# Gravitational Waves and Numerical Relativity: Example Sheet 2 Part III, Easter Term 2025 U. Sperhake

*Comments are welcome and may be sent to U.Sperhake@damtp.cam.ac.uk.* 

#### 1. Series expansion

Consider the leading-order expansion of the functions

$$\gamma(u, r, \theta) = c(u, \theta)r^{-1} + \mathcal{O}(r^{-2}),$$
  
$$\beta(u, r, \theta) = H(u, \theta) - c(u, \theta)r^{-2} + \mathcal{O}(r^{-3})$$

and the second main equation of the Bondi formalism,

$$\partial_r \left[ r^4 e^{2(\gamma - \beta)} \partial_r U \right] - 2r^2 \left[ \partial_r \partial_\theta \beta - \partial_r \partial_\theta \gamma + 2 \partial_r \gamma \, \partial_\theta \gamma - 2 \cot \theta \, \partial_r \gamma - 2 \frac{\partial_\theta \beta}{r} \right] = 0 \,, \tag{1}$$

Taking into account functions of integration, show that the leading-order behaviour of the function  $U(u, r, \theta)$  is given by

$$U(u, r, \theta) = L(u, \theta) + 2e^{2H(u, \theta)} \partial_{\theta} H(u, \theta) r^{-1} + \mathcal{O}(r^{-2}).$$
<sup>(2)</sup>

#### 2. The metric and its determinant

Let  $g_{\alpha\beta}$  be a metric of Lorentzian signature  $g_{\alpha\beta}$  and  $g \equiv \det g_{\alpha\beta}$  its determinant. Show that

$$\begin{split} \frac{\partial g}{\partial g_{\alpha\beta}} &= g g^{\alpha\beta} \,, \\ \frac{\partial g}{\partial g^{\alpha\beta}} &= -g g_{\alpha\beta} \,, \end{split}$$

where  $g^{\alpha\beta}$  denotes the inverse metric. Conclude that the derivative of the determinant g can be written as

$$\partial_{\alpha}g = g g^{\mu\nu}\partial_{\alpha}g_{\mu\nu} = -g g_{\mu\nu}\partial_{\alpha}g^{\mu\nu} = 2g\Gamma^{\mu}_{\mu\alpha}.$$

## 3. Absence of conical singularities

The Bondi metric for an axisymmetric spacetime that is invariant under azimuthal reflection is

$$ds^{2} = \left(-\frac{V}{r}e^{2\beta} + U^{2}r^{2}e^{2\gamma}\right)du^{2} - 2e^{2\beta}du\,dr - 2Ur^{2}e^{2\gamma}du\,d\theta + r^{2}(e^{2\gamma}d\theta^{2} + e^{-2\gamma}\sin^{2}\theta\,d\phi^{2})\,,$$

where  $\beta$ ,  $\gamma$ , U and V are functions of the coordinates  $(u, r, \theta)$ . Consider the case of asymptotically flat spacetimes with no incoming radiation, so that the series expansion of  $\gamma$  at large radius r is given by

$$\gamma(u, r, \theta) = c(u, \theta)r^{-1} + \mathcal{O}(r^{-3}),$$

where  $c(u, \theta)$  is the Bondi news function.

Show that for spacetimes with no conical singularity on the polar axis, the news function satisfies

$$\lim_{\theta \to 0} c = \lim_{\theta \to \pi} c = 0.$$

[Hint: Consider spheres of constant u and r in the limit of large r. Compute the proper circumference and the proper radius of circles with small constant  $\theta = \Delta \theta$  around the North Pole and show that their ratio equals  $2\pi$  if and only if the first of the above conditions is satisfied. Repeat the same for the South Pole.]

### 4. The Bondi mass

Let  $(\mathcal{M}, \boldsymbol{g})$  be an asymptotically flat, axisymmetric spacetime that is invariant under azimuthal reflection and contains no conical simgularities. In the Bondi formalism for such spacetimes, the mass aspect  $M(u, \theta)$  arises as one of the functions of integration. By the supplementary equations, the evolution of the mass aspect is given in terms of the Bondi news  $c(u, \theta)$  by

$$\partial_u M = -\partial_u c + \frac{1}{2} \partial_\theta^2 \partial_u c + \frac{3}{2} \cot \theta \partial_\theta \partial_u c - (\partial_u c)^2.$$

The Bondi mass is defined as the integral of the mass aspect,

$$m(u) := \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} M \sin \theta \mathrm{d}\phi \mathrm{d}\theta \,.$$

Using the result from question 3, show that the Bondi mass evolves according to

$$\partial_u m = -\frac{1}{2} \int_0^\pi (\partial_u c)^2 \sin\theta \mathrm{d}\theta$$

Briefly interprete this result.

#### 5. Spatial covariant derivative

Let  $(\mathcal{M}, \boldsymbol{g})$  be a globally hyperbolic spacetime with Levi Civita connection  $\nabla_{\alpha}$  and spatial hypersurface  $\Sigma$  (i.e. the gradient dt is timelike), and let  $T^{\alpha \dots}{}_{\beta \dots}$  be a rank  $\binom{r}{s}$  tensor that is tangent to  $\Sigma$  in all components, i.e.  $T^{\alpha \dots}{}_{\beta \dots}n_{\alpha} = \dots = T^{\alpha \dots}{}_{\beta \dots}n^{\beta} = \dots = 0$ . The three-dimensional or spatial covariant derivative of  $\boldsymbol{T}$  is defined as the rank  $\binom{r}{s+1}$  tensor

$$D_{\mu}T^{\alpha\ldots}{}_{\beta\ldots} := \bot^{\rho}{}_{\mu}\bot^{\alpha}{}_{\sigma}\bot^{\tau}{}_{\beta}\ldots\nabla_{\rho}T^{\sigma\ldots}{}_{\tau\ldots}$$

(i) Show that  $D_{\mu}$  satisfies the defining criteria for a covariant derivative, i.e. for spatial vector fields X, Y, V and a scalar function f on  $\Sigma$  the following conditions hold,

- (1)  $D_{\mu}f = \partial_{\mu}f$ ,
- (2)  $D_{f\boldsymbol{X}+g\boldsymbol{Y}}\boldsymbol{V} = fD_{\boldsymbol{X}}\boldsymbol{V} + gD_{\boldsymbol{Y}}\boldsymbol{V},$
- (3)  $D_{\boldsymbol{X}}\boldsymbol{V} + D_{\boldsymbol{X}}W = D_{\boldsymbol{X}}(\boldsymbol{V} + \boldsymbol{W}),$

(4) 
$$D_{\boldsymbol{X}}(f\boldsymbol{V}) = fD_{\boldsymbol{X}}\boldsymbol{V} + (D_{\boldsymbol{X}}f)\boldsymbol{V}$$

(ii) Show that  $D_{\mu}$  is compatible with the three-dimensional (spatial) metric  $\gamma_{\alpha\beta}$ , i.e.  $D_{\mu}\gamma_{\alpha\beta} = 0$ .

(iii) The torsion tensor associated with  $D_{\mu}$  is defined by

$$T: (X, Y) \mapsto T(X, Y) = D_X Y - D_Y X - [X, Y]$$

or 
$$T_{\mu\nu}{}^{\alpha}X^{\mu}Y^{\nu} = X^{\mu}D_{\mu}Y^{\alpha} - Y^{\mu}D_{\mu}X^{\alpha} - [\boldsymbol{X},\boldsymbol{Y}]^{\alpha}$$

where X, Y are spatial vector fields and  $[X, Y]^{\alpha} = X^{\mu}\partial_{\mu}Y^{\alpha} - Y^{\mu}\partial_{\mu}X^{\alpha}$  is the commutator of X and Y. Show that  $D_{\mu}$  is torsion free, i.e. its torsion tensor vanishes.

#### 6. Lie derivative of the projector

Given a hypersurface  $\Sigma$  of a globally hyperbolic spacetime with unit normal  $n^{\alpha}$  and projector  $\perp^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + n^{\alpha}n_{\beta}$ , show that

$$\mathcal{L}_{\boldsymbol{n}} \perp^{\alpha \beta} = n^{\alpha} a^{\beta} + n^{\beta} a^{\alpha} + 2K^{\alpha \beta} ,$$
$$\mathcal{L}_{\boldsymbol{n}} \perp^{\alpha}{}_{\beta} = n^{\alpha} a_{\beta} ,$$
$$\mathcal{L}_{\boldsymbol{n}} \perp_{\alpha \beta} = -2K_{\alpha \beta} ,$$

where  $\mathcal{L}_{n}$  is the Lie derivative along  $n^{\alpha}$ . Conclude that for a spatial tensor  $T_{\alpha\beta}$ ,

$$\mathcal{L}_{\boldsymbol{n}} T_{\alpha\beta} = \bot^{\mu}{}_{\alpha} \bot^{\nu}{}_{\beta} \mathcal{L}_{\boldsymbol{n}} T_{\mu\nu} \,,$$

i.e. the tensor's Lie derivative along *n* is also spatial.

#### 7. Ricci equation

Let  $\Sigma$  be a spatial hypersurface of a globally hyperbolic spacetime with unit normal  $n^{\alpha}$ , acceleration vector  $a_{\alpha} = n^{\mu} \nabla_{\mu} n_{\alpha}$  and extrinsic curvature  $K_{\alpha\beta}$ .

(i) By projecting the (spacetime) Ricci identity applied to the unit normal,

$$\nabla_{\rho} \nabla_{\sigma} n^{\mu} - \nabla_{\sigma} \nabla_{\rho} n^{\mu} = R^{\mu}{}_{\nu\rho\sigma} n^{\nu} \,,$$

twice onto space and once onto time, show that

$$\perp_{\alpha\mu} n^{\nu} \perp^{\rho} {}_{\gamma} n^{\sigma} R^{\mu}{}_{\nu\rho\sigma} = -K_{\alpha\sigma} K^{\sigma}{}_{\gamma} + D_{\gamma} a_{\alpha} + a_{\alpha} a_{\gamma} + \perp^{\mu} {}_{\alpha} \perp^{\rho} {}_{\gamma} n^{\sigma} \nabla_{\sigma} K_{\rho\mu} \,. \tag{\dagger}$$

(ii) Compute the spatial projection of the Lie derivative of the extrinsic curvature along the unit normal,  $\perp^{\mu}{}_{\alpha}\perp^{\nu}{}_{\beta}\mathcal{L}_{n}K_{\mu\nu}$ , to substitute for the last term on the right-hand side of Eq. (†), and thus derive the Ricci equation

$$\perp^{\mu}{}_{\alpha}n^{\nu}\perp^{\rho}{}_{\gamma}n^{\sigma}R_{\mu\nu\rho\sigma} = \mathcal{L}_{n}K_{\alpha\gamma} + \frac{1}{\alpha}D_{\alpha}D_{\gamma}\alpha + K_{\rho\gamma}K_{\alpha}{}^{\rho}.$$

You may use without proof that  $a_{\mu} = D_{\mu}(\ln \alpha)$ , where  $\alpha$  is the lapse function.

#### 8. The evolution equation for the energy density

The energy- and momentum density and stress associated with the energy momentum tensor  $T_{\alpha\beta}$  are defined as

$$\begin{split} \rho &:= T_{\mu\nu} n^{\mu} n^{\nu} , \quad j_{\alpha} := - \bot^{\mu}{}_{\alpha} T_{\mu\nu} n^{\nu} , \quad S_{\alpha\beta} := \bot^{\mu}{}_{\alpha} \bot^{\nu}{}_{\beta} T_{\mu\nu} , \\ \Leftrightarrow \quad T_{\alpha\beta} &= \rho n_{\alpha} n_{\beta} + j_{\alpha} n_{\beta} + n_{\alpha} j_{\beta} + S_{\alpha\beta} , \end{split}$$

where  $n^{\mu}$  denotes the unit normal of a spatial hypersurface  $\Sigma$  and  $\perp^{\mu}{}_{\alpha} = \delta^{\mu}{}_{\alpha} + n^{\mu}n_{\alpha}$ . The equation for conservation of energy-momentum is  $\nabla_{\mu}T^{\mu}{}_{\alpha} = 0$ .

By projecting the energy conservation law onto  $n^{\alpha}$ , derive the evolution equation for the energy density,

$$\mathcal{L}_{\boldsymbol{n}}\rho = -2j^{\mu}D_{\mu}(\ln\alpha) + \rho K + S^{\mu\nu}K_{\mu\nu} - D_{\mu}j^{\mu}.$$

Show that in coordinates adapted to the spacetime foliation, this equation becomes

$$\partial_t \rho = \beta^m \partial_m \rho - 2j^m D_m \alpha + \alpha \left(\rho K + S^{mn} K_{mn} - D_m j^m\right).$$

Here *K* is the trace of the extrinsic curvature,  $\alpha$  the lapse and  $\beta^{\mu}$  the shift vector.

## 9. Strong hyperbolicity

Consider the partial differential equation

$$\mathbf{A}\partial_t \boldsymbol{u} + \mathbf{P}^i \partial_i \boldsymbol{u} + \mathbf{C} \boldsymbol{u} = 0 \quad (\text{sum over } i), \qquad (\star)$$

for a function  $\boldsymbol{u} : \Omega \subset \mathbb{R}^{d+1} \to \mathbb{R}^N$ , where  $\boldsymbol{A}, \boldsymbol{P}^i, \boldsymbol{C}$  are real  $N \times N$  matrices and  $\boldsymbol{A}$  is invertible. This PDE is weakly hyperbolic if for all vectors  $\hat{k}^i$  with  $|\hat{k}| = 1$ , all Eigenvalues of

$$\mathbf{Q}(\hat{k}_i) \coloneqq -\mathbf{A}^{-1}\mathbf{P}^m\hat{k}_m$$

are real. The PDE is strongly hyperbolic if furthermore  $\mathbf{Q}(\hat{k}^i)$  is diagonalizable for all  $\hat{k}^i$ .

(i) Let 
$$\mathbf{J}_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$
. Show that  
 $e^{i\mathbf{J}_2\hat{t}} = e^{i\lambda\hat{t}} \begin{pmatrix} 1 & i\hat{t} \\ 0 & 1 \end{pmatrix}$ 

(ii) Assume that the PDE ( $\star$ ) is weakly hyperbolic but that the Jordan normal form of  $\mathbf{Q}(\hat{k}^i)$  contains a Jordan block of the form  $\mathbf{J}_2$ . Argue why for such a PDE, there exists no regular function f(t) such that

$$\left| \left| e^{\mathbf{i} \mathbf{M} t} \right| \right| \le f(t) \,, \tag{\ddagger}$$

where  $\mathbf{M} = \mathbf{A}^{-1}(-\mathbf{P}^m k_m + \mathrm{i}\mathbf{C}).$