Complex Methods: Example Sheet 3

Part IB, Lent Term 2024

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Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Fourier transforms

1. By using the relationship between the Fourier transform and its inverse, show that for real a and b with $a > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2}\, e^{\mathrm{i}\omega t}\, \mathrm{d}\omega = \frac{\pi}{a} e^{-a|t|} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{b}{(\mathrm{i}\omega + a)^2 + b^2}\, e^{\mathrm{i}\omega t}\, \mathrm{d}\omega = 2\pi e^{-at} \sin bt\, H(t)$$

where $H(t)$ is the Heaviside step function. What are the values of the integrals when $a < 0$? What happens when $a = 0$?

2. Show that the convolution of the function $e^{-|x|}$ with itself is given by $f(x) = (1 + |x|)e^{-|x|}$. Use the convolution theorem for Fourier transforms to show that

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}kx}}{(1 + k^2)^2}\, \mathrm{d}k$$

and verify this result by contour integration.

3. Let

$$f(x) = \begin{cases} 1 & |x| < \frac{1}{2}a, \\ 0 & \text{otherwise;} \end{cases} \qquad g(x) = \begin{cases} a - |x| & |x| < a, \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\widetilde{f}(k) = \frac{2}{k} \sin \frac{ak}{2} \quad \text{and} \quad \widetilde{g}(k) = \frac{4}{k^2} \sin^2 \frac{ak}{2}.$$

What is the convolution of f with itself? Use Parseval's identity to evaluate $\int_{-\infty}^{\infty} (\sin^2 x)/x^2\, \mathrm{d}x$. Verify by contour integration the inversion formula for $f(x)$ for all values of x except $\pm\frac{1}{2}a$.

$*$ Verify the inversion formula also at $x = \pm\frac{1}{2}a$.

$*$ **4.** The displacement $x(t)$ of a damped harmonic oscillator obeys the the equation

$$\ddot{x} + 2\gamma\dot{x} + q^2 x = f(t)\,, \quad \text{where } \gamma > 0\,.$$

Assuming that the Fourier transforms $\tilde{x}(\omega)$ and $\tilde{f}(\omega)$ exist, show that

$$x(t) = \int_{-\infty}^{\infty} G(t - t') f(t') \mathrm{d}t'\,, \quad \text{where } G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}\omega t}}{q^2 + 2\mathrm{i}\gamma\omega - \omega^2} \mathrm{d}\omega\,.$$

Show, by differentiation under the integral sign, that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} G(t) + 2\gamma \frac{\mathrm{d}}{\mathrm{d}t} G(t) + q^2 G(t) = \delta(t)\,.$$

Show that for $0 < \gamma < q$,

$$G(t) = \frac{1}{p} e^{-\gamma t} \sin(pt)\, H(t)\,, \quad \text{where } p = \sqrt{q^2 - \gamma^2}\,.$$

[You may use here the results from question 1.]

Laplace transforms

5. Starting from the Laplace transform of 1 (namely s^{-1}), and using only standard properties of the Laplace transform (shifting, etc.), find the Laplace transforms of the following functions: (i) e^{-2t}; (ii) $t^3 e^{-3t}$; (iii) $e^{3t}\sin 4t$; (iv) $e^{-4t}\cosh 4t$; (v) $e^{-t}H(t-1)$, where H is the Heaviside step function.

6. Using partial fractions and expressions for the Laplace transforms of elementary functions, find the inverse Laplace transform of $F(s) = (s+3)/\{(s-2)(s^2+1)\}$. Verify this result using the Bromwich inversion formula.

7. Use Laplace transforms to solve the differential equation

$$\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} - 3\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y}{\mathrm{d}t} - y = t^2 e^t$$

with initial conditions $y(0) = 1$, $\dot{y}(0) = 0$, $\ddot{y}(0) = -2$.

8. Consider a linear system obeying the differential equation

$$\ddot{y} - 3\dot{y} + 2y = u(t)\,, \qquad \dot{y}(0) = y(0) = 0\,.$$

Use Laplace transforms to determine the response of the system to the signal $u(t) = t$. Determine also the response $y(t)$ to a signal $u(t) = \delta(t)$.

[For $\delta(t)$, take the Laplace transform to be $F(s) = \int_{0-}^{\infty} f(t)e^{-st}\mathrm{d}t$, i.e. start "just left of 0".]

9. Solve the *integral equation* $f(t) + 4\displaystyle\int_0^t (t-\tau)f(\tau)\,\mathrm{d}\tau = t$ for the unknown function f. Verify your solution.

$*$ **10.** The zeroth order Bessel function $J_0(x)$ satisfies the differential equation

$$xJ_0'' + J_0' + xJ_0 = 0$$

for $x > 0$, with $J_0(0) = 1$ (and $J_0'(0) = 0$ from the equation). Find the Laplace transform of J_0 and deduce that $\int_0^\infty J_0(x)\,\mathrm{d}x = 1$. Find the convolution of J_0 with itself.

11. Use Laplace transforms to solve the heat equation $\partial T/\partial t = \partial^2 T/\partial x^2$ with boundary conditions $T(x,0) = \sin^3 \pi x$ $(0 < x < 1)$, $T(0,t) = T(1,t) = 0$ $(t > 0)$. [*Hint:* $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.]

12. Using the equality $\int_0^\infty e^{-x^2}\mathrm{d}x = \frac{1}{2}\sqrt{\pi}$, find the Laplace transform of $f(t) = t^{-1/2}$. By integrating around a Bromwich keyhole contour, verify the inversion formula for $f(t)$. What is the Laplace transform of $t^{1/2}$?

$*$ **13.** The gamma and beta functions are defined for $z, w \in \mathbb{C}$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t \quad \text{and} \quad \mathrm{B}(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,\mathrm{d}t$$

when $\mathsf{Re}(z)$, $\mathsf{Re}(w) > 0$. Show that $\Gamma(z+1) = z\Gamma(z)$ and hence that $\Gamma(n+1) = n!$ if n is a non-negative integer. Using the previous question, write down the value of $\Gamma(\frac{1}{2})$.

For a fixed value of z, find the Laplace transform of $f(t) = t^{z-1}$ in terms of $\Gamma(z)$. Find the Laplace transform of the convolution $t^{z-1} * t^{w-1}$. Hence establish that

$$\mathrm{B}(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{$*$}$$

The domain of Γ and B can be extended to the whole of $\mathbb{C}$, apart from isolated singularities, by analytic continuation. Does the relation $(*)$ still hold?