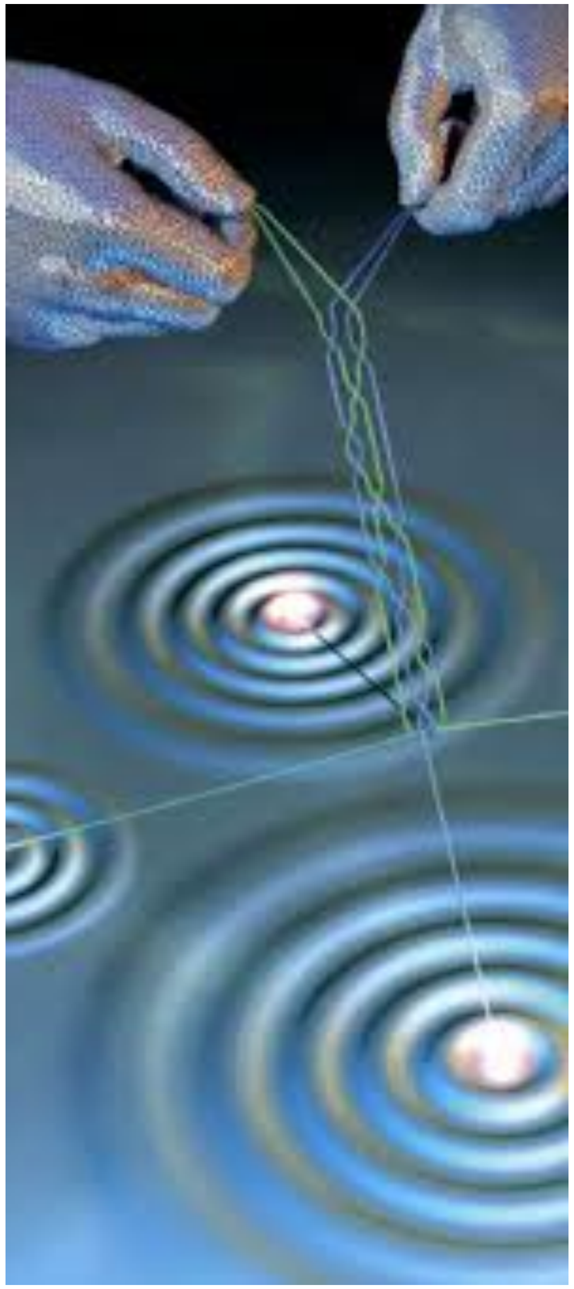
We describe the state by a $2^{N/2}$ dimensional vector ψ

Now move the particles around on some path:



$$\psi \mapsto U_{\text{path}}\, \psi$$

with U_{path} a unitary matrix that depends on the path taken

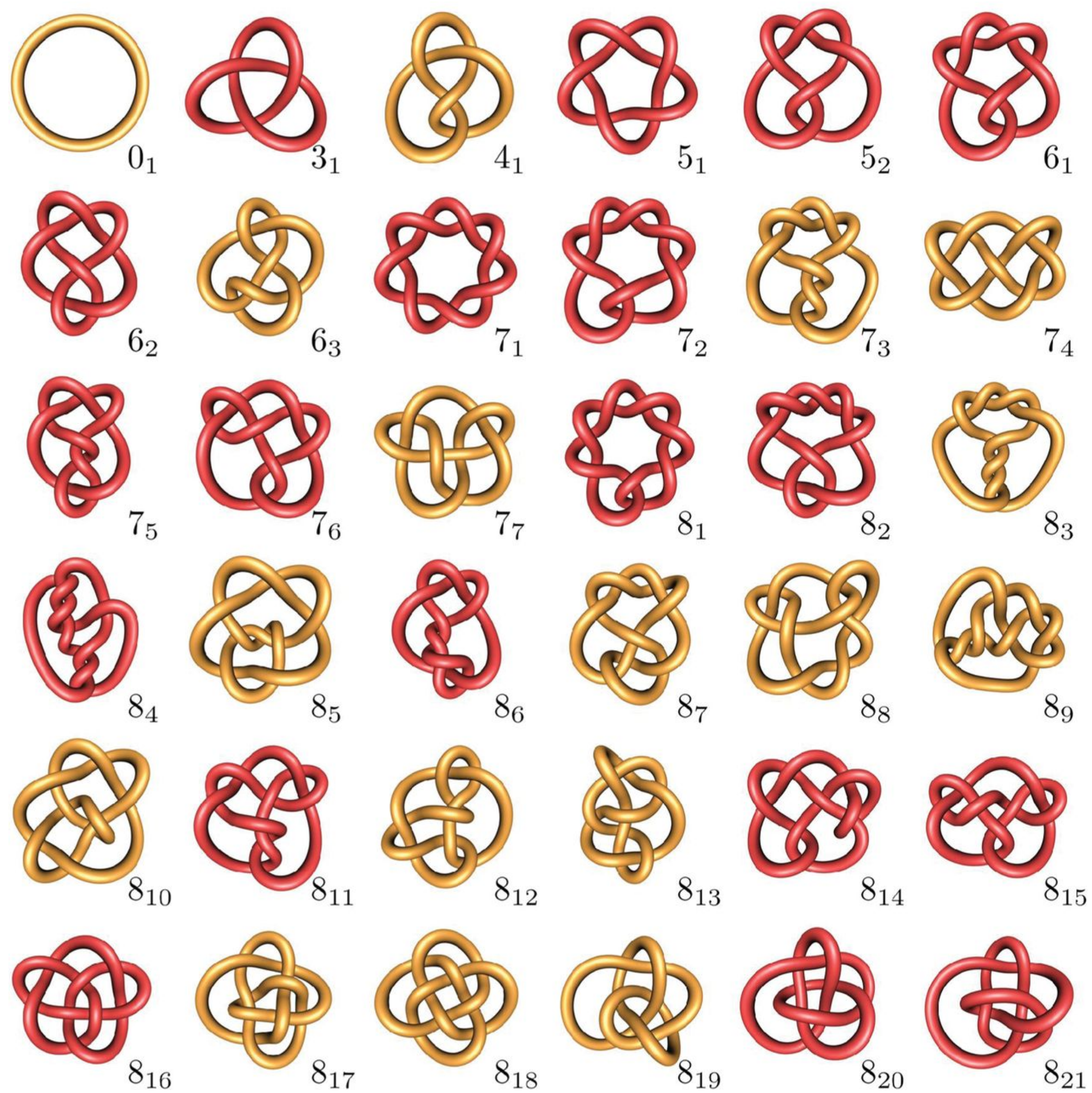


These particles are called *non-Abelian anyons*.

This allows us to do calculations in a quantum computer without errors!

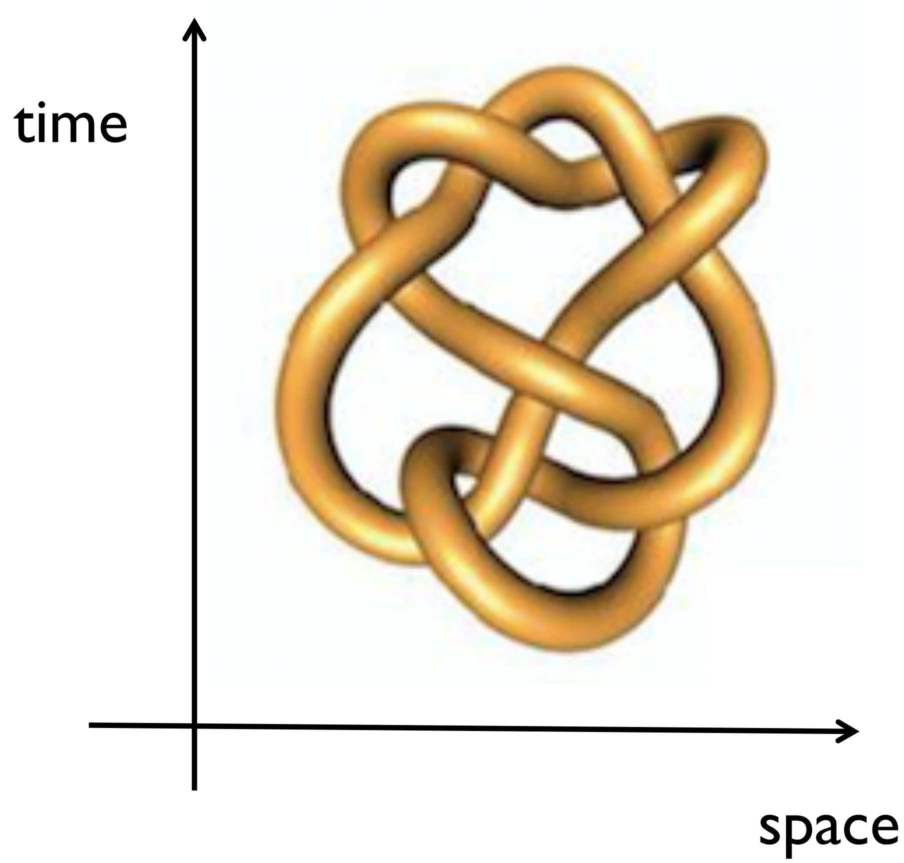
The Hall Effect and Knot Invariants

A question in mathematics: how do you distinguish different types of knots?



The Hall Effect and Knot Invariants

An answer from physics: view this as the worldline of particles in a quantum Hall system



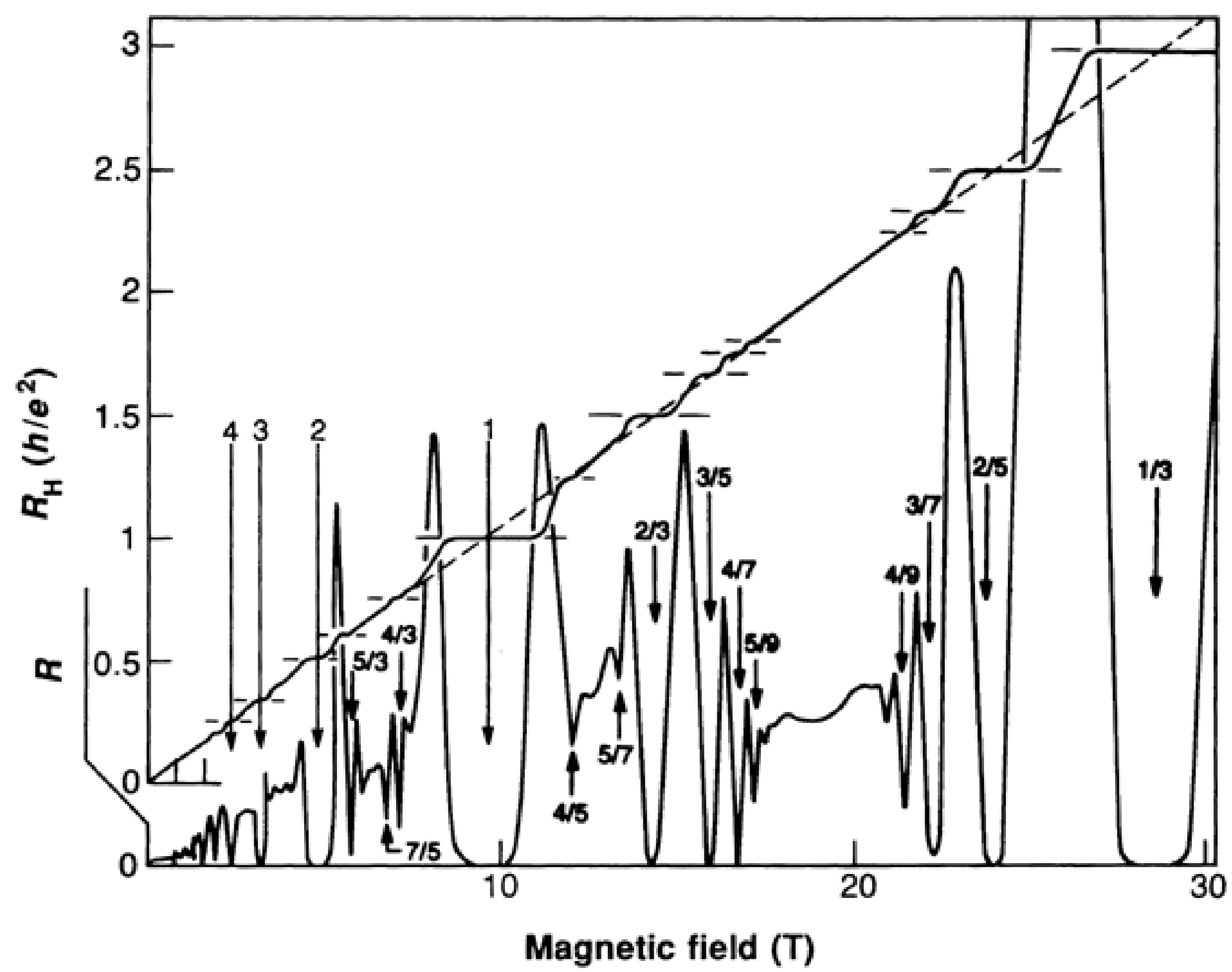
Think of this as particles
and anti-particles appearing
and disappearing.

The quantum probability for this to happen is the knot invariant

Witten's 1990 Fields medal

Summary

There's a lot hiding in this picture!



The End