Gravitational Waves and Numerical Relativity: Example Sheet 1 Part III, Easter Term 2025 U. Sperhake

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1. Riemann tensor in linearized theory

Show that in linearized theory, the components of the Riemann tensor,

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} (\partial_{\rho}\partial_{\nu}h_{\mu\sigma} + \partial_{\sigma}\partial_{\mu}h_{\nu\rho} - \partial_{\rho}\partial_{\mu}h_{\nu\sigma} - \partial_{\sigma}\partial_{\nu}h_{\mu\rho}),$$

are invariant under first-order coordinate transformations $\tilde{x}^{\alpha} = x^{\alpha} - \xi^{\alpha}$ where $\xi^{\alpha} = \mathcal{O}(\epsilon), \ \epsilon \ll 1$.

2. Advection equation

Consider the advection equation

$$\partial_t f + \lambda \partial_x f = 0, \quad \lambda \in \mathbb{R},$$

for a function *f* on the domain $(t, x) \in \mathbb{R}^2$.

- (i) Show that *f* remains constant along curves $x(t) = \lambda t + x_0$, where $x_0 = \text{const.}$ These are the *characteristic curves* of the advection equation.
- (ii) Consider Gaussian initial data

$$f(0,x) = e^{-(x-x_0)^2}$$
.

Sketch the characteristic curves in the (t, x) plane (with t pointing upwards and x horizontally. Also graphically sketch the solution f(t, x). Quantitative precision is not required in this sketch and there are multiple ways to generate a graphic illustration; one figure is sufficient.

(iii) Now consider the case of a varying $\lambda = \lambda(t, x)$ where we write the advection equation in the form

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x} (\lambda(t, x)f) = 0$$

The *characteristic curves* of this differential equation are solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda(t, x)$$

Show that the advection equation can be written as an ordinary differential equation along the characteristic curves.

3. Burgers' equation

A more dramatic variation of the advection equation arises when we allow the *flux* term λf to be a non-linear function of the evolution variable f, i.e. where λ itself depends on f. This results in so-called *quasi-linear* PDEs which are linear in the derivatives but not in the function f itself. The prototypical example for this type of PDEs is *Burgers' equation*, where $\lambda(f) = \frac{1}{2}f$. In a more common notation, this is written as $F := \lambda f = \frac{1}{2}f^2$, so that

$$\partial_t f + \partial_x F = \partial_t f + \partial_x \left(\frac{1}{2}f^2\right) = 0.$$

- (i) Determine the characteristic curves for this PDE, i.e. the curves along which the PDE can be written as an ordinary differential equation. Write down this ordinary differential equation.
- (ii) Sketch the characteristic curves in the (t, x) plane (with *t* pointing upwards and *x* horizontal) for Gaussian initial data

$$f(0,x) = e^{-(x-x_0)^2}$$
.

(iii) Sketch the time evolution of the Gaussian initial data. Compare the result with that obtained for the advection equation in Question 2.

4. Quasi-linear first-order PDE systems

Let $\boldsymbol{f}:\mathbb{R}^N \to \mathbb{R}^M$ and let us use the notation

$$oldsymbol{f} = egin{pmatrix} f_1 \ dots \ f_M \end{pmatrix}, \qquad \partial_lpha oldsymbol{f} := rac{\partial}{\partial x_lpha} oldsymbol{f} = egin{pmatrix} \partial_lpha f_1 \ dots \ \partial_lpha f_M \end{pmatrix}, \quad lpha = 1, \dots, N.$$

The general quasi-linear first-order PDE for f is

$$\mathbf{A}^{\mu}(\boldsymbol{x},\boldsymbol{f})\partial_{\mu}\boldsymbol{f} + \boldsymbol{b}(\boldsymbol{x},\boldsymbol{f}) = 0,$$

$$\Leftrightarrow A^{\mu}_{mn}(x_{\alpha},f_{i})\partial_{\mu}f_{n} + b_{m}(x_{\alpha},f_{i}) = 0 \quad (\text{sum over } \mu, m, n), \qquad (\dagger)$$

where **b** is a vector valued function, each \mathbf{A}^{μ} is an $M \times M$ matrix and initial data for **f** are given on a surface *S* defined as the level set of a function $\theta(x^{\alpha}) = 0$ with $\nabla \theta \neq 0$. We introduce coordinates adapted to this hypersurface *S* by

$$\xi^{\alpha} = \xi^{\alpha}(x^{\mu})$$
 for $\alpha = 1, ..., N - 1$,
 $\xi^{N} = \theta(x^{\mu})$.

(i) Show that the initial data f on S determine all derivatives $\partial f_i / \partial \xi_\alpha$ for $\alpha = 1, ..., N - 1$. Show that the PDE (†) also determines the derivative $\partial f_i / \partial \xi_N$ if and only if

$$\det\left(\mathbf{A}^{\mu}\frac{\partial\theta}{\partial x_{\mu}}\right)\neq0\,.$$

(ii) The *characteristic equation* associated with the PDE ([†]) is

$$\det\left(\mathbf{A}^{\mu}\frac{\partial\theta}{\partial x_{\mu}}\right) = 0, \qquad (*)$$

and a level surface *S* defined through $\theta(x_{\alpha}) = 0$ by a solution to this equation is a *characteristic surface*.

- (a) Determine the matrices \mathbf{A}^{α} for the advection equation $\partial_t f + \partial_x f = 0$ and write down the characteristic equation. Note that the \mathbf{A}^{α} are just scalars in this case.
- (b) Introducing $\psi := \partial_t f$ and $\lambda := \partial_x f$, write the 1+1 dimensional wave equation $\partial_t^2 f c^2 \partial_x^2 f = 0$, c > 0 as a first-order system of two equations, one each for $\partial_t \psi$ and $\partial_t \lambda$. [*Hint: Partial derivatives commute.*] Determine the \mathbf{A}^{α} and the characteristic equation.
- (c) Introducing $\psi := \partial_x f$ and $\lambda := \partial_y f$, write the 2 dimensional Laplace equation $\partial_x^2 f + \partial_y^2 f = 0$ as a first-order system of two equations, one each for $\partial_x \psi$ and $\partial_x \lambda$. Determine the \mathbf{A}^{α} and the characteristic equation.

5. Classification of first-order PDE systems

(This question directly proceeds from question 4 and employs all definitions therefrom.)

For the classification of first-order PDE systems, we define for an N dimensional vector $\boldsymbol{\zeta}$,

$$\mathcal{C}(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{\zeta}) \coloneqq \det \left[\mathsf{A}^{\mu}(\boldsymbol{x}, \boldsymbol{f}) \zeta_{\mu} \right]$$

We furthermore introduce a linear mapping

$$\boldsymbol{\zeta} = \mathbf{M} \boldsymbol{\eta}, \text{ where } \boldsymbol{\eta} = (\eta_1, \ldots, \eta_{N-1}, \kappa),$$

with a non-degenerate matrix **M**, and write

$$\mathcal{C}(\boldsymbol{x},\boldsymbol{f},\boldsymbol{\eta}) = \mathcal{C}(\boldsymbol{x},\boldsymbol{f},\eta_1,\ldots,\eta_{N-1},\kappa) \coloneqq \mathcal{C}(\boldsymbol{x},\boldsymbol{f},\boldsymbol{\zeta}(\eta_1,\ldots,\eta_{N-1},\kappa))$$

This reparametrization of the vector ζ is necessary to single out a specific parameter κ that does not necessarily coincide with a single component of ζ in the coordinates x_{μ} .

We then define the PDE

$$\mathbf{A}^{\mu}(\boldsymbol{x},\boldsymbol{f})\partial_{\mu}\boldsymbol{f} + \boldsymbol{b}(\boldsymbol{x},\boldsymbol{f}) = 0$$

to be

- *hyperbolic* at *x* if there exists a regular linear mapping *ζ* = Mη, such that there exist *M* real roots κ_i = κ_i(*x*, *f*(*x*), η₁,..., η_{N-1}), *i* = 1,..., *M* of C(*x*, *f*, η) = 0 for all (η₁,..., η_{n-1}). Note that the number of roots required equals the number of independent variables in *f*, not the dimensionality *N* of the domain.
- *parabolic* at *x* if there exists a linear mapping *ζ* = Mη such that *C* is independent of *κ*, i.e. depends on fewer than *N* parameters.
- *elliptic* if $C(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{\zeta}) = 0$ only if $\boldsymbol{\zeta} = 0$.

Determine the type of the advection equation, the 2-dimensional wave equation and 2-dimensional Laplace equation from question 4 according to this classification.

6. The constraint equations

Let (\mathcal{M}, g) be a globally hyperbolic spacetime with a foliation Σ_t . Let $x^{\alpha} = (t, x^i)$ denote coordinates adapted to this foliation.

(i) Show that the components of the Ricci tensor can be written as

$$R_{00} = -\frac{1}{2}g^{mn}\partial_0^2 g_{mn} + M_{00}$$

$$R_{0i} = \frac{1}{2}g^{0m}\partial_0^2 g_{im} + M_{0i},$$

$$R_{ij} = -\frac{1}{2}g^{00}\partial_0^2 g_{ij} + M_{ij},$$

where M_{00} , M_{0i} and M_{ij} are a collection of terms that include at most first time derivatives of the metric $g_{\alpha\beta}$.

(ii) Show that the components G_{α}^{0} of the Einstein tensor can be written as

$$G_0^{\ 0} = \frac{1}{2}g^{00}M_{00} - \frac{1}{2}g^{mn}M_{mn},$$

$$G_i^{\ 0} = g^{00}M_{0i} + g^{0m}M_{im},$$

and therefore contain no second time derivative of the metric.

7. The Bondi metric

The Bondi metric is given by

$$g_{\alpha\beta} = \begin{pmatrix} -\frac{V}{r}e^{2\beta} + r^2U^2e^{2\gamma} & -e^{2\beta} & -r^2Ue^{2\gamma} & 0\\ -e^{2\beta} & 0 & 0 & 0\\ -r^2Ue^{2\gamma} & 0 & r^2e^{2\gamma} & 0\\ 0 & 0 & 0 & r^2e^{-2\gamma}\sin^2\theta \end{pmatrix}.$$
 (1)

Use the cofactor matrices to compute the inverse Bondi metric $g^{\alpha\beta}.$

8. Bianchi identities

Le \mathcal{M} be a Lorentzian manifold with metric $g_{\alpha\beta}$. Show that the contracted Bianchi identities $\nabla^{\mu}G_{\alpha\mu}$ can be written as

$$g^{\mu
ho}\left(\partial_{
ho}R_{lpha\mu}-\Gamma^{\sigma}_{\mu
ho}R_{lpha\sigma}-rac{1}{2}\partial_{lpha}R_{\mu
ho}
ight)\,.$$