

Vector Calculus: Example Sheet 2

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1. The vector field $\mathbf{B}(\mathbf{x})$ is everywhere parallel to the normals of a family of surfaces $f(\mathbf{x}) = \text{constant}$. Show that $\mathbf{B} \cdot (\nabla \times \mathbf{B}) = 0$.

The tangent vector at each point on a curve is parallel to a non-vanishing vector field $\mathbf{H}(\mathbf{x})$. Show that the curvature of the curve is given by $\kappa = |\mathbf{H} \times (\mathbf{H} \cdot \nabla)\mathbf{H}|/|\mathbf{H}^3|$.

2. Let $\phi(\mathbf{x})$ be a scalar field and $\mathbf{v}(\mathbf{x})$ a vector field. Show, using suffix notation, that

$$\nabla \cdot (\phi \mathbf{v}) = (\nabla \phi) \cdot \mathbf{v} + \phi(\nabla \cdot \mathbf{v}), \quad \nabla \times (\phi \mathbf{v}) = (\nabla \phi) \times \mathbf{v} + \phi(\nabla \times \mathbf{v}).$$

Evaluate the divergence and curl of the following:

$$r\mathbf{x}, \quad \mathbf{a}(\mathbf{x} \cdot \mathbf{b}), \quad \mathbf{a} \times \mathbf{x}, \quad \mathbf{x}/r^3,$$

where $r = |\mathbf{x}|$ and \mathbf{a}, \mathbf{b} are constant vectors.

3. For vector fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$, use suffix notation to show that,

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - \mathbf{v}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{v},$$

Show also that

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \left(\frac{1}{2}|\mathbf{u}|^2 \right) - \mathbf{u} \times (\nabla \times \mathbf{u})$$

4. Verify directly that the vector field

$$\mathbf{u}(\mathbf{x}) = (e^x(x \cos y + \cos y - y \sin y), e^x(-x \sin y - \sin y - y \cos y), 0)$$

is *irrotational* and express it as the gradient of a scalar field ϕ . Check that \mathbf{u} is *solenoidal* and show that it can be written as the curl of the vector field $\mathbf{v} = (0, 0, \psi)$, for some function ψ .

5. Check that the following vector field is irrotational

$$\mathbf{F} = (3x^2 \tan z - y^2 e^{-xy^2} \sin y, (\cos y - 2xy \sin y)e^{-xy^2}, x^3 \sec^2 z)$$

Find the most general scalar potential $\phi(\mathbf{x})$ such that $\mathbf{F} = \nabla \phi$.

6*. Suppose $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is divergence free, i.e. $\nabla \cdot \mathbf{F} = 0$. Show that $\mathbf{F} = \nabla \times \mathbf{A}$ where

$$\mathbf{A}(\mathbf{x}) = \int_0^1 \mathbf{F}(t\mathbf{x}) \times (t\mathbf{x}) dt.$$

What goes wrong with this formula if \mathbf{F} is not defined on all of \mathbb{R}^3 ?

7. Let (u, v, w) be a set of orthogonal curvilinear coordinates for \mathbb{R}^3 . Show that

$$dV = h_u h_v h_w du dv dw.$$

Confirm that $dV = \rho d\rho d\phi dz$ and $dV = r^2 \sin \theta dr d\theta d\phi$ in cylindrical and spherical polars respectively.

8. If \mathbf{a} is constant vector and $r = |\mathbf{x}|$, verify that

$$\nabla (r^n) = nr^{n-2}\mathbf{x}, \quad \nabla(\mathbf{a} \cdot \mathbf{x}) = \mathbf{a}$$

using (i) Cartesian coordinates and suffix notation, (ii) cylindrical polar coordinates, (iii) spherical polar coordinates, and (iv) a first principles, coordinate-independent approach.

[Note: for parts (ii) and (iii) you will need to be careful with the components of \mathbf{a} with respect to each of the relevant bases.]

9. The vector field $\mathbf{A}(\mathbf{x})$ is, in Cartesian, cylindrical and spherical polar coordinates respectively,

$$\mathbf{A}(\mathbf{x}) = -\frac{1}{2}y \mathbf{e}_x + \frac{1}{2}x \mathbf{e}_y = \frac{1}{2}\rho \mathbf{e}_\phi = \frac{1}{2}r \sin \theta \mathbf{e}_\phi.$$

Compute the $\nabla \times \mathbf{A}$ in each different coordinate system and check that your answers agree.

10. Recall that in cylindrical polar coordinates

$$\nabla = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial \mathbf{e}_\rho}{\partial \phi} = \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_\rho,$$

while all other derivatives are zero. Derive expressions for the $\nabla \cdot \mathbf{A}$ and $\nabla \times \mathbf{A}$ where \mathbf{A} is an arbitrary vector field given in cylindrical polars by $\mathbf{A} = A_\rho \mathbf{e}_\rho + A_\phi \mathbf{e}_\phi + A_z \mathbf{e}_z$. Also derive an expression for the Laplacian of a scalar function $\nabla^2 f$ in this coordinate system

11. By applying the divergence theorem to the vector field $\mathbf{a} \times \mathbf{A}$, where \mathbf{a} is an arbitrary constant vector and $\mathbf{A}(\mathbf{x})$ is a vector field, show that

$$\int_V \nabla \times \mathbf{A} \, dV = \int_S d\mathbf{S} \times \mathbf{A}$$

where $S = \partial V$. Verify this result when $V = \{(x, y, z) : 0 < x < a, 0 < y < b, 0 < z < c\}$ and $\mathbf{A}(\mathbf{x}) = (z, 0, 0)$.

12. Let $\mathbf{F}(\mathbf{x}) = (x^3 + 3y + z^2, y^3, x^2 + y^2 + 3z^2)$ and let S be the *open* surface

$$1 - z = x^2 + y^2, \quad 0 \leq z \leq 1.$$

Use the divergence theorem and cylindrical polar coordinates to evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$. Verify your result by calculating the area integral directly.

[Hint: you should find that $d\mathbf{S} = (2\rho \cos \phi, 2\rho \sin \phi, 1) \rho \, d\rho \, d\phi$.]

13. For the electric and magnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ define the quantities

$$U = \frac{1}{2} \left(\varepsilon_0 |\mathbf{E}|^2 + \frac{1}{\mu_0} |\mathbf{B}|^2 \right), \quad \mathbf{P} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$

Use Maxwell's equations with $\mathbf{J} = 0$ to establish the conservation law $\partial U / \partial t + \nabla \cdot \mathbf{P} = 0$. If $U(\mathbf{x})$ has the interpretation of the energy density stored in electric and magnetic fields, what is the interpretation of the so-called Poynting vector \mathbf{P} ?