Mathematical Tripos Part III Dr U. Sperhake

GENERAL RELATIVITY: Examples 4 Michaelmas 2015

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1. Consider two stars, each of mass M, moving in a circular Newtonian orbit of radius R in the x, y plane centred on the origin. Show that their positions may be taken to be

$$\mathbf{x} = \pm(R\cos\Omega t,\ R\sin\Omega t,\ 0)\,,$$

where $\Omega^2 = M/(4R^3)$. Treating the stars as non-relativistic point masses (in the sense of question 7 on sheet 3), compute the corresponding energy-momentum tensor, the second moment of the energy distribution I_{ij}, and the metric perturbation $\bar{h}_{ij}$. Determine the time average of the power radiated in gravitational waves.

2. Show that the second-order terms in the expansion of the Ricci tensor around Minkowski spacetime are

$$\begin{aligned} R^{(2)}_{\mu\nu}[h] &= \frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} + \frac{1}{4}\partial_\mu h_{\rho\sigma}\partial_\nu h^{\rho\sigma} + \partial^\sigma h^\rho{}_\nu \partial_{[\sigma}h_{\rho]\mu} \\ &\quad + \frac{1}{2}\partial_\sigma(h^{\sigma\rho}\partial_\rho h_{\mu\nu}) - \frac{1}{4}\partial^\rho h \partial_\rho h_{\mu\nu} - \left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h\right)\partial_{(\mu}h_{\nu)\rho}\,. \end{aligned}$$

3. (a) Use the linearized Einstein equations to show that in vacuum

$$\langle \eta^{\mu\nu} R^{(2)}_{\mu\nu}[h]\rangle = 0\,.$$

(b) Show that

$$\langle t_{\mu\nu}\rangle = \frac{1}{32\pi}\langle \partial_\mu \bar{h}_{\rho\sigma}\partial_\nu \bar{h}^{\rho\sigma} - \frac{1}{2}\partial_\mu \bar{h}\,\partial_\nu \bar{h} - 2\partial_\sigma \bar{h}^{\rho\sigma}\partial_{(\mu}\bar{h}_{\nu)\rho}\rangle\,.$$

(c) Show that $\langle t_{\mu\nu}\rangle$ is gauge invariant.

4. Let η be a p-form and ω a q-form on a manifold $\mathcal{N}$. Show that the exterior derivative satisfies the properties $d(d\eta) = 0$, $d(\eta\wedge\omega) = (d\eta)\wedge\omega + (-1)^p\eta\wedge d\omega$ and $d(\phi^*\eta) = \phi^*(d\eta)$ where $\phi : \mathcal{M} \to \mathcal{N}$ for some manifold $\mathcal{N}$.

5. A three-sphere can be parametrized by Euler angles $(\theta,\ \phi,\ \psi)$ where $0 < \theta < \pi$, $0 < \phi < 2\pi$, $0 < \psi < 4\pi$. Define the following 1-forms

$$\sigma_1 = -\sin\psi\, d\theta + \cos\psi\,\sin\theta\, d\phi\,, \qquad \sigma_2 = \cos\psi\, d\theta + \sin\psi\,\sin\theta\, d\phi\,, \qquad \sigma_3 = d\psi + \cos\theta\, d\phi\,.$$

Show that $d\sigma_1 = \sigma_2\wedge\sigma_3$ with analogous results for $d\sigma_2$ and $d\sigma_3$.

6. For this question it may be helpful to recall questions 10 and 11 from example sheet 3. Consider a metric of Lorentzian signature $g_{\alpha\beta}$ and its determinant $g \equiv \det g_{\alpha\beta}$. Show that

$$\frac{\partial g}{\partial g_{\alpha\beta}} = g g^{\alpha\beta}\,,$$

$$\frac{\partial g}{\partial g^{\alpha\beta}} = -g g_{\alpha\beta}\,,$$

where $g^{\alpha\beta}$ denotes the inverse metric. Conclude that the variation of the determinant g can be expressed as

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\,\delta g^{\alpha\beta}\,.$$

7. Let $(\mathcal{N}, g)$ be a spacetime and the covariant derivative be given by the Levi-Civita connection. Let $t : \mathcal{N} \to \mathbb{R}$ be a foliation, Σ_t the spacelike hypersurfaces of this foliation and n be the unit normal field on the Σ_t. We define the *acceleration* as $a_b = n^c \nabla_c n_b$. Show that

$$a_b = D_b \ln \alpha \,,$$

where D_b is the covariant derivative associated with the induced metric γ_{ab} and α denotes the lapse function.

8. Let $(\mathcal{N}, g)$ be a spacetime and the covariant derivative be given by the Levi-Civita connection. Let $t : \mathcal{N} \to \mathbb{R}$ be a foliation, Σ_t the spacelike hypersurfaces of this foliation and n be the unit normal field on the Σ_t. Let γ_{ab} be the induced metric on the hypersurfaces and $m = \alpha n$ the normal evolution vector. Show that

(b) $\mathcal{L}_m \gamma_{ab} = -2\alpha K_{ab}\,,$
(c) $\mathcal{L}_n \gamma_{ab} = -2K_{ab}\,,$
(d) $\mathcal{L}_m \gamma^a{}_b = 0\,,$

where $\mathcal{L}_m$ and $\mathcal{L}_n$ denote the Lie derivative along the vector fields m and n, respectively, and K_{ab} is the extrinsic curvature.

9. The Lagrangian for the electromagnetic field is

$$L = -\frac{1}{16\pi} g^{ab} g^{cd} F_{ac} F_{bd}\,,$$

where F_{ab} is written in terms of a potential A_a as $F = dA$. Show that this Lagrangian reproduces the energy-momentum tensor for the Maxwell field that was discussed in lectures.

10. A test particle of rest mass m has a (timelike) world line $x^\mu(\lambda)$, $0 \le \lambda \le 1$ and action

$$S = -m \int d\tau \equiv -m \int_0^1 \sqrt{-g_{\mu\nu}(x(\lambda))\dot{x}^\mu \dot{x}^\nu}\, d\lambda\,,$$

where τ is proper time and a dot denotes a derivative with respect to λ.
(a) Show that varying this action with respect to $x^\mu(\lambda)$ leads to the geodesic equation.
(b) Show that the energy-momentum tensor of the particle in any chart is

$$T^{\mu\nu}(x) = \frac{m}{\sqrt{-g(x)}} \int u^\mu(\tau)\, u^\nu(\tau)\, \delta^4(x - x(\tau)) d\tau\,,$$

where u^μ is the 4-velocity of the particle.
(c) Conservation of the energy-momentum tensor is equivalent to the statement that

$$\int_R \sqrt{-g} v_\nu \nabla_\mu T^{\mu\nu} d^4x = 0\,,$$

for any vector field v^μ and region R. By choosing v^μ to be compactly supported in a region intersecting the particle world line, show that conservation of $T^{\mu\nu}$ implies that test particles move on geodesics. (This is an example of how the "geodesic postulate" of GR is a consequence of energy-momentum conservation.)

11. The action for *Brans-Dicke* theory of gravity is given by

$$S = \frac{1}{16\pi} \int \left[R\phi - \frac{\omega}{\phi} g^{ab} \phi_{,a} \phi_{,b} + 16\pi L_{\text{matter}} \right] \sqrt{-g} d^4x \,,$$

where ϕ is a scalar field and ω is a constant. Ordinary matter is included in the action L_{matter}. How is the Einstein equation modified, and what is the equation of motion for ϕ? (See Misner, Thorne and Wheeler or Carroll for further discussion of this theory.)

12. Calculate the extrinsic curvature tensor for a surface of constant t in the Schwarzschild spacetime

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \,.$$

Do the same for a surface of constant r.