Gravitational Waves and Numerical Relativity: Example Sheet 3

Part III, Easter Term 2025

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Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

1. The BSSNOK equations 1

The BSSNOK variables are defined by

$$\chi = \gamma^{-1/3}\,, \qquad K = \gamma^{mn}K_{mn},$$

$$\tilde{\gamma}_{ij} = \chi\gamma_{ij} \qquad \Leftrightarrow \quad \tilde{\gamma}^{ij} = \frac{1}{\chi}\gamma^{ij},$$

$$\tilde{A}_{ij} = \chi\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) \qquad \Leftrightarrow \quad K_{ij} = \frac{1}{\chi}\left(\tilde{A}_{ij} + \frac{1}{3}\tilde{\gamma}_{ij}K\right)\,,$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn}\,,$$

Given the ADM equations

$$\mathcal{H} \coloneqq \mathcal{R} + K^2 - K_{mn}K^{mn} - 2\Lambda - 16\pi\rho = 0\,,$$

$$\mathcal{M}_i \coloneqq D_iK - D_mK_i{}^m + 8\pi j_i = 0\,,$$

$$\partial_t\gamma_{ij} = \beta^m\partial_m\gamma_{ij} + 2\gamma_{m(i}\partial_{j)}\beta^m - 2\alpha K_{ij}\,,$$

$$\partial_t K_{ij} = \beta^m\partial_m K_{ij} + 2K_{m(i}\partial_{j)}\beta^m - D_iD_j\alpha + \alpha\left[\mathcal{R}_{ij} + KK_{ij} - 2K_{im}K^m{}_j\right]$$

$$- \alpha\Lambda\gamma_{ij} - 8\pi\alpha\left[S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right]\,,$$

derive the following BSSNOK equations,

$$\mathcal{H} = \mathcal{R} - \tilde{A}^{mn}\tilde{A}_{mn} + \frac{2}{3}K^2 - 2\Lambda - 16\pi\rho = 0\,,$$

$$\mathcal{M}_i = \frac{2}{3}\partial_iK - \tilde{\gamma}^{mn}\tilde{D}_m\tilde{A}_{in} + \frac{3}{2}\tilde{A}_i{}^m\frac{\partial_m\chi}{\chi} + 8\pi j_i = 0\,,$$

$$\partial_t\chi = \beta^m\partial_m\chi - \frac{2}{3}\chi\partial_m\beta^m + \frac{2}{3}\alpha\chi K\,,$$

$$\partial_t\tilde{\gamma}_{ij} = \beta^m\partial_m\tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{\gamma}_{ij}\partial_m\beta^m - 2\alpha\tilde{A}_{ij}\,,$$

$$\partial_t\tilde{A}_{ij} = \beta^m\partial_m\tilde{A}_{ij} + 2\tilde{A}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{A}_{ij}\partial_m\beta^m + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{im}\tilde{A}^m{}_j$$

$$+ \chi\left(\alpha\mathcal{R}_{ij} - D_iD_j\alpha - 8\pi S_{ij}\right)^{\mathrm{TF}}\,.$$

∗ **2. The BSSNOK equations 2**

Derive the remaining BSSNOK equation

$$\partial_t\tilde{\Gamma}^i = \beta^m\partial_m\tilde{\Gamma}^i - \tilde{\Gamma}^m\partial_m\beta^i + \frac{2}{3}\tilde{\Gamma}^i\partial_m\beta^m + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + \frac{1}{3}\tilde{\gamma}^{im}\partial_m\partial_n\beta^n + 2\alpha\tilde{\Gamma}^i_{mn}\tilde{A}^{mn}$$
$$-2\tilde{A}^{im}\partial_m\alpha - \frac{4}{3}\alpha\tilde{\gamma}^{im}\partial_m K - 3\alpha\tilde{A}^{im}\frac{\partial_m\chi}{\chi} - 16\pi\alpha\tilde{\gamma}^{im}j_m - \sigma\mathcal{G}^i\,,$$

where $\mathcal{G}^i := \Gamma^i - \tilde{\gamma}^{mn}\Gamma^i_{mn}$ and σ is some function.

3. Constraints in the conformal traceless split

The Hamiltonian and momentum constraints in the ADM formalism are

$$\mathcal{H} := \mathcal{R} + K^2 - K_{mn}K^{mn} - 2\Lambda - 16\pi\rho = 0\,,$$
$$\mathcal{M}_i := D_iK - D_mK_i{}^m + 8\pi j_i = 0\,.$$

In the conformal-traceless York-Lichnerowicz split, we introduce the conformal spatial metric $\bar{\gamma}_{ij}$ and the conformal traceless extrinsic curvature through the definitions

$$\gamma_{ij} = \psi^4\bar{\gamma}_{ij} \qquad \Leftrightarrow \qquad \gamma^{ij} = \psi^{-4}\bar{\gamma}^{ij}\,,$$
$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K\,,$$
$$A_{ij} = \psi^{-2}\bar{A}_{ij} \qquad \Leftrightarrow \qquad A^{ij} = \psi^{-10}\bar{A}^{ij}\,.$$

We furthermore split the extrinsic curvature variable $\bar{A}_{ij}$ into a transverse traceless and a longitudinal part according to

$$\bar{A}_{ij} = Q_{ij} + (\mathbb{L}X)_{ij} := Q_{ij} + \bar{D}_iX_j + \bar{D}_jX_i - \frac{2}{3}\bar{\gamma}_{ij}\bar{D}_mX^m\,,$$

where Q_{ij} is symmetric, transverse ($\bar{D}_mQ^{mi} = 0$) and traceless ($Q^m{}_m = 0$), and X_i is a vector potential. $\bar{D}_i$ and $\bar{\mathcal{R}}_{ij}$ are the covariant derivative and Ricci tensor associated with $\bar{\gamma}_{ij}$.

Show that the Hamiltonian and momentum constraints expressed in these new variables are given by

$$\bar{\mathcal{H}} = 8\bar{\gamma}^{mn}\bar{D}_m\bar{D}_n\psi - \psi\bar{\mathcal{R}} - \frac{2}{3}\psi^5K^2 + \psi^{-7}\bar{A}_{mn}\bar{A}^{mn} + 2\psi^5\Lambda + 16\pi\psi^5\rho = 0\,,$$
$$\bar{\mathcal{M}}^i = \bar{D}^m\bar{D}_mX^i + \frac{1}{3}\bar{D}^i\bar{D}_mX^m + \bar{\mathcal{R}}^i{}_mX^m - \frac{2}{3}\psi^6\bar{\gamma}^{mi}\partial_mK - 8\pi\psi^{10}j^i = 0\,.$$

You may use without proof general relations from the lecture for conformal transformations but should state these clearly.

4. Isotropic Schwarzschild

(i) Show that the spatial part of the Schwarzschild metric in isotropic coordinates,

$$\mathrm{d}s^2 = -\left(\frac{2r-M}{2r+M}\right)^2\mathrm{d}t^2 + \left(1+\frac{M}{2r}\right)^4(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2)\,,$$

solves the constraint equations in their conformal transverse traceless form (i.e. the constraint equations derived in the previous question) for the case of time symmetry ($K_{ij} = 0$), conformal flatness ($\bar{\gamma}_{ij} = \delta_{ij}$), asymptotic flatness ($\Lambda = 0$) and vacuum ($j_i = 0$).

(ii) Give a brief physical interpretation of the point $r = \sqrt{x^2 + y^2 + z^2} = 0$, for example in terms of its location in the Kruskal diagram. [*Hint: Write the isotropic Schwarzschild metric in spherical coordinates and consider the coordinate transformation* $\varrho = M^2/(4r)$.]

5. Bowen-York data

Consider the momentum constraint in the conformal traceless split,

$$\bar{\mathcal{M}}^i = \bar{D}^m \bar{D}_m X^i + \frac{1}{3}\bar{D}^i \bar{D}_m X^m + \bar{\mathcal{R}}^i{}_m X^m - \frac{2}{3}\psi^6 \bar{\gamma}^{mi}\partial_m K - 8\pi\psi^{10} j^i = 0\,.$$

(i) Write down the momentum constraint for the case of vacuum ($\rho = 0,\ j_i = 0$), asymptotic flatness ($\Lambda = 0$), conformal flatness ($\bar{\gamma}_{ij} = \delta_{ij}$), vanishing trace of the extrinsic curvature and a vanishing transverce traceless part ($K = 0,\ Q_{ij} = 0$).

(ii) We introduce the vector potential

$$X^i = \epsilon^{ijk}\frac{x_j}{r^3}J_k\,, \qquad (\dagger)$$

where ϵ^{ijk} is the totally antisymmetric Levi-Civita symbol, $x_j = (x, y, z)$ are Cartesian coordinates, $r = \sqrt{x^2 + y^2 + z^2}$, and J_k is a constant vector. Show that this vector potential solves the constraint equations on the domain $\mathbb{R}^3 \setminus \{0\}$.

(iii) The total angular momentum of an asymptotically flat spacetimes is given in terms of the extrinsic curvature K_{ij} by

$$J_m^\infty = \frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_r}(K_{ij} - K\gamma_{ij})(\boldsymbol{\phi_m})^i \frac{x^j}{r} r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi\,,$$

where $(\boldsymbol{\phi}_m)^i = \epsilon_{mj}{}^i x^j$. Show that for the extrinsic curvature of the vector potential ($\dagger$), the total angular momentum equals the constant vector J_m, $J_m^\infty = J_m$. [*Hint: You may rotate your Cartesian coordinate system such that* J_k *points in the z direction. Also* $\epsilon_{ijk}\epsilon^{imn} = \delta_j{}^m\delta_k{}^n - \delta_j{}^n\delta_k{}^m$.]

6. Observers falling into a black hole

The Schwarzschild spacetime

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\,,$$

is transformed to Kruskal-Szekeres coordinates by the sequence of coordinate changes

$$\bar{t} = t + 2M\ln|r - 2M|\,, \qquad \tilde{t} = t - 2M\ln|r - 2M|\,,$$

$$v = \bar{t} + r\,, \qquad u = \tilde{t} - r\,,$$

$$\hat{v} = e^{\frac{v}{4M}}\,, \qquad \hat{u} = -e^{-\frac{u}{4M}}\,,$$

$$\hat{t} = \frac{1}{2}(\hat{v} + \hat{u})\,, \qquad \hat{r} = \frac{1}{2}(\hat{v} - \hat{u})\,.$$

The trajectories of observers falling radially into the black hole is given by the geodesic equations

$$\left(1-\frac{2M}{r}\right)\dot{t}=E\,, \qquad\qquad -E^2+\dot{r}^2=-1+\frac{2M}{r}\,,$$

where E is a constant of motion and $\dot{} := \mathrm{d}/\mathrm{d}\tau$ denotes the derivative with respect to the observer's proper time.

(i) Show that for an observer starting at rest at initial position $r_0 > 2M$,

$$\dot{v}=\frac{r}{r-2M}\left[\sqrt{1-\frac{2M}{r_0}}-\sqrt{\frac{2M}{r}-\frac{2M}{r_0}}\right]\,.$$

(ii) Can the resulting function $v(\tau)$ be evaluated beyond the observer's crossing of the event horizon at $r = 2M$? Justify your answer.

7. Christoffel symbols in adapted coordinates

In coordinates adapted to the 3+1 split, the spacetime metric is given by

$$\mathrm{d}s^2=(-\alpha^2+\beta_i\beta^i)\mathrm{d}t^2+2\beta_i\mathrm{d}t\,\mathrm{d}x^i+\gamma_{ij}\mathrm{d}x^i\,\mathrm{d}x^j\,.$$

Show that the components $\Gamma^0_{\alpha\beta}$ of the spacetime Christoffel symbols can be written as

$$\begin{aligned}\Gamma^0_{00} &= \frac{1}{\alpha}\left(\partial_0\alpha+\beta^m\partial_m\alpha\right)-\frac{1}{\alpha}\beta^m\beta^l K_{ml}\,,\\ \Gamma^0_{0i} &= \frac{\partial_i\alpha}{\alpha}-\frac{1}{\alpha}\beta^m K_{im}\,,\\ \Gamma^0_{ij} &= -\frac{1}{\alpha}K_{ij}\,.\end{aligned}$$

8. Time evolution of the volume element

Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma^{1/2}:=(\partial_t-\mathcal{L}_{\boldsymbol{\beta}})\gamma^{1/2}=-\alpha\gamma^{1/2}\,,$$

where α, β^i, γ_{ij} are the ADM variables, $\gamma := \det\gamma_{ij}$, and $\mathcal{L}_{\boldsymbol{\beta}}$ denotes the Lie derivative along the shift vector.

* 9. The distortion tensor

The distortion tensor is defined as

$$\Sigma_{ij}:=\frac{1}{2}\gamma^{1/3}\partial_t\tilde{\gamma}_{ij}=\frac{1}{2\chi}\partial_t\tilde{\gamma}_{ij}\,,$$

where $\tilde{\gamma}_{ij}$ is the conformal metric from the BSSNOK formulation and χ the conformal factor.

(i) Show that the distortion tensor can be written as

$$\Sigma^{ij}=\frac{\chi}{2}\left[-\beta^m\partial_m\tilde{\gamma}^{ij}+\tilde{\gamma}^{jl}\partial_l\beta^i+\tilde{\gamma}^{il}\partial_l\beta^j-\frac{2}{3}\tilde{\gamma}^{ij}\partial_m\beta^m-2\alpha\tilde{A}^{ij}\right]\,,$$

and has zero trace, $\tilde{\gamma}_{mn}\Sigma^{mn}=0$.

(ii) Show that

$$2\partial_j(\chi^{-1}\Sigma^{ij}) \;=\; \frac{2}{\chi}\left(D_j\Sigma^{ij}-\tilde{\Gamma}^i_{jk}\Sigma^{jk}+\frac{3}{2}\frac{\partial_j\chi}{\chi}\Sigma^{ij}\right) \;=\; \partial_t\tilde{\Gamma}^i\,.$$