Vortices, Strings, and Vortex Strings

David Tong
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge,
CB3 OWA, UK
d.tong@damtp.cam.ac.uk

ABSTRACT

We describe recent progress in understanding the classical and quantum dynamics of vortices in supersymmetric gauge theories. We show how D-branes in string theory can be used to provide a new description of the dynamics of the vortex moduli space. We go on to use this to study the quantum theory of vortex strings in four-dimensional gauge theories. We show how the string knows about the exact quantum spectrum of the strongly coupled gauge theory in which it's embedded.

1. Introduction

I'm going to talk about vortices. The motivation for studying vortices should be obvious: they are one of the most ubiquitous objects in physics. On tabletops, gauge theoretic vortices appear as magnetic flux tubes in superconductors and fractionally charged quasi-excitations in quantum Hall fluids. In the sky, vortices in the guise of cosmic strings have been one of the most enduring themes in cosmology research. There is hope that with gravitational wave detectors now on-line, we may be able to see the distinctive signatures of these strings as they twist and snap into cusps. Finally, and more formally, vortices play a crucial role in determining the phases of low-dimensional quantum field theories: from the Kosterlitz-Thouless transition in condensed matter systems, to mirror symmetry of Calabi-Yau manifolds in string theory, the vortex is key.

In this talk I would like to discuss several new aspects of vortex physics. Specifically, I will show:

- How to use string theory to learn about vortices.
- How to use vortices to learn about four-dimensional gauge dynamics.

Let me start by telling you the punchline. We will study a four-dimensional non-abelian gauge theory that admits vortices. In four dimensions, vortices are string-like objects and we'll study the low energy excitations of a straight, infinitely long vortex string. Of course, a string always has trivial fluctuations corresponding to motion in two transverse directions. But, in our case, it will turn out that the vortex string also has a much richer spectrum of internal modes arising from turning on various Higgs-like fields in the vortex core. As we will show in some detail, these modes are described by a variant of the $\mathbf{CP}^N$ sigma-model. At this stage we quantise the low-energy modes of this string – in other words, we quantise the $d = 1 + 1$ dimensional sigma-model on the string worldsheet.

The punchline of this talk is that the spectrum of this $d = 1 + 1$ dimensional theory on the string is *identical* to the spectrum of the $d = 3 + 1$ dimensional non-abelian gauge theory in which the string is embedded. This statement holds exactly, at the quantum level. It holds in weak-coupling and strong-coupling regimes. It holds for both elementary excitations and solitons, such as monopoles, of the theory.

Before we get to this punchline however, we have a journey ahead of us. We start in the following section by introducing the vortices and some of their properties. In section 3, we show how to use D-branes in string theory to provide a description of the low-energy dynamics of vortices. In section 4, we discuss a relationship between vortices and confined magnetic monopoles, while in section 5, we reveal a similar relationship between vortices and Yang-Mils instantons. Finally, in section 6 we explain how the correspondence between the theories in two and four dimensions arises.

The research described in this talk was done in collaboration with Ami Hanany. Full details can be found in the original papers [1,2,3].

2. Vortices

The theory we will consider in $d = 3+1$ dimensions has $\mathcal{N} = 2$ supersymmetry and lies in the class of theories whose low-energy quantum dynamics was solved by Seiberg and Witten [4]. However, we will focus our attention to classical aspects of the dynamics until section 6. The specific theory of interest has a $U(N_c)$ gauge $N_f = N_c \equiv N$ fundamental flavours[a]. These theories have a plethora of fields – both bosons and fermions – but, for the purpose of discussing vortices, we may restrict attention to only two: the gauge field $(A_\mu)^a{}_b$ and the fundamental scalars fields $q^a{}_i$, where $a = 1, \ldots, N_c$ is the colour index and $i = 1, \ldots, N_f$ is the flavour index. Truncating the Lagrangian to these fields, we have simply

$$-\mathcal{L} = \frac{1}{4e^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N_f}|\mathcal{D}q|^2 + \frac{e^2}{2}\mathrm{Tr}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2)^2 \tag{1}$$

where the final term is the supersymmetric D-term, containing the Higgs vev v^2 which is to be thought of as multiplying the $N_c \times N_c$ unit matrix. The theory has a unique vacuum state $q^a{}_i = v\delta^a{}_i$ which exhibits a classical mass gap at the scale ev. In the vacuum the $U(N_c)$ gauge group and the $SU(N_f)$ flavour group are broken by

$$U(N_c) \times SU(N_f) \to SU(N)_{\mathrm{diag}} \tag{2}$$

and the theory lives in the colour-flavour locked phase. The existence of the surviving $SU(N)_{\mathrm{diag}}$ symmetry will prove important later. Note also that the overall $U(1)$ gauge symmetry is broken. This supplies us with the topology $(\Pi_1(U(1)) \cong \mathbf{Z})$ necessary to

[a]Generalisations of our story hold for $N_f > N_c$ and can be found in the original papers.

build vortices. As often happens in supersymmetric theories, in the soliton sector the equations of motion can be integrated once to give the first order vortex equations,

$$\begin{aligned} (B_3)^a{}_b &= e^2\left(\sum_{i=1}^{N_f} q^a{}_i q^\dagger_{ib} - v^2\delta^a{}_b\right) \\ (\mathcal{D}_z q_i)^a &= 0 \end{aligned} \tag{3}$$

which describe a vortex string oriented in the x^3 direction, with the transverse plane parameterised by $z = x^1 + ix^2$. For $N = 1$, these are the well-studied Nielsen-Olesen vortices of the abelian-Higgs model. For $N > 1$, these equations describe simple non-abelian generalisation of this model. Solutions to these equations have tension $T = v^2\mathrm{Tr}\int dx^1dx^2\, B_3 = 2\pi v^2 k$ where $k \in \mathbf{Z}$ is interpreted as the vortex number. The fact that the tension is proportional to k implies that there is no force between far separated vortices, a consequence of supersymmetry.

The rest of this talk will be devoted to exploring properties of these simple vortex equations (3). For example, one surprising property is that no solution is known! This ignorance holds even for a single $k = 1$ vortex in $U(1)$ gauge theory. Of course, this is no big hurdle - one can easily find solutions numerically; but the solution is not expressible in terms of elementary functions. However, one can do better than numerics. It was proven by Taubes that solutions to these equations exist [5]. The most general solution depends on several parameters, giving rise to the concept of the *vortex moduli space* $\mathcal{V}_{k,N}$. This is the space of solutions to the equations (3) for fixed k and N. In [1] it was shown that the number of parameters of a general solution is

$$\dim(\mathcal{V}_{k,N}) = 2kN \tag{4}$$

Let's examine the physical meaning of these parameters. The k in (4) is clear: we have k vortices, each with $2N$ parameters. For example, in the abelian model with $N = 1$, the two parameters are simply the position of the vortex in the (x^1, x^2) plane. But what about for non-abelian gauge groups? In fact, it is trivial to construct solutions to the $U(N)$ theory from the $U(1)$ theory. Suppose that $B_\star$ and $q_\star$ are solutions to the abelian-Higgs model. Then we may embed them in the $N \times N$ matrices by placing them, say, in the upper-left corner:

$$B = \begin{pmatrix} B_\star & & & \\ & 0 & & \\ & & \ldots & \\ & & & 0 \end{pmatrix} \quad , \quad q = \begin{pmatrix} q_\star & & & \\ & v & & \\ & & \ldots & \\ & & & v \end{pmatrix} \tag{5}$$

But there are other embeddings as well. We could act on this solution with the $SU(N)_{\mathrm{diag}}$ vacuum symmetry (2) to generate new solutions. This gives rise to a moduli space of solutions coming from the coset

$$SU(N)_{\mathrm{diag}}/SU(N-1) \times U(1) \cong \mathbf{CP}^{N-1} \tag{6}$$

where we've divided by the stabiliser of the action. For a single vortex, $k=1$, all the further zero modes come from this symmetry action and we have the moduli space [1,6]

$$\mathcal{V}_{1,N} \cong \mathbf{C} \times \mathbf{CP}^{N-1} \tag{7}$$

where $\mathbf{C}$ parameterises the center-of-mass of the vortex and $\mathbf{CP}^{N-1}$ parameterises its orientation in the gauge and flavour groups. For $k>1$ far separated vortices, the moduli space should be asymptotically the symmetric product of $\mathcal{V}_{1,N}$. The question is, what is the description of the moduli space as the vortices approach? We shall provide an answer to this question in the following section.

Let me mention another important property of the vortex moduli space: its metric. It was shown by Manton that the moduli spaces of solitons inherit a natural metric from the parent theory in which they're embedded [7]. The geodesics of this metric track the classical scattering of the solitons. For vortices, this metric is smooth, Kähler, and unknown beyond the asymptotic regime [8]. Determining the full metric on the vortex moduli space appears to be a difficult problem and I shall have little to say about it here. However, it is important to stress that it is quite possible to make progress in understanding the quantum properties of vortices without knowing the full metric. The important point is that if one restricts attention to particular "holomorphic" questions, then the answers depend only on the topology of the vortex moduli space (combined, perhaps, with some information about its asymptotic behaviour). For example, the instanton contributions to chiral operators in $d=1+1$ dimensional theories, and the BPS bound states of vortices in $d=2+1$ dimensional theories fall into this category. We shall see in Section 6 that similar questions exist for vortex strings in $d=3+1$ dimensional theories as well.

2.1. Turning on Masses

There exists a simple deformation of the theory (1) in which we give bare masses to the N_f fundamental flavours. This has an interesting effect on the vortex solutions which will be important for our story.

While in supersymmetric theories, the most general mass parameter is complex, for simplicity let us chose to make the masses real: $m_i = m_i^\dagger$. To describe the resulting Lagrangian, we need to include a field which we neglected in (1): this is the adjoint scalar field ϕ which lives in the vector multiplet with A_μ. By choosing real masses, we may also restrict attention to a real ϕ. The Lagrangian is:

$$-\mathcal{L} = \frac{1}{2e^2}\mathrm{Tr}(\tfrac{1}{2}F^2 + (\mathcal{D}\phi)^2) + \sum_{i=1}^{N_f} |\mathcal{D}q|^2 + \frac{e^2}{2}\mathrm{Tr}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2)^2 + \frac{1}{2}\sum_{i=1}^{N_f} q_i^\dagger(\phi - m_i)^2 q_i \tag{8}$$

We can see from the last term why we need to include ϕ since mixes with the masses m_i. The theory once again has a unique vacuum state: $q^a{}_i = v\delta^a{}_i$ as before, but this is now supplemented by $\phi = \mathrm{diag}(m_1,\ldots,m_{N_c})$. Importantly though, the symmetry breaking structure in the vacuum differs from (6) since the masses explicitly break the flavour group. We now have

$$U(N_c) \times SU(N_f) \to U(N_c) \times U(1)^{N_f-1} \to U(1)^{N-1}_{\rm diag} \tag{9}$$

where the first breaking is explicit due to the masses m_i, and the second breaking is spontaneous due to the vev v^2. Since the overall $U(1)$ remains broken, we still have vortices. The question is: how does the presence of the masses m_i change the allowed vortex solutions? The answer was given in [2]; since we have seen that some of the vortex zero modes are generated by acting with the symmetry (6) which is no longer available when masses are turned on, we certainly expect that much of the vortex moduli space is lifted. In fact, the only surviving solutions to (3) are those which satisfy $(\phi - m_i)q_i = 0$ when evaluated on the solution q_i with ϕ in its vacuum. These are simply the abelian vortex solutions embedded diagonally within the gauge group,

$$(B_3)^a{}_b = B^a_\star \delta^a{}_b \quad , \quad q^a{}_i = q^a_\star \delta^a{}_{i=\sigma(a)} \tag{10}$$

(no sum over a), where each pair $\{B^a_\star, q^a_\star\}$ for $a = 1, \ldots, N_c$ solves the abelian vortex equations. In this way, the single topological quantum number k gets split into N_c distinct quantum numbers. For example, there exist N_c different vortices with winding number $k = 1$, distinguished by the diagonal element of B that carries the flux and by the flavour q_i that winds around the string (the latter is a Weyl invariant statement). From the perspective of the vortex moduli space, the masses m_i can be thought of as inducing a potential whose zeroes correspond to the surviving solutions. This potential can be shown to take a simple geometrical form, being proportional to the length of a particular Killing vector on $\mathcal{V}_{k,N}$ [3]. Thus the low-energy dynamics of the vortex string is described by a massive sigma-model with target space $\mathcal{V}_{k,N}$. In the following section, we shall give an explicit description of this vortex theory.

3. Branes

To develop a useful description of the vortex moduli space, we turn to string theory and D-branes. The technique we use is one developed early in the D-brane game [9]. Firstly we engineer our theory (1) on the worldvolume of D-branes. We then identify the vortices within this setup which appear as additional D-branes. Finally we read off the dynamics of the vortices in terms of a gauge theory living on these new branes. This approach has been applied successfully in the past for Yang-Mills instantons [9] and monopoles [10] where it yields the ADHM [11] and Nahm [12] descriptions of these objects respectively. The application of D-branes to vortices was explained in [1] and results in a new description of the vortex moduli space as we shall now review.

To build our theory on D-branes we work in type IIA string theory and use the well-known construction of $\mathcal{N} = 2$ theories in $d = 3+1$ dimensions realised on the worldvolume of N_c D4-branes suspended between two NS5-branes [13]. A further $N_f = N_c$ D6-branes

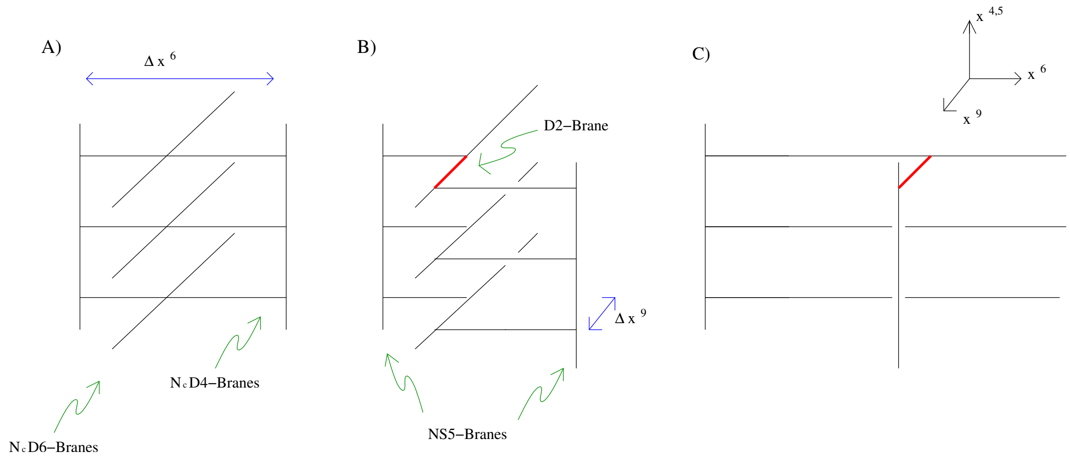

Figure 1: The type IIA brane set-up.

give rise to hypermultiplets coming from $4-6$ strings. The final configuration is drawn in Figure 1. The spatial worldvolume directions of the branes are

$$\begin{aligned} NS5: &\quad 12345 \\ D4: &\quad 1236 \\ D6: &\quad 123789 \\ D2: &\quad 39 \end{aligned}$$

The gauge coupling e^2 and FI parameter v^2 are encoded in the separation Δx of the two NS5-branes,

$$\frac{1}{e^2} = \frac{\Delta x^6}{(2\pi)^2 g_s l_s} \quad , \quad v^2 = \frac{\Delta x^9}{(2\pi)^3 g_s l_s^3} \tag{11}$$

where g_s and l_s are the string coupling and string length respectively. The hypermultiplet masses and the vacuum expectation value of $\phi = \mathrm{diag}(\phi_1, \ldots, \phi_{N_c})$ are encoded in the x^4 and x^5 positions of the D-branes

$$m_i = \left.\frac{x^4 + ix^5}{l_s^2}\right|_{D6_i} \quad , \quad \phi_i = \left.\frac{x^4 + ix^5}{l_s^2}\right|_{D4_i} \tag{12}$$

In Figure 1A) we draw the brane configuration corresponding to the four dimensional theory with $v^2 = 0$ at the point $\phi = \mathrm{diag}(m_1, \ldots, m_{N_c})$. In Figure 1B), we have turned on the FI parameter v^2 by moving the right-hand NS5-brane out of the page in the x^9 direction. At this stage, we also depict the vortex string; it appears as a D2-brane stretched the distance Δx^9 between the NS5-brane and the D3-brane. We have drawn a single D2-brane attached to the upper D4-brane, corresponding to a vortex string with magnetic flux in $B = \mathrm{diag}(1, 0, \ldots, 0)$. It is clear from the brane picture that there exist a further $N_c - 1$ inequivalent vortex configurations in which the D2-brane is attached to one of the other D4-branes, in agreement with our discussion in section 2. In general a k vortex configuration contains k D2-branes distributed between the N_c D4-branes.

To understand the dynamics of vortices, we need to read off the the gauge theory living on the D2-branes. To facilitate this, we first manipulate the branes a little. The field theory cares nothing for the x^6 position of the D6-branes and we may freely move them in this direction. There is one caveat however: they have non-zero linking number with the NS5-branes which ensures that D4-branes are created or destroyed if the two pass through each other [14]. We choose to move the D6-branes to the right. When they pass through the right-hand NS5-brane, the connecting D4-branes disappear by flux conservation and the D6-branes are now attached only to the left-hand NS5-brane. After moving the D6-branes to $x^6 \to \infty$, the resulting configuration is shown in Figure 1C. From this we may read off the gauge theory on the D2-brane using the rules given in [15]:

Vortex Theory: The theory describing the vortex string is [1]: $d = 1+1$ dimensional, $\mathcal{N} = (2,2)$ supersymmetric $U(k)$ gauge theory with a single adjoint chiral multiplet Z and N_c fundamental chiral multiplets ψ_i. It has a gauge coupling constant g^2 and the FI parameter r while the fundamental chiral multiplets have (twisted) masses m_i

Let us denote the $U(k)$ gauge field on the vortex string as G_{03}. The $\mathcal{N} = (2,2)$ vector multiplet also contain a complex adjoint scalar χ. Since we have taken the twisted masses to be real, we may also restrict to real χ. Then the Lagrangian describing the low-energy dynamics of the vortex string is

$$\begin{aligned}\mathcal{L}_{\rm vortex} &= \frac{1}{2g^2}\mathrm{Tr}\,(G_{03}^2 + (\mathcal{D}\chi)^2) + \sum_{i=1}^{N_c}(\mathcal{D}\psi_i)^2 + \mathrm{Tr}\,(\mathcal{D}Z)^2 \\ &\quad -\frac{g^2}{2}\mathrm{Tr}\,(\sum_{i=1}^{N_c}\psi_i\psi_i^\dagger - [Z,Z^\dagger] - r)^2 - \sum_{i=1}^{N_c}\psi_i^\dagger(\chi - m_i)^2\psi_i \end{aligned} \tag{13}$$

The parameters g^2 and r are both determined by the separation of the NS5-branes: $1/g^2 = \Delta x^9 l_s/g_s$ and $r = \Delta x^6/2\pi g_s l_s$. As explained in [1], taking the decoupling limit of the four-dimensional gauge theory from the full string dynamics translates to the requirement that $g^2 \to \infty$, meaning that the gauge field and the scalar χ are auxiliary and the D-term constraint $\psi\psi^\dagger - [Z, Z^\dagger] = r$ is imposed absolutely. Comparing with (11), we find that the FI parameter r is given in terms of the $d = 3+1$ gauge coupling [1]

$$r = \frac{2\pi}{e^2} \tag{14}$$

This description of the vortex dynamics is a little different from that described in the last section. How does the vortex moduli space $\mathcal{V}_{k,N}$ arise from (13)? The answer is that $\mathcal{V}_{k,N}$ is the vacuum moduli space of the theory when the masses m_i vanish. (This space is also referred to as the "Higgs branch" of the theory). Thus $\mathcal{V}_{k,N}$ is parameterised by gauge invariant combinations of ψ and Z, subject to the D-term constraint. This is known as the symplectic quotient construction of $\mathcal{V}_{k,N}$.

When the masses are turned on, we also have the last term in (13): $\psi^\dagger(\chi - m)^2\psi$. This can be thought of as inducing a potential on $\mathcal{V}_{k,N}$ after integrating out χ. It is a short exercise to show that the resulting potential is proportional length of a Killing vector on $\mathcal{V}_{k,N}$ (see, for example, Appendix B of [3] for the case of a single vortex). Such a potential was shown by Alvarez-Gaumé and Freedman to preserve supersymmetry [16]. We therefore find that in the limit $g^2 \rightarrow \infty$, the gauge theory (13) reduces to a massive sigma model on the vortex moduli space.

Let me illustrate this with a simple example: Consider a single $k = 1$ vortex with vanishing masses. The theory (13) is a $U(1)$ gauge theory and the Z field is neutral. The D-term is simply $\sum_i |\psi_i|^2 = r$ which, after quotienting by the $U(1)$ action $\psi_a \rightarrow \exp(i\alpha)\psi_a$ results in the Higgs branch $\mathbf{CP}^{N_c-1}$. Including the neutral field Z, the vacuum moduli space of the theory is given by $\mathcal{M}_{\rm Higgs} \cong \mathbf{C}\times\mathbf{CP}^{N-1}$ in agreement with the vortex moduli space (7). In contrast, when the masses are turned on, the $\mathbf{CP}^{N-1}$ part of the vacuum moduli space of the vortex theory is lifted completely, leaving behind N_c discrete vacuum states given by $\chi = m_i$ and $|\psi_j|^2 - r\delta_{ij}$. These correspond to the N_c different vortex strings described above.

In general, the construction theory (13) gives a correct description of the topology, the asymptotic form, and the symmetry structure of $\mathcal{V}_{k,N}$. (At least in certain cases, it also gives the correct Kähler class – it is not known if this holds for all k and N). The gauge theory (13) also defines a natural metric on the Higgs branch; this does *not* in general coincide with the Manton metric on $\mathcal{V}_{k,N}$. One exception is in the case of a single vortex, when the symmetry structure results in a unique round metric on $\mathbf{CP}^{N-1}$ known as the Fubini-Study metric.

4. Monopoles

It was shown in [2] that the theory (8) contains BPS monopoles. At first glance, this may be somewhat surprising: the theory has a mass gap and there is no photon to carry away the flux of a 't Hooft-Polyakov monopole. Nevertheless, topologically stable, BPS monopoles do exist, with their flux carried away in collimated tubes. In other words, they are confined. In this section we describe the this confined monopole and review the results of [2].

The confined monopole is a 1/4-BPS object. The first order Bogomoln'yi equations which result in solutions to the equations of motion of (8) are

$$\begin{aligned} B_1 = \mathcal{D}_1\phi \quad , \quad B_2 = \mathcal{D}_2\phi \quad , \quad B_3 = \mathcal{D}_3\phi + e^2(\sum_{i=1}^{N} q_i q_i^\dagger - v^2) \\ \mathcal{D}_1 q_i = i\mathcal{D}_2 q_i \quad , \quad \mathcal{D}_3 q_i = -(\phi - m_i)q_i \end{aligned} \tag{15}$$

While no explicit solutions to these equations are known, several properties can be deduced by examining the configuration in various limits [2]. The solutions describe a magnetic monopole emitting two flux tubes in the x^3 direction as sketched in Figure 2. Far from the monopole, these flux tubes are simply the vortex strings we discussed in Section 2. The monopole provides a way to join together two different vortex strings.

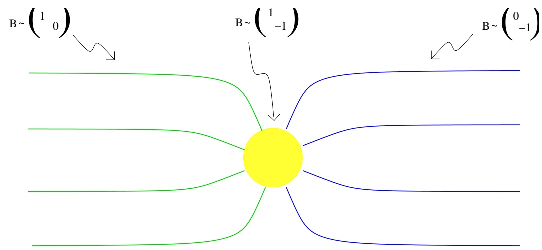

Figure 2: A magnetic monopole emitting two flux tubes.

The energy of the entire configuration is infinite due to the presence of the two semi-infinite vortex strings. This is simply the statement that the monopole is confined. However, it still makes sense to talk about the finite mass $M_{\rm mono}$ of the monopole as an excitation over the energy of a single infinite, straight string. As in the case of a free 't Hooft-Polyakov monopole in the Coulomb phase, the mass of the monopole is given by

$$M_{\rm mono} = \frac{1}{e^2}{\rm Tr}\int d^3x\ \partial\cdot(\phi B) \tag{16}$$

Recall that in the Coulomb phase the magnetic flux of the monopole escapes radially to infinity and is captured by the integral evaluated on the S^2_∞ boundary. In the present case, the magnetic flux does not make it to all points on the boundary, but is confined to two flux tubes which stretch in the $\pm x^3$ direction. Correspondingly, the integral should now be evaluated over two planes $\mathbf{R}^2_{\pm\infty}$ at $x^3 = \pm\infty$. Nevertheless, both integrals yield the same result since they both pick up all of the flux.

As an example, consider the case of the $U(2)$ gauge theory, and set $m_1 = -m_2 = m$. In Section 2, we saw that the theory contains two different vortex strings. The first is supported by the winding of q_1 and has $B \sim {\rm diag}(1,0)$; the second is supported by the winding of q_2 and has $B \sim {\rm diag}(0,1)$. The magnetic monopole is an object which changes one of these objects to the other. This means it must be a sink for magnetic flux $B \sim {\rm diag}(0,1)$, and a source for flux $B \sim {\rm diag}(1,0)$. Flipping the minus signs, we see it carries magnetic $B \sim {\rm diag}(1,-1)$ which lies in the $SU(2) \subset U(2)$. This is precisely the magnetic quantum numbers of the 't Hooft-Polyakov monopole in the Coulomb phase. From (16), the mass of this object is given by,

$$M_{\rm mono} = 2\pi\frac{(\phi_1-\phi_2)}{e^2} = \frac{4\pi m}{e^2} \tag{17}$$

where we have used the fact that the theory lies in the vacuum $\phi = {\rm diag}(m,-m)$. This is precisely the mass of the 't Hooft-Polyakov monopole in the Coulomb phase. Note that the existence of the confined monopole is guaranteed by the topology of the symmetry breaking of the theory. However, the fact that the mass of the monopole in the Higgs phase (as measured as an excitation over the infinite vortex string) is identical to that in the Coulomb phase can be traced to the existence of $\mathcal{N}=2$ supersymmetry (the monopole is a BPS state) and will not hold in more general theories.

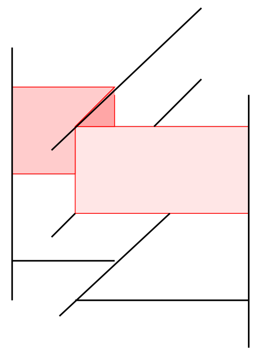

Figure 3: The confined monopole in the type IIA setup.

Finally, it is simple to understand the confined monopole from the D-brane picture of Section 2. We consider a D2-brane worldvolume which starts attached to the upper D4-brane at $x^3 \to -\infty$, and then interpolates down to the middle D4-brane as $x^3 \to +\infty$. At intermediate steps, the D2-brane cannot simply be a line stretching distance Δx^9 as drawn in Figure 1B) since it has no where to end. The only possibility is that the D2-brane bends in the x^6 direction to attach itself to the NS5-branes. The final configuration is drawn in Figure 3. Note that as $v^2 \to 0$, the two NS5-branes align and the configuration returns to that describing a 't Hooft-Polyakov monopole in the Coulomb phase.

4.1. The View from the String

It is interesting to ask how the confined monopole looks from the worldsheet of the vortex string. In the previous section, we saw from a D-brane construction that the low-energy dynamics of a single string is described by a $U(1)$ gauge theory with Lagrangian given by (13). This theory has N_c ground states:

$$\chi = m_i \quad , \quad |\psi_j|^2 = r\delta_{ij} \tag{18}$$

corresponding to the N_c different vortex strings that exist in the theory. We thus have $d = 1+1$ dimensional theory on the string worldsheet with a set of isolated, discrete vacua. This immediately guarantees us that we have a new object: a kink on the string. This kink is the confined monopole [2]. From the perspective of the $d = 1+1$ dimensional theory (13), the kink is a BPS object. For a single string, the Bogomoln'yi equations for the kink are

$$\partial_3\chi = g^2\left(\sum_{i=1}^{N_c}\psi_i\psi_i^\dagger - [Z, Z^\dagger] - r\right) \quad , \quad \mathcal{D}_3\psi_i = (\chi - m_i)\psi_i \tag{19}$$

subject to two different boundary conditions (18), say $\chi = m_i$ at left infinity and $\chi = m_j$ at right infinity. The mass of the kink is given by

$$M_{\rm kink} = r(\chi_{+\infty} - \chi_{-\infty}) = r(m_j - m_i) \tag{20}$$

Since the kink interpolates between two different vortex flux tubes, it is clear that it carries the same quantum numbers as the monopole. It is also instructive to examine the mass of the kink $M_{\rm kink}$ in more detail. To compare with the monopole, let us consider the $U(2)$ gauge theory in $d = 3 + 1$ dimensions with $m_1 = -m_2 = m$. The theory on the vortex string is a $U(1)$ gauge theory with two charged fields ψ_1 and ψ_2 and two ground states. The mass of the kink is $M_{\rm kink} = 2rm$. Using the formula for the FI parameter (14) and comparing to (17), we have the important formula

$$M_{\rm kink} = M_{\rm mono}$$

While we have derived this for monopole in the $U(2)$ theory, it also holds for the richer spectrum of monopoles in higher rank gauge groups. More surprisingly, as we shall review in section 6, it also continues to hold in the full quantum theory.

5. Instantons

We have seen that the kink in the vortex string is interpreted as the confined monopole in $d = 3+1$ dimensions lying in the Higgs phase. When defined on a Euclidean worldsheet, the theory on the vortex string (13) itself admits BPS vortices with action $2\pi r = 4\pi^2/e^2$. What is the interpretation of this "vortex in a vortex string" in the $d = 3+1$ dimensional bulk? These objects are Yang-Mills instantons, trapped inside the vortex string [3].

This correspondence between vortices on the vortex worldsheet and Yang-Mills instantons can be thought of as the underlying reason for a related correspondence between their moduli spaces[b] discovered in [1]. Put simply, the moduli space of vortices is half of the moduli space of instantons.

To make a more precise statement, let us remind ourselves about the moduli space $\mathcal{I}_{k,N}$ of k Yang-Mills instantons in a $U(N)$ gauge theory. $\mathcal{I}_{k,N}$ is a hyperKähler manifold of real dimension $\dim(\mathcal{I}_{k,N}) = 4kN$. Note that this is twice the dimension of the vortex moduli space (4). A description of the instanton moduli space in terms of D-branes [9] results in the ADHM gauge theory whose Higgs branch is $\mathcal{I}_{k,N}$. The theory describing the instanton moduli space $\mathcal{I}_{k,N}$ is

Instanton Theory: $\mathcal{N} = (4,4)$ supersymmetric $U(k)$ gauge theory with a single adjoint hypermultiplet and N_c fundamental hypermultiplets.

Compare with the theory we found in Section 3 describing the vortex moduli space $\mathcal{V}_{k,N}$,

Vortex Theory: $\mathcal{N} = (2,2)$ supersymmetric $U(k)$ gauge theory with a single adjoint chiral multiplet and N_c fundamental chiral multiplets.

[b]To make this argument concrete we would need to discuss vortices in theories with $N_f > N_c$. Details can be found in [1,3].

If we recall that a hypermultiplet contains two chiral multiplets, our statement above becomes clearer: the vortex gauge theory is precisely half of the instanton gauge theory. We can also make the statement at the level of the moduli spaces without recourse to the auxiliary gauge theories. We start with the resolved moduli space of instantons $\mathcal{I}_{k,N}$. Consider rotating the instanton solutions in a given plane; this generates new solutions and so induces a $U(1)$ action on $\mathcal{I}_{k,N}$. Since rotation is a symmetry, this action is described by a Killing vector h on $\mathcal{I}_{k,N}$. The exact result of [1] is that $\mathcal{V}_{k,N}$ is a middle-dimensional, complex sub-manifold of $\mathcal{I}_{k,N}$ defined by the fixed points of this $U(1)$ action,

$$\mathcal{V}_{k,N} \cong \mathcal{I}_{k,N}|_{h=0} \tag{21}$$

Full details of this reduction can be found in [1]. But the basic idea should be clear: start with the ADHM construction of the instanton moduli space; throw away half the fields (H and M can go), and you're left with the AD (= Ami and David) construction of the vortex moduli space.

An interesting open problem remains: the ADHM data also provides a method to construct the explicit instanton solutions. No similar method is currently known for the vortices.

6. Quantum Vortex Strings

So far our discussion has been entirely classical. Now we turn to the quantum theory. We study both the $d = 3+1$ dimensional theory (8) and the $d = 1+1$ dimensional theory (13) on a single vortex string. The punchline of this analysis, indeed of this talk, is that the equation

$$M_{\rm kink} = M_{\rm mono} \tag{22}$$

continues to hold in the full quantum theory. The original observation that the spectra of the theories in $d = 1 + 1$ and $d = 3 + 1$ dimensions coincide is due to Dorey [17], with further examples found in [18]. The physical explanation for this correspondence in terms of an underlying vortex string was presented in [3], and simultaneously by Shifman and Yung in [19]. I will not present all the calculations here, but instead simply give a flavour of the results.

The first hint that the quantum theory on the vortex string knows something about the quantum dynamics of the 4d theory in which it's embedded can be seen by examining the β-functions of both theories. Recall our relationship (14) between the FI parameter of the 2d theory and the gauge coupling of the 4d theory: $r = 2\pi/e^2$. This statement about a classical property of the vortex solutions is, importantly, preserved under RG flow. The one-loop running in both theories is given by

$$r(\mu) = r_0 - \frac{N_c}{2\pi}\log\left(\frac{\mu_{UV}}{\mu}\right) \tag{23}$$

This ensures that both 4d and 2d theories hit strong coupling at the same scale $\Lambda = \mu \exp(-2\pi r/N_c)$.

Let me now stress the meaning of the equation (22). The left-hand side is computed in the $d = 1+1$ dimensional theory. When $(m_i - m_j) \gg \Lambda$, this theory is weakly coupled and $M_{\rm kink}$ receives a one-loop correction (with, obviously, two-dimensional momenta flowing in the loop). Although supersymmetry forbids higher loop corrections, there are an infinite series of worldsheet instanton contributions. The right-hand-side of (22) is computed in the $d = 3+1$ dimensional theory, which is also weakly coupled for $(m_i - m_J) \gg \Lambda$. The monopole mass $M_{\rm mono}$ receives corrections at one-loop (now integrating over four-dimensional momenta), followed by an infinite series of Yang-Mills instanton corrections. *And term by term these two series agree*!

The agreement of the worldsheet and Yang-Mills instanton expansions apparently has its microscopic origin in the results of section 5. Recall that performing an instanton computation requires integration over the moduli space ($\mathcal{V}$ for the worldsheet; $\mathcal{I}$ for Yang-Mills). As shown explicitly by Nekrasov, localization theorems hold when performing the integrals over $\mathcal{I}_{k,N}$ in $\mathcal{N} = 2$ super Yang-Mills, and the final answer contains contributions from only a finite number of points in $\mathcal{I}_{k,N}$. It is simple to check that all of these points lie on $\mathcal{V}_{k,N}$ which, as we have seen, is a submanifold of $\mathcal{I}_{k,N}$. It seems likely therefore that similar localization theorems also hold in the 2d theory. It would be interesting to put these observations on a firmer footing.

The equation (22) also holds in strong coupling regimes of the 2d and 4d theories where no perturbative expansion is available. Nevertheless, exact results allow the masses of BPS states to be computed and successfully compared. Moreover, the quantum correspondence between the masses of kinks and monopoles is not the only agreement between the two theories. Other results include:

• The elementary internal excitations of the string can be identified with W-bosons of the 4d theory. When in the bulk, away from the string, these W-bosons are non-BPS. But they can reduce their mass by taking refuge in the core of the vortex whereupon they regain their BPS status.

• The 4d theory contains dyons. The 2d theory also contains dyonic kink objects, as first noted 10 years ago by Abraham and Townsend [20]. Again, the quantum corrected masses in the two theories are identical.

• Both theories manifest the Witten effect: adding a theta angle to the 4d theory induces an electric charge on the monopole, shifting its mass [21]. It can also be shown to induce a 2d theta angle on the vortex worldsheet, and an analogue of the Witten effect holds for the dyonic kinks [17].

• We have here described the theory with $N_f = N_c$. For $N_f > N_C$, the story can be repeated and again the spectrum of the vortex string coincides with the spectrum of the 4d theory in which it's embedded [18,3].

7. Conclusions

We have known for over 20 years that gauge theories in 4d share many qualitative features with sigma models in 2d, including asymptotic freedom, a dynamically generated mass gap, large N expansions, anomalies and the presences of instantons. In this talk we have seen a *quantitative* relationship between the two: the 2d sigma model and 4d gauge theory share the same quantum spectrum. The link between them is provided by the vortex string.

It is natural to ask if these results can be extended to situations with less supersymmetry. While the agreement of the spectrum of the two theories relied heavily on supersymmetry, it would be very interesting to explore what other properties of 4d gauge theories the vortex knows about. I hope to report on this in the near future.

8. Acknowledgements

The work described here was done in collaboration with Ami Hanany. This talk was presented at the plenary session of SUSY 2004. My thanks to the organisers, and to Paddy Fox whose hotel room was the only one showing the Euro 2004 matches (complete waste of time *that* was). I would like to thank Neil Pappalardo for his support over the last few years, and the Royal Society for their support over the coming few.

9. References

[1] A. Hanany and D. Tong, "*Vortices, instantons and branes*" JHEP **0307**, 037 (2003) [arXiv:hep-th/0306150].

[2] D. Tong, "*Monopoles in the Higgs phase*" Phys. Rev. D **69**, 065003 (2004) [arXiv:hep-th/0307302].

[3] A. Hanany and D. Tong, "*Vortex strings and four-dimensional gauge dynamics*," JHEP **0404**, 066 (2004) [arXiv:hep-th/0403158].

[4] N. Seiberg and E. Witten, "*Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*" Nucl. Phys. B **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [arXiv:hep-th/9407087].

N. Seiberg and E. Witten, "*Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*" Nucl. Phys. B **431**, 484 (1994) [arXiv:hep-th/9408099].

[5] C. H. Taubes, "*Arbitrary N - Vortex Solutions To The First Order Landau-Ginzburg Equations*," Commun. Math. Phys. **72**, 277 (1980).

[6] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, "*Nonabelian Superconductors: Vortices and Confinement in $\mathcal{N} = 2$ SQCD*", arXiv:hep-th/0307287.

[7] N. S. Manton, "*A Remark On The Scattering Of Bps Monopoles*," Phys. Lett. B **110**, 54 (1982).

[8] T. M. Samols, "*Vortex scattering*," Commun. Math. Phys. **145**, 149 (1992).
N. S. Manton and J. M. Speight, "*Asymptotic interactions of critically coupled vortices*," Commun. Math. Phys. **236**, 535 (2003) [arXiv:hep-th/0205307].

[9] M. R. Douglas, "*Branes within branes*," arXiv:hep-th/9512077; "*Gauge Fields and D-branes*," J. Geom. Phys. **28**, 255 (1998) [arXiv:hep-th/9604198].

[10] D. E. Diaconescu, "*D-branes, monopoles and Nahm equations*," Nucl. Phys. B **503**, 220 (1997) [arXiv:hep-th/9608163].

[11] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, "*Construction Of Instantons*," Phys. Lett. A **65**, 185 (1978).

[12] W. Nahm, "*The Construction Of All Selfdual Multi - Monopoles By The ADHM Method*", in Trieste 1981 Proceedings, "*Monopoles In Quantum Field Theory*"

[13] E. Witten, "*Solutions of four-dimensional field theories via M-theory*" Nucl. Phys. B **500**, 3 (1997) [arXiv:hep-th/9703166].

[14] A. Hanany and E. Witten, "*Type IIB superstrings, BPS monopoles, and three dimensional gauge dynamics*" Nucl. Phys. B **492**, 152 (1997) [arXiv:hep-th/9611230].

[15] A. Hanany and K. Hori, "*Branes and N=2 Theories in Two Dimensions*" hep-th/9707192 *Nucl. Phys.* **B513** (1998) 119.

[16] L. Alvarez-Gaume and D. Z. Freedman, "*Potentials For The Supersymmetric Nonlinear Sigma Model*" Commun. Math. Phys. **91**, 87 (1983).

[17] N. Dorey, "*The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms*" JHEP **9811**, 005 (1998) [arXiv:hep-th/9806056].

[18] N. Dorey, T. J. Hollowood and D. Tong, "*The BPS spectra of gauge theories in two and four dimensions*" JHEP **9905**, 006 (1999) [arXiv:hep-th/9902134].

[19] M. Shifman and A. Yung, "*Non-Abelian String Junctions as Confined Monopoles*", arXiv:hep-th/0403149.

[20] E. R. Abraham and P. K. Townsend, "*Q Kinks*", Phys. Lett. B **291**, 85 (1992); "*More On Q Kinks: A (1+1)-Dimensional Analog Of Dyons*", Phys. Lett. B **295**, 225 (1992).

[21] E. Witten, "*Dyons Of Charge $e\theta/2\pi$*" Phys. Lett. B **86**, 283 (1979).