Comments and corrections: e-mail to `U.Sperhake@damtp.cam`.

Feel free to use either commas and semi-colons or ∂ *and* ∇ *for derivatives.*

Unless stated otherwise, the connection may be assumed to be the Levi-Civita one whose components are the Christoffel symbols: $\Gamma^{\alpha}_{\beta\gamma} = \left\{ {\alpha \atop \beta\,\gamma} \right\} = \frac{1}{2} g^{\alpha\mu}(\partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma})$

1 Let $\Gamma^{\alpha}_{\beta\gamma}$ be a connection (torsion free, but not necessarily Levi-Civita) in coordinates x^α. Show that under the coordinate transformation

$$\bar{x}^\delta = x^\delta + \tfrac{1}{2}\Gamma^{\delta}_{\beta\gamma} x^\beta x^\gamma$$

the transformed connection vanishes at the origin. The transformation law for a connection is

$$B_\beta{}^\mu B_\gamma{}^\nu \bar{\Gamma}^{\delta}_{\mu\nu} = \Gamma^{\alpha}_{\beta\gamma}\, B_\alpha{}^\delta - \frac{\partial B_\gamma{}^\delta}{\partial x^\beta} \qquad \text{where} \qquad B_\alpha{}^\delta = \frac{\partial \bar{x}^\delta}{\partial x^\alpha}.$$

What further coordinate transformation is needed to reduce the coordinates to Local Inertial Coordinates at the origin?

2 The Lie derivative $(\mathcal{L}_k Y)^\alpha$ of a vector field Y^α with respect to a vector field k^α is defined by the following conditions:

(a) Let x^α be any coordinate system in which $k^\alpha = (1, 0, 0, 0)$. Then

$$(\mathcal{L}_k Y)^\alpha = \frac{\partial Y^\alpha}{\partial x^0} \equiv k^\beta \frac{\partial Y^\alpha}{\partial x^\beta}.$$

(b) $(\mathcal{L}_k Y)^\alpha$ transforms as a vector.

Show that, in a general coordinate system $\bar{x}^{\bar{\alpha}}$,

$$(\mathcal{L}_{\bar{k}} \bar{Y})^{\bar{\alpha}} = \bar{k}^{\bar{\beta}} \nabla_{\bar{\beta}} \bar{Y}^{\bar{\alpha}} - \bar{Y}^{\bar{\beta}} \nabla_{\bar{\beta}} \bar{k}^{\bar{\alpha}}$$

Hint: You need to write condition (a) in terms of tensor quantities; your solution should be no more than four lines. Work back from the answer if you can't see how to to it.

Given that the Lie derivative $\mathcal{L}_k\phi$ of a scalar field ϕ with respect to a vector field k^α is defined in a general coordinate system x^α by

$$\mathcal{L}_k \phi = k^\alpha \frac{\partial \phi}{\partial x^\alpha}$$

and that the Lie derivative obeys the usual Leibniz rule when applied to a product, find the Lie derivative $(\mathcal{L}_k Z)_\alpha$ of a covector field Z_α with respect to a vector field k^α.

Write down an expression for the Lie derivative with respect to k^α of a $\binom{0}{2}$ tensor and hence write down the Lie derivative of the metric tensor.

3 Show that, if a vector S^α is parallely transported along an affinely parametrized geodesic γ with tangent vector T^α, then $S_\alpha T^\alpha$ is constant on γ.

Consider the parallel transport of a vector S^α round a closed path on the unit 2-sphere consisting of the following four segments:

(i) $\theta = \frac{1}{2}\pi$, $\phi_0 \geq \phi \geq 0$; (ii) $\frac{1}{2}\pi \geq \theta \geq \theta_0$, $\phi = 0$; (iii) $\theta = \theta_0$, $0 \leq \phi \leq \phi_0$; (iv) $\theta_0 \leq \theta \leq \frac{1}{2}\pi$, $\phi = \phi_0$; where θ and ϕ are the normal polar coordinates. The starting point is $(\frac{1}{2}\pi, \phi_0)$ (on the equator), where $S^\alpha = (S^\theta, S^\phi) = (1, 0)$.

(a) Sketch a picture in the case $\theta_0 = 0$ (so the path is a spherical triangle with one vertex at the North pole) using the result of the first paragraph (no further calculation required) and hence show that the angle between the initial and final vectors S^α is proportional to the area enclosed by the path.

(b) Verify that for $0 < \theta_0 < \frac{1}{2}\pi$ the parallel transport equations have solutions:

$S^\alpha = (1, 0)$ on (i); $S^a = (1, 0)$ on (ii); $S^a = (\cos(\phi \cos\theta_0), -\sin(\phi\cos\theta_0)/\sin\theta_0)$ on (iii);

and $S^a = (\cos(\phi_0\cos\theta_0), -\sin(\phi_0\cos\theta_0)/\sin\theta)$ on (iv).

Hence find S^α at the end point. Check that, when $\theta_0 \to 0$, your answer agrees with your answer to part (a).

Note: $\Gamma^\theta{}_{\phi\phi} = -\sin\theta\,\cos\theta$ and $\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta$.

4 The metric $g_{\alpha\beta}(x)$ has the property that, if each point x^a is mapped to $\phi^a(x)$, distances are unaltered. Such a mapping is called an *isometry* for this metric. Show that

$$g_{\alpha\beta}\big(\phi(x)\big)\frac{\partial\phi^\alpha}{\partial x^\gamma}\frac{\partial\phi^\beta}{\partial x^\delta} = g_{\gamma\delta}(x).$$

Setting $\phi^\alpha(x) = x^\alpha + \epsilon\xi^\alpha(x)$, where ϵ is small, show that to first order in ϵ

$$\xi^\gamma\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + g_{\gamma\beta}\frac{\partial\xi^\gamma}{\partial x^\alpha} + g_{\gamma\alpha}\frac{\partial\xi^\gamma}{\partial x^\beta} = 0.$$

Show also, either by calculating the Christoffel symbols or by using locally inertial coordinates, that this condition can be written in tensorial form as

$$\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha = 0.$$

This is known as *Killing's equation* and solutions ξ_α are called *Killing covector fields.*

If $\xi^\alpha = (1, \mathbf{0})$ is a Killing vector field, show that $\dfrac{\partial g_{\alpha\beta}}{\partial x^0} = 0$.

5 Let (x, y) be standard Cartesian coordinates in (flat) $\mathbb{R}^2$, and let $\bar{x} = x - y\cot\alpha$ and $\bar{y} = y\,\mathrm{cosec}\,\alpha$, where α a constant. Calculate the Jacobian matrix of this transformation and use it to obtain the metric in the new coordinates. Show that

$$ds^2 = d\bar{x}^2 + d\bar{y}^2 + 2\cos\alpha\,d\bar{x}d\bar{y}. \qquad (*)$$

Identify the new axes, and obtain $(*)$ geometrically. If $X^i = (\bar{x}, \bar{y})$, what is the geometrical significance of the components of X_i?

6 The elements M_{ab} of a non-singular matrix are functions of t. Show that the derivative of its determinant M is given by $\dot{M} = M(M^{-1})_{ba}\dot{M}_{ab}$, where the dot denotes the derivative with respect to t. (Note that, in the determinant, the term M_{ab} occurs multiplied by its cofactor.)

Use this result to show that the determinant g of $g_{\alpha\beta}$ (the metric tensor) satisfies

$$2g\Gamma^{\beta}_{\beta\alpha} = \frac{\partial g}{\partial x^{\alpha}},$$

where $\Gamma^{\beta}_{\beta\alpha} = \left\{{}_{\beta}{}^{\beta}{}_{\alpha}\right\}$ is the Christoffel symbol.

A *tensor density* of weight n is a quantity that transforms as a tensor under coordinate transformations $x^{\alpha} = x^{\alpha}(\bar{x}^{\bar{\beta}})$ but with an additional factor of A^n, where A is the Jacobian of the transformation:

$$A = \det A_{\bar{\beta}}{}^{\alpha}, \qquad A_{\bar{\beta}}{}^{\alpha} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\bar{\beta}}}.$$

Show that g transforms as a scalar density of weight 2.

The covariant derivative of a scalar density is defined as follows. Let ψ be a scalar density of weight n. Then

$$\nabla_{\alpha}\psi = \frac{\partial \psi}{\partial x^{\alpha}} - n\Gamma^{\beta}_{\beta\alpha}\psi.$$

Show that, with this definition, $\nabla_{\alpha}\psi$ transforms as a covector density of weight n.

Show further that $\nabla_{\alpha} g = 0$.

7 A static space-time has line element

$$\mathrm{d}s^2 = -e^{2\phi/c^2}c^2\mathrm{d}t^2 + h_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

where ϕ and h_{ij} are functions only of $\mathbf{x}$, and i, $j = 1, 2, 3$. Show that

$$\Gamma^0_{0i} = \frac{1}{c^2}\frac{\partial \phi}{\partial x^i}$$

and evaluate Γ^0_{00}, Γ^i_{00} and Γ^0_{ij}, where $\Gamma^{\alpha}_{\beta\gamma}$ is the Levi-Civita connection.

An observer is at rest with 4-velocity $V^{\alpha} = \mathrm{d}x^{\alpha}/\mathrm{d}\tau$, where τ is proper time. Use the normalisation condition $V^{\alpha}V_{\alpha} = -c^2$ to write down V^0 and V_0.

Show that the 4-acceleration A^{α}, defined by $A^{\alpha} = V^{\beta}\nabla_{\beta}V^{\alpha}$, is given by $A_{\alpha} = \dfrac{\partial \phi}{\partial x^{\alpha}}$.

8 In the static space-time of question 7, let $\xi^{\alpha} = (1, \mathbf{0})$. Show that ξ_{α} satisfies

$$\nabla_{\alpha}\xi_{\beta} + \nabla_{\beta}\xi_{\alpha} = 0\,.$$

A massive freely falling particle moves along an affinely parametrized geodesic γ with tangent vector V^{α}. Let $E = \xi_{\alpha}V^{\alpha}$. Show that E is constant along γ, and interpret this result physically.

9 Let ξ_{α} be a Killing covector field as defined in question 4. Use the Ricci identity and $R^{\alpha}{}_{[\beta\gamma\delta]} = 0$ to show that

$$\xi_{\alpha;\beta\gamma} = -R^{\delta}{}_{\gamma\alpha\beta}\,\xi_{\delta}\ .$$

In the case of Minkowski space, integrate this equation twice to obtain the 10 independent Killing vectors.