4 The Integral Theorems

The fundamental theorem of calculus states that integration is the inverse of the differentiation, in the sense that

$$\int_a^b dx\, \frac{df}{dx} = f(b) - f(a)$$

In this section, we describe a number of generalisations of this result to higher dimensional integrals. Along the way, we will also gain some intuition for the meaning of the various vector derivative operators.

4.1 The Divergence Theorem

The *divergence theorem*, also known as *Gauss' theorem*, states that, for a smooth vector field $\mathbf{F}(\mathbf{x})$ over $\mathbb{R}^3$,

$$\int_V \nabla \cdot \mathbf{F}\, dV = \int_S \mathbf{F} \cdot d\mathbf{S} \tag{4.1}$$

where V is a bounded region whose boundary $\partial V = S$ is a piecewise smooth closed surface. The integral on the right-hand side is taken with the normal $\mathbf{n}$ pointing outward.

The Meaning of the Divergence

We'll prove the divergence theorem shortly. But first, let's make good on our promise to build some intuition for the divergence. To this end, integrate $\nabla \cdot \mathbf{F}$ over some region of volume V centred at the point $\mathbf{x}$. If the region is small enough, then $\nabla \cdot \mathbf{F}$ will be roughly constant, and so

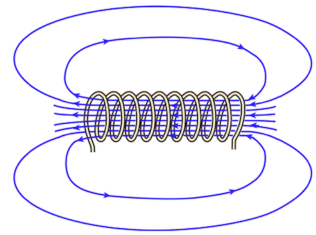
$$\int_V \nabla \cdot \mathbf{F}\, dV \approx V\, \nabla \cdot \mathbf{F}(\mathbf{x})$$

and this becomes exact as the region shrinks to zero size. The divergence theorem then provides a coordinate independent definition of the divergence

$$\nabla \cdot \mathbf{F} = \lim_{V \to 0} \frac{1}{V} \int_S \mathbf{F} \cdot d\mathbf{S} \tag{4.2}$$

This is the result that we advertised in Section 3: the right way to think about the divergence of a vector field is as the net flow into, or out of, a region. If $\nabla \cdot \mathbf{F} > 0$ at some point $\mathbf{x}$, then there is a net flow out of that point; if $\nabla \cdot \mathbf{F} < 0$ at some point $\mathbf{x}$ then there is a net flow inwards.

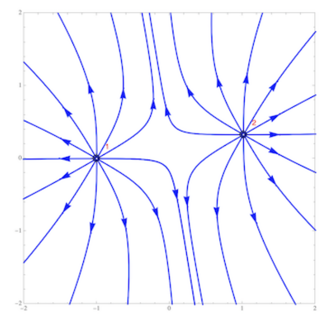
We can illustrate this by looking at a couple of the Maxwell equations (3.7). The magnetic field $\mathbf{B}$ is solenoidal, obeying



$$\nabla \cdot \mathbf{B} = 0$$

This means that the magnetic vector field can't pile up anywhere: at any given point in space, there is as much magnetic field coming in as there is going out. This leads us to draw the magnetic field as continuous, never ending streamlines. For example, the magnetic field lines for solenoid, a long coil of wire carrying a current, is shown in the figure (taken from the website hyperphysics).

Meanwhile, electric field $\mathbf{E}$ obeys

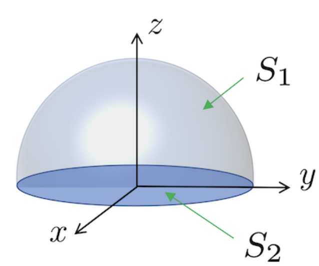


$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

where $\rho(\mathbf{x})$ is the electric charge density. In any region of space where there's no electric charge, so $\rho(\mathbf{x}) = 0$, the electric field lines act just like the magnetic field and can't pile up anywhere. However, the presence of electric charge changes this, and causes the field lines to pile up or disappear. In other words, the electric charge acts as a source or a sink for electric field lines. The electric field lines arising from two pointlike, positive charges which act as sources, are shown in the figure.

Example

Before proving the theorem, we first give an example. Take the volume V to be the solid hemispherical ball, defined as $x^2 + y^2 + z^2 \leq R^2$ and $z \geq 0$. Then boundary of V then has two pieces



$$\partial V = S_1 + S_2$$

where S_1 is the hemisphere and S_2 the disc in the $z = 0$ plane. We'll integrate the vector field

$$\mathbf{F} = (0, 0, z + R)$$

The $+R$ doesn't contribute in the volume integral since we have $\nabla \cdot \mathbf{F} = 1$. Then

$$\int_V \nabla \cdot \mathbf{F}\, dV = \int_V dV = \frac{2}{3}\pi R^3 \tag{4.3}$$

which is the volume of the hemispherical ball. For the surface integral, we work with S_1 and S_2 separately. On the hemisphere S_1, the unit normal vector is $\mathbf{n} = \frac{1}{R}(x, y, z)$ and so

$$\mathbf{F} \cdot \mathbf{n} = \frac{z(z+R)}{R} = R\cos\theta(\cos\theta + 1)$$

where we've used polar coordinates $z = R\cos\theta$. The integral is then

$$\begin{aligned}
\int_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \, (R^2 \sin\theta)\, R\cos\theta(\cos\theta + 1) \\
&= 2\pi R^3 \left[-\frac{1}{3}\cos^3\theta - \frac{1}{2}\cos^2\theta \right]_0^{\pi/2} \\
&= = 2\pi R^3 \left(\frac{1}{3} + \frac{1}{2} \right) = \frac{5\pi R^3}{3}
\end{aligned} \tag{4.4}$$

where the $R^2 \sin\theta$ factor in the first line is the Jacobian that we previously saw in (2.9). Meanwhile, for the integral over the disc S_2, we have the normal vector $\mathbf{n} = (0, 0, -1)$, and so (remembering that the disc sits at $z = 0$),

$$\mathbf{F} \cdot \mathbf{n} = -R \quad \Rightarrow \quad \int_{S_2} \mathbf{F} \cdot d\mathbf{S} = (-R) \times \pi R^2$$

with πR^2 the area of the disc. Adding these together, we have

$$\int_{S_1 + S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{2}{3}\pi R^3$$

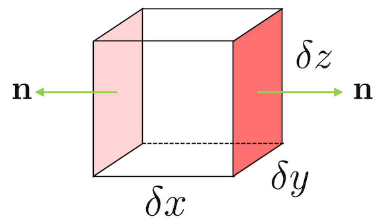
which reproduces the volume integral as promised.

It's worth tracking what became of the $+R$ term in the vector field $\mathbf{F}$. Obviously it didn't contribute to the volume integral. For the surface integral over S_1, it gave the $+1/2$ term in the penultimate expression in (4.4). This was then cancelled by the surface integral over S_2, which only received a contribution from the $+R$ term. We see that this constant vector field when in the top surface, and out the bottom surface, giving no contribution to the overall surface integral. This is how we get agreement with the volume integral which, due to the derivative, is oblivious to any constant (or, indeed, divergent free) components of $\mathbf{F}$.

4.1.1 A Proof of the Divergence Theorem

We start by giving an informal sketch of the basic idea underlying the divergence theorem. We'll then proceed with a more rigorous proof.

To get some intuition for the divergence theorem, take the volume V and divide it up into a bunch of small cubes. A given cube $V_{\mathbf{x}}$ has one corner of the cube sitting at $\mathbf{x} = (x, y, z)$ and sides of lengths δx, δy and δz.

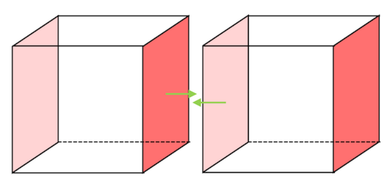


For a small enough cube, we can think of $\mathbf{F} \cdot \mathbf{n}$ as being approximately constant on any given side. To start, we look at the flux of $\mathbf{F}$ through the two sides that lie in the (y, z) plane is given by

$$[F_x(x + \delta x, y, z) - F_x(x, y, z)]\, \delta y\, \delta z \approx \frac{\partial F_x}{\partial x} \delta x\, \delta y\, \delta z \tag{4.5}$$

where the minus sign comes because the flux is calculated using the outward pointing normal and the right-hand side comes from Taylor expanding $F_x(x + \delta x, y, z)$. We get similar expressions for the integrals over the sides that lie in the (x, y) plane and in the (x, z) plane. Summing over six sides, the total flux through the surface of this tiny cube is then

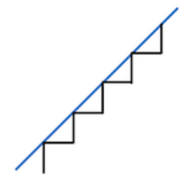
$$\int_{\text{tiny}\,\square} \mathbf{F} \cdot d\mathbf{S} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \delta x\, \delta y\, \delta z = \nabla \cdot \mathbf{F}\, \delta x\, \delta y\, \delta z$$



But now we've tiled our volume V with a whole bunch of these cubes, we can apply the formulae above to each of them. On the right-hand side, we add up the value of $\nabla \cdot \mathbf{F}$ in each cube. This, of course, is the volume integral that we're after. On the left-hand side, something more interesting happens. Now we get a term like the left-hand side of (4.5) for each box, and we sum over all boxes. But this means that all contributions from interior faces cancel out because the outward normal of one box is in the opposite direction to the outward normal from the other box. The upshot is that any interior contribution to the flux vanishes, and we are left only with the contribution from the boundary $S = \partial V$. This then gives us the claimed result

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F}\, dV$$

The derivation above is simple and intuitive, but it might leave you a little nervous. The essence of the divergence theorem is to relate a bulk integral to a boundary integral. But it's not obvious that the boundary can be well approximated by stacking cubes together. To give an analogy, if you try to approximate a 45° line by a series of horizontal and vertical lines, as shown on the right, then the total length of the steps is always going to be $\sqrt{2}$ larger than the length of the horizontal line, no matter how fine you make them. You might worry that these kind of issues afflict the proof above. For that reason, we now give a more careful derivation of the divergence theorem.



Before we proceed, first note that, suitably interpreted, the divergence theorem holds in arbitrary dimension $\mathbb{R}^n$, where a "surface" now means a codimension one subspace. In particular, the divergence theorem holds in $\mathbb{R}^2$, where a surface is a curve. This result, which is interesting in its own right, will serve as a warm-up exercise to proving the general divergence theorem.

The 2d Divergence Theorem: Let $\mathbf{F}$ be a vector field in $\mathbb{R}^2$. Then

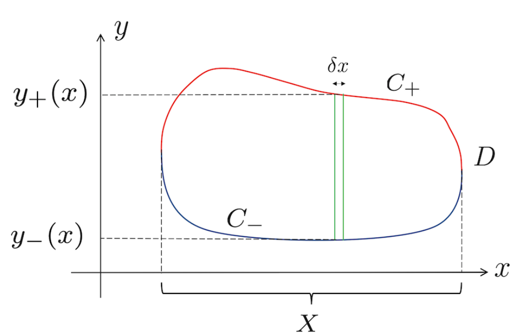
$$\int_D \nabla \cdot \mathbf{F}\, dA = \int_C \mathbf{F} \cdot \mathbf{n}\, ds \tag{4.6}$$

where D is a region in $\mathbb{R}^2$, bounded by the closed curve C and $\mathbf{n}$ is the outward normal to C.

Proof of the 2d Divergence Theorem: For simplicity, we'll assume that $\mathbf{F} = F(x,y)\,\hat{\mathbf{y}}$. The proof that we're about to give also works if $\mathbf{F}$ points solely in the $\hat{\mathbf{x}}$ direction, but a general $\mathbf{F}$ is just a linear sum of the two.

We then have

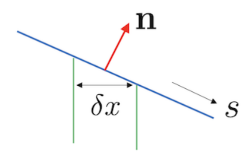
$$\int_D \nabla \cdot \mathbf{F}\, dA = \int_X dx \int_{y_-(x)}^{y_+(x)} dy\, \frac{\partial F}{\partial y}$$



where, as the notation shows, we've chosen to do the area integral by first integrating over y, and then over x. We'll assume, for now, that the region D is convex, as shown in the figure, so that each $\int dy$ is over just a single interval with limits $y_\pm(x)$. These limits trace out an upper curve C_+, shown in red in the figure, and a lower curve C_- shown in blue. We then have

$$\int_D \nabla\cdot\mathbf{F}\,dA = \int_X dx\,\Big(F(x,y_+(x)) - F(x,y_-(x))\Big)$$

We've succeeded in converting the area integral into an ordinary integral, but it's not quite of the line integral form that we need. The next part of the proof is to massage the integral over $\int dx$ into a line integral over $\int ds$. This is easily achieved if we look at the zoomed-in figure to the right. Along the upper curve C_+, the normal $\mathbf{n}$ points upwards and makes an angle $\cos\theta = \hat{\mathbf{y}}\cdot\mathbf{n}$ with the vertical. Moving a small distance δs along the curve is equivalent to moving



$$\delta x = \cos\theta\,\delta s = \hat{\mathbf{y}}\cdot\mathbf{n}\,\delta s \quad \text{along } C_+$$

Along the lower curve, C_-, the normal $\mathbf{n}$ points downwards and so $\hat{\mathbf{y}}\cdot\mathbf{n}$ is negative. We then have

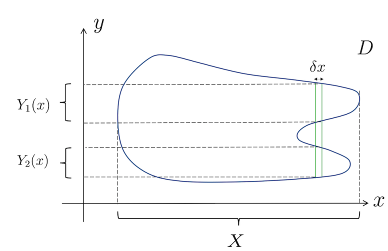
$$\delta x = -\hat{\mathbf{y}}\cdot\mathbf{n}\,\delta s \quad \text{along } C_-$$

The upshot is that we can write the area integral as

$$\begin{aligned}\int_D \nabla\cdot\mathbf{F}\,dA &= \int_X ds\,\Big(\mathbf{n}\cdot\mathbf{F}(x,y_+(x)) + \mathbf{n}\cdot\mathbf{F}(x,y_-(x))\Big)\\ &= \int_{C_+}\mathbf{F}\cdot\mathbf{n}\,ds + \int_{C_-}\mathbf{F}\cdot\mathbf{n}\,ds\\ &= \int_C \mathbf{F}\cdot\mathbf{n}\,ds\end{aligned}$$

with $C = C_+ + C_- = \partial D$ the boundary of the region.

We're left with one small loophole to close: if the region D is not convex, then the range of the integral $\int dy$ may be over two or more disconnected intervals, as shown in the figure. In this case, the boundary curve decomposes into more pieces, but the basic strategy still holds. □



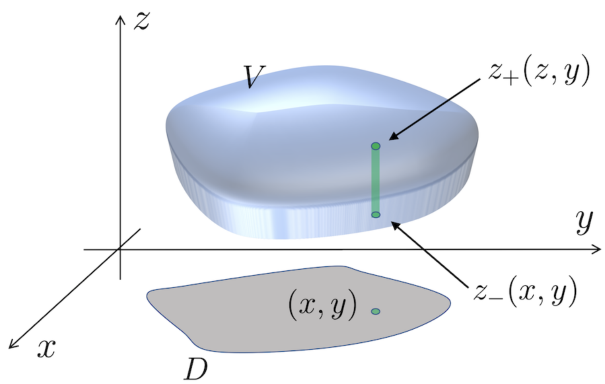


Figure 13. Performing the $\int dz$ integral for the proof of the 3d divergence theorem.

Proof of the 3d Divergence Theorem

The proof of the 3d (or, indeed, higher dimensional) divergence theorem follows using the same strategy. If we focus on $\mathbf{F} = F(x, y, z)\,\hat{\mathbf{z}}$ we have

$$\begin{aligned}\int_V \nabla \cdot \mathbf{F}\, dV &= \int_D dA \int_{z_-(x,y)}^{z_+(x,y)} dz\, \frac{\partial F}{\partial z} \\ &= \int_D dA\, \Big(F(x, y, z_+(x, y)) - F(x, y, z_-(x, y)) \Big)\end{aligned}$$

where the limits of the integral $z_\pm(x, y)$ are the upper and lower surfaces of the volume V. The area integral over D is an integral in the (x, y) plane, while to prove Gauss' theorem we need to convert this into a surface integral over $S = \partial V$. This step of the argument is the same as before: at any given point, the different between $dA = dxdy$ and dS is the angle $\cos\theta = \mathbf{n} \cdot \hat{\mathbf{z}}$ (up to a sign). This then gives the promised result (4.1). □

The Divergence Theorem for Scalar Fields

There is a straightforward extension of the divergence theorem for scalar fields ϕ:

Claim: For $S = \partial V$, we have

$$\int_V \nabla\phi\, dV = \int_S \phi\, d\mathbf{S}$$

Proof: Consider the divergence theorem (4.1) with $\mathbf{F} = \phi\mathbf{a}$ where $\mathbf{a}$ is a constant vector. We have

$$\int_V \nabla \cdot (\phi\mathbf{a}) dV = \int_S (\phi\mathbf{a}) \cdot d\mathbf{S} \quad \Rightarrow \quad \mathbf{a} \cdot \left(\int_V \nabla\phi\, dV - \int_S \phi\, d\mathbf{S} \right) = 0$$

This is true for any constant vector $\mathbf{a}$, and so the expression in the brackets must itself vanish. $\square$

4.1.2 Carl Friedrich Gauss (1777-1855)

Gauss is regarded by many as the greatest mathematician of all time. He made seminal contributions to number theory, algebra, geometry, and physics.

Gauss was born to working class parents in what is now Lower Saxony, Germany. In 1795 he went to study at the university of Göttingen and remained there for the next 60 years.

There are remarkably few stories about Gauss that do not, at the end of the day, boil down to the observation that he was just really good at maths. There is even a website that has collected well over 100 retellings of how Gauss performed the sum $\sum_1^{100} n$ when still a foetus. (You can find an interesting dissection of this story here.)

4.2 An Application: Conservation Laws

Of the many important applications of the divergence theorem, one stands out. In many situations, we have the concept of a conservation law: some quantity that doesn't change over time. There are conservation laws in fundamental physics, including energy, momentum, angular momentum and electric charge and several more that emerge when we look to more sophisticated theories. There are also approximate conservation laws at play when we model more complicated systems. For example, if you're interested in how the population distribution of some species evolves over time then it might well serve you to ignore birth rates and traffic accidents and consider the total number of animals to be fixed.

In all these cases, the quantity is conserved. But we can say something stronger than that: it is conserved *locally*. For example, an electric charge sitting in the palm of your hand can't disappear and turn up on Jupiter. That would satisfy a "global" conservation of charge, but that's not the way the universe works. If the electric charge disappears from your hand, then most likely it has fallen off and is now sitting on the floor. Or, said more precisely, it must have moved to a nearby region of space.

The divergence theorem provides the technology to describe local conservation laws of this type. First, we introduce the *density* $\rho(\mathbf{x}, t)$ of the conserved object. For the purposes of this discussion, we will take this to be the density of electric charge, although it could equally well be the density of any of the other conserved quantities described above. The total electric charge in some region V is then given by the integral

$$Q = \int_V \rho \, dV$$

The conservation of charge is captured by the following statement: there exists a vector field $\mathbf{J}(\mathbf{x}, t)$ such that

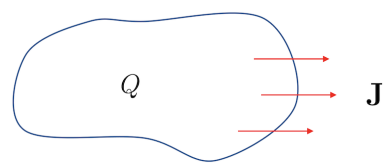
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

This is known as the *continuity* equation and $\mathbf{J}$ is called the *current density*.

The continuity equation doesn't tell us that the density ρ can't change in time; that would be overly prohibitive. But it does tell us that ρ must change only in a certain way. This ensures that the change in the charge Q in a fixed region V is given by

$$\frac{dQ}{dt} = \int_V \frac{\partial \rho}{\partial t} \, dV = -\int_V \nabla \cdot \mathbf{J} \, dV = -\int_S \mathbf{J} \cdot d\mathbf{S}$$

where the second equality follows from the continuity equation and the third from the divergence theorem at some fixed time t. We learn that the charge inside a region can only change if there is a current flowing through the surface of that region. This is how the conservation of charge is enforced locally.



The intuition behind this idea is straightforward. If you want to keep tabs on the number of people in a nightclub, you don't continuously count them. Instead you measure the number of people entering and leaving through the door.

If the current is known to vanish outside some region, so $\mathbf{J}(\mathbf{x}) = 0$ for $|\mathbf{x}| > R$, then the total charge contained inside that region must be unchanging. Often, in such situations, we ask only that $\mathbf{J}(\mathbf{x}, t) \to 0$ suitably quickly as $|\mathbf{x}| \to \infty$, in which case the total charge is unchanging

$$Q_{\text{total}} = \int_{\mathbb{R}^3} \rho \, dV \quad \text{and} \quad \frac{dQ_{\text{total}}}{dt} = 0$$

In later courses, we'll see many examples of the continuity equation. The example of electric charge discussed above will be covered in the lectures on Electromagnetism, where the flux of $\mathbf{J}$ through a surface S is

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

and is what we usually call the electric current.

We will also see the same equation in the lectures on Quantum Mechanics where $\rho(\mathbf{x})$ has the interpretation of the probability density for a particle to be at some point $\mathbf{x}$ and $Q = \int_V \rho\, dV$ is the probability that the particle sits in some region V. Obviously, in this example we must have $Q_{\text{total}} = 1$ which is the statement that particle definitely sits somewhere.

Finally, the continuity equation also plays an important role in Fluid Mechanics where the mass of the fluid is conserved. In that case, $\rho(\mathbf{x}, t)$ is the density of the fluid and the current is $\mathbf{J} = \rho\mathbf{u}$ where $\mathbf{u}(\mathbf{x}, t)$ is the *velocity field*. The continuity equation then reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

In this case the flux is the mass of fluid that passes through a surface S in time t.

In many circumstances, liquids can be modelled as *incompressible*, meaning that $\rho(\mathbf{x}, t)$ is a constant in both space and time. In these circumstances, we have $\dot{\rho} = \nabla \rho = 0$ and the continuity equation tells us that the velocity field is necessarily solenoidal:

$$\nabla \cdot \mathbf{u} = 0 \tag{4.7}$$

This makes sense: for a solenoidal vector field, the flow into any region must be accompanied by an equal outgoing flow, telling us that the fluid can't pile up anywhere, as expected for an incompressible fluid. The statement that fluids are incompressible is a fairly good approximation until we come to think about sound, which arises because of changes in the density which propagate as waves.

4.2.1 Conservation and Diffusion

There is a close connection between conserved quantities and the idea of diffusion. We'll illustrate this with the idea of energy conservation. The story takes a slightly different form depending on the context, but here we'll think of the energy contained in a hot gas. First, since energy is conserved there is necessarily a corresponding continuity equation

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{4.8}$$

where $\mathcal{E}(\mathbf{x}, t)$ is the energy density of the gas, and $\mathbf{J}$ is the *heat current* which tells us how energy is transported from one region of space to another.

At this point we need to invoke a couple of physical principles. First, the energy density in a gas is proportional to the temperature of the gas,

$$\mathcal{E}(\mathbf{x}, t) = c\,T(\mathbf{x}, t) \tag{4.9}$$

where c_V is the specific heat capacity. Next comes a key step: in hot systems, where everything is jiggling around randomly, the heat flow is due to temperature differences between different parts of the system. The relation between the two is captured by the equation

$$\mathbf{J} = -\kappa \nabla T \tag{4.10}$$

where κ is called the *thermal conductivity* and the minus sign ensures that heat flows from hot to cold. This relation is known as *Fick's law*. Neither (4.9) nor (4.10) are fundamental equations of physics and both can be derived from first principles by thinking about the motion of the underlying atoms. (This will be described in the lectures on Statistical Physics and, for Fick's law, the lectures on Kinetic Theory.)

Combining the continuity equation (4.8) with the definition of temperature (4.9) and Fick's law (4.10), we find the heat equation

$$\frac{\partial T}{\partial t} = D\nabla^2 T$$

where the diffusion constant is given by $D = \kappa/c$. This tells us how the temperature of a system evolves. As we mentioned previously, the same heat equation describes the diffusive motion of any conserved quantity.

4.2.2 Another Application: Predator-Prey Systems

We'll see more applications of the divergence theorem in Section 5, mainly in the context of the gravitational and electrostatic forces. However, the uses of the theorem are many and varied and stretch far beyond applications to the laws of physics. Here we give an example in the world of ecology which is modelled mathematically by differential equations. As we'll see, the use of ∇ here is somewhat novel because we're not differentiating with respect to space but with respect to some more abstract variables.

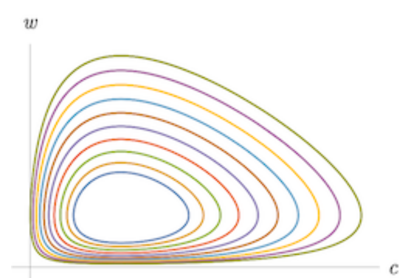
First some background. Predator-prey systems describe the interaction between two species. We will take our predators to be wolves. (Because they're cool.) We will denote the population of wolves at a given time t as $w(t)$. The wolves prey upon something cute and furry. We will denote the population of this cute, furry thing as $c(t)$.

We want to write down a system of differential equations to describe the interaction between wolves and cute furry things. The simplest equations were first written down by Lotka and Volterra and (after some rescaling) take the form

$$\frac{dw}{dt} = w(-\alpha + c)$$
$$\frac{dc}{dt} = c(\beta - w)$$

with $\alpha, \beta > 0$ are some constants. There is a clear meaning to the different terms in these equations. Without food, the wolves die out. That is what the $-\alpha w$ term in the first equation is telling us which, if $c = 0$, will cause the wolf population to decay exponentially quickly. In contrast, without wolves the cute furry things eat grass and prosper. That's what the $+\beta c$ term in the second equation is telling us which, if $w = 0$, ensures that the population of cute furry things grows exponentially. The second term in each equation, $\pm wc$, tells us what happens when the wolves and cute furry things meet. The $\pm$ sign means that it's good news for one, less good for the other.

The Lotka-Volterra equations are straightforward to solve. There is a fixed point at $c = \alpha$ and $w = \beta$ at which the two populations are in equilibrium. Away from this, we find periodic orbits as the two populations wax and wane. To see this, we think of $w = w(c)$ and write the pair of equations as



$$\frac{dw}{dc} = \frac{w(c - \alpha)}{c(\beta - w)}$$

This equation is separable and we have

$$\int \frac{\beta - w}{w}\, dw = \int \frac{c - \alpha}{c}\, dc \quad \Rightarrow \quad \beta \log \omega - \omega + \alpha \log c - c = \text{constant}$$

These orbits are plotted in the (c, w) plane, also known as the phase plane, for different constants in the figure.

So much for the Lotka-Volterra equations. Let's now look at something more complicated. Suppose that there is some intra-species competition: a little wolfy bickering that sometimes gets out of hand, and some cute, furry in-fighting. We can model this by adding extra terms to the original equations:

$$\frac{dw}{dt} = w(-\alpha + c - \mu w)$$
$$\frac{dc}{dt} = c(\beta - w - \nu c) \tag{4.11}$$

where the two new constants are also positive, $\mu, \nu > 0$. Both new terms come with minus signs, which is appropriate because fighting is bad.

What do we do now? There is still a fixed point, now given by $(1 + \mu\nu)w = \beta - \nu\alpha$ and $(1+\mu\nu)c = \alpha+\mu\beta$. But what happens away from this fixed point? Do the periodic orbits that we saw earlier persist? Or does something different happen?

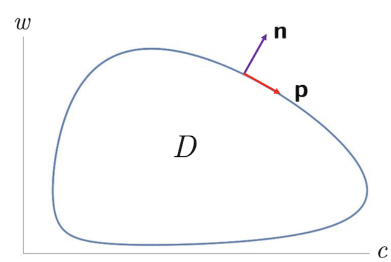
Sadly, we can't just solve the differential equation like we did before because it's no longer separable. Instead, we're going to need a more creative method to understand what's going on. This is where the divergence theorem comes in. We will use it to show that, provided $\mu \neq 0$ or $\nu \neq 0$, the periodic orbits of the Lotka-Volterra equation no longer exist.

We first change notation a little. We write the pair of predator-prey equations (4.11) in vector form

$$\frac{d\mathbf{a}}{dt} = \mathbf{p} \quad \text{with} \quad \mathbf{a} = \begin{pmatrix} w \\ c \end{pmatrix} \quad \text{and} \quad \mathbf{p} = \begin{pmatrix} w(-\alpha + c - \mu w) \\ c(\beta - w - \nu c) \end{pmatrix}$$

Any solution to these equations traces out a path $\mathbf{a}(t)$ in the animal phase plane. The re-writing above makes it clear that $\mathbf{p}$ is the tangent to this path. The question that we wish to answer is: does this path close? In other words, is there a periodic orbit?

It turns out that there are no periodic orbits. To show this, we will suppose that periodic orbits exist and then argue by contradiction. The normal $\mathbf{n}$ to the path $\mathbf{a}(t)$ obeys $\mathbf{n} \cdot \mathbf{p} = 0$, as shown in the figure. This means that if we integrate any function $b(w, c)$ around the periodic orbit we have



$$\oint b(w, c)\, \mathbf{p} \cdot \mathbf{n}\, dt = 0$$

By the 2d divergence theorem, this in turn means that the following integral over the area enclosed by the periodic orbit must also vanish:

$$\int_D \nabla \cdot [b(w, c)\mathbf{p}]\; dA = 0$$

where, in this context, the gradient operator is $\nabla = (\partial/\partial w, \partial/\partial c)$. At this juncture, the trick is to find a cunning choice of function $b(w, c)$. The one that works for us is $b = 1/wc$. This is because we have

$$\nabla \cdot \frac{\mathbf{p}}{wc} = -\frac{\mu}{c} - \frac{\nu}{w}$$

Both of these terms are strictly negative. (For this it is important to remember that populations w and c are strictly positive!) But if $\nabla \cdot (\mathbf{p}/wc)$ is always negative then there's no way to integrate it over a region and get zero. Something has gone wrong. And what's gone wrong was our original assumption of closed orbits. We learn that the nice periodic solutions of the Lotka-Volterra equations are spoiled by any intra-species competition. We're left just with the fixed point which is now stable. All of which is telling us that a little in-fighting may not be so bad after all. It keeps things stable.

The general version of the story above goes by the name of the Bendixson-Dulac theorem and is a powerful tool in the study of dynamical systems.

4.3 Green's Theorem in the Plane

Let $P(x,y)$ and $Q(x,y)$ be smooth functions on $\mathbb{R}^2$. Then

$$\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C P dx + Q dy \tag{4.12}$$

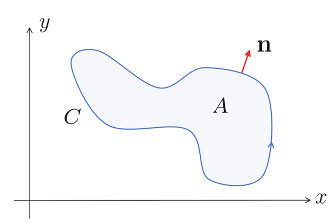
where A is a bounded region in the plane and $C = \partial A$ is a piecewise smooth, non-intersecting closed curve which is traversed anti-clockwise.

Proof: Green's theorem is equivalent to the 2d divergence theorem (4.6). Let $\mathbf{F} = (Q, -P)$ be a vector field in $\mathbb{R}^2$. We then have

$$\int_A \nabla \cdot \mathbf{F}\, dA = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{4.13}$$

If $\mathbf{x}(s) = (x(s), y(s))$ is the parameterised curve C, then the tangent vector is $\mathbf{t}(s) = (x'(s), y'(s))$ and the normal vector $\mathbf{n} = (y'(s), -x'(s))$ obeys $\mathbf{n} \cdot \mathbf{t}$.

You'll need to do a little sketch to convince yourself that, as shown on the right, $\mathbf{n}$ is the outward pointing normal provided that the arc length s increases in the anti-clockwise direction. We then have

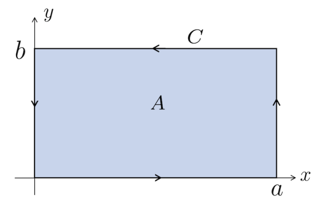


$$\mathbf{F} \cdot \mathbf{n} = Q\frac{dy}{ds} + P\frac{dx}{ds}$$

and so the integral around C is

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \; P dx + Q dy \tag{4.14}$$

The 2d divergence theorem is the statement that the left-hand sides of (4.13) and (4.14) are equal; Green's theorem in the plane is the statement that the right-hand sides are equal. $\square$



Applied to a rectangular region, Green's theorem in the plane reduces to the fundamental theorem of calculus. We take the rectangular region to be $0 \leq x \leq a$ and $0 \leq y \leq b$. Then

$$\begin{aligned}\int_A -\frac{\partial P}{\partial y} dA &= -\int_0^a dx \int_0^b dy\ \frac{\partial P}{\partial y} \\ &= \int_0^a dx\Big(-P(x,b)+P(x,0)\Big) = \int_C P\,dx\end{aligned}$$

where only the horizontal segments contribute, and the minus signs are such that C is traversed anti-clockwise. Meanwhile, we also have

$$\begin{aligned}\int_A \frac{\partial Q}{\partial x} dA &= \int_0^b dy \int_0^a dx\ \frac{\partial Q}{\partial y} \\ &= \int_0^b dy\Big(Q(x,a)-Q(x,0)\Big) = \int_C Q\,dx\end{aligned}$$

where, this time, only the vertical segments contribute.

Green's theorem also holds if the area A has a number of disconnected components, as shown in Figure 14. In this case, the integral should be done in an anti-clockwise direction around the exterior boundary, and in a clockwise direction on any interior boundary. The quickest way to see this is to do the integration around a continuous boundary, as shown in the right-hand figure, with an infinitesimal gap. The two contributions across the gap then cancel.

An Example

Let $P = x^2y$ and $Q = xy^2$. We'll take A to be the region bounded by the parabola $y^2 = 4ax$ and the line $x = a$, both with $-2a \leq y \leq 2a$. Then Green's theorem in the plane tells us that

$$\int_A (y^2 - x^2)\,dA = \int_C x^2y\,dx + xy^2\,dy$$

But this was a problem on the examples sheet, where you found that both give the answer $\frac{104}{105}a^4$.

4.3.1 George Green (1793-1841)

George Green was born in Nottingham, England, the son of a miller. If you were born to a family of millers in the 18th century, they didn't send you to a careers officer at school to see what you want to be when you grow up. You'd be lucky just to be sent to school. Green got lucky. He attended school for an entire year before joining his father baking and running the mill.

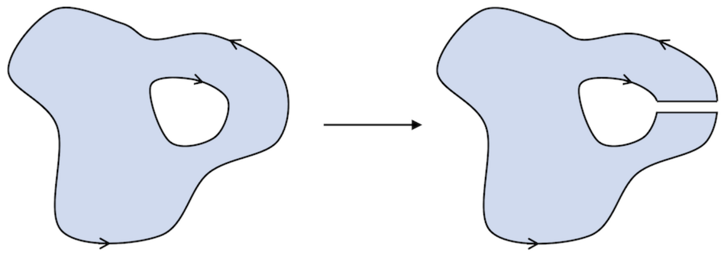


Figure 14. Don't mind the gap. Green's theorem for an area with disconnected boundaries.

It is not known where Green learned his mathematics. The Nottingham subscription library held some volumes, but not enough to provide Green with the background that he clearly gained. Yet, from his mill, Green produced some of the most striking mathematics of his time, including the development of potential theory and, most importantly, the formalism of Green's functions that you will meet in Section 5, as well as in later courses. Much of this was contained in a self-published pamphlet, from 1828, entitled "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism". 51 copies were printed.

Green's reputation spread and, at the age of 40, with no formal education, and certainly no Latin or Greek, Green the miller came to Cambridge as a mathematics undergraduate, clothes covered in flour and pretending it was chalk. (University motto: nurturing imposter syndrome since 1209.) With hindsight, this may not have been the best move. Green did well in his exams, but his published papers did not reach the revolutionary heights of his work in the mill. He got a fellowship at Caius, developed a taste for port, then gout, and died before he reached his 50th birthday.

There are parallels between Green's story and that of Ramanujan who came to Cambridge several decades later. To lose one self-taught genius might be regarded as a misfortune. To lose two begins to look like carelessness.

4.4 Stokes' Theorem

Stokes' theorem is an extension of Green's theorem, but where the surface is no longer restricted to lie in a plane.

Let S be a smooth surface in $\mathbb{R}^3$ with boundary $C = \partial S$ a piecewise smooth curve. *Stokes' theorem* states that, for any smooth vector field $\mathbf{F}(\mathbf{x})$, we have

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{x}$$

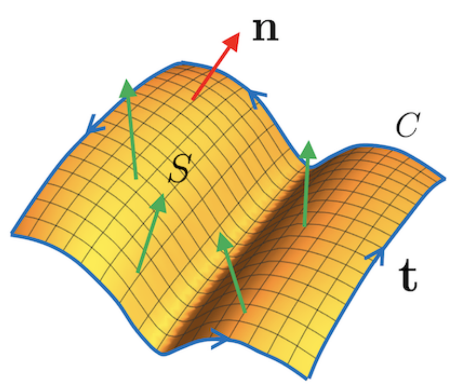


Figure 15. The surface S and bounding curve C for Stokes' theorem. The normal to the surface is shown (at one point) by the red arrow. The theorem invites us to compute the flux of a vector field $\mathbf{F}$, shown by the green arrows, through the surface, and compare it to the line integral around the boundary.

The orientations of **S** and **C** should be *compatible*. The former is determined by the choice of normal vector $\mathbf{n}$ to S; the latter by the choice of tangent vector $\mathbf{t}$ to C. The two are said to be compatible if $\mathbf{t} \times \mathbf{n}$ points out of S. In practice, this means that if you orient the open surface so that $\mathbf{n}$ points towards you, then the orientation of C is anti-clockwise. The general set-up is shown in Figure 15.

Note that there will typically be many surfaces S that share the same boundary C. By Stokes' theorem, the integral of $\nabla \times \mathbf{F}$ over S must give the same answer for all such surfaces. The theorem also holds if the boundary ∂S consists of a number of disconnected components, again with their orientation determined by that of S.

We'll give a proof of Stokes' theorem shortly. But first we put it to some use.

The Meaning of the Curl

Stokes' theorem gives us some new intuition for the curl of a vector field. If we integrate $\nabla \times \mathbf{F}$ over a small enough surface such that $\nabla \times \mathbf{F}$ is approximately constant, then we will have

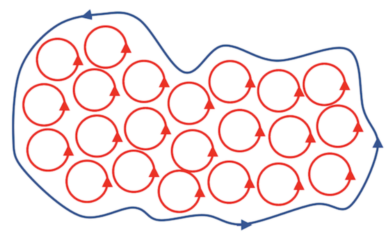
$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \approx A\, \mathbf{n} \cdot (\nabla \times \mathbf{F})$$

where A is the area and $\mathbf{n}$ the normal of the surface. Taking the limit in which this area shrinks to zero, Stokes' theorem then tell us that

$$\mathbf{n} \cdot (\nabla \times \mathbf{F}) = \lim_{A \to 0} \frac{1}{A} \int_C \mathbf{F} \cdot d\mathbf{x} \tag{4.15}$$

In other words, at any given point, the value of $\nabla \times \mathbf{F}$ in the direction $\mathbf{n}$ tells us about the circulation of $\mathbf{F}$ in the plane normal to $\mathbf{n}$

A useful benchmark comes from considering the vector field $\mathbf{u} = \boldsymbol{\omega} \times \mathbf{x}$, which describes a rigid rotation with angular velocity $\boldsymbol{\omega}$. (See, for example, the lectures on Dynamics and Relativity.) In that case, we have $\nabla \times \mathbf{u} = 2\boldsymbol{\omega}$, so twice the angular velocity.



Turning this on its head, we can get some intuition for Stokes' theorem itself. The curl of the vector field tells us about the local circulation of $\mathbf{F}$. When you integrate this circulation over some surface S, most of it cancels out because the circulation going one way is always cancelled by a neighbouring circulation going the other, as shown in the figure. The only thing that's left when you integrate over the whole surface is the circulation around the edge.

A Corollary: Irrotational Implies Conservative

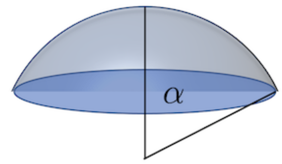
Before we prove Stokes' theorem, we can use it to tie off a thread that we previously left hanging. Recall that in Section 3.2, we proved that $\mathbf{F} = \nabla\phi \implies \nabla \times \mathbf{F} = 0$, but we didn't then have the tools to prove the converse. Now we do. It follows straightforwardly from Stokes' theorem because an irrotational vector field, obeying $\nabla \times \mathbf{F} = 0$, necessarily has

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = 0$$

around any closed curve C. But we showed in Section 1.2 that any such conservative field can be written as $\mathbf{F} = \nabla\phi$ for some potential ϕ.

An Example

Let S be the cap of a sphere of radius R that is covered by the angle $0 \leq \theta \leq \alpha$, as shown in the figure. We'll take



$$\mathbf{F} = (0, xz, 0) \quad \Rightarrow \quad \nabla \times \mathbf{F} = (-x, 0, z) \quad (4.16)$$

This is the example that we discussed in Section 2.2.5, where we computed (see (2.11))

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \pi R^3 \cos\alpha \sin^2\alpha \tag{4.17}$$

That leaves us with the line integral around the rim. This curve C is parameterised by the angle ϕ and is given by

$$\mathbf{x}(\phi) = R(\sin\alpha\cos\phi, \sin\alpha\sin\phi, \cos\alpha) \quad \Rightarrow \quad d\mathbf{x} = R(-\sin\alpha\sin\phi, \sin\alpha\cos\phi, 0)\, d\phi$$

We then have

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_0^{2\pi} d\phi\ Rxz \sin\alpha\cos\phi = R^3 \sin^2\alpha\cos\alpha \int_0^{2\pi} d\phi\ \cos^2\phi = \pi R^3 \sin^2\alpha \cos\alpha$$

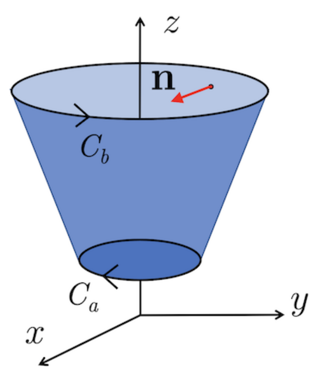
in agreement with the surface integral (4.17).

Another Example

As a second example, consider the conical surface S defined by $z^2 = x^2 + y^2$ with $0 < a \leq z \leq b$. This surface is parameterised, in cylindrical polar coordinates, by

$$\mathbf{x}(\rho, \phi) = (\rho\cos\phi, \rho\sin\phi, \rho) \tag{4.18}$$

with $a \leq \rho \leq b$ and $0 \leq \phi < 2\pi$. We can compute two tangent vectors



$$\frac{\partial \mathbf{x}}{\partial \rho} = (\cos\phi, \sin\phi, 1) \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial \phi} = \rho(-\sin\phi, \cos\phi, 0)$$

and take their cross product to get the normal

$$\mathbf{n} = \frac{\partial \mathbf{x}}{\partial \rho} \times \frac{\partial \mathbf{x}}{\partial \phi} = (-\rho\cos\phi, -\rho\sin\phi, \rho)$$

This points inwards, as shown in the figure. The associated vector area element is

$$d\mathbf{S} = (-\cos\phi, -\sin\phi, 1)\rho d\rho\, d\phi$$

We'll integrate the same vector field (4.16) over this surface. We have

$$\nabla \times \mathbf{F} \cdot d\mathbf{S} = (x \cos\phi + z)\rho \, d\rho \, d\phi = \rho^2(\cos^2\phi + 1) d\rho \, d\phi$$

where we've substituted in the parametric expressions for x and z from (4.18). The integral is then

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_a^b d\rho \int_0^{2\pi} d\phi \; \rho^2 (1 + \cos^2\phi) = \pi(b^3 - a^3) \tag{4.19}$$

Now the surface has two boundaries, and we must integrate over both of them. We write $\partial S = C_b - C_a$ where C_b has radius b and C_a radius a. Note the minus sign, reflecting the fact that the orientation of the two circles is opposite.

For a circle of radius R, we have $\mathbf{x}(\phi) = R(\cos\phi, \sin\phi, 1)$, and so $d\mathbf{x} = R(-\sin\phi, \cos\phi, 0)\, d\phi$ and

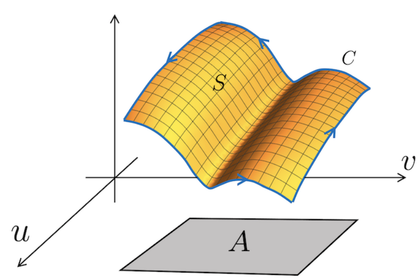
$$\int_{C_R} \mathbf{F} \cdot d\mathbf{x} = \int_0^{2\pi} d\phi \; R^3 \cos^2\phi = \pi R^3$$

Remembering that the orientation of C_a in the opposite direction, we reproduce the surface integral (4.19).

4.4.1 A Proof of Stokes' Theorem

It's clear that Stokes' theorem is a version of Green's theorem in the plane, but viewed through 3d glasses. Indeed, it's trivial to show that the latter follows from the former. Consider the vector field $\mathbf{F} = (P, Q, 0)$ in $\mathbb{R}^3$ and a surface S that lies flat in the $z = 0$ plane. The normal to this surface is $\mathbf{n} = \hat{\mathbf{z}}$, and we have

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dS$$



But Stokes' theorem then tells us that this can also be written as

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C P dx + Q dy$$

However, with a little more work we can also show that the converse is true. In other words, we can lift Green's theorem out of the plane to find Stokes' theorem.

Consider a parameterised surface S defined by $\mathbf{x}(u, v)$ and denote the associated area in the (u, v) plane as A. We parameterise the boundary $C = \partial S$ as $\mathbf{x}(u(t), v(t))$ and the corresponding boundary ∂A as $(u(t), v(t))$. The key idea is to use Green's theorem in the (u, v) plane for the area A and then uplift this to prove Stokes theorem for the surface S.

We start by looking at the integral around the boundary. It is

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C \mathbf{F} \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \, du + \frac{\partial \mathbf{x}}{\partial v} \, dv \right) = \int_{\partial A} F_u \, du + F_v \, dv$$

where $F_u = \mathbf{F} \cdot \partial \mathbf{x} / \partial u$ and $F_v = \mathbf{F} \cdot \partial \mathbf{x} / \partial v$. Now we're in a position to invoke Green's theorem, in the form

$$\int_{\partial A} F_u \, du + F_v \, dv = \int_A \left(\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} \right) dA$$

Now our task is clear. We should look at the partial derivatives on the right hand side. We just need to be careful about what thing depends on what thing:

$$\frac{\partial F_v}{\partial u} = \frac{\partial}{\partial u} \left(\mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial v} \right) = \frac{\partial}{\partial u} \left(F_i \frac{\partial x^i}{\partial v} \right) = \left(\frac{\partial F_i}{\partial x^j} \frac{\partial x^j}{\partial u} \right) \frac{\partial x^i}{\partial v} + F_i \frac{\partial^2 x^i}{\partial u \partial v}$$

Meanwhile, we have

$$\frac{\partial F_u}{\partial v} = \frac{\partial}{\partial v} \left(\mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial u} \right) = \frac{\partial}{\partial v} \left(F_i \frac{\partial x^i}{\partial u} \right) = \left(\frac{\partial F_i}{\partial x^j} \frac{\partial x^j}{\partial v} \right) \frac{\partial x^i}{\partial u} + F_i \frac{\partial^2 x^i}{\partial v \partial u}$$

Subtracting the second expression from the first, the second derivative terms cancel, leaving us with

$$\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} = \frac{\partial x^j}{\partial u} \frac{\partial x^i}{\partial v} \left(\frac{\partial F_i}{\partial x^j} - \frac{\partial F_j}{\partial x^i} \right) = (\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik}) \frac{\partial x^k}{\partial u} \frac{\partial x^l}{\partial v} \frac{\partial F_i}{\partial x^j}$$

At this point we wield everyone's favourite index notation identity

$$\epsilon_{jip}\epsilon_{pkl} = \delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik}$$

We then have

$$\frac{\partial F_v}{\partial u} - \frac{\partial F_u}{\partial v} = \epsilon_{jip}\epsilon_{pkl} \frac{\partial x^k}{\partial u} \frac{\partial x^l}{\partial v} \frac{\partial F_i}{\partial x^j} = (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right)$$

8. If X, Y, Z be functions of the rectangular co-ordinates x, y, z, dS an element of any limited surface, l, m, n the cosines of the inclinations of the normal at dS to the axes, ds an element of the bounding line, shew that

$$\iint\left\{l\left(\frac{dZ}{dy}-\frac{dY}{dx}\right)+m\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)+n\left(\frac{dY}{dx}-\frac{dX}{dy}\right)\right\}dS$$
$$=\int\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)ds,$$

the differential coefficients of X, Y, Z being partial, and the single integral being taken all round the perimeter of the surface.

Figure 16. You may now turn the page. . . the original version of Stokes' theorem, set as an exam question.

Now we're done. Following through the chain of identities above, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{x} &= \int_A \left(\frac{\partial F_v}{\partial u}-\frac{\partial F_u}{\partial v}\right) dudv \\ &= \int_A (\nabla\times\mathbf{F})\cdot\left(\frac{\partial \mathbf{x}}{\partial u}\times\frac{\partial \mathbf{x}}{\partial v}\right) dudv \\ &= \int_S (\nabla\times\mathbf{F})\cdot d\mathbf{S}\end{aligned}$$

This is Stokes' theorem. □

4.4.2 George Gabriel Stokes (1819-1903)

Stokes was born in County Sligo, Ireland, but moved to Cambridge shortly after his 19$^{\text{th}}$ birthday and remained there for the next 66 years, much of it as Lucasian professor. He contributed widely to different area of mathematics and physics, with the Navier-Stokes equation, describing fluid flow, a particular highlight.

What we now call Stokes' theorem was communicated to Stokes by his friend William Thomson, better known by his later name Lord Kelvin. The theorem first appeared in print in 1854 as part of the Smith's prize examination competition, a second set of exams aimed at those students who felt the Tripos wasn't brutal enough.

If you're in Cambridge and looking for a tranquil place away from the tourists to sit, drink coffee, and ponder the wider universe, then you could do worse than the Mill Road cemetery, large parts of which are overgrown, derelict, and beautiful. Stokes is buried there, as is Cayley, although both gravestones were destroyed long ago. You can find Stokes' resting place nestled between the graves of his wife and daughter[1].

4.4.3 An Application: Magnetic Fields

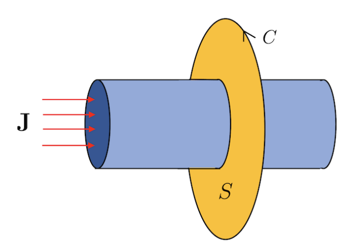
Consider an infinitely long wire carrying a current. What is the magnetic field that is produced? We can answer this by turning to the Maxwell equations (3.7). For time independent situations, like this, one of the equations reads

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{4.20}$$

where $\mathbf{J}$ is the current density and μ_0 is a constant of nature that determines the strength of the magnetic field and has some pretentious name that I can never remember. Another of the Maxwell equations reads $\nabla \cdot \mathbf{B} = 0$ and in most situations we should solve this in conjunction with (4.20) but here it will turn out, somewhat fortuitously, that if we just find the obvious solution to (4.20) then it solves $\nabla \cdot \mathbf{B} = 0$ automatically.

The equation (4.20) provides a simple opportunity to use Stokes' theorem. We integrate both sides over a surface S that cuts through the wire, as shown in the figure to the right. We then have

$$\int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \int_C \mathbf{B} \cdot d\mathbf{x} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S} = \mu_0 I$$



where the integral of the current density gives I, the total current through the wire. This equation tells us that there must be a circulation of the magnetic field around the wire. In particular, there must be a component of $\mathbf{B}$ that lies tangent to any curve C that bounds a surface S.

Let's suppose that the wire lies in the z-direction. (Rotate your head or your screen if you

[1] A long, tree lined avenue runs north off Mill Road. At the end, turn right to enter the cemetery. There is a gravel path immediately off to your left, which you should ignore, but take the first mud track that runs parallel to it. Just after the gravestone bearing the name "Frederick Cooper" you will find the Stokes' family plot.

don't like the z direction to be horizontal.) Then if S is a disc of radius ρ, then the boundary $C = \partial S$ is paramterised by the curve

$$\mathbf{x} = \rho(\cos\phi, \sin\phi, 0) \quad \Longrightarrow \quad \mathbf{t} = \frac{\partial \mathbf{x}}{\partial \phi} = \rho(-\sin\phi, \cos\phi, 0)$$

We'll make the obvious guess that $\mathbf{B}$ lies in the same direction as $\mathbf{t}$ and work with the ansatz

$$\mathbf{B}(\mathbf{x}) = b(\rho)(-\sin\phi, \cos\phi, 0)$$

Then $\mathbf{B}\cdot\mathbf{t} = \rho b(\rho)$. Provided that ρ is bigger than the radius of the wire, Maxwell's equation tells us that

$$\mu_0 I = \int_C \mathbf{B}\cdot d\mathbf{x} = \int_0^{2\pi} d\phi\ \rho b(\rho) \quad \Longrightarrow \quad \mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{2\pi\rho}(-\sin\phi, \cos\phi, 0)$$

You can check that this answer also satisfies the other Maxwell equation $\nabla\cdot\mathbf{B} = 0$. We learn that the magnetic field circulates around the wire, and drops off as $1/\rho$ with ρ the distance from the wire.

4.4.4 Changing Coordinates Revisited

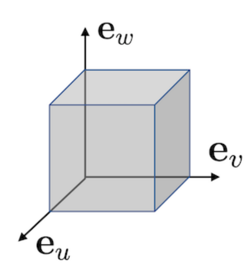
Back in Section 3.3, we wrote down the expressions for the divergence and curl in a general orthonormal curvilinear coordinate system. Now we can offer a proof using the integral theorems above.

Claim: The divergence of a vector field $\mathbf{F}(u, v, w)$ in a general orthogonal, curvilinear coordinate system is given by

$$\nabla\cdot\mathbf{F} = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_u h_w F_v) + \frac{\partial}{\partial w}(h_u h_v F_w)\right) \tag{4.21}$$

Proof: We sketch a proof that works with the integral definition of the divergence (4.2),

$$\nabla\cdot\mathbf{F} = \lim_{V\to 0}\frac{1}{V}\int_S \mathbf{F}\cdot d\mathbf{S}$$



We can take the volume V to consist of a small cuboid at point (u, v, w) with sides parallel to the basis vectors $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$. The volume of the cube is $h_u h_v h_w \delta u\,\delta v\,\delta w$. Meanwhile, the area of, say, the upper face in the figure is roughly $h_u h_v \delta u\,\delta v$. Since h_u and h_v may depend on the coordinates, this could differ from the area of the lower face, albeit only by a small amount δw. Then, assuming that $\mathbf{F}$ is roughly constant on each face, we have

$$\begin{aligned}\int_S \mathbf{F}\cdot d\mathbf{S} &\approx \Big[h_u h_v F_w(u,v,w+\delta w) - h_u h_v F_w(u,v,w)\Big]\delta u\,\delta v + \text{two more terms}\\ &\approx \frac{\partial}{\partial w}(h_u h_v F_w)\delta u\,\delta v\,\delta w + \text{two more terms}\end{aligned}$$

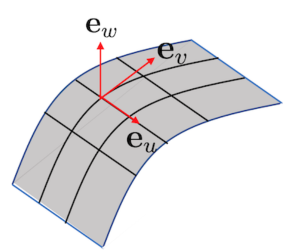
Dividing through by the volume then gives us the advertised expression for $\nabla\cdot\mathbf{F}$. □

Claim: The curl of a vector field $\mathbf{F}(u,v,w)$ in a general orthogonal, curvilinear coordinate system is given by

$$\begin{aligned}\nabla\times\mathbf{F} &= \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\mathbf{e}_u & h_v\mathbf{e}_v & h_w\mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w\end{vmatrix}\\ &= \frac{1}{h_v h_w}\left(\frac{\partial}{\partial v}(h_w F_w) - \frac{\partial}{\partial w}(h_v F_v)\right)\mathbf{e}_u + \text{two similar terms}\end{aligned}$$

Proof: This time we use the integral definition of curl (4.15)

$$\mathbf{n}\cdot(\nabla\times\mathbf{F}) = \lim_{A\to 0}\frac{1}{A}\int_C \mathbf{F}\cdot d\mathbf{x}$$



We'll take a surface S with normal $\mathbf{n} = \mathbf{e}_w$ and integrate over a small region, bounded by one of the squares in the figure on the right. The area of the square $h_u h_v \delta u\,\delta v$ while the length of each side is $h_u\delta u$ and $h_v\delta v$. Assuming that the square is small enough so that $\mathbf{F}$ is roughly constant along any given side, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{x} &\approx h_u F_u(u,v)\delta u + h_v F_v(u+\delta u, v)\delta v - h_u F_u(u, v+\delta v)\delta u - h_v F_v(u,v)\delta v\\ &\approx \left[\frac{\partial}{\partial u}(h_v F_v) - \frac{\partial}{\partial v}(h_u F_u)\right]\delta u\,\delta v\end{aligned}$$

Dividing by the area, this gives

$$\mathbf{e}_w\cdot\nabla\times\mathbf{F} = \frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}(h_v F_v) - \frac{\partial}{\partial v}(h_u F_u)\right]$$

which is one of the three promised terms in the expression for $\nabla\times\mathbf{F}$. □