5 The Poisson and Laplace Equations

Until now, our focus has been very much on understanding how to differentiate and integrate functions of various types. But, with this under our belts, we can now take the next step and explore various differential equations that are written in the language of vector calculus. Our goal in this section is to find solutions to the Poisson equation and the related Laplace equation. This we will do in Section 5.2. But first we will explain why these equations underly two of the most important forces in the universe.

5.1 Gravity and Electrostatics

The first two fundamental forces to be discovered are also the simplest to describe mathematically. Newton's law of gravity states that two masses, m and M, separated by a distance r will experience a force

$$\mathbf{F}(r) = -\frac{GMm}{r^2}\hat{\mathbf{r}} \tag{5.1}$$

with G Newton's constant, a fundamental constant of nature that determines the strength of the gravitational force. Meanwhile, Coulomb's law states that two electric charges, q and Q, separated by a distance r will experience a force

$$\mathbf{F}(r) = \frac{Qq}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{5.2}$$

with the electric constant ϵ_0 a fundamental constant of nature that determines the inverse strength of the electrostatic force. The extra factor 4π reflects the fact that in the century between the Newton and Coulomb people had figured out where factors of 4π should sit in equations.

Most likely it will not have escaped your attention that these two equations are essentially the same. The only real difference is that overall minus sign which tells us that two masses always attract while two like charges repel. The question that we would like to ask is: why are the forces so similar?

Certainly it's not true that there is a deep connection between gravity and the electrostatic force, at least not one that we've uncovered to date. In particular, when masses and charges start to move, both the forces described above are replaced by something different and more complicated – general relativity in the case of gravity, the full Maxwell equations (3.7) in the case of the Coulomb force – and the equations of these theories are very different from each other. Yet, when we restrict to the simple, static set-up, the forces take the same form.

The reason for this is twofold. First, both forces are described by fields. Second, space has three dimensions. The purpose of this section is to explain this in more detail. And, for this, we need the tools of vector calculus.

5.1.1 Gauss' Law

Each of the force equations (5.1) and (5.2) contains some property that characterises the force: mass for gravity and electric charge for the electrostatic force. For our purposes, it will be useful to focus on one of the particles that carries mass m and charge q. We call this a *test particle*, meaning that we'll look at how this particle is buffeted by various forces but won't, in turn, consider its effect on any other particle. Physically, this is appropriate if $m \ll M$ and $q \ll Q$. Then it is useful to write the equation in a way that separates the properties of the test particle from the other. The force experienced by the test particle is

$$\mathbf{F}(\mathbf{x}) = m\mathbf{g}(\mathbf{x}) + q\mathbf{E}(\mathbf{x})$$

where $\mathbf{g}(\mathbf{x})$ is the gravitational and $\mathbf{E}(\mathbf{x})$ is the electric field. Clearly Newton's law is telling us that a particle of mass M sets up a gravitational field

$$\mathbf{g}(\mathbf{x}) = -\frac{GM}{r^2}\hat{\mathbf{r}} \tag{5.3}$$

while a particle with electric charge Q sets up an electric field

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{5.4}$$

So far this is just a trivial rewriting of the force laws. However, we will now reframe these force laws in the language of vector calculus. Instead of postulating the $1/r^2$ force laws (5.3) and (5.4), we will replace them by two properties of the fields from which everything else follows. Here we specify the first property; the second will be explained in Section 5.1.2.

The first property is that if you integrate the relevant field over a closed surface, then it captures the amount of "stuff" inside this surface. For the gravitational field, this stuff is mass

$$\int_S \mathbf{g} \cdot d\mathbf{S} = -4\pi GM \tag{5.5}$$

while for the electric field it is charge

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0} \tag{5.6}$$

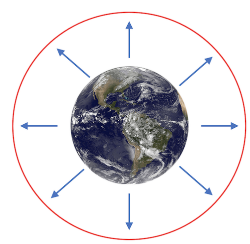
Again, the difference in minus sign signals the important attractive/repulsive difference between the two forces. In contrast, the factors of $4\pi G$ and $1/\epsilon_0$ are simply convention for how we characterise the strength of the fields. These two equations are known as *Gauss' law*. Or, more precisely, "Gauss' law in integrated form". We'll see the other form below.

Examples

For concreteness, let's focus on the gravitational field. We will take a sphere of radius R and total mass M. We will require that the density of the sphere is spherically symmetric, but not necessarily constant. The spherical symmetry of the problem then ensures that the gravitational field itself is spherically symmetric, with $\mathbf{g}(\mathbf{x}) = g(r)\hat{\mathbf{r}}$. If we then integrate the gravitational field over any spherical surface S of radius $r > R$, we have

$$\int_S \mathbf{g} \cdot d\mathbf{S} = \int_S g(r) dS = 4\pi r^2 g(r)$$

where we recognise $4\pi r^2$ as the area of the sphere. From Gauss' law (5.5) we then have



$$\mathbf{g}(r) = -\frac{GM}{r^2}\hat{\mathbf{r}} \tag{5.7}$$

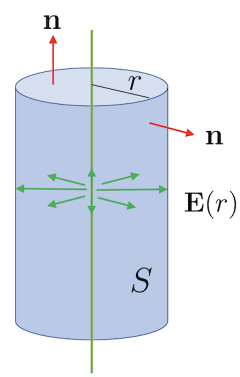
This reproduces Newton's force law (5.1). Note, however, that we've extended Newton's law beyond the original remit of point particles: the gravitational field (5.7) holds for any spherically symmetric distribution of mass, provided that we're outside this mass. For example, it tells us that the gravitational field of the Earth (at least assuming spherical symmetry) is indistinguishable from the gravitational field of a point-like particle with the same mass, sitting at the origin. This way of solving for the vector field is known as the *Gauss flux method*.

Another rather cute consequence of this is that, at least for spherically symmetric mass distributions, you don't feel the mass outside you. According to Gauss' law, the gravitational field at any point is determined only by what lies inside a sphere of a given radius. So if, for example, you were able to hollow out the centre of a planet (unlikely, admittedly) then anyone living there would feel no gravitational force from the mass that surrounds them.

For our second example, we turn to the electric field. Consider an infinite line of charge, with charge per unit length σ. This situation is crying out for cylindrical polar coordinates. Until now, we've always called the radial direction in cylindrical polar coordinates ρ but, for reasons that will become clear shortly, for this example alone we will instead call the radial direction r as shown in the figure. The symmetry of the problem shows that the electric field is radial so takes the form $\mathbf{E}(r) = E(r)\hat{\mathbf{r}}$. Integrating over cylinder S of radius r and length L we have

$$\int_S \mathbf{E} \cdot d\mathbf{S} = 2\pi r L E(r)$$

where there is no contribution from the end caps because $\mathbf{n} \cdot \mathbf{E} = 0$ there, with $\mathbf{n}$ the normal vector. The total charge inside this surface is $Q = \sigma L$. From Gauss' law (5.6), we then have the electric field



$$\mathbf{E}(r) = \frac{\sigma}{2\pi\epsilon_0 r}\hat{\mathbf{r}}$$

Note that the $1/r$ behaviour arises because the symmetry of the problem ensures that the electric field lies in a plane. Said differently, the electric field from an infinite charged line is the same as we would get from a point particle in a flatland world of two dimensions.

More generally, if space were $\mathbb{R}^n$, then the Gauss' law equations (5.5) and (5.6) would still be the correct description of the gravitational and electric fields. Repeating the calculations above would then tell us that a point charge gives rise to an electric field

$$\mathbf{E}(r) = \frac{1}{A_{n-1}\epsilon_0 r^{n-1}}\hat{\mathbf{r}}$$

where $A_n r^n$ is the "surface area" of an n-dimensional sphere S^n of radius r. (For what it's worth, the prefactor is $A_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ where $\Gamma(x)$ is the gamma function which coincides with the factorial function $\Gamma(x) = (x-1)!$ when x is integer.) For the rest of this section, we'll keep our feet firmly in $\mathbb{R}^3$.

Gauss' Law Again

There's a useful way to rewrite the Gauss' law equations (5.5) and (5.6). For the gravitational field, we introduce the *density*, or mass per unit volume, $\rho(\mathbf{x})$. Invoking the divergence theorem then, for any volume V bounded by S, we have

$$\int_V \nabla \cdot \mathbf{g}\, dV = \int_S \mathbf{g} \cdot d\mathbf{S} = 4\pi G M = -4\pi G \int_V \rho(\mathbf{x})\, dV$$

But, rearranging, we have

$$\int_V \Big(\nabla \cdot \mathbf{g} + 4\pi G\rho(\mathbf{x})\Big) dV = 0$$

for any volume V. This can only hold if the integrand itself vanishes, so we must have

$$\nabla \cdot \mathbf{g} = -4\pi G\rho(\mathbf{x}) \tag{5.8}$$

This is also known as *Gauss' law* for the gravitational field, now in differential form. The equivalence with the earlier integrated form (5.5) follows, as above, from the divergence theorem.

We can apply the same manipulations to the electric field. This time we introduce the *charge density* $\rho_e(\mathbf{x})$. We then get Gauss' law in the form

$$\nabla \cdot \mathbf{E} = \frac{\rho_e(\mathbf{x})}{\epsilon_0} \tag{5.9}$$

This is the first of the Maxwell equations (3.7). (In our earlier expression, we denoted the charge density as $\rho(\mathbf{x})$. Here we've added the subscript ρ_e to distinguish it from mass density.) The manipulations that we've described above show that Gauss' law is a grown-up version of the Coulomb force law (5.2).

5.1.2 Potentials

In our examples above, we used symmetry arguments to figure out the direction in which the gravitational and electric fields are pointing. But in many situations we don't have that luxury. In that case, we need to invoke the second important property of these vector fields: they are both conservative.

Recall that, by now, we have a number of different ways to talk about conservative vector fields. Such fields are necessarily irrotational $\nabla \times \mathbf{g} = \nabla \times \mathbf{E} = 0$. Furthermore, their integral vanishes when integrated around any closed curve C,

$$\oint_C \mathbf{g} \cdot d\mathbf{x} = \oint_C \mathbf{E} \cdot d\mathbf{x} = 0$$

You can check that both of these hold for the examples, such as the $1/r^2$ field, that we discussed above (as long as the path C avoids the singular point at the origin).

Here the key property of a conservative vector field is that it can be written in terms of an underlying scalar field,

$$\mathbf{g} = -\nabla\Phi \quad \text{and} \quad \mathbf{E} = -\nabla\phi \tag{5.10}$$

where $\Phi(\mathbf{x})$ is the gravitational potential and $\phi(\mathbf{x})$ the electrostatic potential. Note the additional minus signs in these definitions. We saw in the discussion around (1.18) that the existence of such potentials ensures that test particles experiencing these forces have a conserved energy:

$$\text{energy} = \frac{1}{2}m\dot{\mathbf{x}}^2 + m\Phi(\mathbf{x}) + q\phi(\mathbf{x})$$

Combining the differential form of the Gauss' law (5.8) and (5.9) with the existence of the potentials (5.10), we find that the gravitational and electric fields are determined, in general, by solutions to the following equations

$$\nabla^2\Phi = 4\pi G\rho(\mathbf{x}) \quad \text{and} \quad \nabla^2\phi = -\frac{\rho_e(\mathbf{x})}{\epsilon_0}$$

Equations of this type are known as the *Poisson equation*. In the special case where the "source" $\rho(\mathbf{x})$ on the right-hand side vanishes, this reduces to the *Laplace equation*, for example

$$\nabla^2\Phi = 0$$

These two equations are commonplace in mathematics and physics. Here we have derived them in the context of gravity and electrostatics, but their applications spread much further.

To give just one further example, in fluid mechanics the motion of the fluid is described by a velocity field $\mathbf{u}(\mathbf{x})$. If the flow is irrotational, then $\nabla \times \mathbf{u} = 0$ and the velocity can be described by a potential function $\mathbf{u} = \nabla\phi$. If, in addition, the fluid is incompressible then $\nabla \cdot \mathbf{u} = 0$ and we once again find ourselves solving the Laplace equation $\nabla^2\phi = 0$.

5.2 The Poisson and Laplace Equations

In the rest of this section we will develop some methods to solve the Poisson equation. We change notation and call the potential $\psi(\mathbf{x})$ (to avoid confusion with the polar angle ϕ). We are then looking for solutions to

$$\nabla^2\psi(\mathbf{x}) = -\rho(\mathbf{x})$$

The goal is to solve for $\psi(\mathbf{x})$ given a "source" $\rho(\mathbf{x})$. As we will see, the domain in which $\psi(\mathbf{x})$ lives, together with associated boundary conditions, also plays an important role in the determining $\psi(\mathbf{x})$.

The Laplace equation $\nabla^2\psi = 0$ is linear. This means that if $\psi_1(\mathbf{x})$ is a solution and $\psi_2(\mathbf{x})$ is a solution, then so too is $\psi_1(\mathbf{x}) + \psi_2(\mathbf{x})$. Any solution to the Laplace equation acts as a complementary solution to the Poisson equation. This should then be accompanied by a particular solution for a given source $\rho(\mathbf{x})$ on the right-hand side.

5.2.1 Isotropic Solutions

Bot the Laplace and Poisson equations are partial differential equation. Life is generally much easier if we're asked to solve ordinary differential equations rather than partial differential equations. For the Poisson equation, this is what we get if we have some kind of symmetry, typically one aligned to some polar coordinates.

For example, if we have spherical symmetry then we can look for solutions of the form $\psi(\mathbf{x}) = \psi(r)$. Using the form of the Laplacian (3.15), Laplace equation becomes

$$\begin{aligned} \nabla^2\psi = 0 \quad &\Rightarrow \quad \frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = 0 \\ &\Rightarrow \quad \psi(r) = \frac{A}{r} + B \end{aligned} \tag{5.11}$$

for some constants A and B. Clearly the A/r solution diverges as $r \to 0$ so we should be cautious in claiming that this solves the Laplace equation at $r = 0$. (We will shortly see that it doesn't, but it does solve a related Poisson equation.) Note that the solution A/r is relevant in gravity or in electrostatics, where $\psi(r)$ has the interpretation as the potential for a point charge.

Meanwhile, in cylindrical polar coordinates we will also denote the radial direction as r to avoid confusion with the source ρ in the Poisson equation. The Laplace equation becomes

$$\begin{aligned} \nabla^2\psi = 0 \quad &\Rightarrow \quad \frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = 0 \\ &\Rightarrow \quad \psi(r) = A\log r + B \end{aligned} \tag{5.12}$$

This again diverges at $r = 0$, this time corresponding to the entire z axis.

Note that if we ignore the z direction, as we have above, then cylindrical polar coordinates are the same thing as 2d polar coordinates, and the log form is the rotationally invariant solution to the Laplace equation in $\mathbb{R}^2$. In general, in $\mathbb{R}^n$, the non-constant solution to the Laplace equation is $1/r^{n-2}$. The low dimensions of $\mathbb{R}^2$ and $\mathbb{R}$ are special because the solution grows asymptotically as $r \to \infty$, while for $\mathbb{R}^n$ with $n \geq 3$, the rotationally invariant solution to the Laplace equation decays to a constant asymptotically.

If $\psi(r)$ is a solution to the Laplace equation, then so too is any derivative of $\psi(r)$. For example, if we take the spherically symmetric solution $\psi(r) = 1/r$, then we can construct a new solution

$$\psi_{\text{dipole}}(\mathbf{x}) = \mathbf{d} \cdot \nabla \left(\frac{1}{r} \right) = -\frac{\mathbf{d} \cdot \mathbf{x}}{r^3}$$

for any constant vector $\mathbf{d}$ and, again, with $r \neq 0$. This kind of solution in important in electrostatics where it arises as the large distance solution for a *dipole*, two equal and opposite charges at a fixed distance apart.

Discontinuities and Boundary Conditions

In many situations, we must specify some further data when solving the Poisson equations. Typically this is some kind of boundary condition and, in some circumstances, a requirement of continuity and smoothness on the solution.

This can be illustrated with a simple example. Suppose that we are looking for a spherically symmetric solution to:

$$\nabla^2 \psi = \begin{cases} -\rho_0 & r \leq R \\ 0 & r > R \end{cases}$$

with ρ_0 constant. We will further ask that $\psi(r = 0)$ is non-singular, that $\psi(r) \to 0$ as $r \to \infty$, and that $\psi(\mathbf{x})$ and $\psi'(\mathbf{x})$ are continuous. We will now see that all of these conditions give us a unique solution.

First look inside $r \leq R$. As we mentioned above, a solution to the Poisson equation can be found by adding a complementary solution and a particular solution. Since we're looking for a spherically symmetric particular solution, we can restrict our ansatz to $\psi(r) = r^p$ for some p. It's simple to check that $\nabla^2 r^p = p(p+1)r^{p-2}$. This then gives us the general solution

$$\psi(r) = \frac{A}{r} + B - \frac{1}{6}\rho_0 r^2 \qquad r \leq R$$

But now we can start killing some terms by invoking the boundary conditions. In particular, the requirement that $\psi(r)$ is non-singular at $r = 0$ tells us that we must have $A = 0$. Meanwhile, outside $r > R$ the most general solution is

$$\psi(r) = \frac{C}{r} + D$$

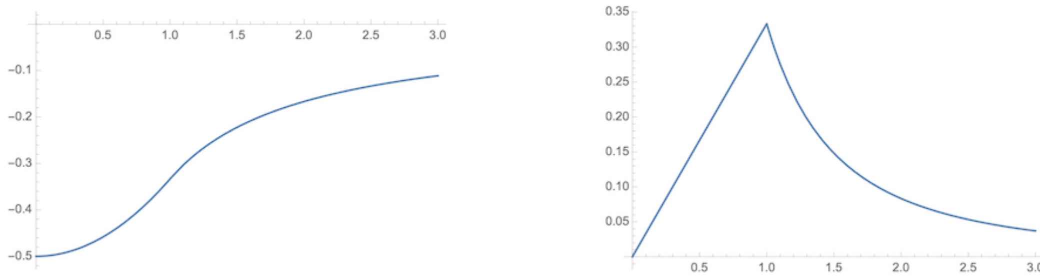


Figure 17. The plot of $\Phi = -4\pi G\psi$ on the left, with the radius $R = 1$ the cross over point. This is more apparent in the gravitational field $g = -\Phi'$ shown on the right.

Now we must have $D = 0$ if $\psi(r) \to 0$ as $r \to \infty$. To finish, we must patch these two solutions at $r = R$, invoking continuity

$$\psi(r = R) = B - \frac{1}{6}\rho_0 R^2 = \frac{C}{R}$$

and smoothness

$$\psi'(r = R) = -\frac{1}{3}\rho_0 R = -\frac{C}{R^2}$$

These determine our last two unknown constants, B and C. Putting this together, we have a unique solution

$$\psi(r) = \begin{cases} \frac{1}{6}\rho_0(3R^2 - r^2) & r \leq R \\ \frac{1}{3}\rho_0 R^3/r & r > R \end{cases}$$

This example has application for the gravitational potential $\Phi = -4\pi G\psi$ of a planet of radius R and density ρ_0. The plot of Φ is shown on the left of Figure 17; the plot of the gravitational field $g = -d\Phi/dr$ is on the right, where we see a linear increase inside the planet, before we get to the more familiar $1/r^2$ fall-off.

5.2.2 Some General Results

So far our solutions to the Poisson equation take place in $\mathbb{R}^3$. (Or, more precisely, $\mathbb{R}^3 - \{0, 0\}$ for the $1/r$ solution (5.11) and $\mathbb{R}^3 - \mathbb{R}$ for the $\log r$ solution (5.12).) In general, we may want to solve the Poisson or Laplace equations $\nabla^2\psi = -\rho$ in some bounded region V. In that case, we must specify boundary conditions on ∂V.

There are two common boundary conditions:

- Dirichlet condition: We fix $\psi(\mathbf{x}) = f(\mathbf{x})$ for some specific $f(\mathbf{x})$ on ∂V.

- Neumann condition: We fix $\mathbf{n}\cdot\nabla\psi(\mathbf{x}) = g(\mathbf{x})$ for some specific $g(\mathbf{x})$ on ∂V, where $\mathbf{n}$ is the outwardly pointing normal of ∂V.

The Neumann boundary condition is sometimes specified using the slightly peculiar notation $\partial\psi/\partial\mathbf{n} := \mathbf{n}\cdot\nabla\psi$. Or even, sometimes, $\partial\psi/\partial n$. We have the following statement of uniqueness:

Claim: There is a unique solution to the Poisson equation on a bounded region V, with either Dirichlet or Neumann boundary conditions specified on each boundary ∂V. (In the case of Neumann boundary conditions everywhere, the solution is only unique up to a constant.)

Proof: Let $\psi_1(\mathbf{x})$ and $\psi_2(\mathbf{x})$ both satisfy the Poisson equation with the specified boundary conditions. Then $\psi(\mathbf{x}) = \psi_1 - \psi_2$ obeys $\nabla^2\psi = 0$ and either $\psi = 0$ or $\mathbf{n}\cdot\nabla\psi = 0$ on ∂V. Then consider

$$\int_V \nabla\cdot(\psi\nabla\psi)\,dV = \int_V \left(\nabla\psi\cdot\nabla\psi + \psi\nabla^2\psi\right)\,dV = \int_V |\nabla\psi|^2 dV$$

But by the divergence theorem, we have

$$\int_V \nabla\cdot(\psi\nabla\psi)\,dV = \int_{\partial V} \psi\nabla\psi\cdot d\mathbf{S} = \int_{\partial V} \psi(\mathbf{n}\cdot\nabla\psi)\,dS = 0$$

where either Dirichlet or Neumann boundary conditions set the boundary term to zero. Because $|\nabla\psi|^2 \geq 0$, the integral can only vanish is $\nabla\psi = 0$ everywhere in V, so ψ must be constant. If Dirichlet boundary conditions are imposed anywhere, then that constant must be zero. $\square$

This result means that if we can find any solution – say an isotropic solution, or perhaps a separable solution of the form $\psi(\mathbf{x}) = \Phi(r)Y(\theta)$ – then this must be the unique solution. By considering the limit of large spheres, it is also possible to extend the proof to solutions on $\mathbb{R}^3$, with the boundary condition $\psi(\mathbf{x}) \to 0$ suitably quickly as $r \to \infty$.

Note, however, that this doesn't necessarily tell us that a solution exists. For example, suppose that we wish to solve the Poisson equation $\nabla^2\psi = \rho(\mathbf{x})$ with a fixed Neumann boundary condition $\mathbf{n}\cdot\nabla\psi = g(\mathbf{x})$ on ∂V. Then there can only be a solution provided that there is a particular relationship between ρ and g,

$$\int_V \nabla^2\psi\,dV = \int_{\partial V} \nabla\psi\cdot d\mathbf{S} \quad\Longleftrightarrow\quad \int_V \rho\,dV = \int_S g\,dS$$

In other situations, there may well be other requirements.

If the region V has several boundaries, it's quite possible to specify a different type of boundary condition on each, and the uniqueness statement still holds. This kind of problem arises in electromagnetism where you solve for the electric field in the presence of a bunch of "conductors" (for now, conductors just means a chunk of metal). The electric field vanishes inside a conductor since, of it didn't the electric charges inside would move around until the created a counterbalancing field. So any attempt to solve for the electric field outside the conductors must take this into account by imposing certain boundary conditions on the surface of the conductor. It turns out that both Dirichlet and Neumann boundary conditions are important here. If the conductor is "grounded", meaning that it is attached to some huge reservoir of charge like the Earth, then then it sits at some fixed potential, typically $\psi = 0$. This is a Dirichlet boundary condition. In contrast, if the conductor is isolated and carries some non-vanishing charge then it will act as a source of electric field, but this field is always emitted perpendicular to the boundary. This, then, specifies $\mathbf{n} \cdot \mathbf{E} = -\mathbf{n} \cdot \nabla\psi$, giving Neumann boundary conditions. You can learn more about this in the lectures on Electromagnetism.

Green's Identities

The proof of the uniqueness theorem used a trick known as Green's (first) identity, namely

$$\int_V \phi \nabla^2 \psi \, dV = -\int_V \nabla\phi \cdot \nabla\psi \, dV + \int_S \phi \nabla\psi \cdot d\mathbf{S}$$

This is essentially a 3d version of integration by parts and it follows simply by applying the divergence theorem to $\phi\nabla\psi$. We used it in the above proof with $\phi = \psi$, but the more general form given above is sometimes useful, as is a related formula that follows simply by anti-symmetrisation,

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dV = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot d\mathbf{S}$$

This is known as Green's second identity.

Harmonic Functions

Solutions to the Laplace equation

$$\nabla^2\psi = 0$$

arise in many places in mathematics and physics. These solutions are so special that they get their own name: they are called *harmonic functions*. Here are two properties of these functions

Claim: Suppose that ψ is harmonic in a region V that includes the solid sphere with boundary $S_R : |\mathbf{x} - \mathbf{a}| = R$. Then $\psi(\mathbf{a}) = \bar{\psi}(R)$ where

$$\bar{\psi}(R) = \frac{1}{4\pi R^2} \int_{S_R} \psi(\mathbf{x})\, dS$$

is the average of ψ over S_R. This is known as the *mean value property.*

Proof: In spherical polar coordinates centred on $\mathbf{a}$, the area element is $dS = r^2 \sin\theta d\theta\, d\phi$, so

$$\bar{\psi}(r) = \frac{1}{4\pi} \int d\phi \int d\theta \;\; \sin\theta\, \psi(r, \theta, \phi)$$

and

$$\begin{aligned} \frac{d\bar{\psi}(R)}{dr} &= \frac{1}{4\pi} \int d\phi \int d\theta \;\; \sin\theta\, \frac{\partial \psi(R)}{\partial r} = \frac{1}{4\pi R^2} \int_{S_R} \frac{\partial \psi(R)}{\partial r} dS \\ &= \frac{1}{4\pi R^2} \int_{S_R} \nabla\psi \cdot d\mathbf{S} = \int_{\text{Ball}} \nabla^2 \psi\, dV = 0 \end{aligned}$$

But clearly $\bar{\psi}(R) \to \psi(\mathbf{a})$ as $R \to 0$ so we must have $\bar{\psi}(R) = \psi(\mathbf{a})$ for all R. $\square$

Claim: A harmonic function can have neither a maximum nor minimum in the interior of a region V. Any maximum of minimum must lie on the boundary ∂V.

Proof: If ψ has a local maximum at $\mathbf{a}$ in V then there exists an ϵ such that $\psi(\mathbf{x}) < \psi(\mathbf{a})$ for all $|\mathbf{x} - \mathbf{a}| < \epsilon$. But, we know that $\bar{\psi}(R) = \psi(\mathbf{a})$ and this contradicts the assumption for any $0 < R < \epsilon$. $\square$

This is consistent with our standard analysis of maxima and minima. Usually we would compute the eigenvalues λ_i of the Hessian $\partial^2\psi/\partial x^i \partial x^j$. For a harmonic function $\nabla^2\psi = \partial^2\psi/\partial x^i \partial x^i = 0$. Since the trace of the Hessian vanishes, we must have eigenvalues of opposite sign since $\sum_i \lambda_i = 0$. Hence, any stationary point must be a saddle. Note that this standard analysis is inconclusive when $\lambda_i = 0$, but the argument using the mean value property closes this loophole.

5.2.3 Integral Solutions

There is a particularly nice way to write down an expression for the general solution to the Poisson equation in $\mathbb{R}^3$, with

$$\nabla^2 \psi = -\rho(\mathbf{x})$$

at least for a localised source $\rho(\mathbf{x})$ that drops off suitably fast, so $\rho(\mathbf{x}) \to 0$ as $r \to \infty$.

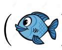
To this end, let's look back to what is, perhaps, our simplest "solution",

$$\psi(\mathbf{x}) = \frac{\lambda}{4\pi r} \tag{5.13}$$

for some constant λ. The question we want to ask is: what equation does this actually solve?! We've seen in (5.11) that it solves the Laplace equation $\nabla^2\psi = 0$ when $r \neq 0$. But clearly something's going on at $r = 0$. In the language of physics, we would say that there is a point particle sitting at $r = 0$, carrying some mass or charge, giving rise to this potential. What is the correct mathematical way of capturing this?

To see that there must be something going on at $r = 0$, let's replay the kind of Gauss flux games that we met in Section 5.1. We integrate $\nabla^2\psi$, with ψ given by (5.13), over a spherical region of radius R, to find

$$\int \nabla^2\psi \, dV = \int_S \nabla\psi \cdot d\mathbf{S} = -\lambda$$

Comparing to (), we see that the function (5.13) must solve the Poisson equation with a source and this source must obey

$$\int_V \rho(\mathbf{x}) \, dV = \lambda$$

This makes sense physically, since $\int \rho dV$ is the total mass, or total charge, which does indeed determine the overall scaling λ of the potential. But what mathematical function obeys $\rho(\mathbf{x}) = 0$ for all $\mathbf{x} \neq 0$ yet, when integrated over all space, gives a non-vanishing constant λ?

The answer is that $\rho(\mathbf{x})$ must be proportional to the 3d *Dirac delta function*,

$$\rho(\mathbf{x}) = \lambda \, \delta^3(\mathbf{x})$$

The Dirac delta function should be thought of as an infinitely narrow spike, located at the origin. It has the properties

$$\delta^3(\mathbf{x}) = 0 \quad \text{for} \quad \mathbf{x} \neq 0$$

and, when integrated against any function $f(\mathbf{x})$ over any volume V that includes the origin, it gives

$$\int_V f(\mathbf{x}) \, \delta^3(\mathbf{x}) \, dV = f(\mathbf{x} = \mathbf{0})$$

The superscript in $\delta^3(\mathbf{x})$ is there to remind us that the delta function should be integrated over a 3-dimensional volume before it yields something finite. In particular, when integrated against a constant function, we get a measure of the height of the spike,

$$\int_V \delta^3(\mathbf{x})\, dV = 1$$

The Dirac delta function is an example of a generalised function, also known as a distribution. And it is exactly what we need to source the solution $\psi \sim 1/r$. We learn that the function (5.13) is not a solution to the Laplace equation, but rather a solution to the Poisson equation with a delta function source

$$\nabla^2\psi = -\lambda\, \delta^3(\mathbf{x}) \quad \Rightarrow \quad \psi(\mathbf{x}) = \frac{\lambda}{4\pi r} \tag{5.14}$$

With this important idea in hand, we can now do something quite spectacular: we can use it to write down an expression for a solution to the general Poisson equation.

Claim: The Poisson equation () has the integral solution

$$\psi(\mathbf{x}) = \frac{1}{4\pi}\int_{V'} \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, dV' \tag{5.15}$$

where the integral is over a region V' parameterised by $\mathbf{x}'$.

Proof: First, some simple intuition behind this formula. A point particle at $\mathbf{x}'$ gives rise to a potential of the form $\psi(\mathbf{x}) = \rho(\mathbf{x}')/4\pi|\mathbf{x}-\mathbf{x}'|$, which is just our solution (5.14), translated from the origin to point $\mathbf{x}'$. The integral solution (5.15) then just takes advantage of the linear nature of the Poisson equation and sums a whole bunch of these solutions.

The technology of the delta function allows us to make this precise. We can evaluate

$$\nabla^2\psi = \frac{1}{4\pi}\int_{V'} \rho(\mathbf{x}')\, \nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\, dV'$$

where you have to remember that ∇^2 differentiates $\mathbf{x}$ and cares nothing for $\mathbf{x}'$. We then have the result

$$\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')$$

which is just a repeat of (5.14), but with the location of the source translated from the origin to the new point $\mathbf{x}'$. Using this, we can continue our proof

$$\nabla^2\psi = -\int_{V'} \rho(\mathbf{x}')\, \delta^3(\mathbf{x}-\mathbf{x}')\, dV' = -\rho(\mathbf{x})$$

which is what we wanted to show. $\square$

The technique of first solving an equation with a delta function source and subsequently integrating to find the general solution is known as the *Green's function* approach. It is a powerful method to solve differential equations and we will meet it again in many further courses.