The Quantum Hall Effect

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(And why these three guys won last week’s Nobel prize)
Electron in a Magnetic Field

\[ m\ddot{x} = -e\dot{x} \times B \]

\[
\begin{align*}
x &= \frac{v}{\omega} \cos \omega t \\
y &= -\frac{v}{\omega} \sin \omega t \\
\omega &= \frac{eB}{m}
\end{align*}
\]
The Classical Hall Effect

\[ m \frac{dv}{dt} = -eE - ev \times B - \frac{mv}{\tau} \]

friction = resistance

applied electric field in x-direction
magnetic field in z-direction

In equilibrium, we solve for velocity \( v \). The solution takes the form

\[ v = \sigma E \]

with \( \sigma \) a 2x2 matrix called the conductivity.
The Classical Hall Effect

We usually plot the resistivity matrix

\[
\rho = \sigma^{-1} = \begin{pmatrix}
\rho_{xx} & \rho_{xy} \\
-\rho_{xy} & \rho_{yy}
\end{pmatrix}
\]

The classical calculation above tells us how the resistivity should change with \( B \)
Integers Quantum Hall Effect

The first experiments exploring the quantum regime of the Hall effect were performed in 1980 by von Klitzing, using samples prepared by Dorda and Pepper. The resistivities look like this:

This is the integer quantum Hall effect. For this, von Klitzing was awarded the 1985 Nobel prize.

Both the Hall resistivity $\rho_{xy}$ and the longitudinal resistivity $\rho_{xx}$ exhibit interesting behaviour. Perhaps the most striking feature in the data is that the Hall resistivity $\rho_{xy}$ sits on a plateau for a range of magnetic field, before jumping suddenly to the next plateau. On these plateau, the resistivity takes the value

$$\rho_{xy} = \frac{2\pi \hbar}{e^2} \frac{1}{\nu} \quad \nu \in \mathbb{Z}$$

The value of $\nu$ is measured to be an integer to an extraordinary accuracy — something like one part in $10^9$. The quantity $\frac{2\pi \hbar}{e^2}$ is called the quantum of resistivity (with $e$, the electron charge). It is now used as the standard for measuring of resistivity.

Moreover, the integer quantum Hall effect is now used as the basis for measuring the ratio of fundamental constants $\frac{2\pi \hbar}{e^2}$ sometimes referred to as the von Klitzing constant. This means that, by definition, the $\nu = 1$ state is exactly an integer!

The centre of each of these plateaux occurs when the magnetic field takes the value

$$B = \frac{2\pi \hbar}{\nu e} = \nu B_0$$

von Klitzing, Dorda and Pepper, 1981. (Nobel prize 1985)
The Fractional Quantum Hall Effect

The value of \( \nu \) sits on a plateau for a range of magnetic field, before jumping suddenly to the next.

As the disorder is decreased, the integer Hall plateaux become less prominent. But not all fractions appear. The most prominent plateaux sit at \( 1/2 \) and \( 2/3 \).

\[ \rho_{xy} = \frac{2\pi \hbar}{e^2} \frac{1}{\nu} \quad \nu \in \mathbb{Q} \]

Tsui, Stormer and Gossard, 1982. (Nobel prize with Laughlin 1998)
A rough explanation:

Particles go in circles but circles must be compatible with de Broglie wavelength.

This is allowed…

…but this is not

The energy of the particle depends on how many wavelengths sit in its orbit.
**Integer Quantum Hall Effect**

\[ E_n = \hbar \omega \left( n + \frac{1}{2} \right) \quad \omega = \frac{eB}{m} \]

\[ \begin{align*}
    n=0 & \quad & n=1 & \quad & n=2 & \quad & n=3 & \quad & n=4 \\
    \text{Circle} & \quad & \text{Doubly Connected} & \quad & \text{Triply Connected} & \quad & \text{Doubly Connected} & \quad & \text{Quadruply Connected}
\end{align*} \]

But we can place these orbits anywhere on the plane.
Understanding the Integer Quantum Hall Effect

The \( n=0 \) orbits can sit in many different positions.

The total number of states with energy \( E_n \) is

\[
\mathcal{N} = \frac{AB}{\Phi_0}
\]

\( \Phi_0 = \frac{2\pi\hbar}{e} \)

The energy levels \( E_n = \hbar\omega \left( n + \frac{1}{2} \right) \) with this degeneracy are called Landau levels.
Integer Quantum Hall Effect

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The centre of each of these plateaux occurs when the magnetic field takes the value

$$B = \frac{2\pi \hbar}{e} \frac{1}{\nu}$$


$\nu \in \mathbb{Z}$ is counting the number of fully filled Landau levels

A full explanation of why the plateau exists needs disorder
An Aside: Topology in Physics

The Nobel Prize for Topology in Physics

David Thouless
Duncan Haldane
Michael Kosterlitz
An Aside: Topology in Physics

Here’s a different way of thinking about the integer quantum Hall effect. First some basic quantum mechanics

Space is periodic: ▶️ Momentum is discrete

$p = \frac{\hbar n}{R}$

But the converse is also true

Space is discrete ▶️ Momentum is periodic

$p \in \left[ -\frac{\hbar \pi}{a}, \frac{\hbar \pi}{a} \right]$ 

A periodic momentum space is called the **Brillouin zone**
An Aside: Topology in Physics

An electron in a material lives on a two-dimensional lattice

Its momentum lives on a two-dimensional torus

• Each point on this torus is a state of the electron, described by a wavefunction $\psi(p)$

• This wavefunction has a complex phase

• This phase can “wind” as we go around the torus.
An Aside: Topology and Physics

The TKNN formula (where T = Thouless) relates this winding to the Hall conductivity

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} C$$

First Chern number
An Aside: Topology and Physics

This idea lay almost-dormant for around 30 years!

- Until topological insulators were discovered in 2007
- No magnetic fields in sight, but the same idea of winding around the Brillouin zone
- Wonderful things happen on their surface
Back to…the Fractional Quantum Effect

\[ \rho_{xy} = \frac{2\pi\hbar}{e^2} \frac{1}{\nu} \]

Now \( \nu \) is the filling fraction of the lowest Landau level

e.g. \( \nu = 1/3 \)

What picks the correct choice of filled states?.......Interactions!
The Fractional Quantum Effect

Solving problems with many interacting electrons is hard

Laughlin wavefunction: \[ \psi(z_i) = \prod_{i<j} (z_i - z_j)^3 e^{-\sum_{i=1}^{n} |z_i|^2/4l_B^2} \]

This describes a liquid of electrons

\[ z = x + iy \]

\[ l_B = \sqrt{\frac{\hbar}{eB}} \]

* hard = no one knows how to do it.
The Fractional Quantum Hall Effect

Laughlin’s wavefunction predicts many surprising things. Here is the most startling

The excitations of the $\nu=1/3$ quantum Hall state have charge

$$q = \pm \frac{e}{3}$$

The indivisible electron has split into three pieces!

Moreover, this particle is neither boson nor fermion…it is an anyon.
Non-Abelian Quantum Hall States

The is one more stage in the quantum Hall story…

The most interesting physics is in the excitations above the ground state

The $N$ excitations do not have a unique state. The number of states is $2^{N/2}$
Global Properties of Non-Abelian Anyons

- $2^{N/2}$ states is a strange number
- If each particle had two different states (e.g. spin up/down), we would get $2^N$

The state is a global property of the system. If we only have access to a subset of the system, there’s no way of telling which state we’re in.

This makes these quantum states robust…
Topological Quantum Computing

We describe the state by a $2^{N/2}$ dimensional vector $\psi$

Now move the particles around on some path:

$$\psi \mapsto U_{\text{path}} \psi$$

with $U_{\text{path}}$ a unitary matrix that depends on the path taken

These particles are called non-Abelian anyons.

This allows us to do calculations in a quantum computer without errors!
The Hall Effect and Knot Invariants

A question in mathematics: how do you distinguish different types of knots?
The Hall Effect and Knot Invariants

An answer from physics: view this as the worldline of particles in a quantum Hall system

Think of this as particles and anti-particles appearing and disappearing.

The quantum probability for this to happen is the knot invariant

Witten’s 1990 Fields medal
Summary

There's a lot hiding in this picture!
The End