General Relativity, Examples sheet 1 Dr U Sperhake

Comments and corrections: e-mail to `U.Sperhake@damtp.cam`.
Feel free to set $G = c = 1$ in any of your calculations, and reintroduce it if required using dimensional analysis.
Unless stated otherwise, the connection may be assumed to be the Levi-Civita one whose components are the Christoffel symbols: $\Gamma^{\alpha}_{\beta\gamma} = \left\{ {\alpha \atop \beta\,\gamma} \right\} = \frac{1}{2} g^{\alpha\mu}(\partial_{\beta} g_{\gamma\mu} + \partial_{\gamma} g_{\mu\beta} - \partial_{\mu} g_{\beta\gamma})$

1 By considering the motion of light in an accelerating lift in deep space and using the strong principle of equivalence, obtain an expression for the deflection of light moving approximately horizontally in a lift fixed on the surface of the Earth. Give your answer in the form

$$\frac{\Delta\theta}{\Delta\phi} = \frac{kGM}{Rc^2}$$

where M is the mass of the Earth, R is the radius of the Earth, k is a constant (to be determined), $\Delta\theta$ is the angle through which light is deflected when the lift subtends an angle of $\Delta\phi$ at the centre of the Earth.

2 Bob stays at home (assume that home is on the surface of the Earth, which is a non-rotating sphere of radius R_E) and his twin Alice orbits the Earth in a circular orbit of radius R. Show that using the gravitational redshift formula and the special relativistic time dilation formula leads to the result that Bob ages faster than Alice by a factor of

$$\frac{1 - \frac{GM}{R_E c^2}}{\left(1 - \frac{GM}{Rc^2}\right)^{\frac{3}{2}}}\,,$$

and hence that Bob and Alice age at the approximately the same rate if $R = \frac{3}{2} R_E$.

3 Write down the line element of (flat) Euclidian 3-space in cylindrical polar coordinates (r, ϕ, z). Show that the 2-dimensional line element

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - r_s/r} + r^2 \mathrm{d}\phi^2,$$

where r_s is a constant, can be regarded as the line element on a surface of revolution $z = f(r)$ in Euclidean 3-space (where the function $f(r)$ is to be found). Sketch the surface, and comment on the behaviour as r is reduced towards r_s.

4 Starting with the flat space-time line element in cylindrical polar coordinates (t, r, ϕ, z), obtain the *Langevin* line element for a rotating observer

$$ds^2 = -dt^2(c^2 - r^2\omega^2) + dr^2 + 2r^2\omega d\theta dt + r^2 d\theta^2 + dz^2,$$

where $\theta = \phi - \omega t$ and ω is constant.

A perfect fibre-optic cable is laid round the Earth's equator ($r = R$, $z = 0$) and two photons travel round the cable in opposite directions at the speed of light (so that $ds^2 = 0$ on their trajectories), starting from the same point. Show that one photon will arrive back at its starting point (for the first time) a time Δt before the other, where

$$\Delta t = \frac{4\pi\omega R^2}{c^2 - \omega^2 R^2}.$$

.

5 A curve $x^{\alpha}(\mu)$ is said to be *geodesic* if there exists a function $f(\mu)$ such that

$$\ddot{x}^{\alpha} + \left\{ {\alpha \atop \beta\,\gamma} \right\} \dot{x}^{\beta} \dot{x}^{\gamma} = f(\mu)\, \dot{x}^{\alpha}\,,$$

where the dot denotes denotes differentiation with respect to μ and $\left\{ {\alpha \atop \beta\,\gamma} \right\}$ is the Christoffel symbol. If in addition $f(\mu) = 0$ the geodesic is said to be *affinely parametrized*.

Show that, by changing to a new paramater $\lambda(\mu)$, any geodesic can be affinely parametrized. If λ is an affine parameter, show that any other affine parameter can be written in the form $A\lambda + B$, where A and B are constants (and $A \neq 0$).

6 Suppose that a system is described by a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ and that L is homogeneous of degree k in the 'velocities' $\dot{q}_i$, i.e.

$$\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = kL\,. \tag{1}$$

Show for $k \neq 1$ that L is conserved (i.e., $\mathrm{d}L/\mathrm{d}\lambda = 0$) along the extremal curves (solutions of the Euler-Lagrange equations) $\mathbf{q}(\lambda)$. In the case $L > 0$, show that L^2 is an equivalent Lagrangian. Here "equivalent" means that the extremal curves of L and L^2 are the same.

7 Obtain the geodesic equations for the Langevin metric (in question 4) in the form of four first integrals. What is their physical significance?

Show that for a particle moving initially with speed v in the radial direction the initial accelerations (measured in coordinate time t) in the radial and θ directions are $r\omega^2$ and $-2\omega v$. How should this be interpreted?

8 Two metrics $g_{\alpha\beta}$ and $\widehat{g}_{\alpha\beta}$ are *conformally related* if $\widehat{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ for some scalar function Ω. Show that their Christoffel symbols $\left\{{}^{\,\alpha}_{\beta\,\gamma}\right\}$ and $\widehat{\left\{{}^{\,\alpha}_{\beta\,\gamma}\right\}}$ are related by

$$\widehat{\left\{{}^{\,\alpha}_{\beta\,\gamma}\right\}} = \left\{{}^{\,\alpha}_{\beta\,\gamma}\right\} + \Omega^{-1}(\delta^{\alpha}{}_{\beta}\Omega_{,\gamma} + \delta^{\alpha}{}_{\gamma}\Omega_{,\beta} - g^{\alpha\delta}g_{\beta\gamma}\Omega_{,\delta}).$$

In Nordstrøm's theory of gravity the metric is given by $g_{\alpha\beta} = e^{2\varphi}\eta_{\alpha\beta}$, where φ is a scalar function of position and $\eta_{\alpha\beta}$ is the Minkowski metric. Compute the equation of a geodesic in Nordstrøm's theory and use your result to show that $T^\alpha T^\beta g_{\alpha\beta}$, where T^α is the tangent vector corresponding to an affine parameter, is constant on the geodesic.

Using the results of question 5, show that a null geodesic is also a null geodesic in Minkowski spacetime, and hence deduce that light rays are not subject to gravitational deflection.

Show that for any time-like geodesic, with suitably chosen parameter μ,

$$\frac{d^2x^\alpha}{d\mu^2} = -\eta^{\alpha\gamma}\psi_{,\gamma}$$

for some function ψ.

9 2-dimensional *de Sitter space-time* has the line element

$$\mathrm{d}s^2 = -\mathrm{d}u^2 + \cosh^2 u\, \mathrm{d}\phi^2\,,$$

where $-\infty < u < \infty$ and $0 \leq \phi < 2\pi$. Compute the Christoffel symbols and hence the geodesic equations. Verify that the equations of an affinely parametrized geodesic $x^\alpha = x^\alpha(\lambda)$ can be derived from the variational principle

$$\delta \int (\dot{u}^2 - \cosh^2 u\, \dot{\phi}^2)\, \mathrm{d}\lambda = 0,$$

where $\dot{u} = \mathrm{d}u/\mathrm{d}\lambda$ etc. Verify from the variational principle that there are two first integrals

$$\cosh^2 u\, \dot{\phi} = K,$$
$$\cosh^2 u\, \dot{u}^2 = K^2 + L\cosh^2 u,$$

along the geodesic, where K and L are constants.

Show that if $K = 0$ the geodesics are $\phi = \text{const}$. If $K \neq 0$, show that λ may be eliminated in favour of ϕ as a parameter along the geodesic and obtain the equation

$$v'^2 = M^2 - v^2,$$

where $v = \tanh u$, $v' = \mathrm{d}v/\mathrm{d}\phi$ and M is a constant depending on L and K. Hence show that the $K \neq 0$ geodesics are given by

$$\tanh u = M\sin(\phi - \phi_0),$$

where ϕ_0 is a constant. Show also that $M^2 > 1$ for timelike geodesics, $M^2 = 1$ for null geodesics and $M^2 < 1$ for spacelike geodesics. Regarding u and ϕ as cartesian coordinates, sketch the set of geodesics starting from $(0, 0)$.

Show from your diagram that no two such timelike geodesics will meet again, but that spacelike geodesics may recross each other. Demonstrate also that there are pairs of points which cannot be joined by a geodesic. Which, if any, of these statements would be valid in Minkowski spacetime?