Gravitational Waves and Numerical Relativity: Example Sheet 1

Part III, Easter Term 2024

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1. Riemann tensor in linearized theory

Show that in linearized theory, the components of the Riemann tensor,

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\mu\rho})\,,$$

are invariant under first-order coordinate transformations $\tilde{x}^\alpha = x^\alpha - \xi^\alpha$ where $\xi^\alpha = \mathcal{O}(\epsilon)$, $\epsilon \ll 1$.

2. Advection equation

Consider the *advection equation*

$$\partial_t f + \lambda \partial_x f = 0\,, \qquad \lambda \in \mathbb{R}\,,$$

for a function f on the domain $(t, x) \in \mathbb{R}^2$.

(i) Show that f remains constant along curves $x(t) = \lambda t + x_0$, where $x_0 = \mathrm{const}$. These are the *characteristic curves* of the advection equation.

(ii) Consider Gaussian initial data

$$f(0, x) = e^{-(x-x_0)^2}\,.$$

Sketch the characteristic curves in the (t, x) plane (with t pointing upwards and x horizontally. Also graphically sketch the solution $f(t, x)$. Quantitative precision is not required in this sketch and there are multiple ways to generate a graphic illustration; one figure is sufficient.

(iii) Now consider the case of a varying $\lambda = \lambda(t, x)$ where we write the advection equation in the form

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\big(\lambda(t, x) f\big) = 0\,.$$

The *characteristic curves* of this differential equation are solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda(t, x)\,.$$

Show that the advection equation can be written as an ordinary differential equation along the characteristic curves.

3. Burgers' equation

A more dramatic variation of the advection equation arises when we allow the *flux* term λf to be a non-linear function of the evolution variable f, i.e. where λ itself depends on f. This results in so-called *quasi-linear* PDEs which are linear in the derivatives but not in the function f itself. The prototypical example for this type of PDEs is *Burgers' equation*, where $\lambda(f) = \frac{1}{2}f$. In a more common notation, this is written as $F := \lambda f = \frac{1}{2}f^2$, so that

$$\partial_t f + \partial_x F = \partial_t f + \partial_x\left(\tfrac{1}{2}f^2\right) = 0\,.$$

(i) Determine the characteristic curves for this PDE, i.e. the curves along which the PDE can be written as an ordinary differential equation. Write down this ordinary differential equation.

(ii) Sketch the characteristic curves in the (t, x) plane (with t pointing upwards and x horizontal) for Gaussian initial data

$$f(0, x) = e^{-(x-x_0)^2}\,.$$

(iii) Sketch the time evolution of the Gaussian initial data. Compare the result with that obtained for the advection equation in Question 2.

4. Quasi-linear first-order PDE systems

Let $\boldsymbol{f} : \mathbb{R}^N \to \mathbb{R}^M$ and let us use the notation

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_M \end{pmatrix}, \qquad \partial_\alpha \boldsymbol{f} := \frac{\partial}{\partial x_\alpha}\boldsymbol{f} = \begin{pmatrix} \partial_\alpha f_1 \\ \vdots \\ \partial_\alpha f_M \end{pmatrix}, \quad \alpha = 1, \ldots, N\,.$$

The general *quasi-linear first-order PDE* for $\boldsymbol{f}$ is

$$\mathbf{A}^\mu(\boldsymbol{x}, \boldsymbol{f})\partial_\mu \boldsymbol{f} + \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{f}) = 0\,,$$

$$\Leftrightarrow A^\mu_{mn}(x_\alpha, f_i)\partial_\mu f_n + b_m(x_\alpha, f_i) = 0 \quad (\text{sum over } \mu,\, m,\, n)\,, \qquad (\dagger)$$

where $\boldsymbol{b}$ is a vector valued function, each $\mathbf{A}^\mu$ is an $M \times M$ matrix and initial data for $\boldsymbol{f}$ are given on a surface S defined as the level set of a function $\theta(x^\alpha) = 0$ with $\nabla\theta \neq 0$. We introduce coordinates adapted to this hypersurface S by

$$\xi^\alpha = \xi^\alpha(x^\mu) \qquad \text{for } \alpha = 1, \ldots, N-1\,,$$

$$\xi^N = \theta(x^\mu)\,.$$

(i) Show that the initial data $\boldsymbol{f}$ on S determine all derivatives $\partial f_i/\partial \xi_\alpha$ for $\alpha = 1, \ldots, N-1$. Show that the PDE (†) also determines the derivative $\partial f_i/\partial \xi_N$ if and only if

$$\det\left(\mathbf{A}^\mu \frac{\partial\theta}{\partial x_\mu}\right) \neq 0\,.$$

(ii) The *characteristic equation* associated with the PDE (†) is

$$\det\left(\mathbf{A}^\mu \frac{\partial\theta}{\partial x_\mu}\right) = 0\,, \qquad (*)$$

and a level surface S defined through $\theta(x_\alpha) = 0$ by a solution to this equation is a *characteristic surface*.

(a) Determine the matrices $\mathbf{A}^\alpha$ for the advection equation $\partial_t f + \partial_x f = 0$ and write down the characteristic equation. Note that the $\mathbf{A}^\alpha$ are just scalars in this case.

(b) Introducing $\psi := \partial_t f$ and $\lambda := \partial_x f$, write the 1+1 dimensional wave equation $\partial_t^2 f - c^2\partial_x^2 f = 0,\ c > 0$ as a first-order system of two equations, one each for $\partial_t\psi$ and $\partial_t\lambda$. [*Hint: Partial derivatives commute.*] Determine the $\mathbf{A}^\alpha$ and the characteristic equation.

(c) Introducing $\psi := \partial_x f$ and $\lambda := \partial_y f$, write the 2 dimensional Laplace equation $\partial_x^2 f + \partial_y^2 f = 0$ as a first-order system of two equations, one each for $\partial_x\psi$ and $\partial_x\lambda$. Determine the $\mathbf{A}^\alpha$ and the characteristic equation.

5. Classification of first-order PDE systems

(*This question directly proceeds from question 4 and employs all definitions therefrom.)*

For the classification of first-order PDE systems, we define for an N dimensional vector $\boldsymbol{\zeta}$,

$$\mathcal{C}(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{\zeta}) := \det\left[\mathbf{A}^{\mu}(\boldsymbol{x}, \boldsymbol{f})\zeta_{\mu}\right] .$$

We furthermore introduce a linear mapping

$$\boldsymbol{\zeta} = \mathbf{M}\boldsymbol{\eta} , \quad \text{where} \quad \boldsymbol{\eta} = (\eta_1, \ldots, \eta_{N-1}, \kappa) ,$$

with a non-degenerate matrix $\mathbf{M}$, and write

$$\mathcal{C}(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{\eta}) = \mathcal{C}(\boldsymbol{x}, \boldsymbol{f}, \eta_1, \ldots, \eta_{N-1}, \kappa) := \mathcal{C}\big(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{\zeta}(\eta_1, \ldots, \eta_{N-1}, \kappa)\big) .$$

This reparametrization of the vector $\boldsymbol{\zeta}$ is necessary to single out a specific parameter κ that does not necessarily coincide with a single component of $\boldsymbol{\zeta}$ in the coordinates x_{μ}.

We then define the PDE

$$\mathbf{A}^{\mu}(\boldsymbol{x}, \boldsymbol{f})\partial_{\mu}\boldsymbol{f} + \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{f}) = 0$$

to be

- *hyperbolic* at $\boldsymbol{x}$ if there exists a regular linear mapping $\boldsymbol{\zeta} = \mathbf{M}\boldsymbol{\eta}$, such that there exist M *real* roots $\kappa_i = \kappa_i(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}), \eta_1, \ldots, \eta_{N-1})$, $i = 1, \ldots, M$ of $\mathcal{C}(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{\eta}) = 0$ for all $(\eta_1, \ldots, \eta_{n-1})$. Note that the number of roots required equals the number of independent variables in $\boldsymbol{f}$, not the dimensionality N of the domain.
- *parabolic* at $\boldsymbol{x}$ if there exists a linear mapping $\boldsymbol{\zeta} = \mathbf{M}\boldsymbol{\eta}$ such that $\mathcal{C}$ is independent of κ, i.e. depends on fewer than N parameters.
- *elliptic* if $\mathcal{C}(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{\zeta}) = 0$ only if $\boldsymbol{\zeta} = 0$.

Determine the type of the advection equation, the 2-dimensional wave equation and 2-dimensional Laplace equation from question 4 according to this classification.

6. The constraint equations

Let $(\mathcal{M}, \boldsymbol{g})$ be a globally hyperbolic spacetime with a foliation Σ_t. Let $x^{\alpha} = (t, x^i)$ denote coordinates adapted to this foliation.

(i) Show that the components of the Ricci tensor can be written as

$$\begin{aligned} R_{00} &= -\frac{1}{2}g^{mn}\partial_0^2 g_{mn} + M_{00} , \\ R_{0i} &= \frac{1}{2}g^{0m}\partial_0^2 g_{im} + M_{0i} , \\ R_{ij} &= -\frac{1}{2}g^{00}\partial_0^2 g_{ij} + M_{ij} , \end{aligned}$$

where M_{00}, M_{0i} and M_{ij} are a collection of terms that include at most first time derivatives of the metric $g_{\alpha\beta}$.

(ii) Show that the components $G_{\alpha}{}^{0}$ of the Einstein tensor can be written as

$$\begin{aligned} G_0{}^0 &= \frac{1}{2}g^{00}M_{00} - \frac{1}{2}g^{mn}M_{mn} , \\ G_i{}^0 &= g^{00}M_{0i} + g^{0m}M_{im} , \end{aligned}$$

and therefore contain no second time derivative of the metric.

7. The Bondi metric

The Bondi metric is given by

$$g_{\alpha\beta} = \begin{pmatrix} -\frac{V}{r}e^{2\beta} + r^2U^2e^{2\gamma} & -e^{2\beta} & -r^2Ue^{2\gamma} & 0 \\ -e^{2\beta} & 0 & 0 & 0 \\ -r^2Ue^{2\gamma} & 0 & r^2e^{2\gamma} & 0 \\ 0 & 0 & 0 & r^2e^{-2\gamma}\sin^2\theta \end{pmatrix}. \tag{1}$$

Use the cofactor matrices to compute the inverse Bondi metric $g^{\alpha\beta}$.

8. Bianchi identities

Le $\mathcal{M}$ be a Lorentzian manifold with metric $g_{\alpha\beta}$. Show that the contracted Bianchi identities $\nabla^\mu G_{\alpha\mu}$ can be written as

$$g^{\mu\rho}\left(\partial_\rho R_{\alpha\mu} - \Gamma^\sigma_{\mu\rho}R_{\alpha\sigma} - \frac{1}{2}\partial_\alpha R_{\mu\rho}\right).$$