Gravitational Waves and Numerical Relativity: Example Sheet 2

Part III, Easter Term 2024
U. Sperhake

Comments are welcomed and may be sent to U.Sperhake@damtp.cam.ac.uk.

1. Series expansion

Consider the leading-order expansion of the functions

$$\begin{aligned}\gamma(u,r,\theta) &= c(u,\theta)r^{-1} + \mathcal{O}(r^{-2})\,,\\ \beta(u,r,\theta) &= H(u,\theta) - c(u,\theta)r^{-2} + \mathcal{O}(r^{-3})\,,\end{aligned}$$

and the second main equation of the Bondi formalism,

$$\partial_r\left[r^4 e^{2(\gamma-\beta)}\partial_r U\right] - 2r^2\left[\partial_r\partial_\theta\beta - \partial_r\partial_\theta\gamma + 2\partial_r\gamma\,\partial_\theta\gamma - 2\cot\theta\,\partial_r\gamma - 2\frac{\partial_\theta\beta}{r}\right] = 0\,, \qquad (1)$$

Taking into account functions of integration, show that the leading-order behaviour of the function $U(u,r,\theta)$ is given by

$$U(u,r,\theta) = L(u,\theta) + 2e^{2H(u,\theta)}\partial_\theta H(u,\theta)\,r^{-1} + \mathcal{O}(r^{-2})\,. \qquad (2)$$

2. The metric and its determinant

Let $g_{\alpha\beta}$ be a metric of Lorentzian signature $g_{\alpha\beta}$ and $g \equiv \det g_{\alpha\beta}$ its determinant. Show that

$$\begin{aligned}\frac{\partial g}{\partial g_{\alpha\beta}} &= g g^{\alpha\beta}\,,\\ \frac{\partial g}{\partial g^{\alpha\beta}} &= -g g_{\alpha\beta}\,,\end{aligned}$$

where $g^{\alpha\beta}$ denotes the inverse metric. Conclude that the derivative of the determinant g can be written as

$$\partial_\alpha g = g\,g^{\mu\nu}\partial_\alpha g_{\mu\nu} = -g\,g_{\mu\nu}\partial_\alpha g^{\mu\nu} = 2g\Gamma^\mu_{\mu\alpha}\,.$$

3. Absence of conical singularities

The Bondi metric for an axisymmetric spacetime that is invariant under azimuthal reflection is

$$\mathrm{d}s^2 = \left(-\frac{V}{r}e^{2\beta} + U^2r^2e^{2\gamma}\right)\mathrm{d}u^2 - 2e^{2\beta}\mathrm{d}u\,\mathrm{d}r - 2Ur^2e^{2\gamma}\mathrm{d}u\,\mathrm{d}\theta + r^2(e^{2\gamma}\mathrm{d}\theta^2 + e^{-2\gamma}\sin^2\theta\,\mathrm{d}\phi^2)\,,$$

where β, γ, U and V are functions of the coordinates (u,r,θ). Consider the case of asymptotically flat spacetimes with no incoming radiation, so that the series expansion of γ at large radius r is given by

$$\gamma(u,r,\theta) = c(u,\theta)r^{-1} + \mathcal{O}(r^{-3})\,,$$

where $c(u,\theta)$ is the Bondi news function.

Show that for spacetimes with no conical singularity on the polar axis, the news function satisfies

$$\lim_{\theta\to 0} c = \lim_{\theta\to\pi} c = 0\,.$$

[*Hint: Consider spheres of constant u and r in the limit of large r. Compute the proper circumference and the proper radius of circles with small constant $\theta = \Delta\theta$ around the North Pole and show that their ratio equals 2π if and only if the first of the above conditions is satisfied. Repeat the same for the South Pole.*]

4. The Bondi mass

Let $(\mathcal{M}, \boldsymbol{g})$ be an asymptotically flat, axisymmetric spacetime that is invariant under azimuthal reflection and contains no conical simgularities. In the Bondi formalism for such spacetimes, the mass aspect $M(u,\theta)$ arises as one of the functions of integration. By the supplementary equations, the evolution of the mass aspect is given in terms of the Bondi news $c(u,\theta)$ by

$$\partial_u M = -\partial_u c + \frac{1}{2}\partial_\theta^2\partial_u c + \frac{3}{2}\cot\theta\,\partial_\theta\partial_u c - (\partial_u c)^2\,.$$

The Bondi mass is defined as the integral of the mass aspect,

$$m(u) := \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} M\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,.$$

Using the result from question 3, show that the Bondi mass evolves according to

$$\partial_u m = -\frac{1}{2}\int_0^\pi (\partial_u c)^2\sin\theta\,\mathrm{d}\theta\,.$$

Briefly interprete this result.

5. Spatial covariant derivative

Let $(\mathcal{M}, \boldsymbol{g})$ be a globally hyperbolic spacetime with Levi Civita connection ∇_α and spatial hypersurface Σ (i.e. the gradient $\mathbf{d}t$ is timelike), and let $T^{\alpha\ldots}{}_{\beta\ldots}$ be a rank $\binom{r}{s}$ tensor that is tangent to Σ in all components, i.e. $T^{\alpha\ldots}{}_{\beta\ldots}n_\alpha = \ldots = T^{\alpha\ldots}{}_{\beta\ldots}n^\beta = \ldots = 0$. The three-dimensional or spatial covariant derivative of $\boldsymbol{T}$ is defined as the rank $\binom{r}{s+1}$ tensor

$$D_\mu T^{\alpha\ldots}{}_{\beta\ldots} := \perp^\rho{}_\mu \perp^\alpha{}_\sigma \perp^\tau{}_\beta \ldots \nabla_\rho T^{\sigma\ldots}{}_{\tau\ldots}\,.$$

(i) Show that D_μ satisfies the defining criteria for a covariant derivative, i.e. for spatial vector fields $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{V}$ and a scalar function f on Σ the following conditions hold,

$$(1)\qquad D_\mu f = \partial_\mu f\,,$$

$$(2)\qquad D_{f\boldsymbol{X}+g\boldsymbol{Y}}\boldsymbol{V} = fD_{\boldsymbol{X}}\boldsymbol{V} + gD_{\boldsymbol{Y}}\boldsymbol{V}\,,$$

$$(3)\qquad D_{\boldsymbol{X}}\boldsymbol{V} + D_{\boldsymbol{X}}W = D_{\boldsymbol{X}}(\boldsymbol{V}+\boldsymbol{W})\,,$$

$$(4)\qquad D_{\boldsymbol{X}}(f\boldsymbol{V}) = fD_{\boldsymbol{X}}\boldsymbol{V} + (D_{\boldsymbol{X}}f)\boldsymbol{V}\,.$$

(ii) Show that D_μ is compatible with the three-dimensional (spatial) metric $\gamma_{\alpha\beta}$, i.e. $D_\mu\gamma_{\alpha\beta} = 0$.

(iii) The torsion tensor associated with D_μ is defined by

$$\boldsymbol{T} : (\boldsymbol{X}, \boldsymbol{Y}) \mapsto \boldsymbol{T}(\boldsymbol{X}, \boldsymbol{Y}) = D_{\boldsymbol{X}}\boldsymbol{Y} - D_{\boldsymbol{Y}}\boldsymbol{X} - [\boldsymbol{X}, \boldsymbol{Y}]$$

$$\text{or} \quad T_{\mu\nu}{}^{\alpha} X^\mu Y^\nu = X^\mu D_\mu Y^\alpha - Y^\mu D_\mu X^\alpha - [\boldsymbol{X}, \boldsymbol{Y}]^\alpha ,$$

where $\boldsymbol{X}$, $\boldsymbol{Y}$ are spatial vector fields and $[\boldsymbol{X}, \boldsymbol{Y}]^\alpha = X^\mu \partial_\mu Y^\alpha - Y^\mu \partial_\mu X^\alpha$ is the commutator of $\boldsymbol{X}$ and $\boldsymbol{Y}$. Show that D_μ is torsion free, i.e. its torsion tensor vanishes.

6. Lie derivative of the projector

Given a hypersurface Σ of a globally hyperbolic spacetime with unit normal n^α and projector $\perp^\alpha{}_\beta = \delta^\alpha{}_\beta + n^\alpha n_\beta$, show that

$$\begin{aligned} \mathcal{L}_{\boldsymbol{n}} \perp^{\alpha\beta} &= n^\alpha a^\beta + n^\beta a^\alpha + 2K^{\alpha\beta} , \\ \mathcal{L}_{\boldsymbol{n}} \perp^\alpha{}_\beta &= n^\alpha a_\beta , \\ \mathcal{L}_{\boldsymbol{n}} \perp_{\alpha\beta} &= -2K_{\alpha\beta} , \end{aligned}$$

where $\mathcal{L}_{\boldsymbol{n}}$ is the Lie derivative along n^α. Conclude that for a spatial tensor $T_{\alpha\beta}$,

$$\mathcal{L}_{\boldsymbol{n}} T_{\alpha\beta} = \perp^\mu{}_\alpha \perp^\nu{}_\beta \mathcal{L}_{\boldsymbol{n}} T_{\mu\nu} ,$$

i.e. the tensor's Lie derivative along $\boldsymbol{n}$ is also spatial.

7. Ricci equation

Let Σ be a spatial hypersurface of a globally hyperbolic spacetime with unit normal n^α, acceleration vector $a_\alpha = n^\mu \nabla_\mu n_\alpha$ and extrinsic curvature $K_{\alpha\beta}$.

(i) By projecting the (spacetime) Ricci identity applied to the unit normal,

$$\nabla_\rho \nabla_\sigma n^\mu - \nabla_\sigma \nabla_\rho n^\mu = R^\mu{}_{\nu\rho\sigma} n^\nu ,$$

twice onto space and once onto time, show that

$$\perp_{\alpha\mu} n^\nu \perp^\rho{}_\gamma n^\sigma R^\mu{}_{\nu\rho\sigma} = -K_{\alpha\sigma} K^\sigma{}_\gamma + D_\gamma a_\alpha + a_\alpha a_\gamma + \perp^\mu{}_\alpha \perp^\rho{}_\gamma n^\sigma \nabla_\sigma K_{\rho\mu} . \tag{†}$$

(ii) Compute the spatial projection of the Lie derivative of the extrinsic curvature along the unit normal, $\perp^\mu{}_\alpha \perp^\nu{}_\beta \mathcal{L}_{\boldsymbol{n}} K_{\mu\nu}$, to substitute for the last term on the right-hand side of Eq. (†), and thus derive the Ricci equation

$$\perp^\mu{}_\alpha n^\nu \perp^\rho{}_\gamma n^\sigma R_{\mu\nu\rho\sigma} = \mathcal{L}_{\boldsymbol{n}} K_{\alpha\gamma} + \frac{1}{\alpha} D_\alpha D_\gamma \alpha + K_{\rho\gamma} K_\alpha{}^\rho .$$

You may use without proof that $a_\mu = D_\mu(\ln \alpha)$, where α is the lapse function.

8. The evolution equation for the energy density

The energy- and momentum density and stress associated with the energy momentum tensor $T_{\alpha\beta}$ are defined as

$$\rho := T_{\mu\nu} n^\mu n^\nu , \qquad j_\alpha := -\perp^\mu{}_\alpha T_{\mu\nu} n^\nu , \qquad S_{\alpha\beta} := \perp^\mu{}_\alpha \perp^\nu{}_\beta T_{\mu\nu} ,$$

$$\Leftrightarrow \quad T_{\alpha\beta} = \rho n_\alpha n_\beta + j_\alpha n_\beta + n_\alpha j_\beta + S_{\alpha\beta} ,$$

where n^μ denotes the unit normal of a spatial hypersurface Σ and $\perp^\mu{}_\alpha = \delta^\mu{}_\alpha + n^\mu n_\alpha$. The equation for conservation of energy-momentum is $\nabla_\mu T^\mu{}_\alpha = 0$.

By projecting the energy conservation law onto n^α, derive the evolution equation for the energy density,

$$\mathcal{L}_{\boldsymbol{n}}\rho = -2j^\mu D_\mu(\ln\alpha) + \rho K + S^{\mu\nu}K_{\mu\nu} - D_\mu j^\mu\,.$$

Show that in coordinates adapted to the spacetime foliation, this equation becomes

$$\partial_t\rho = \beta^m\partial_m\rho - 2j^m D_m\alpha + \alpha\big(\rho K + S^{mn}K_{mn} - D_m j^m\big)\,.$$

Here K is the trace of the extrinsic curvature, α the lapse and β^μ the shift vector.

9. Strong hyperbolicity

Consider the partial differential equation

$$\mathbf{A}\partial_t\boldsymbol{u} + \mathbf{P}^i\partial_i\boldsymbol{u} + \mathbf{C}\boldsymbol{u} = 0 \quad \text{(sum over } i)\,, \tag{$\star$}$$

for a function $\boldsymbol{u} : \Omega \subset \mathbb{R}^{d+1} \to \mathbb{R}^N$, where $\mathbf{A}$, $\mathbf{P}^i$, $\mathbf{C}$ are real $N\times N$ matrices and $\mathbf{A}$ is invertible. This PDE is weakly hyperbolic if for all vectors $\hat{k}^i$ with $|\hat{k}| = 1$, all Eigenvalues of

$$\mathbf{Q}(\hat{k}_i) := -\mathbf{A}^{-1}\mathbf{P}^m\hat{k}_m$$

are real. The PDE is strongly hyperbolic if furthermore $\mathbf{Q}(\hat{k}^i)$ is diagonalizable for all $\hat{k}^i$.

(i) Let $\mathbf{J}_2 = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$. Show that

$$e^{\mathrm{i}\mathbf{J}_2\hat{t}} = e^{\mathrm{i}\lambda\hat{t}}\begin{pmatrix}1 & \mathrm{i}\hat{t}\\ 0 & 1\end{pmatrix}\,.$$

(ii) Assume that the PDE ($\star$) is weakly hyperbolic but that the Jordan normal form of $\mathbf{Q}(\hat{k}^i)$ contains a Jordan block of the form $\mathbf{J}_2$. Argue why for such a PDE, there exists no regular function $f(t)$ such that

$$\left|\left|e^{\mathrm{i}\mathbf{M}t}\right|\right| \leq f(t)\,, \tag{‡}$$

where $\mathbf{M} = \mathbf{A}^{-1}(-\mathbf{P}^m k_m + \mathrm{i}\mathbf{C})$.