NST Part IB Complex Methods

Lecture Notes

Abstract

This document contains the long version of the lecture notes for the IB course *Complex Methods*. These notes are extensive and written such that they can be consumed either together with the lecture or on their own. In drafting these notes, I am indebted to previous lecturers of this course and, in particular, the notes by R. E. Hunt (see also Dexter Chua's site https://dec41.user.srcf.net/notes/) and Gary Gibbons. I also thank Owain Salter Fitz-Gibbon and Miren Radia for discussions and comments on the lecture notes.

These notes assume that readers are already familiar with complex numbers, calculus in multi-dimensional real space $\mathbb{R}^n$, and Fourier transforms. The most important features of these three areas are briefly summarized, but not at the level of a dedicated lecture. This course is primarily aimed at applications of complex methods; readers interested in a more rigorous treatment of proofs are refered to the IB Complex Analysis lecture. Some more extensive discussions may also be found in the following books, though readers are not required to have studied them.

- M. J. Ablowitz and A. S. Fokas, *Complex variables: introduction and applications.* Cambridge University Press (2003)
- G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists.* Elsevier (2013)
- G. J. O. Jameson, *A First Course in Complex Functions.* Chapman and Hall (1970)
- T. Needham, *Visual complex analysis*, Clarendon (1998)
- H. A. Priestley, *Introduction to Complex Analysis.* Clarendon (1990)
- K. F. Riley, M. P. Hobson, and S. J. Bence, *Mathematical Methods for Physics and Engineering: a comprehensive guide.* Cambridge University Press (2002)

Example sheets will be on Moodle and on

`http://www.damtp.cam.ac.uk/user/examples`

Lectures Webpage:

`http://www.damtp.cam.ac.uk/user/us248/Lectures/lectures.html`

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Contents

A Background material

In this section, we briefly review some basic properties of general calculus, complex numbers and functions of them. While many readers will be familiar with the content of this section, it will serve as a convenient reference throughout these notes. We will also introduce some notational conventions.

A.1 Complex numbers

We begin our discussion of complex numbers with the imaginary unit.

<u>Def. :</u> The "imaginary unit" is $\mathrm{i} := \sqrt{-1}$.

A complex number $z \in \mathbb{C}$ can then be written as

$$z = x + \mathrm{i}y = re^{\mathrm{i}\phi} = r\cos\phi + \mathrm{i}\,r\sin\phi\,. \tag{A.1}$$

We may regard (x, y) and (r, ϕ) as Cartesian and polar coordinates on the $\mathbb{R}^2$, respectively, and they are related by the transformation

$$\begin{aligned} x &= \mathsf{Re}(z) = r\cos\phi\,, & y &= \mathsf{Im}(z) = r\sin\phi\,, \\ r &= |z| = \sqrt{x^2+y^2}\,, & \phi &= \arg z\,, \end{aligned} \tag{A.2}$$

where $\mathsf{Re}()$ and $\mathsf{Im}()$ denote the real and imaginary part. The argument $\arg z$ is only defined up to integer multiples of 2π and we therefore define the following.

<u>Def. :</u> The "principal argument" of a complex number z is the value $\phi = \arg z$ that falls into the range $(-\pi, \pi]$.

One may be tempted to compute the argument of $z = x + \mathrm{i}y$ from

$$\phi = \arctan\left(\frac{y}{x}\right)\,, \tag{A.3}$$

but this does not work because the arctan function only maps to the interval $(-\pi/2, \pi/2)$ as illustrated in Fig. 1. The arctan function therefore returns the principle argument only for points with positive real part but is off by π radians when $\mathsf{Re}(z) < 0$. For example, for $\phi = \arg z = \pi/3$,

$$z = \cos\phi + \mathrm{i}\sin\phi = \frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2} \qquad \text{and} \qquad \arctan\frac{y}{x} = \arctan(\sqrt{3}) = \frac{\pi}{3}\,,$$

gives us the correct result, but for $\phi = \arg z = 2\pi/3$, we obtain

$$z = \cos\phi + \mathrm{i}\sin\phi = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2} \qquad \text{and} \qquad \arctan\frac{y}{x} = \arctan(-\sqrt{3}) = -\frac{\pi}{3}\,.$$

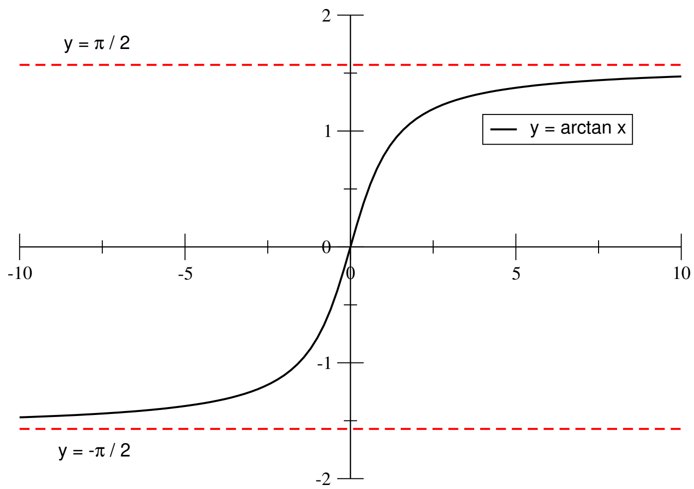

Figure 1: The arctan function.

Programming languages are aware of this shortcoming of the arctan function and therefore often provide a separate function

$$\texttt{atan2} : \mathbb{R} \to (-\pi, \pi]\,, \qquad (x, y) \mapsto \arg(x + \mathrm{i}y)\,. \tag{A.4}$$

In `Fortran`, `C` and `Python` this function is indeed called `atan2`, but other names cannot be ruled out in other languages.

Def. : The "complex conjugate" of $z = x + \mathrm{i}y$ is defined as

$$\bar{z} = x - \mathrm{i}y\,. \tag{A.5}$$

From this definition, we immediately find

$$|z|^2 = r^2 = z\bar{z}\,, \qquad \mathsf{Re}(z) = \frac{z + \bar{z}}{2}\,, \qquad \mathsf{Im}(z) = \frac{z - \bar{z}}{2i}\,. \tag{A.6}$$

Complex numbers obey the triangle inequality

$$|z_1 + z_2| \le |z_1| + |z_2| \quad \text{for all} \quad z_1,\ z_2 \in \mathbb{C}\,. \tag{A.7}$$

For any two complex numbers $\zeta_1,\ \zeta_2 \in \mathbb{C}$, we therefore find,

$$\begin{aligned} z_1 &= \zeta_1 + \zeta_2,\ z_2 = -\zeta_2 &\quad\Rightarrow\quad& |\zeta_1| - |\zeta_2| \le |\zeta_1 + \zeta_2|\,, \\ z_1 &= \zeta_1 + \zeta_2,\ z_2 = -\zeta_1 &\quad\Rightarrow\quad& |\zeta_2| - |\zeta_1| \le |\zeta_1 + \zeta_2|\,, \end{aligned}$$

and, hence,

$$\big||\zeta_1| - |\zeta_2|\big| \le |\zeta_1 + \zeta_2| \qquad \text{for all} \quad \zeta_1,\ \zeta_2 \in \mathbb{C}\,. \tag{A.8}$$

Throughout these notes, we will frequently make use of the geometrical series.

Proposition: For $z \in \mathbb{C}$, $z \neq 1$ and $n \in \mathbb{N}_0$,

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z} \,. \tag{A.9}$$

Proof. We proof this by induction. Evidently, for $n = 0$, we have $z^0 = 1 = (1 - z)/(1 - z)$. So let us assume Eq. (A.9) holds for n. Then

$$\sum_{k=0}^{n+1} z^k = z^{n+1} + \sum_{k=0}^{n} z^k = z^{n+1} + \frac{1 - z^{n+1}}{1 - z} = \frac{z^{n+1} - z^{n+2} + 1 - z^{n+1}}{1 - z} = \frac{1 - z^{n+2}}{1 - z}$$

□

Note that Eq. (A.9) is correct for any $z \neq 1$, irrespective of whether or not the series converges. For $|z| < 1$, we furthermore obtain the limit

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z} \,. \tag{A.10}$$

As we shall see later, this equation may also be interpreted as the Taylor expansion of the function $1/(1 - z)$ around $z = 0$.

Finally, let us recall the definition of open sets. Equation (A.2) provides us with a norm on $\mathbb{C}$ and we can define,

Def. : A set $\mathcal{D} \subset \mathbb{C}$ is an *open set* if for all $z_0 \in \mathcal{D}$, there exist a $\epsilon > 0$ such that the ϵ sphere of points with $|z - z_0| < \epsilon$ is contained in $\mathcal{D}$. More prosaically, an open set does not contain its boundary.

Def. : A "neighbourhood of a point $z \in \mathbb{C}$" is an open set $\mathcal{D}$ that contains the point z.

A.2 Trigonometric and hyperbolic functions

Complex numbers provide a close connection between trigonometric and hyperbolic functions through the relations

$$e^{\mathrm{i}\phi} = \cos\phi + \mathrm{i}\sin\phi \qquad \wedge \qquad e^{-\mathrm{i}\phi} = \cos\phi - \mathrm{i}\sin\phi$$

$$\Leftrightarrow \quad \cos\phi = \frac{e^{\mathrm{i}\phi} + e^{-\mathrm{i}\phi}}{2} \qquad \wedge \qquad \sin\phi = \frac{e^{\mathrm{i}\phi} - e^{-\mathrm{i}\phi}}{2\mathrm{i}} \,. \tag{A.11}$$

Trigonometric functions satisfy a vast range of relations and an impressive collection can be found on [7]. Here we list a few that will be used later on,

$$\cos(\alpha+\beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta\,, \tag{A.12}$$

$$\sin(\alpha+\beta) = \sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta\,, \tag{A.13}$$

$$\cos\big(\theta\pm\tfrac{\pi}{2}\big) = \mp\sin\theta\,, \qquad \cos(\theta+\pi) = -\cos\theta\,, \qquad \cos(\theta+2k\pi) = \cos\theta\,, \tag{A.14}$$

$$\sin\big(\theta\pm\tfrac{\pi}{2}\big) = \pm\sin\theta\,, \qquad \sin(\theta+\pi) = -\sin\theta\,, \qquad \sin(\theta+2k\pi) = \sin\theta\,, \tag{A.15}$$

where $k\in\mathbb{Z}$. The hyperbolic functions are given by

$$\begin{aligned} & \cosh\phi = \frac{e^{\phi}+e^{-\phi}}{2} \qquad \wedge \qquad \sinh\phi = \frac{e^{\phi}-e^{-\phi}}{2}\,. \\ \Leftrightarrow\quad & e^{\phi} = \cosh\phi+\sinh\phi \qquad \wedge \qquad e^{-\phi} = \cosh\phi-\sinh\phi\,. \end{aligned} \tag{A.16}$$

Equations (A.11) and (A.16) imply

$$\cos(\mathrm{i}x) = \frac{e^{-x}+e^{x}}{2} = \cosh x \qquad \Leftrightarrow \qquad \cosh(\mathrm{i}x) = \cos(-x) = \cos x\,, \tag{A.17}$$

$$\sin(\mathrm{i}x) = \frac{e^{-x}-e^{x}}{2\mathrm{i}} = \mathrm{i}\sinh x \qquad \Leftrightarrow \qquad \sinh(\mathrm{i}x) = \mathrm{i}\sin x\,. \tag{A.18}$$

Similar relations can be established for tan, cot, tanh, coth etc., but these are all direct consequences of Eqs. (A.17) and (A.18).

The hyperbolic functions satisfy addition theorems analogous to (A.12) and (A.13),

$$\cosh(x+y) = \cosh x\,\cosh y + \sinh y\,\sinh x\,, \tag{A.19}$$

$$\sinh(x+y) = \sinh x\,\cosh y + \cosh x\,\sinh y\,. \tag{A.20}$$

A.3 Calculus of real functions in ≥ 1 variables

We shall occasionally regard a function of a complex variable as a function of two real variables, the real and imaginary parts $x=\mathsf{Re}(z)$, $y=\mathsf{Im}(z)$. The partial derivative $\partial f/\partial x_i$ is defined in the usual way keeping all the other x_j constant. For an open subset $\Omega\in\mathbb{R}^n$, we define

Def. : $C^m(\Omega)$ is the set of all functions $f:\Omega\Rightarrow\mathbb{R}$ whose partial derivatives up to order m exist and are continuous.

Note that the existence of the partial derivatives is not a particularly strong requirement. In fact, it does not even imply that the function f is continuous.

Example : Let

$$f(x,y) = \begin{cases} x & \text{for } y = 0 \\ y & \text{for } x = 0 \\ \text{arbitrary} & \text{for } x \neq 0 \text{ and } y \neq 0 \end{cases} \tag{A.21}$$

We immediately find satisfactory partial derivatives at the origin,

$$\frac{\partial f}{\partial x}(0,0) = 1 = \frac{\partial f}{\partial y}(0,0)\,. \tag{A.22}$$

Differentiability of a function f on $\Omega \subset \mathbb{R}^n$ is a stronger criterion that can be defined as follows.

Def. : A function $f : \Omega \to \mathbb{R}$ is *differentiable* if there exists a linear function $A : \mathbb{R}^n \to \mathbb{R}$ such that for every $\boldsymbol{x} \in \Omega$

$$f(\boldsymbol{x} + \Delta\boldsymbol{x}) - f(\boldsymbol{x}) = A(\Delta\boldsymbol{x}) + r(\Delta\boldsymbol{x}) \quad \text{with} \quad \lim_{\Delta\boldsymbol{x}\to 0} \frac{r(\Delta\boldsymbol{x})}{||\Delta\boldsymbol{x}||} = 0\,. \tag{A.23}$$

Furthermore, this definition extends to vector-valued functions $f : \Omega \to \mathbb{R}^m$ by simply considering each component function f_i separately.

Finally, the function f is *continuously differentiable* if furthermore $f \in C^1$, i.e. if its first partial derivatives are continuous.

One can then show that the following implications (but in general not their reverse!) hold,

$$\begin{aligned} & f \text{ is continuously differentiable} \quad \Leftrightarrow \quad \text{all partial derivatives } \frac{\partial f_i}{\partial x_j} \text{ are continuous} \\ \Rightarrow \quad & f \text{ is differentiable} \\ \Rightarrow \quad & f \text{ is continuous and all partial derivatives } \frac{\partial f_i}{\partial x_j} \text{ exist}\,. \end{aligned} \tag{A.24}$$

For our purposes, the important consequence is that continuity of the partial derivatives $\partial f_i/\partial x_j$ is sufficient to ensure all differentiable properties we may require from our function f.

We next recapitulate some properties of sequences and series of functions.

Def. : Let $f_k : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, $k \in \mathbb{N}$ be a sequence of functions. The sequence f_n is *uniformly convergent* on Ω with limit $f : \Omega \to \mathbb{R}$ if

$$\forall_{\epsilon>0} \;\; \exists_{N\in\mathbb{N}} \;\; \forall_{k\geq N,\; x\in\Omega} \;\; |f_k(x) - f(x)| < \epsilon\,. \tag{A.25}$$

One can show that for a uniformly converging sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x\,. \tag{A.26}$$

Uniform convergence of complex functions is defined in complete analogy. The geometric series (A.10), for example, is uniformly convergent in $\mathbb{C}$ for $|z| < 1$. This will become important in our derivation of the Laurent series in Sec. D.1, where we need to swap the integral and the sum.

B Analytic functions

B.1 The Extended Complex Plane and the Riemann Sphere

In Sec. A.1, we have seen that a complex number z is defined in terms of its real and imaginary parts (x, y). This mapping $z \leftrightarrow (x, y)$ is bijective and we can therefore identify the complex plane with the 2-dimensional plane of ordered pairs of real numbers. We can likewise identify addition and multiplication and summarize this in the following definition.

Def. : We denote the set of all complex numbers by $\mathbb{C}$. A complex number $z = x + \mathrm{i}y$ is uniquely identified with a pair of real numbers $(x,\ y) \in \mathbb{R}^2$.
We can furthermore represent addition and multiplication of two complex numbers $z_1 = x_1 + \mathrm{i}y_1$, $z_2 = x_2 + \mathrm{i}y_2$ as operations on $\mathbb{R}^2$ according to

$$z = z_1 + z_2 \qquad \Leftrightarrow \qquad (x,\ y) = (x_1 + x_2, y_1 + y_2)\,, \tag{B.1}$$

$$z = z_1\, z_2 \qquad \Leftrightarrow \qquad (x,\ y) = (x_1\, x_2 - y_1\, y_2,\ x_1\, y_2 + x_2\, y_1)\,. \tag{B.2}$$

The latter relation is probably easier to remember by simply using the imaginary unit and standard rules of multiplication,

$$z_1\, z_2 = (x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) = x_1\, x_2 + \mathrm{i}y_1\, x_2 + x_1\, \mathrm{i}y_2 + \mathrm{i}^2 x_2\, y_2 = x_1\, y_1 - x_2\, y_2 + \mathrm{i}(x_1\, y_2 + x_2\, y_1)\,. \tag{B.3}$$

The identification of $\mathbb{C}$ with $\mathbb{R}^2$ can sometimes be helpful, but we will not carry it too far; after all, much of the beauty of complex numbers arises from treating them as numbers in their own right.

It is often convenient to include ∞ as part of the complex domain and therefore define:

Def. : The *extended complex domain* is $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

The fact that ∞ is indeed a single point is best understood in terms of the Riemann sphere which we display in Fig. 2. There we consider a sphere resting on the complex plane. A bijective mapping between points P on the sphere and points z in the plane is obtained by drawing a line from the North Pole through P until it intersects the plane; the point of intersection is z. The origin $z = 0$ is thus identified with the South Pole and we reach infinity by moving P ever closer to the North Pole which represents $z = \infty$. In particular, infinity represents merely one single point and we reach the same point by going towards infinity in any direction. The symbols ∞ and $-\infty$ therefore denote the same point in $\mathbb{C}^*$ and we will only use the expression $-\infty$ if we wish to emphasize that we proceed towards infinity in a specific direction, namely along the negative real axis. The Riemann sphere may appear as a nice but impractical device, but we will soon encounter a very useful application.

An alternative viewpoint of great value for investigating properties of infinity consists in the substitution $\zeta = 1/z$. We then ascribe to a function $f(z)$ a specific property at $z = \infty$, if the function $f(1/\zeta)$ has this property at $\zeta = 0$. If this procedure appears somewhat vague at

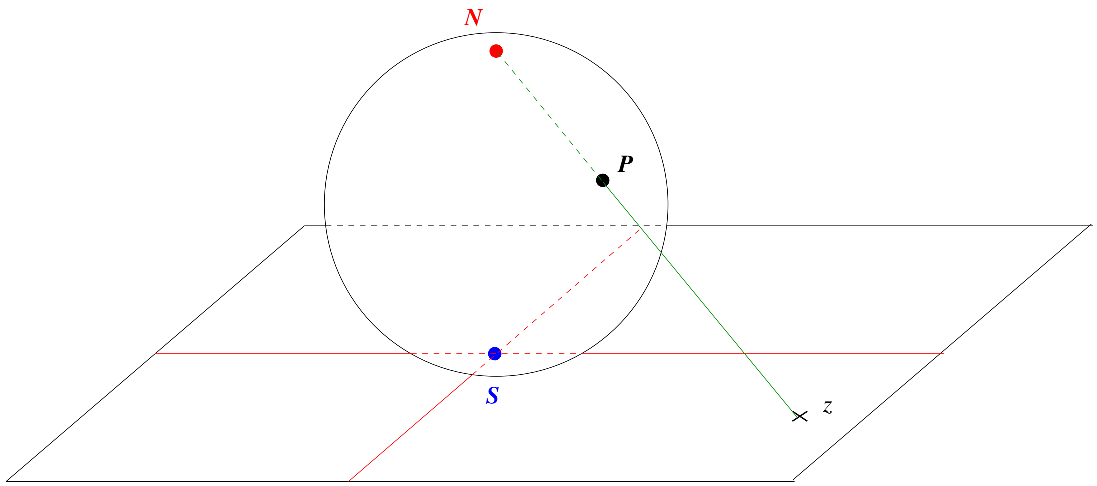

Figure 2: The Riemann sphere. A sphere rests on a plane. Any point P on the sphere is uniquely identified with a point z in the plane by drawing a line from the North Pole through P which intersects the plane at z. We thus identify the origin $z = 0$ with the South Pole while the North Pole itself represents infinity.

present, it will become clear when we discuss concrete examples; there will be plenty throughout these notes.

B.2 Complex differentiation and analytic functions

We define the derivative of a complex function on $\mathbb{C}$ in complete analogy to the derivative of a real function.

A function $f : \mathbb{C} \to \mathbb{C}$ is *complex differentiable* (or *differentiable* for short) at $z \in \mathbb{C}$, if

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}\,, \tag{B.4}$$

exists and, by implication, is independent of the direction of approach.

The key difference to real differentiability arises from the fact that the limit (B.4) can now be taken in an infinite number of directions. In order to see the fundamental importance of this feature, let us briefly return to the case of real functions and consider the simple example $f(x) = |x|$. At $x = 0$, calculating the derivative from the left side, $\delta x \to 0^-$ returns $f'(0^-) = -1$, while approaching from the right through $\delta x \to 0^+$ yields $f'(0^+) = 1$. In consequence, the function $|x|$ is not (real) differentiable at $x = 0$. For complex functions, the requirement that $f'(z)$ be independent of the direction of approaching z, turns out to be an exceptionally strong requirement.

Differentiability throughout a subset of $\mathbb{C}$ is such an important property that it is given its own name or, rather, names.

Def. : A complex function f is called *analytic at a point* $z \in \mathbb{C}$ if there exists a neighbourhood $\mathcal{D}$ of z such that f is differentiable throughout $\mathcal{D}$. Sometimes, the equivalent terms *regular* or *holomorphic* are used to denote the same property. A function that is analytic on all $\mathbb{C}$ is called *entire*.

Analyticity of a complex function implies a lot; a good deal more that we are used to from real functions. For example, a function that is analytic at z can be differentiated infinitely many times. This is clearly not generally true for real functions: $F(x) = \int |x|\, dx$, for example, can be differentiated once but no further. Also, a complex function f that is analytic on all $\mathbb{C}$ and bounded must be a constant. We will discuss these properties in more detail further below.

Equation (B.4) provides us with an expression for the derivative of a complex function, but using it in practice is often cumbersome. Just like for real functions, life will be a good deal easier with a comprehensive set of rules to compute complex derivatives. As it turns out, many of these rules turn out to be the same as those familiar from real functions. First, however, we are looking for a simpler criterion to determine whether a given function is differentiable or not. For this purpose, we select two out of the infinite number of directions by which we can approach the limit in (B.4), the x direction (parallel to the real axis) and the y direction (parallel to the imaginary direction). Let us write the complex function as

$$f(z) = u(x, y) + \mathrm{i}\, v(x, y)\,, \tag{B.5}$$

and insert into (B.4). First, we take the limit in the real direction, by setting $\delta z = \delta x$ and obtain

$$\begin{aligned} f'(z) &= \lim_{\delta x \to 0} \frac{f(z + \delta x) - f(z)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{u(x + \delta x, y) + \mathrm{i} v(x + \delta x, y) - u(x, y) - \mathrm{i} v(x, y)}{\delta x} \\ &= \frac{\partial u}{\partial x} + \mathrm{i} \frac{\partial v}{\partial x}\,. \end{aligned} \tag{B.6}$$

Next, we repeat the game along the imaginary axis and set $\delta z = \mathrm{i}\delta y$ (recall $z = x + \mathrm{i}\, y$),

$$\begin{aligned} f'(z) &= \lim_{\delta y \to 0} \frac{f(z + \mathrm{i}\delta y) - f(z)}{\mathrm{i}\delta y} \\ &= \lim_{\delta y \to 0} \frac{u(x, y + \delta y) + \mathrm{i} v(x, y + \delta y) - u(x, y) - \mathrm{i} v(x, y)}{\mathrm{i}\delta y} \\ &= \frac{\partial v}{\partial y} - \mathrm{i} \frac{\partial u}{\partial x}\,. \end{aligned} \tag{B.7}$$

If f is differentiable, the result must be the same for every direction $\delta z \to 0$ and, in particular, for the two choices we have made; for a differentiable function the results (B.6) and (B.7) must agree. Since two complex numbers are equal if and only if their real and their imaginary parts are, we obtain the Cauchy-Riemann equations,

Def. : For a complex function $f(z) = u(x,y) + \mathrm{i}v(x,y)$, the *Cauchy-Riemann equations* are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,, \tag{B.8}$$

and have proven the following result,

Proposition: If $f(z) = u(x,y) + \mathrm{i}v(x,y)$ is differentiable at $z = z_0$, it satisfies the Cauchy-Riemann equations (B.8) at this point.

The Cauchy-Riemann equations thus constitute a necessary condition for differentiability. Are they also sufficient? In general, no. For the reverse implication to be true, we also need to demand that u and v are themselves differentiable functions, and we have seen in Sec. A.3 that differentiability of u and v is a stronger condition than the mere existence of their partial derivatives $u_x,\ u_y,\ v_x,\ v_y$. We have also seen, however, that u and v are differentiable if their partial derivatives exist *and* are continuous. This provides us with a practical criterion for assessing the differentiability of a complex function.

Proposition: If a function $f(z) = u(x,y) + \mathrm{i}v(x,y)$ satisfies

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,,$$

at a point $z = z_0$ and the partial derivatives are continuous in a neighbourhood of z_0, then $f(z)$ is differentiable at $z = z_0$.

The complete proof of this result is outside the scope of our course, but will be discussed in the IB Complex Analysis lecture.

Our discussion of the differentiability conditions for a complex function has been based on writing

$$f(z) = u(x,y) + \mathrm{i}v(x,y)\,, \qquad z = x + \mathrm{i}y\,, \tag{B.9}$$

and considering the differentiability of the functions u and v. An alternative and equivalent approach consists in recalling the complex conjugate $\bar{z}$ and the relations

$$x = \frac{z+\bar{z}}{2}\,, \qquad y = \frac{z-\bar{z}}{2\mathrm{i}} = -\mathrm{i}\,\frac{z-\bar{z}}{2}\,. \tag{B.10}$$

This allows us to write any complex function in the form $g(z,\bar{z})$ and it turns out that g is differentiable if it does not depend on $\bar{z}$, i.e. $\partial g/\partial\bar{z} = 0$. This will be explored in more detail in the first example sheet.

Complex differentiation obeys the product, quotient and chain rule we are already familiar with from real calculus.

Corollary: "Product rule": The product of two analytic functions $f\,g$ is analytic with the derivative

$$(f\,g)'(z) = f'(z)\,g(z) + f(z)\,g'(z)\,. \tag{B.11}$$

Proof. Given that f and g are differentiable, we can directly evaluate the derivative of the product fg from Eq. (B.4). We first define

$$\varpi := \frac{f(z+h) - f(z)}{h} - f'(z)\,,$$
$$w := \frac{g(z+h) - g(z)}{h} - g'(z)\,,$$

Note that both $\varpi \to 0$ and $w \to 0$ as $h \to 0$. It follows that

$$\begin{aligned}
(f\,g)' &= \lim_{h\to 0} \frac{f(z+h)g(z+h) - f(z) - g(z)}{h} \\
&= \lim_{h\to 0} \frac{\{g(z) + [g'(z) + w]h\}\,\{f(z) + [f'(z) + \varpi]h\} - g(z)f(z)}{h} \\
&= \lim_{h\to 0} \frac{[g'(z) + w]h\,f(z) + [f'(z) + \varpi]h\,g(z) + [g'(z) + w]h\,[f'(z) + \varpi]h}{h} \\
&= g'(z)\,f(z) + f'(z)\,g(z)\,.
\end{aligned}$$

□

Corollary: "Chain rule": The composition $g \circ f$ of two analytic functions $f\,g$ is analytic with the derivative

$$(f \circ g)'(z) = f'(g)\,g'(z)\,. \tag{B.12}$$

Proof. As for the product rule, we can directly evaluate $(f \circ g)'$ from Eq. (B.4) using that f and g are differentiable. With the definitions

$$\begin{aligned}
\varpi &:= \frac{f(\zeta + k) - f(\zeta)}{k} - f'(\zeta)\,, \\
w &:= \frac{g(z+h) - g(z)}{h} - g'(z)\,,
\end{aligned}$$

we find

$$\begin{aligned}
& f(\zeta + k) = f(\zeta) + [f'(\zeta) + \varpi]k \qquad \wedge \qquad g(z+h) = g(z) + [g'(z) + w]h \\
\Rightarrow\ & f\big(g(z+h)\big) = f\big(\underbrace{g(z)}_{=:\zeta} + \underbrace{[g'(z) + w]h}_{=:k}\big) = f\big(g(z)\big) + [f'\big(g(z)\big) + \varpi]\,[g'(z) + w]h \\
\Rightarrow\ & \frac{f\big(g(z+h)\big) - f\big(g(z)\big)}{h} = [f'(g) + \varpi]\,[g'(z) + w] \\
\Rightarrow\ & (f \circ g)'(z) = \lim_{h\to 0} \frac{f\big(g(z+h)\big) - f\big(g(z)\big)}{h} = f'(g)\,g'(z)\,.
\end{aligned} \tag{B.13}$$

□

The use of the Cauchy-Riemann conditions is best illustrated with some examples of analytic and non-analytic functions.

Examples of analytic functions

(1) The function $f(z) = z$ is entire, i.e. analytic everywhere. Here, we have $u(x, y) = x$ and $v(x, y) = y$ and the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial y}\,,$$

are satisfied everywhere. Furthermore, the partial derivatives are clearly continuous. Alternatively, we could have plugged $f(z) = z$ directly into Eq. (B.4) to prove its differentiability which also gives us directly $f'(z) = 1$.

(2) The function $e^z = e^x(\cos y + \mathrm{i}\sin y)$ is also entire, since

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x} \tag{B.14}$$

Again, the partial derivatives are continuous everywhere. Having demonstrated its differentiability, we are free to compute the derivative f' along any direction we like. We choose the x direction in Eq. (B.4) and obtain

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = e^x \cos y + \mathrm{i}e^x \sin y = e^x\, e^{\mathrm{i}y} = e^z\,, \tag{B.15}$$

as expected.

(3) For $n \in \mathbb{N}$, the function $f(z) = z^n$ is also entire. One could check this by writing $z^n = r^n(\cos\theta + \mathrm{i}\sin\theta)^n$ which leads to $u = r^n\cos(n\theta)$ and $v = r^n\sin(n\theta)$, but this becomes rather cumbersome since we still need to switch from polar to Cartesian coordinates. A simpler proof uses induction to show $f'(z) = nz^{n-1}$. We have done this already for $n = 1$ in item (1) above. Let's assume then the result is true for n. We then write

$$\begin{aligned}\frac{(z+\delta z)^{n+1} - z^{n+1}}{\delta z} &= \frac{(z+\delta z)^n(z+\delta z) - z^n(z+\delta z) + z^n\delta z}{\delta z} \\ &= (z+\delta z)\frac{(z+\delta z)^n - z^n}{\delta z} + z^n\end{aligned}$$

and therefore

$$f'(z) = \lim_{\delta z\to 0}\frac{(z+\delta z)^{n+1} - z^{n+1}}{\delta z} = z\, nz^{n-1} + z^n = (n+1)z^n\,. \tag{B.16}$$

Since a linear combination $\alpha f + \beta g$, α, $\beta \in \mathbb{C}$ of two analytic functions f and g is manifestly analytic by the definition (B.4), polynomials are also entire functions.

(4) The function $f(z) = 1/z$ is analytic everywhere except at $z = 0$. Writing

$$f(z) = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - \mathrm{i}y}{x^2 + y^2}\,, \tag{B.17}$$

we find

$$\partial_x u = \frac{y^2 - x^2}{(x^2+y^2)^2}\,, \quad \partial_y u = \frac{-2xy}{(x^2+y^2)^2}\,, \quad \partial_x v = \frac{2xy}{(x^2+y^2)^2}\,, \quad \partial_y v = \frac{y^2 - x^2}{(x^2+y^2)^2}\,,$$

and the Cauchy-Riemann conditions hold everywhere except at the origin. Furthermore, the partial derivatives are continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$. We evaluate f' along the x direction using differentiation rules for real functions,

$$\frac{\partial}{\partial x}\frac{x - \mathrm{i}y}{x^2+y^2} = \frac{-x^2 + y^2 + 2\mathrm{i}xy}{(x^2+y^2)^2} = \frac{(\mathrm{i}x + y)^2}{(x^2+y^2)^2} = \frac{-(x - \mathrm{i}y)^2}{(x^2+y^2)^2} = -\frac{\bar{z}^2}{z^2\bar{z}^2} = -\frac{1}{z^2}\,.$$

Combined with the product and chain rule, this result furthermore gives us the quotient rule, $(f/g)' = (f'\,g - g'\,f)/g^2$ which, for analytic functions f and g, is analytic provided $g(z) \neq 0$. We furthermore conclude that any rational function $f(x) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials, is analytic except at points where $Q(z) = 0$. For example,

$$f(z) = \frac{z}{z^2 + 1}\,, \tag{B.18}$$

is analytic everywhere except $z = \pm\mathrm{i}$.

(5) Many standard functions turn out to have the same derivatives when extended to the complex domain. The most important examples are as follows.

- The trigonometric functions

$$\cos z = \frac{e^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2}\,, \qquad \sin z = \frac{e^{\mathrm{i}z} - e^{-\mathrm{i}z}}{2i} \tag{B.19}$$

are differentiable everywhere with derivatives

$$(\sin)'(z) = \cos(z)\,, \qquad (\cos)'(z) = -\sin(z)\,. \tag{B.20}$$

The quotient rule gives us $(\tan)'(z) = 1/\cos^2 z$, though this is analytic only at points where $\cos z \neq 0$. Useful expressions for the cos and sin functions in terms of (x, y) are obtained from the addition theorems (A.12) and (A.13),

$$\begin{aligned} \sin z &= \sin(x + \mathrm{i}y) = \sin x\,\cos(\mathrm{i}y) + \cos x\,\sin(\mathrm{i}y) = \sin x\,\cosh y + \mathrm{i}\cos x\,\sinh y\,, \\ \cos z &= \cos(x + \mathrm{i}y) = \cos x\,\cos(\mathrm{i}y) - \sin x\,\sin(\mathrm{i}y) = \cos x\,\cosh y - \mathrm{i}\sin x\,\sinh y\,. \end{aligned} \tag{B.21}$$

- Similarly, the hyperbolic functions cosh, sinh etc. differentiate as expected, $\cosh' = \sinh$, $\sinh' = \cosh$.

- The logarithm $f(z) = \log z = \log|z| + \mathrm{i}\arg z$ has derivative $\frac{1}{z}$; we will discuss the logarithm in more detail in Sec. B.4 under the heading of multi-valued functions and branch cuts.

It is also instructive to consider non-analytic functions and see how this is reflected in the Cauchy-Riemann conditions. Some examples are given in the following.

Examples of non-analytic functions

(1) The function $f(z) = \mathsf{Re}(z)$ has $u = x\,,\ v = 0$ and, hence, $\partial_x u = 1 \neq \partial_y v = 0$, so $\mathsf{Re}(z)$ is nowhere analytic.

(2) For $f(z) = |z|$, we obtain $u = \sqrt{x^2+y^2}$ and $v = 0$ and therefore

$$\partial_x u = \frac{x}{\sqrt{x^2+y^2}}\,,\quad \partial_y v = 0\,,\quad \partial_y u = \frac{y}{\sqrt{x^2+y^2}}\,,\quad \partial_x v = 0\,. \tag{B.22}$$

The Cauchy-Riemann equations are nowhere satisfied, so $|z|$ is nowhere analytic.

(3) The complex conjugate $f(z) = \bar{z}$ implies $u = x\,,\ v = -y$, so that $\partial_x u = 1 \neq \partial_y v$, and $|z|$ is nowhere analytic either. This also follows by inserting $\bar{z}$ directly into Eq. (B.4) as will be discussed in example sheet 1.

(4) The function $f(z) = |z|^2 = x^2 + y^2$ is more subtle. The Cauchy Riemann conditions yield

$$\begin{aligned} \partial_x u &= 2x\,, & \partial_y v &= 0\,, \\ \partial_y u &= 2y\,, & \partial_x v &= 0\,. \end{aligned} \tag{B.23}$$

The Cauchy Riemann conditions are therefore satisfied at $z = 0$, but nowhere else. $|z|^2$ is therefore not analytic, not even at $z = 0$, since analyticity requires differentiability in a neighbourhood around the point in question.

B.3 Harmonic functions

In this subsection, we will discuss the relation between real and imaginary parts of an analytic function and solutions to the Laplace equation

$$\triangle\, f = \nabla^2 f = \partial_x^2 f + \partial_y^2 f = 0\,. \tag{B.24}$$

Def. : A function is *harmonic* if it satisfies the Laplace equation in an open set $U \subset \mathbb{R}^n$.

Def. : Two functions u, v satisfying the Cauchy Riemann equations (B.8) are called *harmonic conjugates*.

The double appearance of the word “harmonic” in these two definitions is, of course, no coincidence, as we will show next. Clearly the Cauchy Riemann equations relate the two functions u and v; more specifically, if we know one of them, the other can be determined up to an additive constant. This is best seen by considering an example. If $u(x,y) = x^2 - y^2$, the first Cauchy Riemann equation tells us that

$$\begin{aligned} & \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \\ \Rightarrow \quad & v(x,y) = 2xy + g(x)\,, \end{aligned}$$

for some function $g(x)$. From the second Cauchy Riemann equation, we obtain

$$\begin{aligned} & \frac{\partial u}{\partial y} = -2y \stackrel{!}{=} -\frac{\partial v}{\partial x} = -2y - g'(x) \\ \Rightarrow \quad & g'(x) = 0 \qquad \Rightarrow \qquad g(x) = c_0 = \text{const}\,. \end{aligned} \tag{B.25}$$

Up to the constant c_0, we have thus obtained the analytic complex function

$$f(z) = u + \mathrm{i}v = x^2 - y^2 + 2\mathrm{i}xy + \mathrm{i}c_0 = (x + \mathrm{i}y)^2 + \mathrm{i}c_0 = z^2 + \mathrm{i}c_0\,. \tag{B.26}$$

When asked to find the analytic function f with a given real part u (or a given imaginary part v), the resulting analytic function needs to be expressed in terms of z, rather than x, y. This also serves as a double check to ensure the result is indeed analytic, i.e. depends on z but not on $\bar{z}$.

This relation between the real and imaginary part also brings us to the Laplace equation. Since partial derivatives commute, the Cauchy Riemann equations give us

$$\begin{aligned} & \partial_x^2 u = \partial_x(\partial_x u) = \partial_x(\partial_y v) = \partial_y(\partial_x v) = \partial_y(-\partial_y u) = -\partial_y^2 u\,, \\ \Rightarrow \quad & \triangle u = \partial_x^2 u + \partial_y^2 u = 0\,, \end{aligned} \tag{B.27}$$

i.e. u is a solution to Laplace's equation. One likewise shows that $\triangle v = 0$ which completes the proof of the following.

Proposition: The real and imaginary parts of any analytic complex function are harmonic.

B.4 Multi-valued functions and branch cuts

B.4.1 Single branch cuts

When we discussed the differentiability of the log function in Sec. B.2 we already indicated that its multi-valued character leads to some complication. It is now time to discuss this in more detail. The logarithm is defined as the inverse of exponentiation: for $z = re^{\mathrm{i}\theta}$, we define $\log z = \log r + \mathrm{i}\theta$. There are, however, infinitely many values $\log z$, since we can add an arbitrary $2n\pi$, $n \in \mathbb{Z}$ to θ and still get the same $z = re^{\mathrm{i}\theta}$. For example, we find

$$\log \mathrm{i} = \mathrm{i}\frac{\pi}{2} \quad \text{or} \quad \mathrm{i}\frac{5\pi}{2} \quad \text{or} \quad -\mathrm{i}\frac{3\pi}{2} \quad \text{or} \quad \ldots \tag{B.28}$$

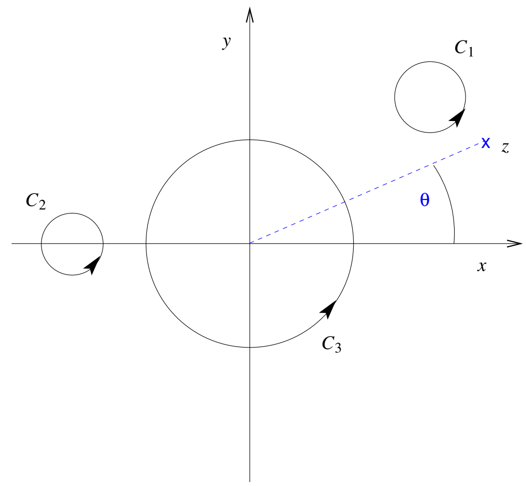

Figure 3: Of the three circles C_1, C_2 and C_3, only C_3 encircles the origin. For the points along C_1 and C_2, the argument θ is bounded, but not along C_3.

At first glance, one might think this problem can be solved by simply requiring the argument θ to be within some specific range, just as we defined in Sec. A.1 the principle argument of a complex number to fall into the range $\theta \in (-\pi, \pi]$. Note that for $r = 1$, the log function reduces to the arg function. This apparent solution, however, obeys the *conservation law of misery*; while the restriction to a specific interval eliminates the problem of multiple values, it introduces a discontinuity to the log function.

In order to see this more clearly, we will first define the notion of a branch point. Consider for this purpose Fig. 3, which shows three circles in the complex plane. Clearly the argument of points z on circle C_1 always falls into the range $\theta \in (0, \frac{\pi}{2})$ while the argument for points on C_2 is bounded by $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$. Along the circles C_1 and C_2, the functions $\log z$ and $\arg z$ are therefore continuous and single-valued.

This cannot be achieved, however, for circle C_3 which encircles the origin $z = 0$. The argument $\arg z$ cannot be made single-valued *and* continuous at the same time. Either $\arg z$ is discontinuous somewhere along the circle C_3 or the value increases by 2π every time we go around the circle and $\arg z$ is no longer single-valued. The same problem holds for $\log z$ along C_3. In this case, the origin is the source of the problem and it is called a branch point of the functions $\arg z$ and $\log z$. More generally, we define:

Def. : A *branch point of a function* $f(z)$ is a point that cannot be encircled by a curve C such that the function $f(z)$ is both, single-valued and continuous along C. The function $f(z)$ is then said to have a *branch point singularity* at this point.

Branch points are not uncommon in complex calculus and not at all restricted to the origin of the complex plane, as is illustrated by the following examples.

Examples of branch points

(1) The function $\log(z-a)$, $a = \text{const} \in \mathbb{C}$, has a branch point at $z = a$.

(2) $f(z) = \log\left(\frac{z-1}{z+1}\right) = \log(z-1) - \log(z+1)$ has two branch points at ± 1.

(3) The function $f(z) = z^\alpha = r^\alpha e^{\mathrm{i}\alpha\theta}$, $\alpha = \text{const}$ is more complicated. Consider a circle of radius r_0 around the origin and let us start at $\theta = 0$ where we have $f = r_0^\alpha$. Now go around the circle once; as we approach $\theta = 2\pi$, we find $f = r_0^\alpha e^{\mathrm{i}\alpha 2\pi}$ which is equal to our starting value only if

$$e^{\mathrm{i}\alpha 2\pi} = 1 \qquad \Leftrightarrow \qquad \alpha 2\pi = 2n\pi \qquad \Leftrightarrow \qquad \alpha \in \mathbb{Z}\,, \tag{B.29}$$

i.e. for integer values of α. For non-integer α, the function z^α has a branch point at the origin.

(4) Recall that we extended the complex domain by including ∞ and that we planned to study the behaviour of functions at $z = \infty$ by considering the behaviour of $f(z = \frac{1}{\zeta})$ at $\zeta = 0$. In this case, $\log z$ has a branchpoint at $z = \infty$ if $\log\frac{1}{\zeta}$ has a branch point at $\zeta = 0$. This is indeed the case, since $\log\frac{1}{\zeta} = -\log\zeta$ and the latter has a branch point at $\zeta = 0$. We likewise show that $f(z) = z^\alpha$ has a branch point at $z = \infty$ for $\alpha \notin \mathbb{Z}$.

(5) Sometimes, we may think at first glance that a function has a specific branch point when in fact it does not. For example, the function $f(z) = \log\left(\frac{z-1}{z+1}\right)$ does not have a branch point at infinity. With $z = 1/\zeta$, we find

$$f(z = 1/\zeta) = \log\left(\frac{z-1}{z+1}\right) = \log\left(\frac{1-\zeta}{1+\zeta}\right)\,.$$

For ζ close to zero, $\frac{1-\zeta}{1+\zeta}$ stays close to 1 and thus stays clear of the branch point 0 of the log function; in other words, we can encircle $\zeta = 0$ without encountering any discontinuity of $\log\frac{1-\zeta}{1+\zeta}$ along the curve.

Having identified the potentially problematic branch points, the obvious questions is how we can handle them. The answer are branch cuts.

In order to obtain single-valued and continuous functions, we need to constrain the regime of the complex domain we are allowed to consider; more specifically, we must exclude curves that encircle branch points. For the $\log z$ function, this means we have to prevent curves from encircling the origin and this is achieved by drawing a red line, a *branch cut* which no curve is allowed to cross. For the function $f(z) = \log z$, for example, we can introduce a branch cut along the nonpositive real axis as shown in Fig. 4. Once we have decided where to place the branch cut, we can specify the range the argument θ is allowed to take, for example $\theta \in (-\pi, \pi]$. This defines a *branch* of the function $\log z$ and on this branch, $\log z$ is single-valued and continuous along any curve that does not cross the branch cut. This branch of $\log z$ is furthermore analytic everywhere with derivative $\frac{\mathrm{d}}{\mathrm{d}z}\log z = \frac{1}{z}$ except on the nonpositive real axis, where this branch is not even continuous. Let us take a closer look at this discontinuity. For points just below the branch cut, we have $\theta = -\pi + \epsilon$ where $0 < \epsilon \ll 1$ and therefore $\log z = \log r_0 - \mathrm{i}(\pi - \epsilon)$. For points just above the branch cut, we have $\theta = \pi - \epsilon$ and $\log z = \log r_0 + \mathrm{i}(\pi - \epsilon)$. The function $\log z$ therefore jumps by $\mathrm{i}2\pi$ across the cut, as expected.

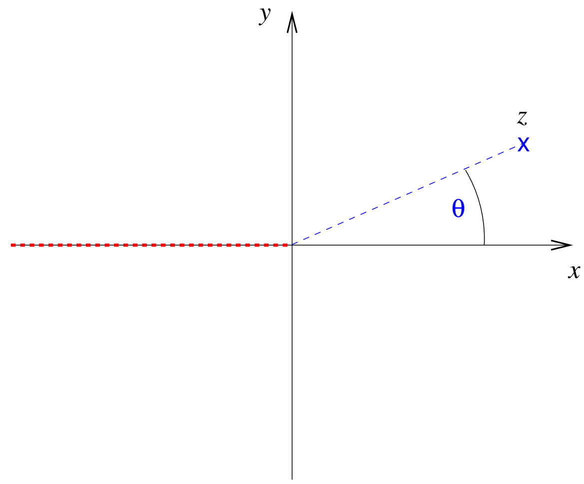

Figure 4: Example of a branch cut for the function $\log z$ which has a branch point at the origin. Here the cut consists of the nonpositive real axis.

Note that we have three different objects here, all with the string "branch" in their names:

- A *branch point* of $f(z)$ is a point we cannot enclose with a curve C such that $f(z)$ is single-valued and continuous along C.
- A *branch cut* is a line or a set of lines we introduce in the complex domain that potentially problematic curves are not allowed to cross.
- The *branch* of $f(z)$ is the particular choice of values $f(z)$ is allowed to take on.

Clearly, we have freedom in choosing branch cuts and branches (though not for branch points). The branch cut of Fig. 4 is the *canonical* branch cut for the log function and the branch $(-\pi, \pi]$ is called the *principal value* of the logarithm. For the same branch cut, we could have chosen a different, but equally legitimate branch, say $(\pi, 3\pi)$. We could also have chosen a different branch cut such as the nonnegative imaginary axis in the left panel of Fig. 5. In that case, we could choose, for example, the branch $(-\frac{3\pi}{2}, \frac{\pi}{2}]$. Note that the branch cut need not be a straight line; a wiggly curve as in the right panel of Fig. 5 will do just as well, as long as it stops curves from encircling the origin. Of course, we may have a harder time in this case to write down the range of valid values for θ.

Note that we can specify the branch of a function in two alternative ways.

(i) We can define the function and the range of values explicitly as for example in

$$f(z) = \log z = \log|z| + \mathrm{i} \arg z\,, \qquad \arg z \in (-\pi, \pi]\,. \tag{B.30}$$

(ii) We can specify the function, the location of the branch cut and the value of the function at a single point not on the cut. The entire branch is then defined by continuity of the function. The previous branch would then be specified by $f(z) = \log z$ with a branch cut along the nonpositive real axis $\mathbb{R}^{\leq 0}$, and $\log 1 = 0$. Setting instead $\log 1 = \mathrm{i}2\pi$ would correspond to the branch $\arg z \in (\pi, 3\pi]$.

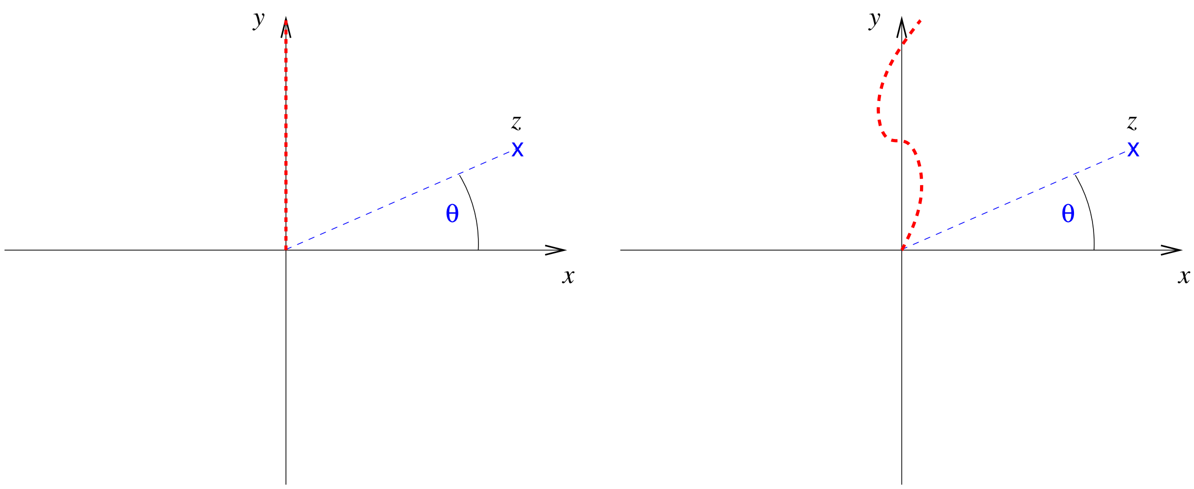

Figure 5: Two further examples of branch cuts for $\log z$.

All we have discussed here about branches and cuts for $\log z$ applies in the same way for the function $f(z) = z^\alpha$, $\alpha \notin \mathbb{Z}$.

B.4.2 Riemann surfaces*

This subsection is not examinable and represents complementary material to the lecture only.

If you think that the introduction of branch cuts and simply forbidding the crossing of these cuts appears a rather crude and not entirely satisfactory measure to control potentially multi-valued functions, you are in good company. Riemann introduced a different idea where the different branches of a function are regarded as separate copies of the complex plane $\mathbb{C}$ stacked onto each other and each connected to its neighbours at the respective branch cuts. This construction is called a *Riemann surface* and illustrated for our log example in Fig. 6.

Here the branches are represented by sheets of different color and with height given by the argument $\theta = \arg z$. The sheets are joined together at the branch cuts indicated in the figure by thick dark lines. Crossing a branch cut is not forbidden in this picture, but will carry us from one sheet or branch to the next. There is a good deal more to say about Riemann surfaces, but this has to be reserved for a dedicated Part IID course titled “Riemann Surfaces”.

B.4.3 Multiple branch cuts

In the examples we have discussed so far, it was rather evident how to place a branch cut; there may have been multiple options, but each of those looked obvious. Sometimes it is less clear and the main goal of this section is to introduce a set of rules for constructing legitimate and complete branch cuts.

Let us first consider the example

$$g(z) = [z(z-1)]^{\frac{1}{3}}\,. \tag{B.31}$$

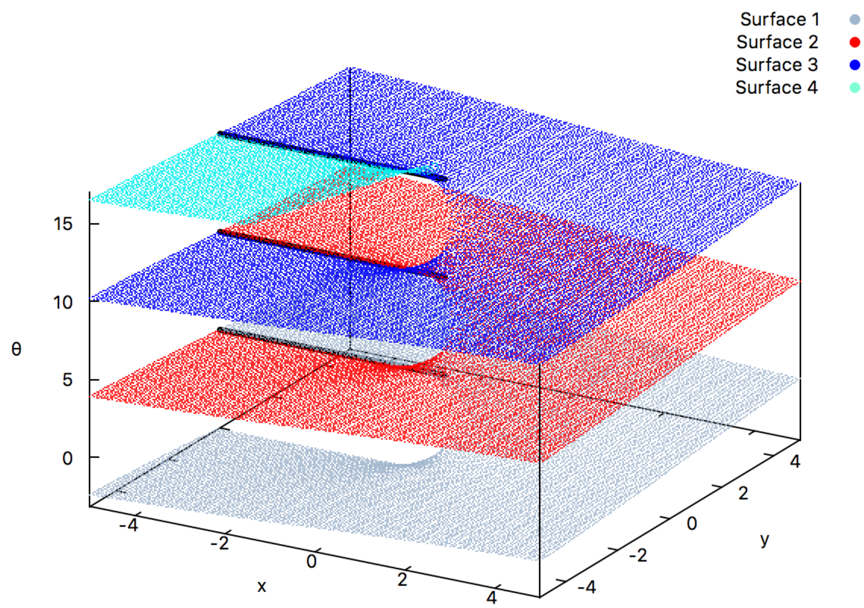

Figure 6: Riemann surfaces for the function $f(z) = \log z$. The different branches of $\log z$ are represented here in terms of sheets of different color measuring the argument $\theta = \arg z$ in the vertical direction. The sheets are joined together at the branch cuts indicated by the dark thick lines.

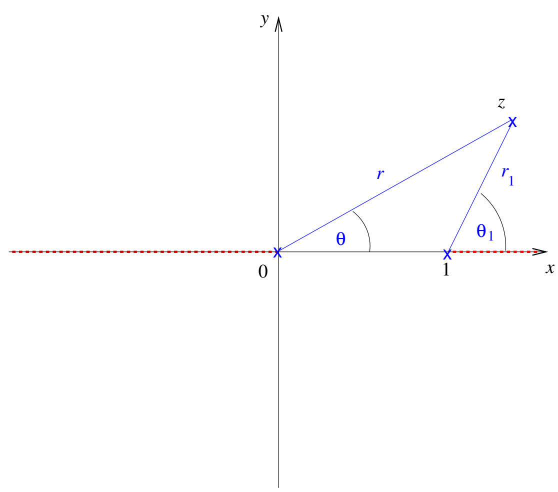

Figure 7: Branch cuts for the function $g(z) = [z(z-1)]^{\frac{1}{3}}$.

This function has two branch points at $z = 0$ and $z = 1$ and we may choose the branch cut illustrated in Fig. 7; it consists of the non-positive x axis as well as the set of points $x \geq 1$.

In order to see that this choice works, we express the function in polar coordinates $z = re^{\mathrm{i}\theta}$ and $z - 1 = r_1 e^{\mathrm{i}\theta_1}$; cf. also Fig. 7. We thus find

$$g(z) = \sqrt[3]{r r_1} e^{\mathrm{i}(\theta+\theta_1)/3} \,. \tag{B.32}$$

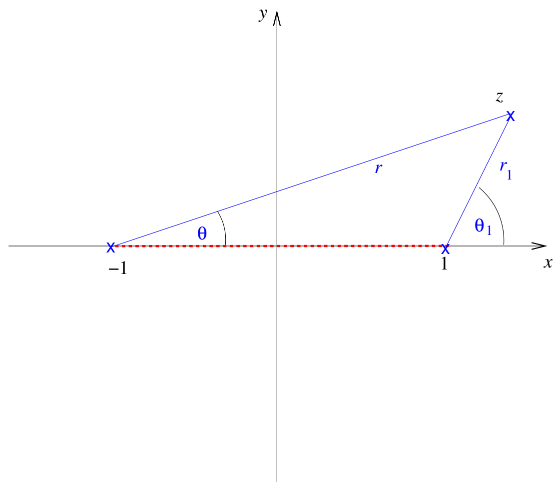

Figure 8: An alternative branch cut for the function $f(z) = \log \frac{z-1}{z+1}$.

We therefore need to avoid either θ or θ_1 covering a complete circle and this is clearly achieved by the two-segment branch cut in Fig. 7.

Sometimes, however, we have other, simpler options to draw a branch cut. Let us consider the example

$$f(z) = \log \frac{z-1}{z+1}\,, \tag{B.33}$$

which has branch points at $z = \pm 1$. Similar to the previous example, we introduce polar coordinates according to $z + 1 = re^{\mathrm{i}\theta}$ and $z - 1 = r_1 e^{\mathrm{i}\theta_1}$, so that

$$f(z) = \log(z-1) - \log(z+1) = \log\left(\frac{r_1}{r}\right) + \mathrm{i}(\theta_1 - \theta)\,. \tag{B.34}$$

We could now introduce a branch cut in complete analogy to Fig. 7 drawing one line from -1 to $-\infty$ and a second from 1 to ∞. This is indeed correct, but not the only option. An alternative branch cut for this function is shown in Fig. 8. It is instructive to see why this works. Let us consider for this purpose the range of angles θ, $\theta_1 \in [0, 2\pi)$. This range may not appear intuitve at first glance but it works. Suppose we cross the branch cut. Then θ passes through the branch cut at $\theta = 0$ while θ_1 varies smoothly across $\theta_1 = \pi$. Clearly, the value of $f(z)$ as given by Eq. (B.34) jumps because θ jumps. This is the expected behaviour at a branch cut. Next suppose that we cross the negative real axis at some $x < -1$. Here $f(z)$ varies smoothly as expected, since both θ and θ_1 smoothly pass through π. But what about crossing the positive real axis at $x > 1$ to the right of the branch cut? Now both θ and θ_1 jump by 2π. This is not a problem, however, since $f(z)$ depends only on $\theta_1 - \theta$ and therefore does *not* jump even though θ and θ_1 do.

That leaves us with the question which branch cut is preferable, the single line in Fig. 8 or the two-segment branch cut consisting of the line $x \le -1$ and the line $x \ge 1$ analogous to Fig. 7? That depends on the situation we wish to study. The former has the advantage that it allows us to encircle the point $z = \infty$ where $f(z) = \log \frac{z-1}{z+1}$ does *not* have a branch point. On the other hand, the cut of Fig. 8 does not allow us encircle the origin with a small circle, say $|z| = 1/2$. The alternative branch cut analogous to Fig. 7 is exactly the other way around: it prevents us from encircling $z = \infty$, but allows us to encircle $z = 0$ with a small circle such as $|z| = 1/2$. The optimal choice therefore depends on what we wish to calculate.

Finally, let us ask the question whether we could have used the shorter branch cut analogous to Fig. 8 also for our previous example $g(z) = [z(z-1)]^{\frac{1}{3}}$ from Eq. (B.31)? The answer is no and the reason will guide us towards the set of rules we use for determining legitimate branch cuts. So what is the difference between $g(z) = [z(z-1)]^{\frac{1}{3}}$ and $f(z) = \log \frac{z-1}{z+1}$ and why does the line $0 \le x \le 1$ in Fig. 7 not constitute a proper branch cut where as $-1 \le z \le 1$ in Fig. 8 does? The answer is that $g(z)$ also has a branch point at $z = \infty$ whereas $f(z)$ does not. The line $0 \le x \le 1$ in Fig. 7 would therefore still allow us to encircle the branch point $z = \infty$ of $g(z)$; that's why it is not a complete branch cut. How can we figure out then whether we have a proper branch cut or not? This is addressed by the following proposition which we state without proof.

Proposition: Let $f(z)$ be a function with branch points z_1, z_2, ... A branch cut for this function is given by a set of curves or "cuts" with the following properties. (i) Every branch point has a cut ending on it. (ii) Both ends of each cut end on a branch point. (iii) Any closed curve in $\mathbb{C}$ not intersecting the branch cut encloses either all or none of the branch points.

Merely choosing the line $0 \le x \le 1$ in Fig. 7 fails to satisfy the first of these requirement, since the branch point $z = \infty$ has no cut ending on it. Of course, we could complete the branch cut by adding a second cut, say the line $1 \le x < \infty$.

Note that it is important in this discussion to regard $z = \infty$ as merely one single point in accordance with the extension of the complex plane discussed in Sec. B.1. The above set of rules for a branch cut is indeed best visualized in terms of the Riemann sphere of Fig. 2 where $z = \infty$ is represented by the North Pole. By mapping the two-segment branch cut of Fig. 7 onto the Riemann sphere, we indeed obtain just a single line on the sphere; it happens to pass through the North Pole, but there is nothing special about that point. In case we have > 1 curves in a branch cut (candidate) marching off to infinity, and we are too lazy to draw a Riemann sphere, it is helpful to regard these different lines as joining together at infinity, even if they march towards it in completely different directions and irrespective of whether or not the function in question has a branch point at infinity.

In summary, when we wish to draw a branch cut of a function, we first need to determine all branch points, including $z = \infty$. Then we draw individual cuts according to the above set of rules. While the branch points are uniquely determined by the function in question, we have some degree of freedom in choosing the branch cut and in thereafter choosing a branch of the function.

B.5 Möbius maps

The remainder of this section is concerned with maps between copies of the complex plane $\mathbb{C}$. We start our discussion with Möbius maps.

Def. : A *Möbius map* $\mathcal{M}$ is defined as a map

$$\mathcal{M} : \mathbb{C} \to \mathbb{C}\,, \qquad z \mapsto w = \frac{az+b}{cz+d}\,, \tag{B.35}$$

with a, b, c, $d \in \mathbb{C}$ satisfying $ad \neq bc$.

The condition $ad - bc \neq 0$ is required to have a non-trivial map. If $ad = bc \neq 0$, we find

$$d = \frac{bc}{a} \quad \Rightarrow \quad w = \frac{az+b}{cz+d} = \frac{az+b}{cz+\frac{bc}{a}} = \frac{a}{c}\frac{az+b}{az+b} = \frac{a}{c} = \text{const}\,, \tag{B.36}$$

while for $a = c = 0$ ($a = b = 0$, $d = b = 0$, $d = c = 0$) we get constants $w = \frac{b}{d}$ ($w = 0$, $w = \frac{a}{c}$, "$w = \infty$").

The Möbius map (B.35) is analytic except at $z = -\frac{d}{c}$. We can, however, regard it as a map within the extended complex plane $\mathbb{C}^* \to \mathbb{C}^* := \mathbb{C} \cup \{\infty\}$ (recall Sec. B.1). This map $\mathcal{M} : \mathbb{C}^* \to \mathbb{C}^*$ is bijective with inverse

$$\mathcal{M}^{-1} : \mathbb{C}^* \to \mathbb{C}^*\,, \qquad w \mapsto z = \frac{-dw+b}{cw-a}\,, \tag{B.37}$$

which is also a Möbius map. Interpreted as maps between $\mathbb{C}^*$ and itself, both $\mathcal{M}$ and $\mathcal{M}^{-1}$ are analytic everywhere.

The most important property of Möbius maps is its mapping between circles and lines.

Def. : A "circline" is either a circle or a line.

Proposition: The image of a circline under a Möbius map is a circline.

Proof. The proof is based on the representation of a circline as a "circle of Apollonius", namely that any circline can be expressed as the set of points $z \in \mathbb{C}$ with the following property,

$$|z - z_1| = \lambda |z - z_2|\,, \tag{B.38}$$

where $z_1 \neq z_2 \in \mathbb{C}$ are reference points in the complex plane and $\lambda \in \mathbb{R}^+$ is a positive real number. In words, this expression states that a circline is made up of points with a constant distance ratio λ to two fix points z_1, z_2 in the plane. The proof of this property of circlines is not really the topic of our lectures; we therefore derive it separately in the next, nonexaminable subsection. Note that the case $\lambda = 1$ corresponds to a line and $\lambda \neq 1$ to a circle.

We will show that the image of the set of points z satisfying Eq. (B.38) under the map (B.35) results in a set of points w that also satisfies an equation of the type (B.38). Let us therefore substitute in Eq. (B.38) for z in terms of w according to Eq. (B.37),

$$\begin{aligned} &\left|-\frac{dw-b}{cw-a}-z_1\right| = \lambda\left|-\frac{dw-b}{cw-a}-z_2\right| \qquad\qquad \Bigg|\ \cdot|cw-a| \\ \Rightarrow\quad &|-dw+b-z_1(cw-a)| = |dw-b+z_1(cw-a)| = \lambda|dw-b+z_2(cw-a)| \\ \Rightarrow\quad &|w(cz_1+d)-(az_1+b)| = \lambda|w(cz_2+d)-(az_2+b)|\,. \end{aligned} \tag{B.39}$$

If $cz_1+d=0$ or $cz_2+d=0$, this trivially gives a circle. Otherwise,

$$\left|w-\frac{az_1+b}{cz_1+d}\right| = \lambda\left|w-\frac{az_2+b}{cz_2+d}\right|\,, \tag{B.40}$$

and we have another circline. □

Geometrically, we can determine a circline by specifying three points in $\mathbb{C}^*$; a line is obtained for the special case where one of the points is $z=\infty$. We obtain a similar property for Möbius maps.

Proposition: Let $\alpha \neq \beta \neq \gamma \neq \alpha \in \mathbb{C}^*$ and $\tilde\alpha \neq \tilde\beta \neq \tilde\gamma \neq \tilde\alpha \in \mathbb{C}^*$ denote 6 points in the extended complex domain. Then there exists a Möbius map that sends $\alpha \mapsto \tilde\alpha$, $\beta \mapsto \tilde\beta$ and $\gamma \mapsto \tilde\gamma$.

Proof. We define a first Möbius map

$$\mathcal{M}_1(z) = \frac{\beta-\gamma}{\beta-\alpha}\frac{z-\alpha}{z-\gamma}\,, \tag{B.41}$$

that maps $\alpha \mapsto 0$, $\beta \mapsto 1$ and $\gamma \mapsto \infty$. Next we define a second Möbius map

$$\mathcal{M}_2(z) = \frac{\tilde\beta-\tilde\gamma}{\tilde\beta-\tilde\alpha}\frac{z-\tilde\alpha}{z-\tilde\gamma}\,, \tag{B.42}$$

which clearly maps $\tilde\alpha \mapsto 0$, $\tilde\beta \mapsto 1$ and $\tilde\gamma \mapsto \infty$. As we have seen in Eq. (B.37), a Möbius map is invertible, so that $\mathcal{M}_2^{-1} \circ \mathcal{M}_1$ is the required map. This is also a Möbius map, since Möbius maps form a group. □

This is a convenient result; it enables us to construct for any given pair of cirlines a Möbius map that takes one circline to the other.

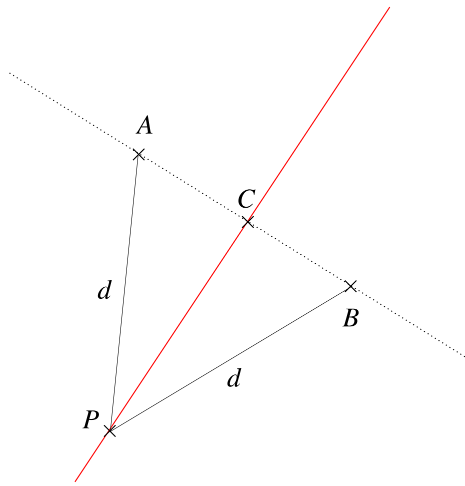

Figure 9: Illustration of the construction of a straight line as the set of points that have a unit distance ratio from two fixed points A and B.

B.6 The circle of Apollonius*

This section is not examinable and represents complementary material to the lecture only.

Before continuing our discussion of maps in the complex plane, we add here a proof for the construction of circlines according to Apollonius' method. While not trivial, this proof employs tools from elementary geometry only, and readers can safely jump to Sec. B.7 without missing material necessary for the remainder of this course.

The standard definition of a circle is the set of all points that have the same distance from some reference point. As early as about 200 years BC, the Greek astronomer Apollonius of Perga realized that a circle can be defined in an alternative way.

Proposition. : A circle is given by the set of points that have a specified ratio of distances to two fixed points. These fixed points are called the *foci* of the circle.

Before we proof this theorem, we consider the special case of a circle with infinite radius, which is simply a straight line.

Proposition. : A straight line is given by the set of points that have equal distance to two fixed points.

This case is illustrated in Fig. 9 and much easier to understand than that of a circle. Consider for this purpose two fixed points A and B. Draw a line through A and B and define C as the midway point between A and B along this line. Draw a second line perpendicular to the first through C. By symmetry, every point P on this second line (red in the figure) has the same distance d from A as from B. One may already imagine that this construction can be deformed into a circle by moving point C away from the center towards either A or B; the distance ratio

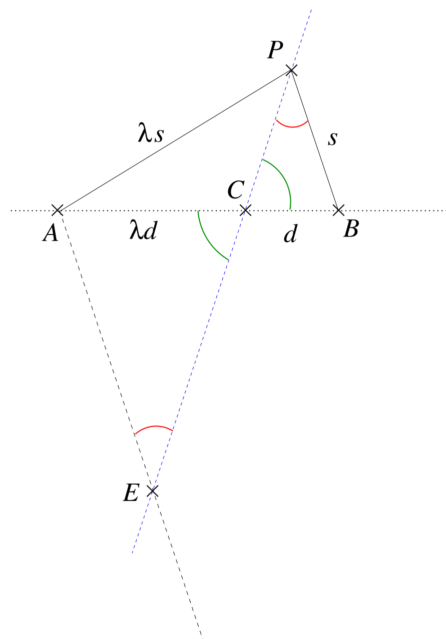

Figure 10: The geometrical derivation of Eq. (B.47).

will then differ from unity. This is indeed the case, as we will show next.

For this purpose, however, we need to do some preliminary geometrical calculations. Let us again start with two points A and B and draw a straight line through these points. Again we choose point C on this line, but now a bit off-center, say a bit closer to B than to A, but still in between A and B. This is displayed in Fig. 10 where we have labeled the distances as [1]

$$\overline{CB} = d \qquad \text{and} \qquad \overline{AC} = \lambda d\,, \tag{B.43}$$

with $\lambda > 1$. In mathematical language, point C is said to *internally divide* the line segment AB in the ratio $\lambda : 1$ or just λ.

Now let P be a point away from the line, but with the same distance ratio to A and B, say

$$\overline{PB} = s \qquad \text{and} \qquad \overline{AP} = \lambda s\,. \tag{B.44}$$

Next, we draw the line through points P and C and then we draw a line through point A that is parallel to the line segment PB; cf. the dashed curves in Fig. 10. The intersection of these two dashed lines gives us point E. By construction, the angles $\sphericalangle CPB$ and $\sphericalangle CEA$ (the red angles in the figure) are equal, and the angles $\sphericalangle ACE$ and $\sphericalangle BCP$ (green angles in the figure) are equal. Hence, the triangles ACE and BCP are similar and we obtain

$$\frac{\overline{BC}}{\overline{PB}} = \frac{\overline{AC}}{\overline{AE}} \quad \Rightarrow \quad \frac{\overline{AE}}{\overline{PB}} = \frac{\overline{AC}}{\overline{BC}} \quad \Rightarrow \quad \overline{AE} = \lambda\,\overline{PB}\,. \tag{B.45}$$

[1] We denote the length of a line segment between points A and B by an overline as in $\overline{AB}$. In contrast, we denote the vector from A to B with an arrow as in $\overrightarrow{AB}$. Note that the order of the points matters in the latter case but not in the former, i.e. $\overline{BA} = \overline{AB}$ but $\overrightarrow{BA} = -\overrightarrow{AB}$.

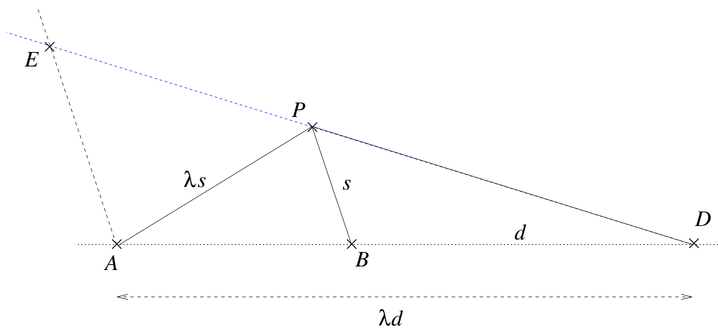

Figure 11: The geometrical derivation of Eq. (B.49).

The triangle PAE is therefore half of a diamond and, incidentally, we have shown that the line PC bisects the angle $\sphericalangle APB$. But that's on the side and we instead continue employing the similarity of our triangles. We find

$$\frac{\overline{AC}}{\overline{EC}} = \frac{\overline{BC}}{\overline{PC}} \quad \Rightarrow \quad \frac{\overline{EC}}{\overline{PC}} = \frac{\overline{AC}}{\overline{BC}} \quad \Rightarrow \quad \overline{EC} = \lambda\,\overline{PC} \quad \Rightarrow \quad (1+\lambda)\overline{PC} = \overline{PE}\,. \tag{B.46}$$

Combining this result with Eq. (B.45), we obtain the vectorial relation

$$\overrightarrow{PE} = \overrightarrow{PA} + \overrightarrow{AE} \quad \Rightarrow \quad \boxed{(1+\lambda)\overrightarrow{PC} = \overrightarrow{PA} + \lambda\,\overrightarrow{PB}} \tag{B.47}$$

In analogy to the internal division of the line segment AB, the *external division* in the ratio λ is defined by the point D outside of the segment that satisfies

$$\frac{\overline{AD}}{\overline{BD}} = \lambda\,. \tag{B.48}$$

This is illustrated in Fig. 11. As in Fig. 10, we consider a point P with a distance ratio λ from the points A and B but not located on the line AB. The triangle ABP is therefore the same as in Fig. 10, but instead of the internal division point C, we now have the external division point D. Let us next extend the line DP and draw the parallel to PB through the point A. Point E is defined as the intersection between the two resulting lines. From the interception theorems, we conclude

$$\frac{\overline{ED}}{\overline{PD}} = \frac{\overline{AD}}{\overline{BD}} \stackrel{!}{=} \lambda \qquad \text{and} \qquad \frac{\overline{AE}}{\overline{BP}}\,\frac{\overline{AD}}{\overline{BD}} = \lambda$$

$$\Rightarrow \quad \overrightarrow{PE} = \overrightarrow{PA} + \overrightarrow{AE} = \overrightarrow{PA} + \lambda\,\overrightarrow{BP} \qquad \wedge \qquad \overrightarrow{PE} = \overrightarrow{DE} - \overrightarrow{DP} = (\lambda-1)\,\overrightarrow{DP}$$

$$\Rightarrow \quad \boxed{(\lambda-1)\,\overrightarrow{PD} = \lambda\,\overrightarrow{PB} - \overrightarrow{PA}}\,. \tag{B.49}$$

Now we return to our original question, namely to prove that points with a fixed distance ratio to two fixed points A and B lie on a circle. For this purpose, we combine in Fig. 12

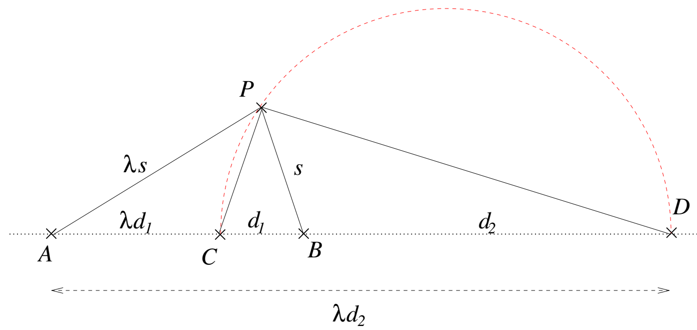

Figure 12: The internal and external division of the line segment AB. Any point P that has the same distance ration λ to A and B that the internal division point C and the external division point D have, lies on a circle with C and D; cf. Eq. (B.50).

the internal and external divisions of the line segment AB of Figs. 10 and 11. For clarity, let us summarize what we have done in this figure. We started with two fixed points A and B. We then chose a distance ratio $\lambda > 1$ and defined the internal and external division points C and D as the two points on the extended line through A and B that have the distance ratio $\overline{AC}/\overline{BC} = \overline{AD}/\overline{BD} = \lambda$. We argued why these two points are uniquely specified and one (C in our case) is inevitably located between A and B and the other outside the segment AB. We then considered an (otherwise arbitrary) point P also with distance ratio $\overline{AP}/\overline{BP} = \lambda$ from A and B. For this point we concluded that Eqs. (B.47) and (B.49) hold. The rest is easy, as we find

$$\begin{aligned}\overrightarrow{PC}\cdot\overrightarrow{PD} &\overset{(B.47),\,(B.49)}{=} \frac{\overrightarrow{PA}+\lambda\,\overrightarrow{PB}}{1+\lambda}\cdot\frac{\lambda\,\overrightarrow{PB}-\overrightarrow{PA}}{\lambda-1} = \frac{1}{\lambda^2-1}(\lambda\,\overrightarrow{PB}+\overrightarrow{PA})\cdot(\lambda\,\overrightarrow{PB}-\overrightarrow{PA}) \\ &= \frac{1}{\lambda^2-1}\left(\lambda^2\,\overrightarrow{PB}^2-\overrightarrow{PA}^2\right) = \frac{1}{\lambda^2-1}\left(\lambda^2\,\overline{PB}^2-\overline{PA}^2\right) = 0\,, \end{aligned} \tag{B.50}$$

since $\overline{PA} = \lambda\,\overline{PB}$. So $\overrightarrow{PC}$ and $\overrightarrow{PD}$ are perpendicular and, by Thales' theorem, on a circle with diameter $\overline{CD}$. Note that all points P thus chosen must lie on the same circle since, for fixed points A and B and distance ratio λ, the internal and external division points are uniquely determined. In consequence all P lie on the circle which has a diameter $\overline{CD}$ and includes the points C and D. We complete the proof with the following remarks.

1. Our restriction to distance ratios $\lambda > 1$ is without loss of generality; for $0 < \lambda < 1$, we simply swap the points A and B, work with the distance ratio $\mu := 1/\lambda > 1$ and apply the proof as above.
2. Note that the set of all points P with distance ratio λ from A and B includes the division points C and D. We have merely restricted our discussion to points P away from the axis AB to make possible the geometrical constructions in Figs. 10 and 11. Equation (B.50) is trivially satisfied for $P = C$ or $P = D$ and C and D are perfectly respectable members of the circle.

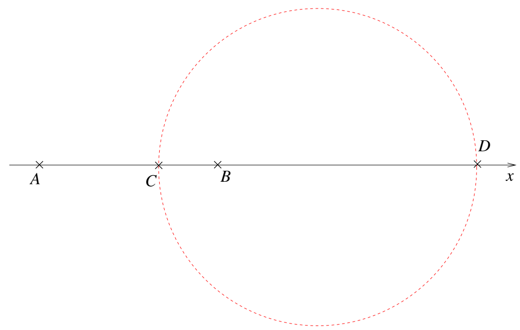

Figure 13: For an arbitrary circle, we choose to opposite points C and D on the circle and define the x axis by the line from C to D. For a given $\lambda > 1$, we then obtain points A and B according to Eq. (B.51) such that C and D are the internal and external division points of AB.

3. Strictly speaking, our proof has shown that for two fixed points A, B and a distance ratio $\lambda > 1$, the set of all points P with the same distance ratio from A and B form a circle. The reverse is also true: Let us be given an arbitrary circle. Choose a pair of opposite points on the circle, call these C and D, and define the x axis by the line from C to D. Choose an arbitrary distance ratio $\lambda > 1$. We now seek points A and D such that C and D internally, respectively externally, divide the line segment AB. By definition, this imposes the conditions (cf. Fig. 13),

$$\begin{aligned}
& x_C - x_A = \lambda(x_B - x_C) && \wedge && x_D - x_A = \lambda(x_D - x_B) \\
\Rightarrow\quad & x_D + x_C - 2x_A = \lambda(x_D - x_C) && \wedge && x_D - x_C = \lambda(x_D + x_C - 2x_B) \\
\Rightarrow\quad & x_A = \frac{1}{2}\left[x_D + x_C - \lambda(x_D - x_C)\right] && \wedge && x_B = \frac{1}{2}\left(x_D + x_C - \frac{x_D - x_C}{\lambda}\right).
\end{aligned} \tag{B.51}$$

So the points A and B are uniquely determined and by our extended derivation above, the points on the given circle are points with distance ratio λ to A and B.

4. Note that we have a one-parameter mapping between circles on the one side and the foci A and B on the other. For a given circle, the points A and B are determined by Eq. (B.51) only up to our choice of $\lambda > 1$. Likewise, for a given set of points A and B, the division points C and D and, hence, the radius of the circle, are only determined up to our choice of λ; cf. Fig. 12.

B.7 Conformal mappings

Conformal maps are a convenient tool for solving problems on complicated subdomains of the complex plane $U \subseteq \mathbb{C}$ or the $\mathbb{R}^2$. The idea is to first transform the problem onto simpler domain V, say an open disk, solve the resulting set of equations on V and then transform back to the original domain U. A remarkable application of this technique has been used by Ansorg *et al* [2] in numerically computing initial data for black-hole binary systems with particularly

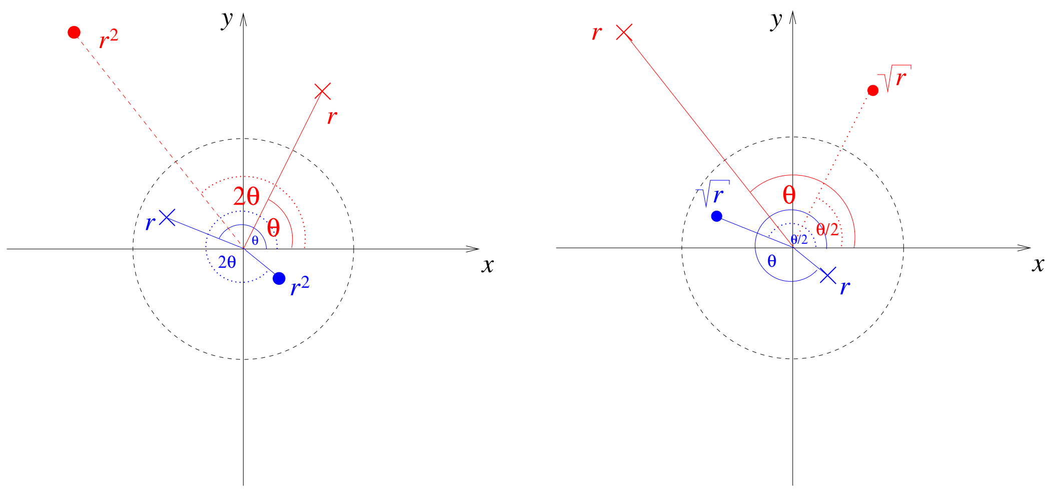

Figure 14: *Left:* Geometric illustration of the mapping $z \mapsto x^2$ in the complex plane. The unit circle is shown in black dashed line style. The red (blue) point z marked by a $\times$ symbol gets mapped to the red (blue) filled circle. The effect of this map is a rotation to twice the original angle θ and a push away from the unit circle, either away from the origin (for points with $r = |z| > 1$) or towards the origin (for points with $r = |z| < 1$). Points on the unit circle stay on the unit circle but are still rotated. *Right:* Illustration of the mapping $z \mapsto \sqrt{z}$, which is the inverse of the square mapping. Points now get rotated to half their original angle and are drawn towards the unit circle. Note that we need to choose a branch for the square root; for the points in the figure we have chosen a branch cut given by the nonnegative real axis ($x \geq 0$).

high accuracy and efficiency; this code is still the gold standard for many numerical relativity simulations of black hole systems such as GW150914 whose Nobel-Prize winning detection by LIGO in 2015 [1] marked the dawn of gravitational-wave astronomy. Further below, we will discuss conformal maps in terms of graphical representations that look strikingly similar to Fig. 3 in Ref. [2]. First, however, let us start by developing some intuition for the geometric effect of simple operations on complex numbers.

B.7.1 Simple maps in the complex plane

We are quite used to mappings among real numbers and generally have few problems picturing the effects; for instance the exponential function maps $\mathbb{R}$ to the positive real numbers, and the square function will conjure the image of a parabola. Mappings in the complex plane often represent systematic geometric operations, too, and the purpose of this introduction is to acquire familiarity with these systematics for the most important functions.

Examples of simple mappings

(1) Let us first consider the square mapping $z \mapsto f(z) = z^2$. Its effect is best understood by writing $z = re^{\mathrm{i}\theta}$, so that

$$f : re^{\mathrm{i}\theta} \mapsto r^2 e^{\mathrm{i}2\theta} \,. \tag{B.52}$$

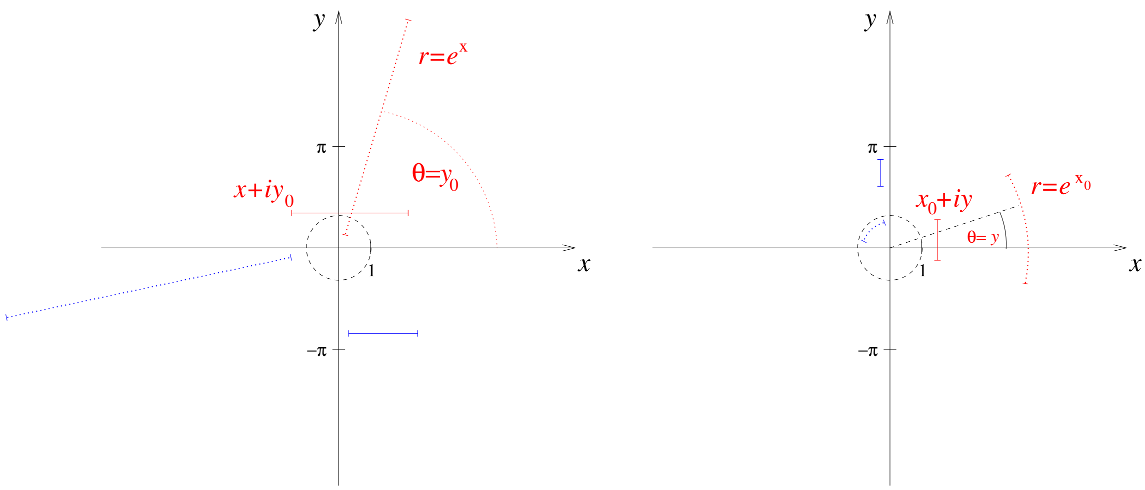

Figure 15: *Left:* The effect of the exponential map on a horizontal line segment, i.e. a set of points with constant imaginary part. The set of points $x + \mathrm{i}y_0$ with $y_0 \in \mathbb{R}$ and $a < x < b$ is shown as the solid red line. The image of this line consists of the points with constant angle $\theta = y_0$ and distance $r = e^x \in [e^a, e^b]$ as shown by the red dotted line. Note that points with $x < 0$ are mapped to points inside the unit circle while a positive real part $x > 0$ leads to points outside the unit circle. As a second example, we consider the solid blue line with negative y_0 a bit short of $-\pi$. Its image is the blue dotted line segment pointing just shy off 09:00 o'clock. *Right:* The effect of the exponential map on vertical line segments with constant real part x_0. All image points have distance $r = e^{x_0}$ from the origin and thus form part of a circle. The length of this arc is determined by the range of the complex part of the original set of points. The solid blue curve represents a second example, now with $x_0 < 0$; its image lies inside the unit circle.

The mapping therefore has two effects. First, the point is rotated around the origin from its original angle θ to twice that angle. Second, the squaring of the point's modulus r implies that points outside the unit circle are pushed to larger radii and points inside the unit circle are pushed closer to the origin. Points on the unit circle stay on the circle but are still rotated. These effects are illustrated for two example points in the left panel of Fig. 14. A similar effect is obtained for mappings $z \mapsto z^\alpha$ with $\alpha > 1$, but note that we need to choose a branch for non-integer α.

(2) The second map we consider is the square root $z \mapsto \sqrt{z}$. Again, this is best understood writing $z = re^{\mathrm{i}\theta}$ and

$$f : re^{\mathrm{i}\theta} \mapsto \sqrt{r}e^{\mathrm{i}\frac{1}{2}\theta}\,. \tag{B.53}$$

This map rotates a point to half its original angle θ and draws it towards the unit circle. These effects are illustrated in the right panel of Fig. 14. The figure also illustrates that the square root is the inverse of the square mapping. Maps of the form $z \mapsto z^\alpha$ with $0 < \alpha < 1$ have a similar effect. Note that for all these cases we need to choose a branch; for the points of Fig. 14 we have chosen the nonnegative real axis as the branch cut. We can summarize the effect of power law mappings $z \mapsto z^\alpha$ as a rotation and a stretching or squeezing of the original point's distance from the origin.

(3) The exponential map is best understood by writing $z = x + \mathrm{i}y$, whence

$$f : (x + \mathrm{i}y) \mapsto e^{x+\mathrm{i}y} = re^{\mathrm{i}y} \quad \text{with} \quad r = e^x\,. \tag{B.54}$$

The real part of the input number thus determines the distance r of the image point from the origin and the imaginary part determines the argument or angle of the image. It is instructive to consider the images of horizontal or vertical line segments under the exponential map as illustrated in Fig. 15. Horizontal lines with constant imaginary part y_0 and real range $c \le x \le d$ are mapped to points with constant angle $\theta = y_0$ and form part of a radial ray with extent $e^a \le r \le e^b$. A vertical line, in contrast, has constant real part and is therefore mapped to a segment of a circle of radius $r = e^{x_0}$ with the angular range determined by the imaginary extent of the original line. Note that vertical curves to the right of the y axis with $x_0 > 0$ result in images outside the unit sphere and those to the left of the y axis are mapped inside the unit circle; cf. the red and blue curves in the right panel of Fig. 15. A Rectangle is bounded by two vertical and two horizontal line segments, which are mapped to two circle segments and two ray segments, respectively. The exponential function thus maps rectangles to sectors of annuli ("annulus" is Latin for "ring" which I remember best by recalling the Spanish title of J.R.R. Tolkien's epic "El señor de los anillos").

(4) The logarithmic map $z \mapsto \ln z$ is the inverse operation of the exponential mapping, provided we have appropriately chosen a branch. It therefore maps sectors of annuli to rectangles.

B.7.2 Conformal maps

Having established the effects of simple maps, we turn our attention to the specific class of conformal maps.

Def. : A *conformal map* is a map $f : U \to V$ between *open* subsets U, V of $\mathbb{C}$ which is analytic and whose derivative f' is non-zero throughout U. In practice, we often require this map to be bijective; this requirement is sometimes made explicit by calling the map a *conformal equivalence*.

One can alternatively define a conformal map in terms of its action on the angle between intersecting curves.

Proposition: A conformal map preserves the angle in magnitude *and* orientation between intersecting curves.

Proof. Let $z_1(t)$ be a curve in $\mathbb{C}$, parametrized by $t \in \mathbb{R}$, that passes through a point $z_0 := z(t_0)$ with nonzero derivative $z_1'(t_0)$. The curve then makes an angle $\theta = \arg z_1'(t_0)$ relative to the x direction at z_0; cf. Fig. 16.

Now consider the image of this curve under a conformal map f, $\zeta_1(t) = f(z_1(t))$. The image's tangent at t_0 is given by chainrule,

$$\zeta_1'(t_0) = \left.\frac{\mathrm{d}f}{\mathrm{d}z_1}\right|_{t=t_0} \left.\frac{\mathrm{d}z_1}{\mathrm{d}t}\right|_{t=t_0} = f'(z_0)\, z_1'(t_0)\,, \tag{B.55}$$

and it makes an angle ϑ with the x direction given by

$$\vartheta = \arg(\zeta_1'(t_0)) = \arg\left(z_1'(t_0)\, f'(z_0)\right) = \theta + \arg f'(z_0)\,. \tag{B.56}$$

Here we need the conformal nature of the map f which ensures that $f'(z_0) \neq 0$, so that its argument $\arg f'(z_0)$ is well defined.

In words, Eq. (B.56) tells us that the tangent direction of the image has been rotated by $\arg f'(z_0)$ relative to the original curve $z_1(t)$. Now let $z_2(t)$ be another curve passing through z_0. Because the rotation angle $\arg f'(z_0)$ is independent of the original curve, the tangent direction of $z_2(t)$ will be rotated by the same angle as that of $z_1(t)$, and the angle between $z_1(t)$ and $z_2(t)$ at z_0 remains unchanged under the conformal map f. □

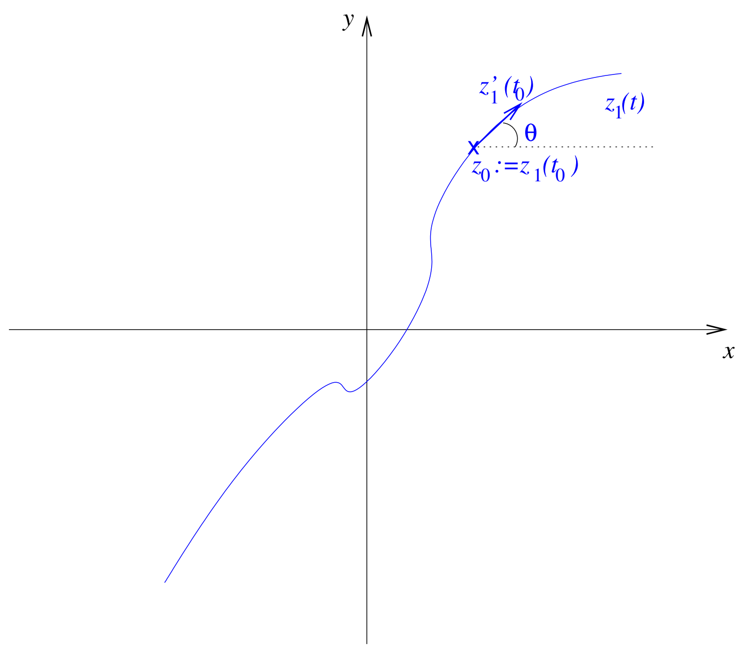

Figure 16: A curve $z_1(t)$ with tangent $z_1'(t_0)$ at the point $z_0 \coloneqq z_1(t_0)$. The tangent makes an angle $\theta = \arg z_1'(t_0)$ with the (parallel of the) x axis at z_0.

The reverse of our proposition is also true, though we omit the proof. In consequence conformal maps can be equivalently defined as maps that preserve the angle between intersecting curves.

In practice one often determines the image V of a conformal map f acting on U by computing the image of the boundary $f(\partial U)$ which forms the boundary ∂V of V. This procedure does not tell us on which side of ∂V the image V is located; this difficulty is easily overcome, though, by selecting a convenient point of U whose image will then tell us which side V lies on.

These ideas are best illustrated by some examples.

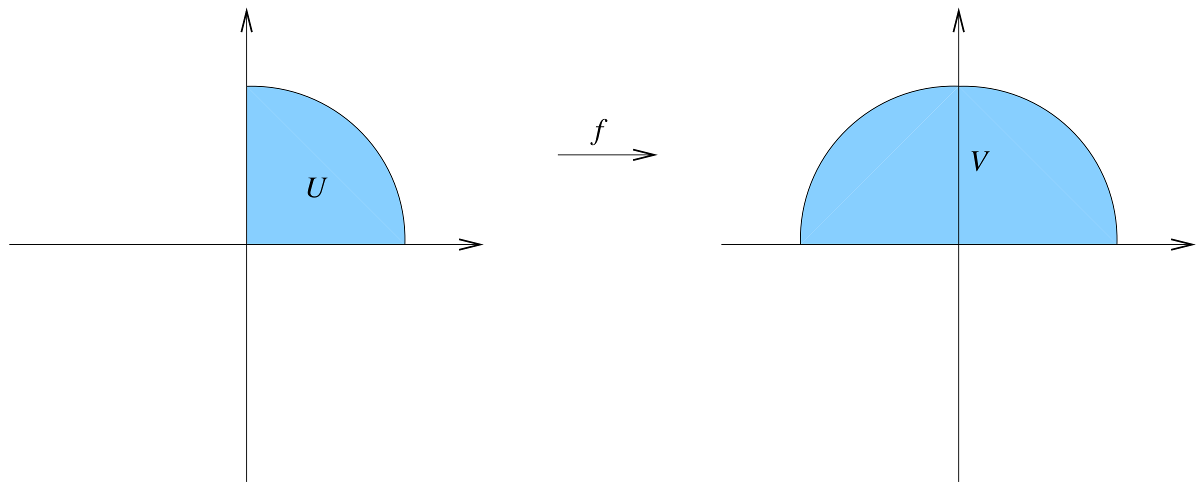

Figure 17: The map $f(z) = z^2$ maps the upper right quarter U of the unit disk to the upper half unit disk (both without boundaries).

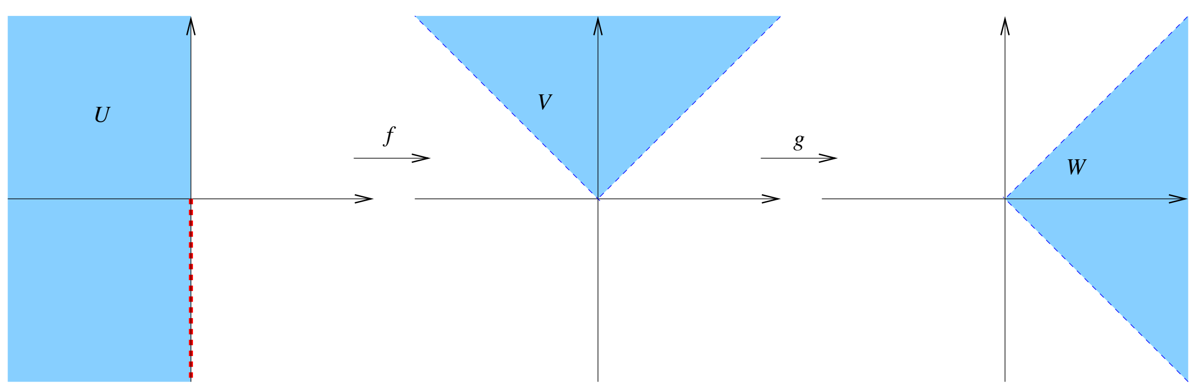

Figure 18: The left half plane U is mapped to the "right" wedge W in two steps: $f(z) = z^{\frac{1}{2}}$ halves the angle and carries us to the "upper" wedge V which is then rotated to W by $g(\zeta) = -\mathrm{i}\zeta$. For the square root $z^{\frac{1}{2}}$, we need to choose a branch and make sure the branch cut does not intersect U; this is satisfied by the non-positive imaginary axis marked by the red dotted line.

Examples of conformal maps

(1) The map $f(z) = az + b$ with $a,\ b \in \mathbb{C}$ and $a \neq 0$ is clearly analytic everywhere with non-zero derivative $f'(z) = a$ and hence conformal. It rotates points by $\arg a$, enlarges or reduces their distance from the origin by a factor $|a|$ and translates by b.

(2) The function $f(z) = z^2$ is analytic everywhere, but its derivative vanishes at $z = 0$, so it is a conformal map as long as we exclude the origin. For example, it is a conformal map from the upper right quadrant of the unit disk,

$$U = \left\{ z \in \mathbb{C} \;\middle|\; 0 < |z| < 1 \quad \wedge \quad 0 < \arg z < \frac{\pi}{2} \right\}$$

to the upper half disk

$$V = \left\{ w \in \mathbb{C} \;\middle|\; 0 < |w| < 1 \quad \wedge \quad 0 < \arg w < \pi \right\} ,$$

as illustrated in Fig. 17. Note that the three boundary segments of U form right angles at the points $z = 0$, $z = 1$ and $z = \mathrm{i}$. The latter two points are mapped to $w = 1$, $w = -1$ where the boundary segments of V also form right angles, as they should since conformal maps preserve angles. The right angle at $z = 0$, however, is not preserved at $w = 0$; this is not surprising, since the map f is *not* conformal at $z = 0$.

(3) Often, one is looking for a conformal map that transforms between two given subsets of the complex plane. Let us determine, for example, which conformal map carries us from the left half plane

$$U = \left\{ z \in \mathbb{C} \;\middle|\; \mathsf{Re}(z) < 0 \right\}$$

to the "right" wedge

$$W = \left\{ w \in \mathbb{C} \;\middle|\; -\frac{\pi}{4} < \arg w < \frac{\pi}{4} \right\} .$$

From the graphical illustration of U and W in Fig. 18, we see that we need to half the angular range. This is achieved by taking the square root $f(z) = z^{1/2}$, but for this function with non-integer exponent we need to choose an appropriate branch. Because we require $f(z)$ to be analytic on U, the branch cut must not intersect U which excludes the principal branch $\arg z \in (-\pi, \pi]$. Instead, we can put the branch cut on the non-positive imaginary axis as shown by the red dotted line in the figure and choose, for example, $\arg z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Under the mapping f, however, this gives us the "upper" wedge V displayed in the middle of Fig. 18. We need a rotation by $-\pi/2$ to arrive at our destination W, which is achieved by multiplication with $e^{-\mathrm{i}\pi/2} = -\mathrm{i}$, i.e. an additional map $g(\zeta) = -\mathrm{i}\zeta$. The complete map from U to W is then

$$g \circ f(z) = -\mathrm{i} z^{\frac{1}{2}} .$$

(4) We have already seen in Fig. 15 that the exponential function maps rectangles to sectors of annuli. The exponential map is always conformal, since $f(z) = e^z = f'(z)$ is analytic and non-zero everywhere. Its mapping is illustrated in Fig. 19. With an appropriate choice of a branch, the log function does the reverse.

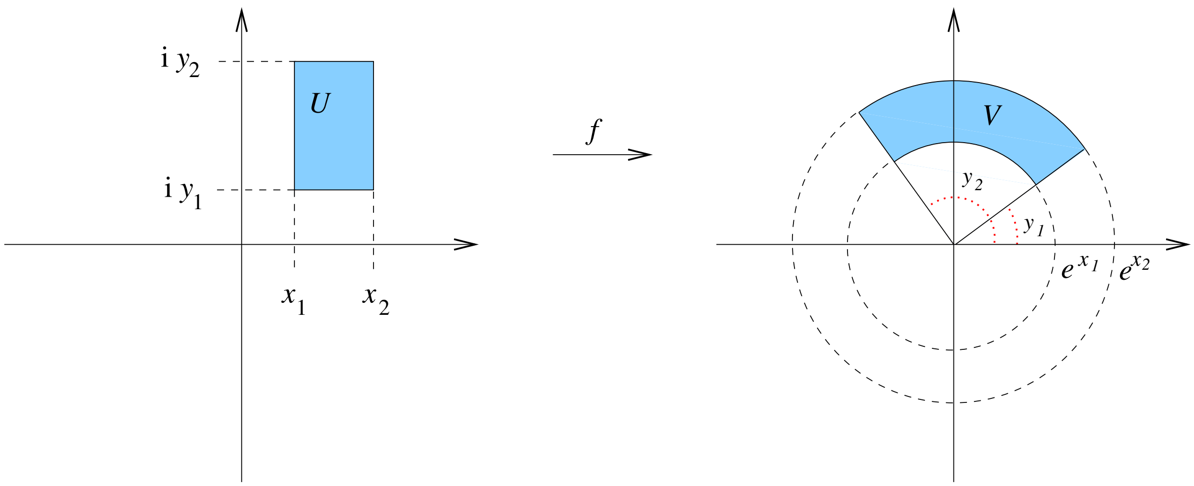

Figure 19: The exponential map $f(z) = e^z$ takes rectangles to sectors of annuli. With appropriate choice of branch, the log function does the inverse operation.

(5) Möbius maps (B.35) are conformal everywhere except at the point that is sent to ∞. They have very useful properties for the mapping between circles and straight lines (recall that they map circlines to circlines).

Let us consider the example $f(z) = \frac{z-1}{z+1}$ acting on the unit disk

$$U = \left\{ z \in \mathbb{C} \;\middle|\; |z| < 1 \right\} . \tag{B.57}$$

The boundary ∂U is the unit circle and includes the points $z = -1,\ \mathrm{i},\ +1$ which are mapped to $w = f(z) = \infty,\ \mathrm{i},\ 0$, respectively. The image ∂V of the boundary is therefore given by the imaginary axis. Furthermore $f(z = 0) = -1$ and the image $V = f(U)$ is the *left* half-plane $\mathsf{Re}(w) < 0$; see Fig. 20. This can also be shown by recalling the inverse of a Möbius map (B.37), whence

$$w = \frac{z-1}{z+1} \qquad \Leftrightarrow \qquad z = -\frac{w+1}{w-1} , \tag{B.58}$$

so that

$$|z| < 1 \qquad \Leftrightarrow \qquad |w+1| < |w-1| \tag{B.59}$$

and w has to be closer to -1 than to $+1$, i.e. lies to the left of the imaginary axis.

We can now compute further images of points located either on the unit circle or the real or imaginary axis, as for example,

$$f\big(z = (1+\mathrm{i})/\sqrt{2}\big) = \ldots = \frac{1-\sqrt{2}+\mathrm{i}}{1+\sqrt{2}+\mathrm{i}} = \ldots = \frac{\mathrm{i}}{1+\sqrt{2}} . \tag{B.60}$$

If we furthermore divide the complex plane into the 8 sectors shown in Fig. 21 – 4 quadrants of the unit disk shown in light blue and the 4 quadrant regions exterior to

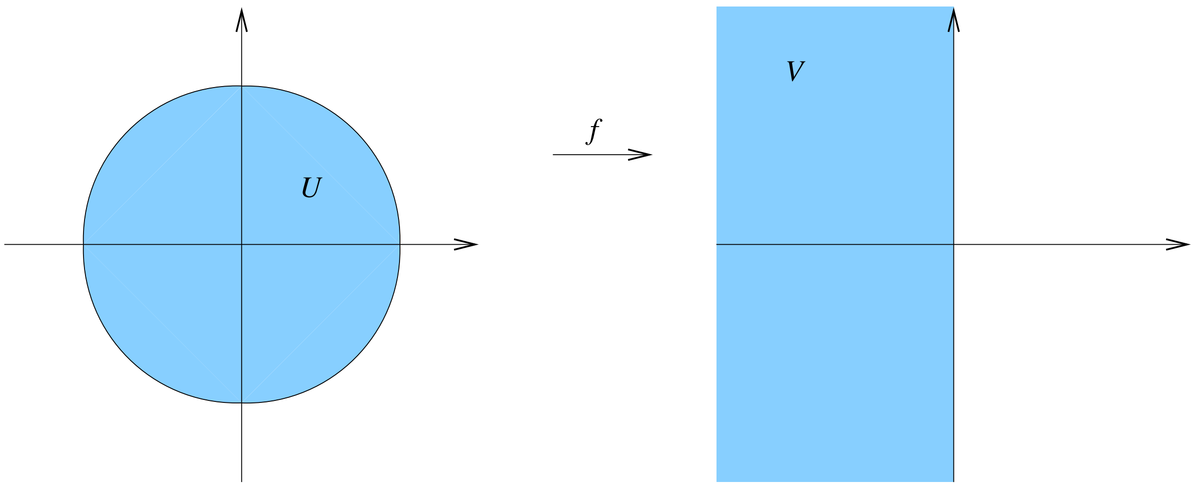

Figure 20: The Möbius map $f(z) = \frac{z-1}{z+1}$ takes the unit disk to the left half-plane.

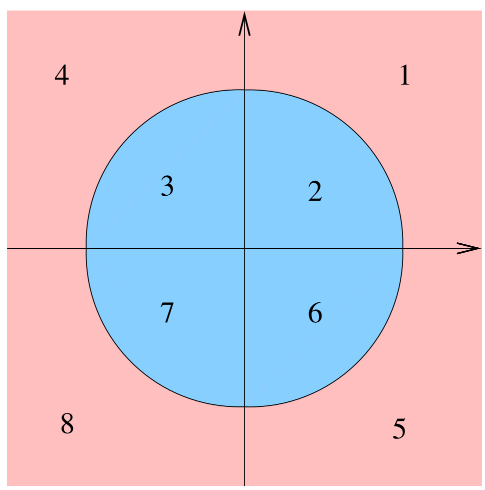

Figure 21: We divide the complex plane into 8 sectors as denoted by the numbers 1 to 8; note that region 1 includes only points in the upper right quadrant *exterior* to the unit disk and likewise for regions 4, 5 and 8. With this division, $f(z) = \frac{z-1}{z+1}$ maps $1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 1 \mapsto \ldots$ and $5 \mapsto 6 \mapsto 7 \mapsto 8 \mapsto 5 \mapsto \ldots$.

the unit disk shown in light red – we will eventually find that $f(z) = \frac{z-1}{z+1}$ systematically maps between these 8 sectors. More specifically the set of regions denoted 1 to 4 in Fig. 21 is mapped according to the cycle $1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 1 \mapsto \ldots$, and the regions 5 to 8 are mapped as $5 \mapsto 6 \mapsto 7 \mapsto 8 \mapsto 5 \mapsto \ldots$. This agrees, in particular with the above case where we mapped the unit disk to the left half-plane.

(6) $f(z) = \frac{1}{z}$ is another Möbius map which turns out useful for acting on vertical and horizontal lines. More details will be explored in the first example sheet.

(7) We have already seen in example (3) that it can be convenient to work with compositions of conformal maps. This can achieve quite remarkable results.

Suppose we want to map the upper half disk $|z| < 1$, $\mathsf{Im}(z) > 0$ to the full disc $|z| < 1$. One might naively choose $f(z) = z^2$, but this fails: no point gets mapped to the non-negative real axis, say to $w = \frac{1}{2}$, since the candidate points $z \sim e^{i\pi}$ are not contained in our original set.

We achieve our goal, however, by the following sequence of maps, as illustrated in Fig. 22.

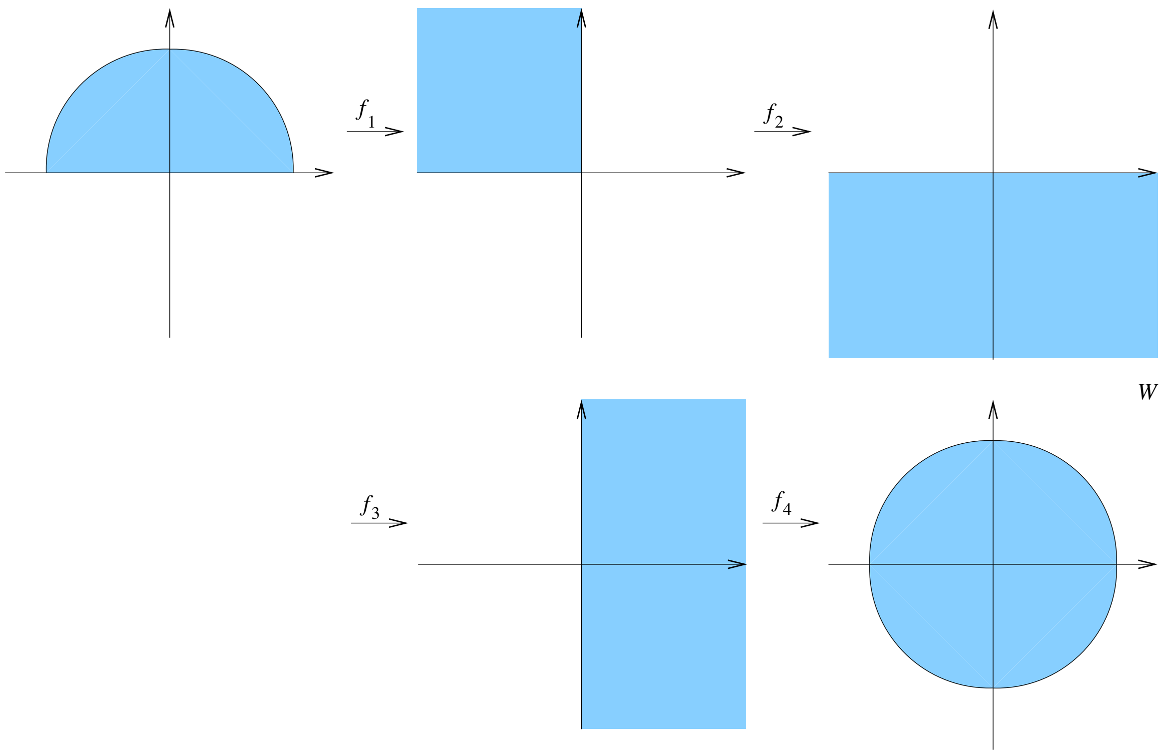

Figure 22: We wish to map the upper half disk to the full disk. The obvious attempt $f(z) = z^2$ fails as it does not map any point to the positive real axis. The desired mapping can be obtained, however, by the sequence of maps $f_4 \circ f_3 \circ f_2 \circ f_1$, as detailed in the text.

1. First, we apply $f_1(z) = \frac{z-1}{z+1}$ to take the upper half-disk to the second quadrant; recall the mapping pattern of Fig. 21.
2. We then apply $f_2(z) = z^2$ in order to expand the quadrant into the lower half plane.
3. We rotate this half-plane by $\pi/2$, i.e. apply the multiplication $f_3(z) = \mathrm{i}z$.
4. By looking once again at the mapping pattern of Fig. 21, we see that $\frac{z-1}{z+1}$ maps the right half-plane to the full unit disk. The final step is therefore $f_4(z) = \frac{z-1}{z+1} = f_1(z)$.

If you think that's black magic, that makes two of us. In order to shed at least some light on how the boundary problem has been overcome in this composite map, let us reverse engineer how we got to the point 1/2 that eluded us in the simpler $f(z) = z^2$

attempt. The last three steps towards $1/2$ are rather easily inverted,

$$f_4^{-1}(\tfrac{1}{2}) = 3\,, \qquad f_3^{-1}(3) = -3\mathrm{i} = 3e^{-\mathrm{i}3\pi/2}\,, \qquad f_2^{-1}(-3\mathrm{i}) = \sqrt{3}e^{-\mathrm{i}3\pi/4}\,. \tag{B.61}$$

We could plug the final value into f_1^{-1}, but that would be more work and not necessary; the key point is that $\sqrt{3}e^{-\mathrm{i}3\pi/4}$ is located somewhere in the interior of region 4 in Fig. 21; in particular, it is *not* located on the boundary of that region and therefore has to originate somewhere in the interior of region 3.

B.7.3 Laplace's equation and conformal maps

We have seen in Sec. B.3 that the real and imaginary parts of an analytic function are harmonic and solutions to the Laplace equation. By chain rule, the composition of an analytic function and a conformal map is also analytic, so that its real and imaginary part are also harmonic. One can therefore use conformal maps to carry solutions of Laplace's equation in some (conveniently chosen "simple") domain to a solution of Laplace's equation in another (more complicated) domain.

For this purpose, we will identify the complex domain $\mathbb{C}$ with $\mathbb{R}^2$ according to $z \leftrightarrow x + \mathrm{i}y$. Let $U \subseteq \mathbb{R}^2$ be some "tricky" domain and $f : U \to V$ a conformal map to a much nicer domain V as illustrated in Fig. 23. Our goal is to find a solution $\phi(x, y)$ of the Laplace equation

$$\triangle\phi = \partial_x^2\phi + \partial_y^2\phi = 0 \tag{B.62}$$

in U subject to some Dirichlet boundary conditions on ∂U. This can be achieved as follows.

1. The conformal map f carries points from U to points in V according to

$$f : U \to V\,, \qquad z = x + \mathrm{i}y \mapsto \zeta = u + \mathrm{i}v\,. \tag{B.63}$$

2. The Dirichlet boundary conditions $\phi(x, y) = \phi_0(x, y)$ on ∂U translate into boundary conditions $\Phi(u, v) = \Phi_0(u, v)$ on ∂V.
3. Now we solve the Laplace equation $\triangle\Phi = 0$ on the nice domain V subject to the boundary conditions on ∂V.
4. Then the solution on U is given by

$$\phi(x, y) = \Phi\big(\mathsf{Re}(f(x + \mathrm{i}y)), \mathsf{Im}(f(x + \mathrm{i}y))\big)\,. \tag{B.64}$$

The final expression looks more complicated than it really is, and we can write it in much simpler from if we (formally incorrectly) set "$z = (x, y)$" and "$\zeta = (u, v)$". This improper version of Eq. (B.64) becomes "$\phi(z) = \Phi\big(f(z)\big)$".

The proof of Eq. (B.64) is remarkably simple given the knowledge we already have. As we have seen in Sec. B.3, a solution $\Phi(u, v)$ to the Laplace equation is harmonic and, hence, the real part of an analytic complex function $G(\zeta)$. The conformal map f is analytic by definition, so that $g = G \circ f$ is a complex analytic function on U. Finally $g(z) = G(f(z))$, so that the real

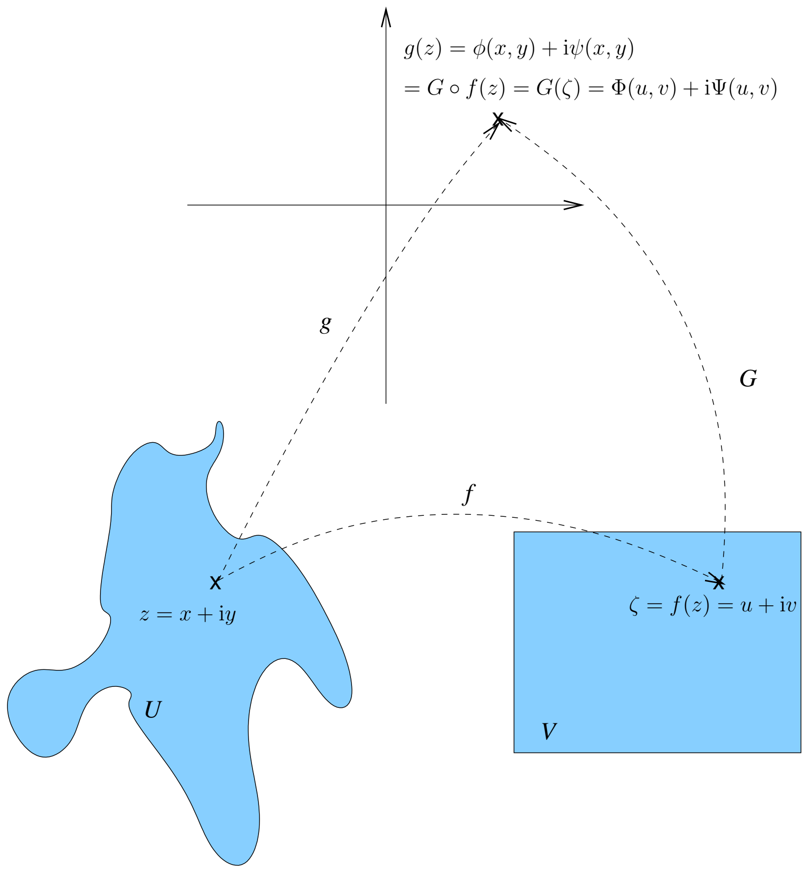

Figure 23: When a tricky domain U and a nice domain V are related by a conformal map f, an analytic function G on V defines an analytic function g on U. The real part of G is harmonic and gives us a solution to the Laplace equation on U according to Eq. (B.64).

part of g is given by the right-hand side of Eq. (B.64) which therefore is a solution to Laplace's equation on U with the required boundary conditions on ∂U.

We illustrate this method with the following example. The goal is to determine a bounded solution of $\triangle\phi = 0$ on the first quadrant of $\mathbb{R}^2$ with boundary conditions $\phi(x, 0) = 0$ and $\phi(0, y) = 1$; cf. Fig. 24. Admittedly, this domain U is not very tricky, but it serves its purpose as an example. With the choice $f(z) = \log z$, we map the quadrant to V given by the strip $0 < \mathsf{Im}(z) < \frac{\pi}{2}$. Recall that the log function maps sectors of annuli to rectangles. Here we have the special case of an annulus with zero inner and infinite outer radius (the first quadrant) which is mapped to an infinitely wide rectangle.

According to our recipe, we carry the boundary conditions as illustrated in Fig. 24. We next solve the Laplace equation $\triangle\Phi = \partial_u^2\Phi + \partial_v^2\Phi = 0$ on V with the boundary conditions

$$\Phi(u, 0) = 0\,, \qquad \Phi\big(u, \tfrac{\pi}{2}\big) = 1\,. \tag{B.65}$$

Clearly, we can solve Laplace's equation with a linear function whose coefficients are obtained

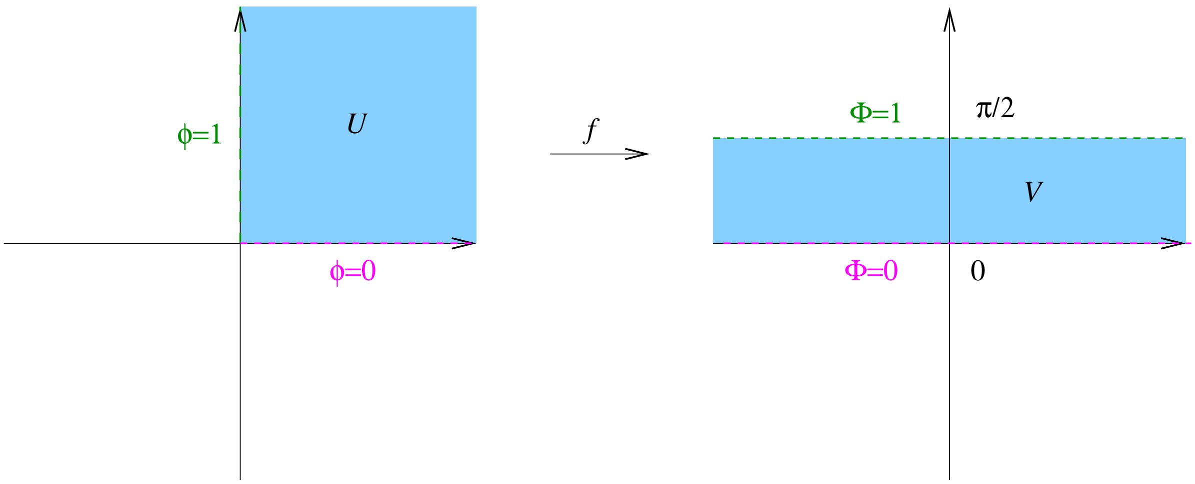

Figure 24: The first quadrant U is mapped to the strip V by $f(z) = \log z$. The boundary conditions on ∂U are mapped to those on ∂V as indicated.

from the boundary conditions, so that

$$\Phi(u, v) = \frac{2}{\pi} v\,. \tag{B.66}$$

This gives us the solution on U as

$$\phi(x, y) = \Phi\big(\mathsf{Re}(\log z),\ \mathsf{Im}(\log z)\big) = \frac{2}{\pi}\mathsf{Im}(\log z) = \frac{2}{\pi}\arg z = \frac{2}{\pi}\arctan\frac{y}{x}\,. \tag{B.67}$$

We were allowed here to equate the argument with the arctan, since $\arg z \in \left(0, \frac{\pi}{2}\right)$.

C Contour Integration and Cauchy's theorem

The remainder of these notes will focus on the integration of complex functions. We have already seen that the differentiation of complex functions reveals some remarkable features quite different from those known for real functions. This will be even more pronounced in the case of integration. In fact, complex integration will allow us to compute some very tricky real integrals by pretending they were complex.

C.1 Contours and integrals

When differentiating complex functions, we noted that there were infinite directions along which we could evaluate the differences in Eq. (B.4), rather than just one or two for a function on $\mathbb{R}$. Likewise, there is only one path for integrating a real function from a to be b, but any number of paths to get from a point A to a point B in the complex domain. In the case of complex differentiation, we required all directions to yield the same derivative – otherwise we discarded a complex function as not differentiable. For integration, we do not make such a restriction, but instead have to carefully specify the path of integration. If you have done calculations in thermodynamics, you will recognize the analogy to do calculations in the pressure-volume diagram; path independence has crucial consequences in thermodynamics and leads to the concept of entropy, Carnot engines etc.

We therefore start our discussion with defining numerous elements of the toolbox we need for complex integration.

Def. : A *curve* γ is a continuous map $\gamma : [0,1] \to \mathbb{C}$.

In some abuse of notation, we will sometimes also denote by γ the image of the curve; the distinction will be marked explicitly if not evident from the context. Without loss of generality, we have used in our definition the interval $[0,1]$ of real numbers; we can easily employ an arbitrary interval $I = [a,b]$ through an additional linear function $\lambda : [0,1] \to [a,b]\,, \quad x \mapsto a + (b-a)x$. In practice, $[0,1]$ will do nicely and sometimes turns out a particularly convenient choice.

Def. : A *closed curve* is a curve γ with $\gamma(0) = \gamma(1)$.

Def. : A *simple curve* is a curve γ that does not intersect itself except at the end points. For a simple curve and $a \neq b$, $\gamma(a) = \gamma(b)$ implies that $(a = 0 \vee a = 1)$ and $(b = 0 \vee b = 1)$.

Def. : A *contour* is a piecewise differentiable (i.e. piecewise smooth) curve.

Note that the definition of a curve and, thus, of a contour implies a direction, namely from the point $\gamma(0)$ to $\gamma(1)$. We will sometimes use the notation $-\gamma$ for the contour γ traversed in the opposite direction. More formally, we write

$$(-\gamma) : [0,1] \to \mathbb{C}\,, \qquad t \mapsto (-\gamma)(t) = \gamma(1-t)\,. \tag{C.1}$$

Contours γ_1 and γ_2 can be joined together if $\gamma_1(1) = \gamma_2(0)$. We denote the joint contour by $\gamma_1 + \gamma_2$ or, more formally,

$$(\gamma_1+\gamma_2) : [0,1] \to \mathbb{C}\,, \qquad t \mapsto (\gamma_1+\gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{for } t < \frac{1}{2} \\ \gamma_2(2t-1) & \text{for } t \geq \frac{1}{2} \end{cases}\,. \tag{C.2}$$

Def. : The *contour integral* $\int_\gamma f(z)\mathrm{d}z$ of a function f along the contour γ is defined to be the real integral

$$\int_\gamma f(z)\mathrm{d}z = \int_0^1 f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t\,. \tag{C.3}$$

Note the similarity to the integration of a vector field $\boldsymbol{F}(\boldsymbol{r})$ along a curve $\mathcal{C}$ in the $\mathbb{R}^n$,

$$\int_\mathcal{C} \boldsymbol{F}(\boldsymbol{r})\mathrm{d}\boldsymbol{r} = \int_A^B \boldsymbol{F}\big(\boldsymbol{r}(t)\big)\cdot\boldsymbol{r}'(t)\,\mathrm{d}t\,. \tag{C.4}$$

The only difference lies in the inner product; for vectors in $\mathbb{R}^2$ we have $\binom{x}{y}\cdot\binom{u}{v} = xu + yv$ and for complex numbers $(x+\mathrm{i}y)(u+\mathrm{i}v) = (xu - yv) + \mathrm{i}(xv + yu)$. Alternative to Eq. (C.3) and equivalently, we can define the integral of a complex function as the limit of a summation as in standard Riemann integration. For this purpose, we dissect $[0,1]$ into $0 = t_0 < t_1 < \ldots < t_n = 1$, write $z_k := \gamma(t_k)$ and define

$$\delta t_k = t_{k+1} - t_k\,, \qquad \delta z_k = z_{k+1} - z_k\,. \tag{C.5}$$

The integral is then given by

$$\int_\gamma f(z)\,\mathrm{d}z = \lim_{\Delta\to 0}\sum_{k=0}^{n-1} f(z_n)\,\delta z_n\,, \qquad \text{where} \quad \Delta = \max_{k=0,\ldots,n-1}\delta t_k\,. \tag{C.6}$$

Of course, the limit $\Delta \to 0$ implies $n \to \infty$.

Before we discuss general properties of complex integrals, it is instructive to illustrate the path dependence – which will play such an important role later on – with an explicit example. Consider for this purpose,

$$I_1 = \int_{\gamma_1}\frac{\mathrm{d}z}{z}\,, \qquad I_2 = \int_{\gamma_2}\frac{\mathrm{d}z}{z}\,, \tag{C.7}$$

where γ_1 and γ_2 are paths from -1 to $+1$ along the upper and lower half of the unit circle as illustrated in Fig. 25. Both paths are conveniently parametrized by the argument θ which leads to $z = e^{\mathrm{i}\theta}$ and $\mathrm{d}z = \mathrm{i}e^{\mathrm{i}\theta}\mathrm{d}\theta$ and

$$I_1 = \int_\pi^0 \frac{\mathrm{i}e^{\mathrm{i}\theta}\mathrm{d}\theta}{e^{\mathrm{i}\theta}} = -\mathrm{i}\pi\,.$$

$$I_2 = \int_{-\pi}^0 \frac{\mathrm{i}e^{\mathrm{i}\theta}\mathrm{d}\theta}{e^{\mathrm{i}\theta}} = \mathrm{i}\pi\,. \tag{C.8}$$

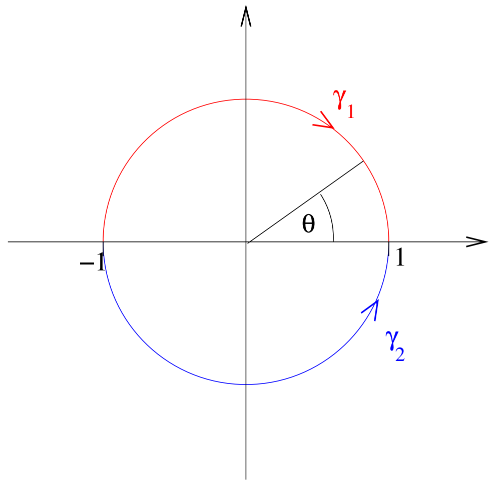

Figure 25: γ_1 and γ_2 are two paths from $z_1 = -1$ to $z_2 = 1$; the former proceeds along the upper unit circle, the latter along the lower half.

We have departed here from the above convention whereby a curve parameter lies in $[0, 1]$, but this makes no difference in the calculation. We will see later that the path dependence of this integral is intricately related to the fact that the integrand $1/z$ has a singularity inside the contours, namely at $z = 0$.

Integrals obey the following rules.

Proposition:

1. An integral along a joint contour is obtained from the sum
$$\int_{\gamma_1+\gamma_2} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z\,. \tag{C.9}$$
Note that this is completely analogous to the corresponding rule of integrals along the real line, $\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x$.

2. For an integral along the reversed contour, we have
$$\int_{-\gamma} f(z)\mathrm{d}z = -\int_{\gamma} f(z)\mathrm{d}z\,. \tag{C.10}$$
Again, this is analogous to the real case $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$.

(to be continued)

(continued)

3. If f is differentiable on all points of a contour γ that connects two points a, $b \in \mathbb{C}$, then

$$\int_\gamma f'(z)\mathrm{d}z = f(b) - f(a)\,. \tag{C.11}$$

At first glance, this appears to contradict the above example of integrating $\frac{1}{z}$ to demonstrate that complex integrals are path dependent; cf. Eq. (C.8). There is no contradiction, however, since the function f has to be differentiable along the entire contour. In the above example we have $f'(z) = \frac{1}{z}$ and, hence, $f(z) = \log z$. In order to ensure that this function is differentiable along the paths γ_1 and γ_2 considered in Fig. 25, we need to choose a branch for $\log z$, and we cannot choose the same branch (why?); hence the difference by $\mathrm{i}2\pi$ in the results of I_1 and I_2 in Eq. (C.8).

4. Integration by parts and substitution work for complex integrals in the same way as for real integrals.

5. The length of a curve γ is given by

$$L = \int_\gamma |\mathrm{d}z| = \int_0^1 |\gamma'(t)|\mathrm{d}t\,. \tag{C.12}$$

If f is a function bounded on γ by f_0, i.e. $|f(z)| \le f_0$, then

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \le \int_\gamma |f(z)|\,|\mathrm{d}z| \le f_0 \int_\gamma |\mathrm{d}z| = f_0 L\,. \tag{C.13}$$

For proofs of the non-obvious results of this list, readers are refered to the IB Complex Analysis course.

Many applications of complex integration employ closed contours. This is often denoted by adding a circle to the integral symbol,

$$\oint_\gamma f(z)\mathrm{d}z = \ldots\,, \tag{C.14}$$

or `$\oint$` for LaTeXusers. The result of an integral over a closed contour does not depend on the starting point as long as we go around all the way, but we need to specify the orientation in which γ is traversed. The convention is to proceed in the mathematically positive sense, i.e. counter-clockwise. This looks simple enough, but it may turn out rather hard to determine this direction for highly complicated contours; in this case, one can employ the equivalent definition that the positive orientation is the direction that keeps the interior of the contour on the *left*. With the direction specified, the meaning of $\oint_\gamma f(z)\mathrm{d}z$ is unambiguous.

For much of our remaining work, we will also need to define a simply connected domain. In simple terms, this is a subset $\mathcal{D} \subseteq \mathbb{C}$ that does not have holes; cf. Fig. 26 for a counter example. More formally we define this as follows.

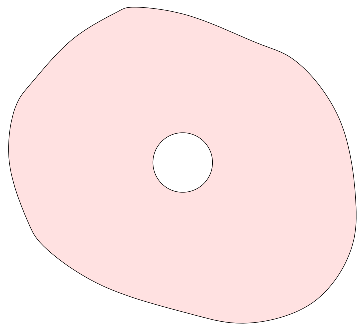

Figure 26: A simply connected domain does not have holes. The shaded region shown in this figure is *not* simply connected.

Def. : An open subset $\mathcal{D} \subseteq \mathbb{C}$ is a *connected domain* if each pair of points $z_1,\ z_2 \in \mathbb{C}$ can be connected by a curve γ whose image lies in $\mathcal{D}$. The subset $\mathcal{D}$ is a *simply connected domain* if it is connected and every closed curve in $\mathcal{D}$ encloses only points that are also in $\mathcal{D}$.

Note that the "holes" need not be extended; a single point is enough to break a simply connected domain.

C.2 Cauchy's theorem

Cauchy's theorem is probably the single most important result of these lectures. In spite of its seeming simplicity, a lot of the remainder of our work depends on it.

Theorem : If $f(z)$ is analytic in a simply connected domain $\mathcal{D}$, and γ is a closed contour inside $\mathcal{D}$, then

$$\oint_\gamma f(z)\mathrm{d}z = 0\,. \tag{C.15}$$

This theorem also tells us that analytic functions are not all that interesting from the viewpoint of integration.. As we shall see later, it is indeed functions with singularities, such as $\frac{1}{z}$, that lead to the most interesting results.

Proof. There exist two different proofs for this theorem. The simpler one – the version provided by Cauchy himself – assumes continuity of the partial derivatives and employs vector calculus. This is the version we shall follow here.

Let us first recall Green's theorem: If γ is a simple closed contour in $\mathbb{R}^2$, $\mathcal{M}$ is the region bounded by γ, and P and Q are functions on an open region $\mathcal{D} \supseteq \mathcal{M}$ with continuous partial derivatives, then

$$\oint_\gamma (P\mathrm{d}x + Q\mathrm{d}y) = \int\int_{\mathcal{M}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y\,. \tag{C.16}$$

Returning to the complex function, we write, as usual, $f(z) = u(x, y) + \mathrm{i}v(x, y)$, and obtain

$$\begin{aligned}\oint_\gamma f(z)\mathrm{d}z &= \oint_\gamma (u + \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) = \oint_\gamma (u\mathrm{d}x - v\mathrm{d}y) + \mathrm{i}\oint_\gamma (v\mathrm{d}x + u\mathrm{d}y)\\ &\overset{(C.16)}{=} \int\int_{\mathcal{M}} \Big(\underbrace{-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{=0}\Big)\mathrm{d}x\,\mathrm{d}y + \mathrm{i}\int\int_{\mathcal{M}} \Big(\underbrace{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}}_{=0}\Big)\mathrm{d}x\,\mathrm{d}y = 0\,,\end{aligned}$$

where the integrands vanish thanks to the Cauchy-Riemann conditions.

An independent proof, which dispenses with the need to assume C^1 smoothness of f, has been given in 1883 by Goursat. It is elegant, but a fair deal longer and outside the scope of our course. □

C.3 Deforming contours

We continue right away with the first application of Cauchy's theorem; it allows us to deform contours without changing the result of the integral.

Proposition: Let γ_1 and γ_2 be contours from a to b, $a,\ b \in \mathbb{C}$, and f be a function that is analytic on both contours *and* inside the region bounded by the contours; cf. Fig. 27. Then

$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z\,. \tag{C.17}$$

Proof. Let us assume that γ_1 and γ_2 do not intersect each other except at the endpoints a and b. Then $\gamma_1 - \gamma_2 := \gamma_1 + (-\gamma_2)$ is a simple closed contour and by Cauchy's theorem,

$$\oint_{\gamma_1-\gamma_2} f(z)\mathrm{d}z = 0\,, \tag{C.18}$$

and the result follows.

If γ_1 and γ_2 intersect each other, then we dissect the resulting closed curve at each crossing point and apply the above proof to each individual closed curves. □

We may regard Cauchy's theorem as the complex analog of the path-independent integrability of functions with an exact differential in the $\mathbb{R}^2$. This relation becomes explicit if we treat $\int f(z)\mathrm{d}z$ as an integral in $\mathbb{R}^2$ and write

$$\mathrm{d}f = f(z)\mathrm{d}z = (u + \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) = \underbrace{(u + \mathrm{i}v)}_{=:P}\,\mathrm{d}x + \underbrace{(-v + \mathrm{i}u)}_{=:Q}\,\mathrm{d}y \tag{C.19}$$

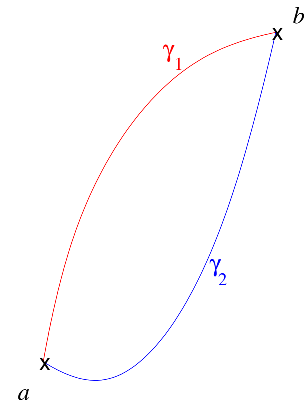

Figure 27: Two contours γ_1 and γ_2 connect two points a, $b \in \mathbb{C}$.

By the Cauchy-Riemann conditions,

$$\partial_y P = \frac{\partial}{\partial y}(u + \mathrm{i}v) = \frac{\partial}{\partial x}(-v + \mathrm{i}u) = \partial_x Q\,, \tag{C.20}$$

so there exists a function F such that $P = \partial_x F$ and $Q = \partial_y F$, i.e. $\mathrm{d}f$ is an exact differential and the integral is path-independent.

Cauchy's theorem also allows us to deform closed contours without changing the value of the integral. Consider, for this purpose, a closed contour γ_1 that can be continuously deformed into another contour γ_2 inside it. Let us further assume that f is analytic on both contours and inside the region between them (but not necessarily inside γ_2!). This is illustrated in the left of Fig. 28. We now cut out a thin piece from each contour and join the resulting end points together as indicated in the right of Fig. 28. This results in the closed contour γ to which we can apply Cauchy's theorem, whence $\int_\gamma f(z)\mathrm{d}z = 0$. In the limiting process where the cut out region approaches zero width, the contributions along the "cross cuts" compensate each other as is most easily seen with the definition of the integral as an infinite sum according to Eq. (C.6). In consequence, we obtain

$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z\,. \tag{C.21}$$

C.4 Cauchy's integral formula

Cauchy's theorem has informed us that the really interesting functions to integrate over closed contours are those with singularities somewhere inside. In fact, it is a particular type of singularities that really matters, those of $1/z$ type or, to allow for arbitrary location, terms $\propto 1/(z - z_0)$. This is a consequence of Cauchy's integral formula (this part of the lectures really feels like a one-man show...).

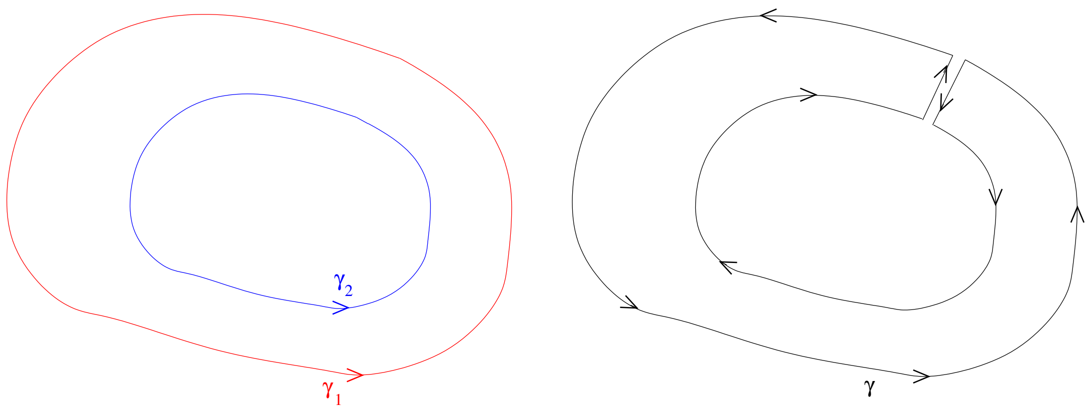

Figure 28: Two closed contours γ_1 and γ_2 (left). We then cut out a thin piece from each of the contours and join the resulting end points as shown on the right to obtain a single closed contour γ.

Theorem : Let $f(z)$ be analytic on an open domain $\mathcal{D} \subseteq \mathbb{C}$, $z_0 \in \mathcal{D}$, and γ a simple closed contour in $\mathcal{D}$ that encircles z_0 counter-clockwise. Then *Cauchy's integral formula* is given by

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{z - z_0} \mathrm{d}z\,, \tag{C.22}$$

and

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z\,, \tag{C.23}$$

for the derivatives of f at z_0.

Proof. (non-examinable) Consider z_0 and the contour γ as displayed in Fig. 29, and let γ_ϵ be the contour along the circle of radius ϵ about z_0 traversed in counter-clockwise direction. For sufficiently small ϵ, this circle is contained completely within the contour γ. The function $\frac{f(z)}{z-z_0}$ is analytic everywhere on and inside γ except for the point $z = z_0$, so that by Eq. (C.21)

$$\oint_\gamma \frac{f(z)}{z - z_0} \mathrm{d}z = \oint_{\gamma_\epsilon} \frac{f(z)}{z - z_0} \mathrm{d}z\,. \tag{C.24}$$

The integral on the right can be evaluated directly by substituting $z = z_0 + \epsilon e^{\mathrm{i}\theta} \Rightarrow \mathrm{d}z = \mathrm{i}\epsilon e^{\mathrm{i}\theta}\mathrm{d}\theta$,

$$\oint_{\gamma_\epsilon} \frac{f(z)}{z - z_0} \mathrm{d}z = \int_0^{2\pi} \frac{f(z_0 + \epsilon e^{\mathrm{i}\theta})}{\epsilon e^{\mathrm{i}\theta}} \mathrm{i}\epsilon e^{\mathrm{i}\theta} \mathrm{d}\theta = \mathrm{i} \int_0^{2\pi} f(z_0 + \epsilon e^{\mathrm{i}\theta}) \mathrm{d}\theta\,. \tag{C.25}$$

This is true for any ϵ circle inside γ, so that

$$\oint_{\gamma_\epsilon} \frac{f(z)}{z - z_0} \mathrm{d}z = \mathrm{i} \lim_{\epsilon \to 0} \int_0^{2\pi} f(z_0 + \epsilon e^{\mathrm{i}\theta}) \mathrm{d}\theta \overset{(B.4)}{=} \mathrm{i} \int_0^{2\pi} f(z_0) \mathrm{d}\theta = 2\pi \mathrm{i} f(z_0)\,. \tag{C.26}$$

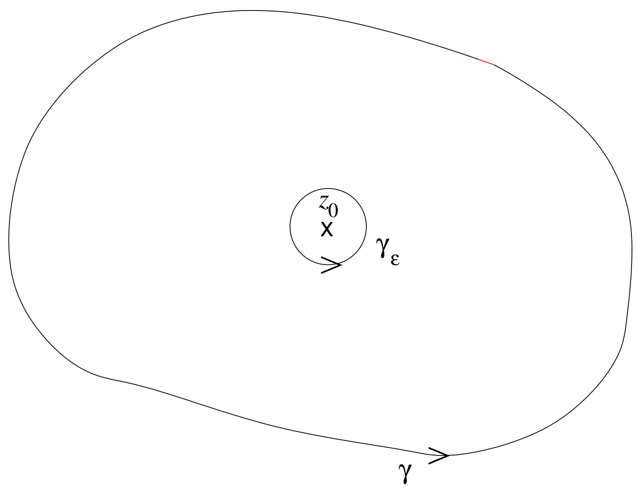

Figure 29: Two contours γ and γ_ϵ around the point z_0. γ_ϵ follows a circle of radius ϵ around z_0 and, for sufficiently small ϵ, lies inside of γ.

This proves (C.22). Differentiating this equation with respect to z_0 gives us

$$f'(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{(z-z_0)^2} \mathrm{d}z \,, \tag{C.27}$$

and Eq. (C.23) follows directly from induction: we are allowed to swap the operators $\mathrm{d}/\mathrm{d}z_0$ and $\oint_\gamma$ since the integrand is analytic on γ, and then use $\left((z-z_0)^{-(n+1)}\right)' = -(n+1)(z-z_0)^{-(n+2)}$. □

Cauchy's integral formula is quite remarkable in at least two respects.

(1) Suppose that we only know the function values $f(z)$ on γ, then the function values at all points inside γ are determined by Eq. (C.22). This holographic property may appear magical at first glance, but can be explained in terms of the harmonic nature of the real and imaginary parts of our analytic function $f(z) = u(x,y) + \mathrm{i}v(x,y)$. u and v are solutions of Laplace's equation with Dirichlet boundary conditions given by the values of $f(z)$ on γ. These solutions are unique, so f is completely determined by its values on the boundary. The one additional piece of information that Cauchy's integral formula does provide is an explicit formula, Eq. (C.22), to evaluate the function values $f(z)$ inside γ without having to solve Laplace's differential equation. Note, however, that this reconstruction of $f(z)$ only works for points inside γ; if z_0 lies outside of γ, we obtain from Cauchy's theorem (C.15)

$$\frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{z-z_0} \mathrm{d}z = 0 \,. \tag{C.28}$$

(2) Equation (C.23) furthermore tells us that if a function is analytic at a point z_0 (recall that this implies differentiability in an open neighbourhood of z_0), then it is differentiable infinitely many times at z_0, with all derivatives given by (C.23). This proves the claim made above in Sec. B.2.

We conclude this discussion with deriving the second claim made in that section, which is known as *Liouville's theorem*.

Proposition: If $f(z)$ is an entire function (analytic on all $\mathbb{C}$) and bounded, then it is a constant.

Proof. (non-examinable) Since f is bounded, there exists $c_0 \in \mathbb{R}$ such that $|f(z)| \leq c_0$ for all $z \in \mathbb{C}$. Let $z_0 \in \mathbb{C}$ be an arbitrary point and γ_r the contour along a circle of radius r around z_0 traversed counter-clockwise. Then

$$f'(z_0) = \frac{1}{2\pi\mathrm{i}} \oint_{\gamma_r} \frac{f(z)}{(z-z_0)^2} \mathrm{d}z$$

$$\Rightarrow \quad |f'(z_0)| \leq \frac{1}{2\pi} \oint_{\gamma_r} \frac{c_0}{r^2} \, |\mathrm{d}z| = \frac{c_0}{r} \,. \tag{C.29}$$

for any value r. In particular, we obtain for $r \to \infty$ that $f'(z_0) = 0$ and f is a constant. □

D Laurent series and singularities

D.1 Taylor series and Laurent series

The Taylor series, familiar from calculus in $\mathbb{R}$, enables us to evaluate a function at points x by merely knowing the function and all of its derivatives at some fixed point x_0. More specifically, if $f(x)$ is an infinitely differentiable function on $\mathcal{D} \subseteq \mathbb{R}$ with a convergent Taylor series and $x,\ x_0 \in \mathcal{D}$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n = f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\frac{1}{3!}f'''(x_0)(x-x_0)^3+\ldots \tag{D.1}$$

As will shortly see, this also holds for complex functions $f(z)$. On the other hand, we have seen in the last two sections, that the really interesting integrals are obtained when $f(z)$ is not everywhere differentiable, but has some singularities instead. Clearly, a Taylor expansion of the type (D.1) does not work if f is singular at z_0. However, we can still construct a series expansion if we include negative powers of $(z - z_0)$; this generalization is the *Laurent series.*

Proposition: Let the function $f(z)$ be analytic in an annulus $R_1 < |z - z_0| < R_2$. Then it has the *Laurent Series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n\,, \tag{D.2}$$

which is convergent within the annulus and uniformly convergent in any compact subset of the annulus.
If $f(z)$ is analytic at z_0, then it has a *Taylor series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{with} \qquad a_n = \frac{f^{(n)}(z_0)}{n!}\,, \tag{D.3}$$

in a neighbourhood of z_0.

We will see in the proof below how the coefficients of the Laurent series can be evaluated formally. In practice, however, we will use a more practical tool box which will be illustrated further below when we consider examples.

Proof. (non-examinable) Without loss of generality, we set $z_0 = 0$. Let z be inside the annulus and choose $r_1,\ r_2$ such that

$$R_1 < r_1 < |z| < r_2 < R_2\,.$$

We denote by γ_1 and γ_2, respectively, the contours traversed in counter-clockwise direction along $|z| = r_1$ and $|z| = r_2$, and approximate them by the closed contour γ as displayed in Fig. 30. We can now apply Cauchy's integral formula (C.22) (with $z \mapsto \zeta$ and $z_0 \mapsto z$) to the

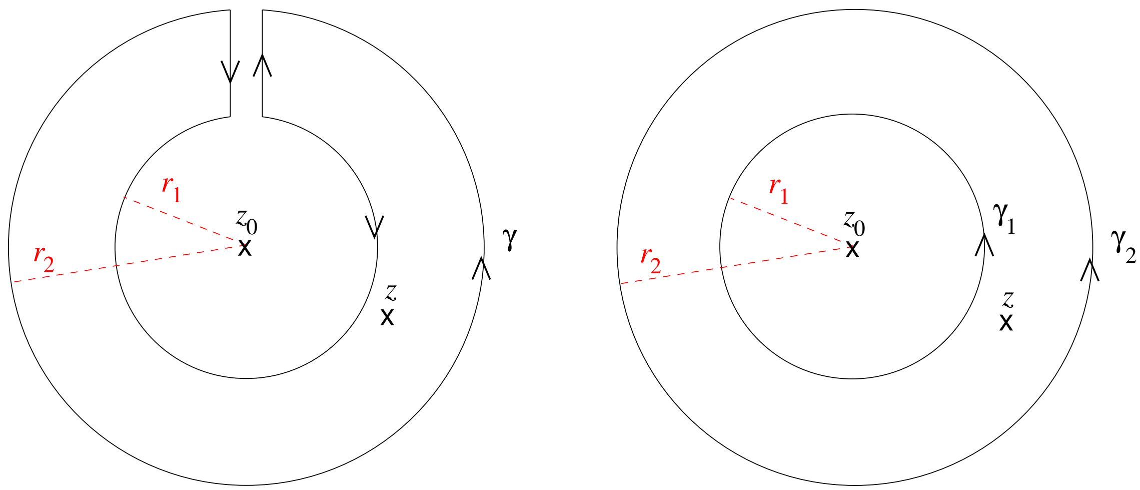

Figure 30: In the limit of infinitesimal width of the cross cut, the closed contour γ approaches two separate circular contours γ_1 and γ_2 around z_0 with radius r_1 and r_2, respectively. The point z is located within the annulus $r_1 < |z| < r_2$.

contour γ with the point z inside the countour,

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_2} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_1} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \,, \end{aligned} \tag{D.4}$$

where the second equality comes from letting the distance between the cross cuts go to zero, and γ_1 has a minus sign because it is traversed in the clockwise direction. Even without further evaluation, we can already infer some properties of the two integrals.

- By Cauchy's theorem (C.15), non-zero results for the integrals can only arise from singularities of the integrand *inside* the respective contours γ_1, γ_2.
- Since the point z is inside the annulus, it is not encircled by γ_1. The integral $\int_{\gamma_1}$ will therefore only pick up singularities of the function $f(\zeta)$.
- The integral contribution $\oint_{\gamma_2}$ will pick up any singularities of $f(\zeta)$ inside γ_1 (since f is analytic in the annulus, it cannot have singularities between γ_1 and γ_2). $\oint_{\gamma_2}$ will also pick up the singularity of $\frac{1}{\zeta - z}$; this term now has a singularity since γ_2 encircles z.
- If f is analytic at z_0 and we have chosen a sufficiently small annulus, f has no singularities inside γ_2. Then the integral contribution $\oint_{\gamma_1}$ vanishes completely and $\oint_{\gamma_2}$ only contains the contribution from the $\frac{1}{\zeta - z}$ singularity within the annulus; this is the case where we will recover the simpler Taylor series.

Let us then evaluate the two integrals on the right-hand side of Eq. (D.4); for the second integral, we find

$$\oint_{\gamma_1} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = -\frac{1}{z} \oint_{\gamma_1} \frac{f(\zeta)}{1 - \frac{\zeta}{z}} \mathrm{d}\zeta \,. \tag{D.5}$$

Now we rewrite the term $\left(1-\frac{\zeta}{z}\right)^{-1}$ using the geometric series (A.10). Since $\left|\frac{\zeta}{z}\right|<1$, the geometric series converges uniformly, so that we are allowed to swap the summation and the integral, and obtain

$$\begin{aligned}
-\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_1}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta &= \frac{1}{2\pi\mathrm{i}z}\oint_{\gamma_1}f(\zeta)\sum_{m=0}^{\infty}\left(\frac{\zeta}{z}\right)^m\mathrm{d}\zeta \\
&= \frac{1}{2\pi\mathrm{i}}\sum_{m=0}^{\infty}\left[z^{-m-1}\oint_{\gamma_1}f(\zeta)\zeta^m\mathrm{d}\zeta\right] \\
&= \sum_{n=-\infty}^{-1}a_n z^n\,,\qquad\text{where}\qquad a_n=\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_1}f(\zeta)\zeta^{-n-1}\mathrm{d}\zeta\,.
\end{aligned}\tag{D.6}$$

On γ_2, we have $\left|\frac{z}{\zeta}\right|<1$, and the first term on the right-hand side of Eq. (D.4) becomes

$$\begin{aligned}
\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_2}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta &= \frac{1}{2\pi\mathrm{i}}\oint_{\gamma_2}\frac{1}{\zeta}\frac{f(\zeta)}{1-\frac{z}{\zeta}}\mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}}\oint_{\gamma_2}\frac{f(\zeta)}{\zeta}\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^n\mathrm{d}\zeta \\
&= \frac{1}{2\pi\mathrm{i}}\sum_{n=0}^{\infty}\left[z^n\oint_{\gamma_2}\frac{f(\zeta)}{\zeta^{n+1}}\mathrm{d}\zeta\right] \\
&= \sum_{n=0}^{\infty}a_n z^n\,,\qquad\text{where}\qquad a_n=\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_2}f(\zeta)\zeta^{-n-1}\mathrm{d}\zeta\,.
\end{aligned}\tag{D.7}$$

Thus, the two terms of (D.4) together give us (D.2); note that the coefficients a_n with negative or non-negative n only differ in the contour in their integral term, γ_2 for $n\geq 0$ and γ_1 for $n<0$.

We will ommit the proof for uniform convergence of the Laurent series, since time is prescious; it is the subject of the IB Complex Analysis course.

The Taylor series (D.3) is obtained as the special case of the Laurent series where f is analytic at z_0. We can then find an annulus around z_0 where f is analytic and repeat the proof for the Laurent series. For a suffiently small annulus, however, f is now also analytic inside γ_1 and, for $n<0$, so is the function $f(\zeta)\zeta^{-n-1}$ inside the integral of Eq. (D.6). In consequence $a_n=0$ for $n<0$. Of course, the a_n in Eq. (D.7) do *not* vanish since now $n\geq 0$ for which $f(\zeta)\zeta^{-n-1}$ has singularities inside γ_2. For f analytic at z_0, the Laurent series therefore reduces to (D.7). From Eq. (C.23), we furthermore see that (for $n\geq 0$)

$$a_n=\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_2}f(\zeta)\zeta^{-n-1}\mathrm{d}\zeta\overset{!}{=}\frac{1}{n!}f^{(n)}(0)\,,\tag{D.8}$$

and we have indeed recovered the Taylor series (D.3); recall that we translated without loss of generality to $z_0=0$ in our proof. □

Our proof has demonstrated the existence of the Laurent series, but it can also be shown to be unique. In particular, the Laurent series of a function f does not depend on the specific annulus we use in the evaluation of its coefficients. This is highly convenient; suppose we have two annuli with non-empty overlap. Both annuli must then yield the same coefficients, because that is the only way the Laurent series can be unique in their region of overlap. In particular, we can choose an arbitrarily small annulus around the expansion point z_0; the resulting Laurent series is valid inside any larger annulus as long as it overlaps with the small one and f is analytic inside of it. As promised in Sec. A.1, we have now also shown that the geometric series is indeed the Taylor expansion of $f(z) = \frac{1}{z}$. Finally, Eq. (D.3) tells us that complex functions have the same Taylor series as their real counterparts.

Enough theory for now. Let us consider some examples, which will also tell us how we can evaluate Laurent series in practice.

Examples of Laurent series

(1) We wish to compute the Laurent series of $f(z) = e^z/z^3$ around $z_0 = 0$. Here, the starting point is that we already know the Taylor series of e^z, which we simply divide by z^3,

$$\frac{e^z}{z^3} = \sum_{n=0}^{\infty} \frac{z^{n-3}}{n!} = \sum_{n=-3}^{\infty} \frac{z^n}{(n+3)!} \,. \tag{D.9}$$

(2) What is the Laurent series of $e^{1/z}$ about $z_0 = 0$? Again, we start with the Taylor series of e^ζ with $\zeta = \frac{1}{z}$,

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} \ldots = \sum_{n=-\infty}^{0} a_n z^n \,, \qquad \text{with} \qquad a_n = \frac{1}{(-n)!} \,. \tag{D.10}$$

(3) Consider the function

$$f(z) = \frac{1}{z-a} \,, \tag{D.11}$$

where $a \in \mathbb{C}$. Then f is analytic in the disk $|z| < |a|$ and its Taylor series around $z_0 = 0$ can be obtained from the geometric series,

$$\frac{1}{z-a} = -\frac{1}{a}\frac{1}{1-\frac{z}{a}} = -\frac{1}{a}\sum_{m=0}^{\infty}\left(\frac{z}{a}\right)^m = -\sum_{n=0}^{\infty}\frac{1}{a^{n+1}} z^n \,. \tag{D.12}$$

So far so good, but what about points where $|z| > |a|$? Now we have an annulus where f is analytic; the annulus stretches from some $R_1 > |a|$ to infinity. We must therefore be able to find a Laurent series (though not a Taylor series since f has a singularity inside R_1). Again, we use the geometric series,

$$\frac{1}{z-a} = \frac{1}{z}\frac{1}{1-\frac{a}{z}} = \frac{1}{z}\sum_{m=0}^{\infty}\left(\frac{a}{z}\right)^m = \sum_{n=-\infty}^{-1} a^{-n-1} z^n \,. \tag{D.13}$$

(4) The function

$$f(z) = \frac{e^z}{z^2 - 1}\,, \tag{D.14}$$

has two singularities at $z = \pm 1$. Clearly f is analytic in an annulus $0 < |z - z_0| < 2$ around $z_0 = 1$; above that radius we hit the other singularity at -1. We wish to calculate the Laurent series of f about $z_0 = 1$. This is achieved by using a trick that often turns out useful in evaluating Laurent series: we rewrite the function in terms of $\zeta = z - z_0 = z - 1$. Using the Taylor expansion of e^z and the geometric series (A.10), we obtain

$$\begin{aligned}
f(z) &= \frac{e^\zeta\, e}{\zeta(\zeta+2)} = e\frac{e^\zeta}{2\zeta}\frac{1}{1+\frac{\zeta}{2}} \\
&= \frac{e}{2\zeta}\left(1 + \zeta + \frac{1}{2!}\zeta^2 + \ldots\right)\left[1 - \frac{\zeta}{2} + \left(\frac{\zeta}{2}\right)^2 \mp \ldots\right] \\
&= \frac{e}{2\zeta}\left(1 + \frac{1}{2}\zeta + \ldots\right) = \frac{e}{2}\left(\frac{1}{z-1} + \frac{1}{2} + \ldots\right).
\end{aligned} \tag{D.15}$$

Note that the geometric series in $\zeta/2$ in the second line converges inside the annulus $|\zeta| < 2$. The leading coefficients in the Laurent series are given by $a_{-1} = \frac{e}{2}$ and $a_0 = \frac{e}{4}$. In practice it is often sufficient to obtain the leading-order terms and, in particular, a_{-1}.

(5) For the function $f(z) = z^{-1/2}$, we cannot compute the Laurent series around $z_0 = 0$. The reason is that $z_0 = 0$ and ∞ are branch points and must be connected by a branch cut. This branch cut intersects any annulus we could draw around $z_0 = 0$, so that there exists no annulus around z_0 on which f is analytic.

We conclude this discussion of Laurent series with its radius of convergence. More specifically, suppose know the Laurent expansion of a function around a point z_0 inside an annulus $0 < |z - z_0| < R_2$, what is the maximum value R_2 for which the Laurent series is convergent inside the annulus? One can show that the answer is the distance of the point z_0 to the nearest other singularity of $f(z)$. In simple terms, the reason is that we can choose R_2 to be equal to that distance; the Laurent series obtained in that larger annulus must, by uniqueness, be the same as that obtained in any smaller annulus inside.

D.2 Zeros and singularities

The complex factorization theorem of algebra tells us that any polynomial $P(z)$ of degree $n \geq 1$ has exactly n zeros, counting multiplicities, and can be factorized according to

$$P(z) = a(z - z_1)^{m_1}(z - z_2)^{m_2} \;\cdots\; (z - z_k)^{m_k}\,, \tag{D.16}$$

where a is the leading coefficient of the polynomial, m_k denotes the multiplicity of the zero z_k and $m_1 + m_2 + \ldots + m_k = n$. We now wish to extend this concept of zeros and their orders to a wider range of functions. The tool we use for this purpose is the Taylor expansion.

Def. : The *zeros* of an analytic function $f(z)$ are the points z_0 where $f(z_0) = 0$. A zero z_0 is of *order* n if in its Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$, the first non-zero coefficient is a_n.
Equivalently, a *zero* z_0 of a function $f(z)$ is of *order* n if

$$f(z_0) = f'(z_0) = \ldots = f^{(n-1)}(z_0) = 0 \quad \text{but} \quad f^{(n)}(z_0) \neq 0\,.$$

A *simple zero* is a zero of order $n = 1$.

This is best discussed with a handful of examples.

Examples of zeros

(1) The function $f(z) = z^3 + \mathrm{i}z^2 + z + \mathrm{i} = (z^2+1)(z+\mathrm{i}) = (z-\mathrm{i})(z+\mathrm{i})^2$ has a simple zero at $z = \mathrm{i}$ and a zero of order 2 at $z = -\mathrm{i}$.

(2) The zeros of sinh are obtained from writing

$$\sinh z = \frac{1}{2}(e^z - e^{-z}) = 0 \qquad \Leftrightarrow \qquad e^{2z} = 1 \qquad \Leftrightarrow \qquad z = \mathrm{i}n\pi \;\; \text{with} \;\; n \in \mathbb{Z}\,.$$

All these zeros are simple since $\cosh(\mathrm{i}n\pi) = \cos(n\pi) = \pm 1 \neq 0$.

(3) Given that $\sinh z$ has a simple zero at $z = \mathrm{i}\pi$, we conclude that $f(z) = \sinh^3 z$ has a zero of order 3 at $z = \mathrm{i}\pi$. We can either derive this by expanding the Taylor series of $\sinh^3 z$ using $\sinh z = a_1(z - \mathrm{i}\pi) + \ldots$ or evaluating $f'(z)$, $f''(z)$, $f'''(z)$ at $z = \mathrm{i}\pi$ using product rule.
In order to determine the first terms of the Taylor series of $\sinh^3 z$ about $z = \mathrm{i}\pi$, we use the above trick of writing $\zeta = z - \mathrm{i}\pi$, which gives us

$$\begin{aligned}
\sinh^3 z &= [\sinh(\zeta + \mathrm{i}\pi)]^3 \overset{(A.20)}{=} [\sinh\zeta \underbrace{\cosh(\mathrm{i}\pi)}_{=\cos\pi=-1} + \cosh\zeta \underbrace{\sinh(\mathrm{i}\pi)}_{=0}]^3 \\
&= (-\sinh\zeta)^3 = -\left(\zeta + \frac{1}{3!}\zeta^3 + \ldots\right)^3 = -\zeta^3 - \frac{1}{2}\zeta^5 + \ldots \\
&= -(z - \mathrm{i}\pi)^3 - \frac{1}{2}(z - \mathrm{i}\pi)^5 + \ldots\,.
\end{aligned} \tag{D.17}$$

In practice, we are interested in singularities rather than zeros. But in a sense, we can regard singularities as the inverse of zeros, and this is often how we determine them.

Def. : A *singularity* of a function f is a point $z = z_0$ where f is not analytic. If f has a singularity at z_0, but f is analytic in a neighbourhood of z_0 except at z_0 itself, then z_0 is an *isolated singularity* of the function f. If there exists no such neighbourhood, z_0 is a *non-isolated* singularity.

Again, this is best studied with examples...

Examples of singularities

(1) The function $f(z) = \text{cosechz} := \frac{1}{\sinh z}$ has isolated singularities at $z = \mathrm{i}n\pi$, $n \in \mathbb{Z}$, because sinh has zeroes at these points.

(2) The function $f(z) = \text{cosech}\frac{1}{z} = \frac{1}{\sinh \frac{1}{z}}$ has isolated singularities at $z = \frac{1}{\mathrm{i}n\pi}$ for $n \in \mathbb{Z}$, $n \neq 0$. f also has a non-isolated singularity at $z = 0$; for any ϵ disk around $z = 0$ we can find a large enough n such that the disk contains another singularity at $z = \frac{1}{\mathrm{i}n\pi}$.

(3) cosech$z$ has a non-isolated singularity at $z = \infty$ because cosech$\frac{1}{z}$ has one at $z = 0$.

(4) $f(z) = \log z$ has a non-isolated singularity at $z = 0$ because $z = 0$ must be connected to a branchcut. Every ϵ disk around $z = 0$ therefore contains a point on the branch cut where f is not analytic. This type of singularity is often refered to as a *branch point singularity.*

The important feature of isolated singularities is that we can find an annulus $0 < |z - z_0| < R$ for some $R > 0$ where f is analytic and therefore has a Laurent series. Given a candidate point z_0 for a singularity of a function f, we typically analyze this point according to the following recipe.

1. Check whether z_0 is a branch point singularity.
2. Determine whether z_0 is a non-isolated singularity.
3. If neither is the case, determine the coefficients a_n of the Laurent series of $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$.
   (a) If $a_n = 0$ for all $n < 0$, then f has a *removable singularity* at z_0. Such a function has a Laurent series given by $f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$ and we can remove the singularity by simply redefining $f(z_0) := a_0 = \lim\limits_{z \to z_0} f(z)$.
   (b) If there exists an $N > 0$ such that all $a_n = 0 \;\; \forall\, n \leq -N - 1$ and $a_{-N} \neq 0$, then f has a *pole of order* N at z_0. For $N = 1,\ 2,\ 3$, this is also refered to as a *simple*, *double* or *triple pole*, respectively.
   (c) If no such N exists, then f has an *essential isolated singularity* at z_0.

Once more, this is best understood by discussing examples.

Examples of isolated singularities

(1) $f(z) = \frac{1}{z-\mathrm{i}}$ has a simple pole at $z = \mathrm{i}$; its Laurent series is just $\frac{1}{z-\mathrm{i}}$.

(2) $f(z) = \frac{\cos z}{z}$ has a singularity at $z = 0$; its Laurent series around this point directly follows from the Taylor series of cos,

$$\frac{\cos z}{z} = z^{-1} - \frac{1}{2}z + \frac{1}{24}z^3 \mp \ldots\,,$$

and $z = 0$ is a simple pole.

(3) The function $g(z) = \frac{z^2}{(z-1)^3(z-\mathrm{i})^2}$ looks like having a triple pole at $z = 1$ and a double pole at $z = \mathrm{i}$. How can we show this more formally? Consider the pole $z_0 = \mathrm{i}$, and write

$$g(z) = G(z)\frac{1}{(z-\mathrm{i})^2} \qquad \text{with} \qquad G(z) = \frac{z^2}{(z-1)^3}\,.$$

Clearly, $G(z)$ is analytic at $z = \mathrm{i}$ and therefore can be Taylor expanded according to

$$G(z) = b_0 + b_1(z-\mathrm{i}) + b_2(z-\mathrm{i})^2 + \ldots\,,$$

with $b_0 \neq 0$ since $G(\mathrm{i}) \neq 0$. This enables us to Laurent expand $g(z)$ as

$$g(z) = \frac{b_0}{(z-\mathrm{i})^2} + \frac{b_1}{z-\mathrm{i}} + b_2 + b_3(z-\mathrm{i}) + \ldots\,,$$

so $z = \mathrm{i}$ is indeed a double pole of g.

(4) More generally, if $f(z)$ has a zero of order n at z_0, then $\frac{1}{f(z)}$ has a pole of order n at z_0, and vice versa. This can be shown by writing

$$f(z) = (z-z_0)^n F(z)\,,$$

where $\frac{1}{F(z)}$ is analytic at z_0 and can be Taylor expanded in analogy to the specific function discussed in the previous example. The function $f(z)$ need not be a polynomial; for example, $\tan z$ has a simple zero at $z = 0$ and $\cot z$ has a simple pole at $z = 0$.

(5) $f(z) = z^2$ has a double pole at ∞, since $\frac{1}{\zeta^2}$ has a double pole at $\zeta = 0$.

(6) The Laurent expansion for $e^{\frac{1}{z}}$ about $z = 0$ is given by Eq. (D.10); since all $a_k \neq 0$ for $k \leq 0$, $e^{\frac{1}{z}}$ has an essential isolated singularity at $z = 0$.

(7) Likewise, $\sin\frac{1}{z}$ has an essential isolated singularity at $z = 0$. In analogy to Eq. (D.10), we use the Taylor series of $\sin z$ to obtain the Laurent series of $\sin\frac{1}{z}$ which has infinitely many $a_k \neq 0$, $k < 0$.

(8) $f(z) = \frac{e^z-1}{z}$ has a removable singularity at $z = 0$ because the Taylor expansion of e^z gives us

$$\frac{e^z-1}{z} = 1 + \frac{1}{2!}z + \frac{1}{3!}z^2 + \ldots\,. \tag{D.18}$$

We can the redefine $f(0) = 1 = \lim\limits_{z\to 0} f(z)$ and the resulting function is analytic everywhere.

(9) We can likewise regularize $f(z) = \frac{\sin z}{z}$, which also has a removable singularity at $z = 0$, by redefining $f(0) = 1$.

(10) Now consider a rational function $f(z) = \frac{P(z)}{Q(z)}$ with polynomials P and Q. If $Q(z)$ has a zero of order $n > 0$ and $P(z)$ a zero of order $m \geq n$ at z_0, then f has a removable singularity at z_0 where we can redefine $f(z_0) = \frac{P^{(n)}(z_0)}{Q^{(n)}(z_0)}$. This follows from l'Hôpital's rule.

Proposition: Let $f(z)$ have an essential isolated singularity at $z = z_0$ and $\mathcal{D}$ be an (arbitrarily small) neighbourhood of z_0. Then in $\mathcal{D}$, $f(z)$ takes on all possible complex values except at most one.

For example, $e^{\frac{1}{z}}$ takes on any value except 0 in any given neighbourhood of $z = 0$. The proof of this proposition is beyond the scope of our lectures, and we instead move on to the next topic.

D.3 Residues

The calculation of all coefficients in the Laurent series (D.2) of a function f can be challenging at times. As it turns out, however, the integration of complex function only requires a single coefficient, a_{-1}.

Def. : The *residue* $\operatorname*{Res}_{z=z_0} f$ of a function f with an isolated singularity at $z = z_0$ is the coefficient a_{-1} in the Laurent expansion of f about z_0.

The notation "Res" seems to be the most common one, but do not be overly surprised if you encounter alternative notations elsewhere.

Proposition: Let f have a pole of order n at $z = z_0$. Then the residue is given by

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z-z_0)^n f(z)\right] . \tag{D.19}$$

If f has a simple pole at z_0, this reduces to

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \left[(z-z_0) f(z)\right] . \tag{D.20}$$

Proof. Consider the case of a simple pole, n=1. Then f has a Laurent series starting with $a_{-1}\frac{1}{z-z_0}$ and we can expand the right-hand side of Eq. (D.20),

$$\lim_{z\to z_0} \left[(z-z_0)\left(\frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \ldots\right)\right] = \lim_{z\to z_0} \left(a_{-1} + a_0(z-z_0) + \ldots\right) = a_{-1} .$$

The case $n > 1$ follows, by direct exploration of the series or induction and will be discussed on example sheet 2. □

There exists a variety of techniques to determine residues and there is no concrete recipe for selecting the optimal method; we have here a situation where mathematics touches art. Once more, the most instructive method is to consider examples.

Examples for the calculation of residues

(1) The residue of $f(z) = \frac{e^z}{z^3}$ at $z_0 = 0$ is most conveniently computed by using the Taylor expansion of e^z,

$$\frac{e^z}{z^3} = z^{-3} + z^{-2} + \frac{1}{2} z^{-1} + \frac{1}{3!} + \ldots \qquad \Rightarrow \qquad \operatorname*{Res}_{z=0} f(z) = \frac{1}{2} . \tag{D.21}$$

Alternatively, f has a pole of order 3 and Eq. (D.19) with $n = 3$ gives us

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{2}\lim_{z\to 0}\frac{\mathrm{d}^2}{\mathrm{d}z^2}[z^3 f(z)] = \frac{1}{2}\lim_{z\to 0}\frac{\mathrm{d}^2}{\mathrm{d}z^2}e^z = \frac{1}{2}\,. \tag{D.22}$$

(2) We can often compute residues without knowing the entire Laurent series. Consider, for example, $f(z) = \frac{e^z}{z^2-1}$. at $z = 1$. We have already computed the relevant coefficients of its Laurent series in Eq. (D.15) and find

$$\operatorname*{Res}_{z=1}\frac{e^z}{z^2-1} = \frac{e}{2}\,. \tag{D.23}$$

Alternatively, formula (D.20) gives us

$$\operatorname*{Res}_{z=1}\frac{e^z}{z^2-1} = \lim_{z\to 1}(z-1)\frac{e^z}{z^2-1} = \lim_{z\to 1}\frac{e^z}{z+1} = \frac{e}{2}\,. \tag{D.24}$$

(3) Brute force does not always work well. Consider the function

$$f(z) = \frac{1}{z^8 - w^8}\,, \qquad w = \text{const} \in \mathbb{C}\,. \tag{D.25}$$

This function has 8 simple poles obtained from $z^8 = w^8$ which gives us $z = we^{\mathrm{i}k\pi/4}$ with $k = 0,\ 1,\ \ldots,\ 7$. We might try to compute the residue at $z = w$ from

$$\begin{aligned}\operatorname*{Res}_{z=w} f(z) &= \lim_{z\to w}\frac{z-w}{(z-w)(z-we^{\mathrm{i}\pi/4})\cdots(z-we^{\mathrm{i}7\pi/4})} \\ &= \frac{1}{(w-we^{\mathrm{i}\pi/4})\cdots(w-we^{\mathrm{i}7\pi/4})}\,.\end{aligned} \tag{D.26}$$

This is not wrong, but does not give us a helpful answer without a good deal of further manipulation. It is much easier to use l'Hôpital's rule,

$$\operatorname*{Res}_{z=w} f(z) = \lim_{z\to w}\frac{z-w}{z^8-w^8} = \lim_{z\to w}\frac{1}{8z^7} = \frac{1}{8w^7}\,. \tag{D.27}$$

(4) $\sinh(\pi z)$ has simple zeroes at $z = n\mathrm{i}$ and therefore, $f(z) = \frac{1}{\sinh(\pi z)}$ has simple poles at these points. We could construct its Laurent series from the Taylor expansion of sinh, but the residues at $n\mathrm{i}$ are more easily obtained by using Eq. (D.20) and l'Hôpital's rule,

$$\operatorname*{Res}_{z=n\mathrm{i}} f(z) = \lim_{z\to n\mathrm{i}}\frac{z-n\mathrm{i}}{\sinh(\pi z)} = \lim_{z\to n\mathrm{i}}\frac{1}{\pi\cosh(\pi z)} = \frac{(-1)^n}{\pi}\,.$$

(5) Finally, consider $f(z) = \frac{1}{\sinh^3 z}$, which has a pole of order 3 at $z = \mathrm{i}\pi$. We already know the Taylor expansion of $\sinh^3 z$ from Eq. (D.17) and thus obtain the Laurent series of

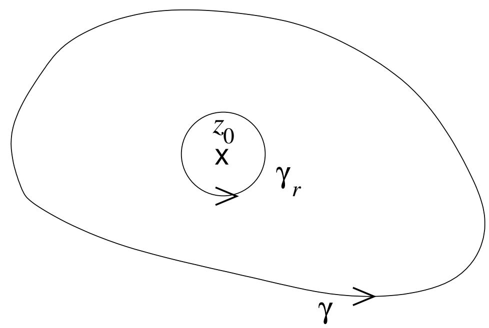

Figure 31: A circular contour γ_r inside a contour γ.

f around $\mathrm{i}\pi$,

$$\begin{aligned}\frac{1}{\sinh^3 z} &= -(z-\mathrm{i}\pi)^{-3}\left[1+\frac{1}{2}(z-\mathrm{i}\pi)^2+\ldots\right]^{-1} \\ &= -(z-\mathrm{i}\pi)^{-3}\left[1-\frac{1}{2}(z-\mathrm{i}\pi)^2+\ldots\right] \\ &= -(z-\mathrm{i}\pi)^{-3}+\frac{1}{2}(z-\mathrm{i}\pi)^{-1}+\ldots\,, \qquad \text{(D.28)}\end{aligned}$$

and the residue is $\frac{1}{2}$.

Why then, is the residue so important? What sets a_{-1} above all the other coefficients of the Laurent series? The answer is that a_{-1}, and a_{-1} alone, determines the integrals of the form $\oint f(z)\mathrm{d}z$, where f has an isolated singularity inside the contour.

Theorem : Let γ be a simple closed contour traversed in counter-clockwise direction and $f(z)$ be analytic within γ except for an isolated singularity z_0. Then

$$\oint_\gamma f(z)\mathrm{d}z = \mathrm{i}2\pi a_{-1} = \mathrm{i}2\pi \operatorname*{Res}_{z=z_0} f(z)\,. \qquad \text{(D.29)}$$

Proof. As we have seen in Sec. C.3, we are allowed to deform the contour within a region where f is analytic, without changing the value of the integral. As illustrated in Fig. 31, we choose for this purpose a circular contour γ_r around z_0 with sufficiently small radius r such that γ_r lies within γ. Furthermore, f has a Laurent series about z_0 which converges uniformly, so that we are allowed to swap integral and summation operations, and thus obtain

$$\oint_\gamma f(z)\mathrm{d}z = \int_{\gamma_r} f(z)\mathrm{d}z = \oint_{\gamma_r}\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n\mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n\oint_{\gamma_r}(z-z_0)^n\mathrm{d}z\,.$$

The crux of the matter is the last integral, which we evaluate using polar coordinates $(z - z_0) = re^{\mathrm{i}\theta}$,

$$\oint_{\gamma_r} (z - z_0)^n \mathrm{d}z = \int_0^{2\pi} r^n e^{\mathrm{i}n\theta} \mathrm{i} r e^{\mathrm{i}\theta} \mathrm{d}\theta = \mathrm{i} r^{n+1} \int_0^{2\pi} e^{\mathrm{i}(n+1)\theta} \mathrm{d}\theta$$
$$= \begin{cases} \mathrm{i}2\pi\,, & \text{for} \quad n = -1 \\ \frac{r^{n+1}}{n+1} \left[e^{\mathrm{i}(n+1)\theta} \right]_0^{2\pi} = 0 & \text{for} \quad n \neq -1 \end{cases} . \tag{D.30}$$

Only the a_{-1} term survives and we have recovered Eq. (D.29). □

In the next section, we will generalize this result to an arbitrary number of singular points and see numerous applications of the theorem.

E The calculus of residues

E.1 The residue theorem

Equation (D.29) generalizes straightforwardly to the case of multiple singularities.

Theorem : Let $f(z)$ be analytic in a simply connected domain $\mathcal{D}$ except at a finite number of isolated singularities $z_1, \;\ldots, z_n$. Let γ be a simple closed counter-clockwise contour in $\mathcal{D}$ that encircles all $z_k,\; k = 1\ldots n$. Then

$$\oint_\gamma f(z)\mathrm{d}z = \mathrm{i}2\pi \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)\,. \tag{E.1}$$

Proof. The case $n = 1$ is simply Eq. (D.29) which we have already proved. For the general case, let us surround each singularity with a circular contour γ_1 as illustrated in the right panel of Fig. 32. We then approximate the contour γ (left panel in the figure) by cutting out out small pieces from γ as well as the circles and connect these with cross cuts such that we obtain the single contour $\tilde{\gamma}$ in the right panel. The key point is that $\tilde{\gamma}$ does not encicle any of the singularities. In the limit of vanishing width of the cross cuts, we thus obtain

$$0 \overset{!}{=} \oint_{\tilde{\gamma}} f(z)\mathrm{d}z = \oint_\gamma f(z)\mathrm{d}z + \sum_{k=1}^{n} \oint_{\gamma_k} f(z)\mathrm{d}z\,. \tag{E.2}$$

Note that each of the circular contours γ_k is traversed in clockwise direction when we regard them as part of $\tilde{\gamma}$. When evaluating each of their contributions according to Eq. (D.29), we therefore obtain a minus sign and find

$$\oint_\gamma f(z)\mathrm{d}z = -\sum_{k=1}^{n} \oint_{\gamma_k} f(z)\mathrm{d}z = \sum_{k=1}^{n} \mathrm{i}2\pi \operatorname*{Res}_{z=z_k} f(z)\,. \tag{E.3}$$

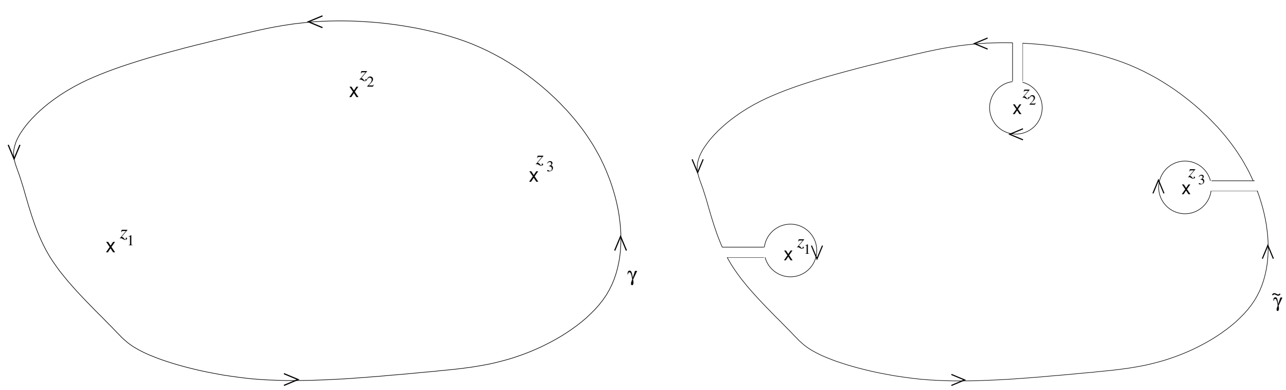

Figure 32: *Left panel*: The contour γ encircles several isolated singularities z_k of a function f. *Right panel*: We approximate this contour with $\tilde{\gamma}$ obtained by adding a circular contour around each singularity, removing thin segments from γ and connecting the remainder to the circles via cross cuts. The resulting $\tilde{\gamma}$ does not encircle any of the singularities.

□

We will now apply this theorem to calculate numerous integrals. We have loosely grouped these applications into a handful of categories, but note that this classification is not a rigorous one.

E.2 Integrals along the real axis

One of the most important applications of complex contour integration arises in the calculation of real integrals. It is one of these recurring mysteries of mathematics that such a detour from the real to the complex domain actually leads to remarkable simplifications and allows us to solve integrals for which we have no handle in the real domain.

The trick to use for these integrals is rather straightforward. The goal is to evaluate a real integral along the real axis or part thereof, say the integral in Eq. (E.4) below. We then extend from the real axis to the complex domain and complement the path of the integral such that the total contour is closed. Typically, we use a circle or segment of a circle for this purpose, as illustrated in Fig. 33. Next, we determine the residues of the integrand inside this closed contour and thus evaluate the total integral. Finally, we consider the limit of infinite radius of the circle; the artificially added contributions to the integral often vanish in this limit and the closed integral equals the real integral we set out to compute.

Examples of contour integration of real integrals

(1) Consider the integral

$$I = \int_0^\infty \frac{\mathrm{d}x}{1+x^2}\,. \tag{E.4}$$

As a matter of fact, this integral can be evaluated straightforwardly with standard techniques in the real domain, namely trigonometric substitution. It is well worth studying from a complex perspective, however, to illustrate our new approach, which, unlike the standard methods, remains effective for many cousins of the integral I.

Let γ_0 be the curve from $-R$ to R along the real axis, as illustrated in Fig. 33. The integral I is related to the contour integral along γ_0 by using $\gamma_0(t) = t$ in Eq. (C.3) or, more specifically,

$$\lim_{R\to\infty}\int_{\gamma_0}\frac{\mathrm{d}z}{1+z^2} = \int_{-\infty}^0 \frac{\mathrm{d}t}{1+t^2} + \int_0^\infty \frac{\mathrm{d}t}{1+t^2} = \int_\infty^0 \frac{-\mathrm{d}\tilde{t}}{1+\tilde{t}^2} + \int_0^\infty \frac{\mathrm{d}t}{1+t^2} = 2I\,. \tag{E.5}$$

We now close the contour by adding the upper half circle, denoted by γ_R in the left panel of the figure, such that $\gamma = \gamma_0 + \gamma_R$ is closed. The only singularity of $\frac{1}{1+z^2}$ inside γ is the single pole $z = \mathrm{i}$ and the residue is

$$\operatorname*{Res}_{z=\mathrm{i}} \frac{1}{1+z^2} = \lim_{z\to\mathrm{i}} \frac{1}{z+\mathrm{i}} = \frac{1}{2\mathrm{i}} \qquad \Rightarrow \qquad \oint_{\gamma_0+\gamma_R} \frac{\mathrm{d}z}{1+z^2} = 2\pi\mathrm{i}\frac{1}{2\mathrm{i}} = \pi\,. \tag{E.6}$$

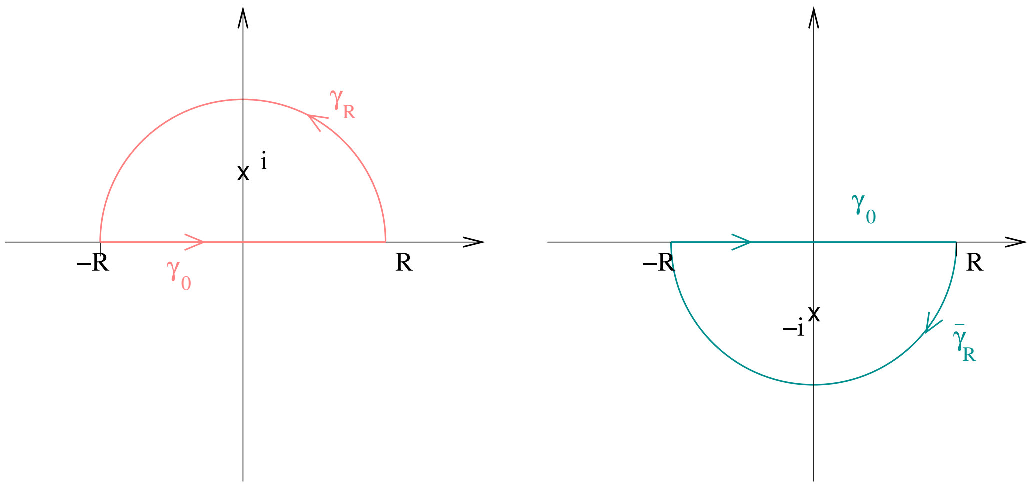

Figure 33: The goal is to evaluate the integral (E.4) along the real axis. We denote this path by γ_0 and complement it by a semi-circular path γ_R such that the contour $\gamma_0 + \gamma_R$ is closed. The integrand $1/(1+z^2)$ has only one residue inside γ, the integral contribution along γ_R vanishes as $R \to \infty$ and we obtain the required integral along γ_0. We can perform this trick by either closing the path in the upper (left panel) or lower (right panel) half plane.

Next we show that the contribution along γ_R vanishes: setting $\zeta_1 = 1$, $\zeta_2 = z^2$ in the triangle inequality (A.8), we find

$$\begin{aligned}&\left|1 - |z|^2\right| \le |1 + z^2| \\ \Rightarrow\quad &\lim_{R\to\infty} \left| \int_{\gamma_R} \frac{\mathrm{d}z}{1+z^2} \right| \le \lim_{R\to\infty} \int_{\gamma_R} \frac{|\mathrm{d}z|}{\left|1 - |z|^2\right|} = \lim_{R\to\infty} \int_{\gamma_R} \frac{|\mathrm{d}z|}{R^2 - 1} = \lim_{R\to\infty} \frac{\pi R}{R^2 - 1} = 0\,.\end{aligned} \tag{E.7}$$

We often write this type of evaluation in the less formal way

$$\lim_{R\to\infty} \left| \int_{\gamma_R} \frac{\mathrm{d}z}{1+z^2} \right| \le \pi R \sup_{z\in\gamma_R} \left| \frac{1}{1+z^2} \right| = \pi R\, \mathcal{O}(R^{-2}) = \mathcal{O}(R^{-1}) = 0\,. \tag{E.8}$$

Combing this result with Eq. (E.6), we obtain

$$2I \stackrel{!}{=} \lim_{R\to\infty} \int_{\gamma_0} \frac{\mathrm{d}z}{1+z^2} = \pi - \lim_{R\to\infty} \int_{\gamma_R} \frac{\mathrm{d}z}{1+z^2} = \pi \qquad \Rightarrow \qquad \int_0^\infty \frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{2}\,.$$

Alternatively, we could have closed the contour in the lower half plane as shown in the right panel of Fig. 33. The residue would then be

$$\operatorname*{Res}_{z=-\mathrm{i}} \frac{1}{1+z^2} = \lim_{z\to-\mathrm{i}} \frac{1}{z-\mathrm{i}} = -\frac{1}{2\mathrm{i}}\,. \tag{E.9}$$

The closed contour $\gamma_0 + \bar{\gamma}_R$ would now be traversed clockwise, however, adding another minus sign and we would obtain the same result $I = \pi/2$ as expected.

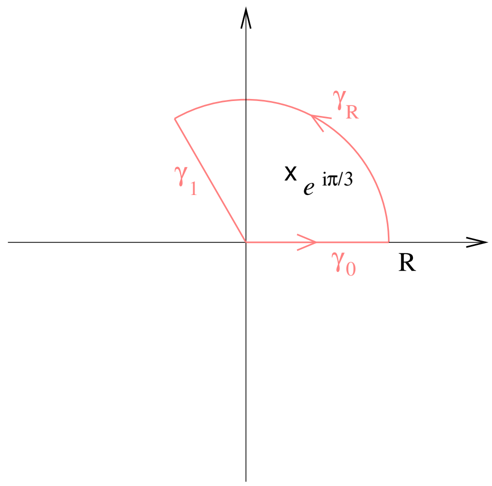

Figure 34: A closed contour constructed for integrating $f(z) = \frac{1}{z^3}$ along the positive real axis. Since $f(z)$ is invariant under rotations by $2\pi/3$, the integral contributions along γ_1 and γ_0 are equal whereas the contribution along γ_R vanishes in the limit $R \to \infty$. The only singularity inside the contour is $e^{\mathrm{i}\pi/3}$.

(2) The previous example was helped by the fact that $\frac{1}{1+x^2}$ is an even function of x and we could therefore relate its integral to the complex integral along γ_0 in Eq. (E.5). With a little variation, however, the same trick works for less convenient symmetries, as for example in the evaluation of

$$I = \int_0^\infty \frac{\mathrm{d}x}{1+x^3} \,. \tag{E.10}$$

The point is that the function $f(z) = \frac{1}{1+z^3}$ is invariant under rotations of z by $2\pi/3$ or $120°$. We then construct a close contour from three pieces, γ_1, γ_0 and γ_R as shown in Fig. 34. Using $\gamma_0(t) = t$ and $\gamma_1(t) = e^{\mathrm{i}2\pi/3}t$ in Eq. (C.3), we find

$$\int_{\gamma_0} \frac{\mathrm{d}z}{1+z^3} = \int_0^\infty \frac{1}{1+t^3}\gamma_0'(t)\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}t}{1+t^3} = I \,,$$

$$\int_{\gamma_1} \frac{\mathrm{d}z}{1+z^3} = \int_\infty^0 \frac{1}{1+(e^{\mathrm{i}2\pi/3}t)^3} e^{\mathrm{i}2\pi/3}\mathrm{d}t = \int_\infty^0 \frac{1}{1+t^3} e^{\mathrm{i}2\pi/3}\mathrm{d}t = -e^{\mathrm{i}2\pi/3} I \,.$$

The contribution along γ_R vanishes as before,

$$\lim_{R\to\infty} \left| \int_{\gamma_R} \frac{\mathrm{d}z}{1+z^3} \right| \le \pi R \sup_{z\in\gamma_R} \left| \frac{1}{1+z^3} \right| = \pi R\, \mathcal{O}(R^{-3}) = \mathcal{O}(R^{-2}) = 0 \,.$$

$f(z)$ has 3 singularities: single poles at $e^{\mathrm{i}k\pi/3}$ with $k = 1,\ 3,\ 5$. Only the first is inside our contour and we find the residue using l'Hôpital's rule,

$$\operatorname*{Res}_{z=e^{\mathrm{i}\pi/3}} \frac{1}{1+z^3} = \lim_{z\to e^{\mathrm{i}\pi/3}} \frac{z - e^{\mathrm{i}\pi/3}}{1+z^3} = \lim_{z\to e^{\mathrm{i}\pi/3}} \frac{1}{3z^2} = \frac{1}{3} e^{-\mathrm{i}2\pi/3} \,.$$

Putting all together, we obtain

$$\oint_{\gamma_1+\gamma_0+\gamma_R} \frac{\mathrm{d}z}{1+z^3} = -e^{\mathrm{i}2\pi/3} I + I + 0 = \mathrm{i}2\pi \operatorname*{Res}_{z=e^{\mathrm{i}\pi/3}} \frac{1}{1+z^3} = \mathrm{i}\frac{2\pi}{3} e^{-\mathrm{i}2\pi/3}$$

$$\Rightarrow \quad I = \mathrm{i}\frac{2\pi}{3}\frac{e^{-\mathrm{i}2\pi/3}}{1-e^{\mathrm{i}2\pi/3}} = \mathrm{i}\frac{2\pi}{3}\frac{e^{-\mathrm{i}2\pi/3}}{1-e^{\mathrm{i}2\pi/3}}\,\frac{1+e^{-\mathrm{i}2\pi/3}}{1+e^{-\mathrm{i}2\pi/3}} = \mathrm{i}\frac{2\pi}{3}\,\frac{e^{-\mathrm{i}2\pi/3}+e^{-\mathrm{i}4\pi/3}}{e^{-\mathrm{i}2\pi/3}-e^{\mathrm{i}2\pi/3}}$$

$$= \mathrm{i}\frac{2\pi}{3} e^{-\mathrm{i}\pi}\,\frac{e^{\mathrm{i}\pi/3}+e^{-\mathrm{i}\pi/3}}{-e^{\mathrm{i}2\pi/3}+e^{-\mathrm{i}2\pi/3}} = -\mathrm{i}\frac{2\pi}{3}\,\frac{2\cos\frac{\pi}{3}}{-2\mathrm{i}\sin\frac{\pi}{3}} = \frac{2\pi}{3}\,\frac{1/2}{\sqrt{3}/2} = \frac{2\pi}{3\sqrt{3}}\,.$$

We can likewise integrate $1/(1+x^4)$ using a quarter circle connected to the origin along the positive real and imaginary axes and so forth.

(3) Let us next compute

$$I = \int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}\,,$$

where $a > 0$ is a real constant. We use the same contour γ as displayed in the left panel of Fig. 33 for the first example. The function $f(z) = 1/(z^2+a^2)^2$ has singularities at $z = \mathrm{i}a$ and $z = -\mathrm{i}a$, and both are poles of order 2. Note, however, that this does not mean their residues are zero! The simple function $1/z^2$ indeed has a zero residue at $z = 0$ because the remainder after factoring out the pole is constant. That is not the case for our more complicated function, though, and we instead find from Eq. (D.19) with $n = 2$ that

$$\operatorname*{Res}_{z=\mathrm{i}a} \frac{1}{(z^2+a^2)^2} = \lim_{z\to\mathrm{i}a} \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{(z+\mathrm{i}a)^2} = \lim_{z\to\mathrm{i}a} \frac{-2}{(z+\mathrm{i}a)^3} = \frac{1}{4\mathrm{i}a^3}\,.$$

Once more, the contribution of the integral along γ_R vanishes for $R \to \infty$,

$$\left|\int_{\gamma_R} \frac{\mathrm{d}z}{(z^2+a^2)^2}\right| \le \pi R\,\mathcal{O}(R^{-4}) = \mathcal{O}(R^{-3}) \to 0 \qquad \text{as} \quad R\to\infty\,.$$

Since our function is symmetric under $x \to -x$, we therefore have

$$2I = \oint_{\gamma_0+\gamma_R} \frac{\mathrm{d}z}{(z^2+a^2)^2} = \mathrm{i}2\pi\frac{1}{4\mathrm{i}a^3} = \frac{\pi}{2a^3} \qquad \Rightarrow \qquad I = \frac{\pi}{4a^3}\,.$$

E.3 Integrals of trigonometric functions

We now look at integrals of the form

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\mathrm{d}\theta\,. \tag{E.11}$$

With appropriate substitutions, such as $u = \tan\theta$, one can translate the problem into the integration of rational functions and attempt a solution by means of partial fractions. The method of residue calculus often provides a much simpler method, however. This is achieved with the standard substitution

$$z = e^{\mathrm{i}\theta}\,, \qquad \cos\theta = \frac{1}{2}(z + z^{-1})\,, \qquad \sin\theta = \frac{1}{2\mathrm{i}}(z - z^{-1})\,. \tag{E.12}$$

The integral (E.11) then becomes a contour integral along the unit circle and we parametrize the contour γ according to

$$\gamma : \theta \mapsto z = e^{\mathrm{i}\theta} \qquad \Rightarrow \qquad \frac{\mathrm{d}z}{\mathrm{d}\theta} = \mathrm{i}e^{\mathrm{i}\theta} = \mathrm{i}z \qquad \Rightarrow \qquad \mathrm{d}\theta = -\mathrm{i}\frac{\mathrm{d}z}{z}\,. \tag{E.13}$$

Examples of contour integration of functions $f(\cos\theta, \sin\theta)$

(1) Let us consider the integral

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos\theta}\,, \tag{E.14}$$

where $a > 1$ is a real constant. With the substitution (E.12), (E.13), this becomes

$$I = \oint_\gamma \frac{-\mathrm{i}\mathrm{d}z}{z\big(a + \frac{1}{2}(z + z^{-1})\big)} = -2\mathrm{i}\oint_\gamma \frac{\mathrm{d}z}{z^2 + 2az + 1}\,, \tag{E.15}$$

along the contour γ shown in Fig. 35. The singularities follow from the zeroes of the denominator $z^2 + 2az + 1$ and are given by

$$z_\pm = -a \pm \sqrt{a^2 - 1}\,. \tag{E.16}$$

For $a > 1$, $z_- < -a < -1$ clearly lies outside the unit circle. For z_+, we find $z_+(a = 1) = -1$ and $\mathrm{d}z_+/\mathrm{d}a = -1 + a/\sqrt{a^2 - 1} > 0$ for all $a > 1$, so that z_+ always lies within

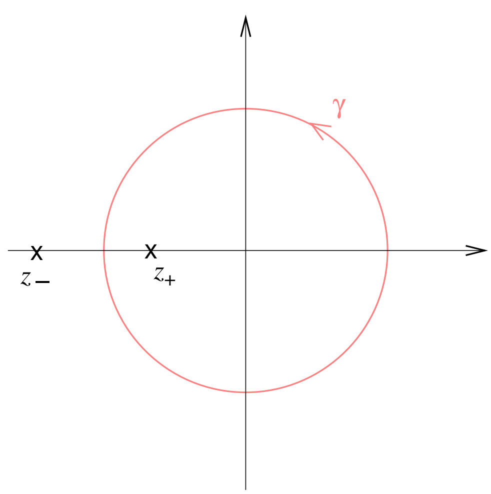

Figure 35: The contour γ is traversed along the unit circle in counter-clockwise direction. The singularity z_+ lies inside γ and z_- outside.

the unit circle. We therefore need the residue

$$\operatorname*{Res}_{z=z_+}\frac{1}{z^2+2az+1}=\operatorname*{Res}_{z=z_+}\frac{1}{(z-z_+)(z-z_-)}=\lim_{z\to z_+}\frac{1}{z-z_-}=\frac{1}{z_+-z_-}=\frac{1}{2\sqrt{a^2-1}}\,.$$

The integral is therefore given by

$$I=-2\mathrm{i}\frac{\mathrm{i}2\pi}{2\sqrt{a^2-1}}=\frac{2\pi}{\sqrt{a^2-1}}\,. \tag{E.17}$$

E.4 Branch cuts and keyhole contours

The integration of functions with a branch cut requires additional care because we cannot use any closed contour that crosses the branch cut. A common method to avoid crossings of the branch cut consists in taking lines parallel to the cut, connecting these by segments of circles and then taking the limit where the parallel lines approach the branch cut. The resulting contours acquire the shape of keyholes and are accordingly refered to as "keyhole contours"; see Fig. 36 for an example that we will discuss in more detail now.

Examples of contour integration around branch cuts

(1) We wish to compute the integral

$$I=\int_0^\infty\frac{x^\alpha}{1+\sqrt{2}x+x^2}\mathrm{d}x\,,\qquad\text{with}\quad-1<\alpha<1\,. \tag{E.18}$$

When treating the integrand as a complex function, we need to choose a branch for z^α except for the special case $\alpha=0$; recall example (3) of Sec. B.4.1. The branch point is $z=0$ and we put the branch cut along the positive real axis. Finally, we fix the branch by defining $z=re^{\mathrm{i}\theta}$ and

$$z^\alpha=r^\alpha e^{\mathrm{i}\alpha\theta}\qquad\text{with}\ \ 0\le\theta<2\pi\,. \tag{E.19}$$

We next wish to construct a closed contour that contains in approximate form the path of the real integral (E.18), i.e. the positive real axis. We achive this by first taking two lines γ_1 and γ_2 parallel to the positive real axis, one just above and the other just below. We then connect the ends of these lines with two circle segments γ_ϵ and γ_R as shown in Fig. 36 and simultaneously take the limits $R\to\infty$, $\epsilon\to0$, such that the circles traverses the complete angular range from 0 to 2π.

The complex integral along the closed contour then consists of four individual contributions. The first along γ_R is

$$\int_{\gamma_R}\frac{z^\alpha}{1+\sqrt{2}z+z^2}\mathrm{d}z=2\pi R\,\mathcal{O}(R^{\alpha-2})=\mathcal{O}(R^{\alpha-1})\quad\overset{R\to\infty}{\longrightarrow}\quad0\,,$$

since $\alpha<1$. Along γ_ϵ, we substitute $z=\epsilon e^{\mathrm{i}\theta}$ and obtain

$$\int_{\gamma_\epsilon}\frac{z^\alpha}{1+\sqrt{2}z+z^2}\mathrm{d}z=\int_{2\pi}^{0}\frac{\epsilon^\alpha e^{\mathrm{i}\alpha\theta}}{1+\sqrt{2}\epsilon e^{\mathrm{i}\theta}+\epsilon^2e^{\mathrm{i}2\theta}}\mathrm{i}\epsilon e^{\mathrm{i}\theta}\mathrm{d}\theta=\mathcal{O}(\epsilon^{\alpha+1})\quad\overset{\epsilon\to0}{\longrightarrow}0\,.$$

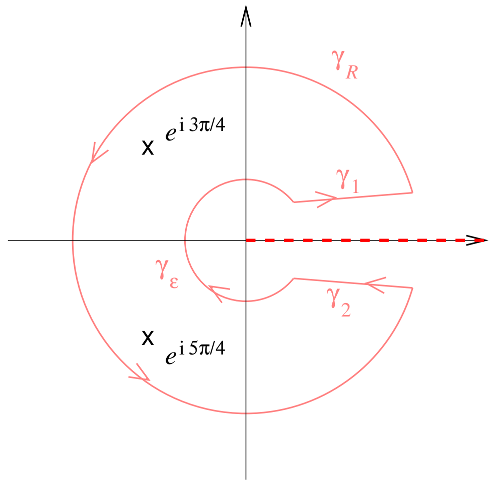

Figure 36: The keyhole contour constructed for calculating the integral (E.18) consists of two nearly complete circle segements γ_ϵ and γ_R of radius ϵ and R, respectively, connected by two lines γ_1 and γ_2 parallel to the branch cut along the positive real axis.

Next, we parametrize the curve γ_1 by $\gamma_1 : t \mapsto z = te^{\mathrm{i}\delta\theta}$ with the limit $\delta\theta \to 0$. The integral contribution then becomes

$$\int_{\gamma_1} \frac{z^\alpha}{1+\sqrt{2}z+z^2}\mathrm{d}z = \lim_{\delta\theta\to 0}\int_0^\infty \frac{t^\alpha e^{\mathrm{i}\alpha\delta\theta}}{1+\sqrt{2}te^{\mathrm{i}\delta\theta}+t^2e^{\mathrm{i}2\delta\theta}}e^{\mathrm{i}\delta\theta}\mathrm{d}t = \int_0^\infty \frac{t^\alpha}{1+\sqrt{2}t+t^2}\mathrm{d}t = I\,.$$

Along γ_2, we likewise parametrize with $\gamma_2 : t \mapsto z = te^{\mathrm{i}\delta\theta}$, but now take the limit $\delta\theta \to 2\pi$ and bear in mind that γ_2 is traversed from right to left, so that

$$\int_{\gamma_2} \frac{z^\alpha}{1+\sqrt{2}z+z^2}\mathrm{d}z = \int_\infty^0 \frac{t^\alpha e^{\mathrm{i}2\alpha\pi}}{1+\sqrt{2}t+t^2}\mathrm{d}t = -e^{\mathrm{i}2\alpha\pi} I\,.$$

In summary, we have

$$\oint_{\gamma_1+\gamma_R+\gamma_2+\gamma_\epsilon} \frac{z^\alpha}{1+\sqrt{2}z+z^2}\mathrm{d}z = \left(1-e^{\mathrm{i}2\alpha\pi}\right) I\,, \tag{E.20}$$

and merely need the residues to evaluate the left-hand side. Noting

$$\begin{aligned} z^2+1+\sqrt{2}z &= z^2+1-\frac{z}{\sqrt{2}}(-1+\mathrm{i})(1+\mathrm{i}) &= z^2+1-ze^{\mathrm{i}3\pi/4}(1+e^{\mathrm{i}\pi/2}) \\ &= z^2+e^{\mathrm{i}8\pi/4}-z(e^{\mathrm{i}3\pi/4}+e^{\mathrm{i}5\pi/4}) &= (z-e^{\mathrm{i}3\pi/4})(z-e^{\mathrm{i}5\pi/4})\,, \end{aligned}$$

and

$$e^{\mathrm{i}3\pi/4}-e^{\mathrm{i}5\pi/4} = e^{\mathrm{i}3\pi/4}(1-e^{\mathrm{i}\pi/2}) = \frac{1}{\sqrt{2}}(-1+\mathrm{i})(1-\mathrm{i}) = -\frac{1}{\sqrt{2}}(1-\mathrm{i})^2 = \sqrt{2}\mathrm{i}\,,$$

we find two first order poles $z = e^{\mathrm{i}3\pi/4}$ and $z = e^{\mathrm{i}5\pi/4}$ with residues

$$\operatorname*{Res}_{z=e^{\mathrm{i}3\pi/4}} \frac{z^\alpha}{1+\sqrt{2}z+z^2} = \frac{e^{\mathrm{i}3\alpha\pi/4}}{e^{\mathrm{i}3\pi/4}-e^{\mathrm{i}5\pi/4}} = \frac{e^{\mathrm{i}3\alpha\pi/4}}{\sqrt{2}\mathrm{i}}\,,$$

$$\operatorname*{Res}_{z=e^{\mathrm{i}5\pi/4}} \frac{z^\alpha}{1+\sqrt{2}z+z^2} = \frac{e^{\mathrm{i}5\alpha\pi/4}}{e^{\mathrm{i}5\pi/4}-e^{\mathrm{i}3\pi/4}} = -\frac{e^{\mathrm{i}5\alpha\pi/4}}{\sqrt{2}\mathrm{i}}\,,$$

and, plugging these into Eq. (E.20),

$$2\pi\mathrm{i}\left(\frac{e^{\mathrm{i}3\alpha\pi/4}}{\sqrt{2}\mathrm{i}} - \frac{e^{\mathrm{i}5\alpha\pi/4}}{\sqrt{2}\mathrm{i}}\right) = \sqrt{2}\pi e^{\mathrm{i}3\alpha\pi/4}(1-e^{\mathrm{i}\alpha\pi/2}) \overset{!}{=} (1-e^{\mathrm{i}2\alpha\pi})I$$

$$\Rightarrow \quad Ie^{\mathrm{i}\alpha\pi}(e^{-\mathrm{i}\alpha\pi}-e^{\mathrm{i}\alpha\pi}) = \sqrt{2}\pi e^{i\alpha\pi}(e^{-\mathrm{i}\alpha\pi/4}-e^{\mathrm{i}\alpha\pi/4})$$

$$\Rightarrow \quad I = \sqrt{2}\pi\frac{\sin\left(\frac{\alpha\pi}{4}\right)}{\sin(\alpha\pi)}\,. \tag{E.21}$$

One final word of caution: When we identified the two poles as $e^{\mathrm{i}3\pi/4}$ and $e^{\mathrm{i}5\pi/4}$, we had to be careful to choose their arguments such that they lie within our selected branch $0 \le \theta < 2\pi$ as specified in Eq. (E.19). Of course,

$$e^{\mathrm{i}5\pi/4} = e^{-\mathrm{i}3\pi/4}\,,$$

denote the same pole, but the latter would have given us a different residue, namely $-e^{-\mathrm{i}3\alpha\pi/4}/(\sqrt{2}\mathrm{i})$, and, hence, a wrong result for the integral I. Once we select a branch for a function, we need to stick to it!

E.5 Rectangular contours

So far, we have employed circles for the closure of contours. Sometimes, however, it is more convenient to use rectangular contours. The reason is that in the limit of infinite radius, circles extend arbitrarily far in all directions. If we face a function with an infinite number of poles, they will all lie inside infinitely large circles and we have a hard time summing over all the residues. This problem can be overcome by rectangular contours that stretch arbitrarily far only in selected directions.

Examples of rectangular contours

(1) We wish to calculate the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\cosh x}\mathrm{d}x\,, \qquad \text{with} \quad -1<\alpha<1\,. \tag{E.22}$$

Recalling that $\cosh(\mathrm{i}z) = \cos z$, we see that the integrand has singularities at $z = \mathrm{i}\left(n+\frac{1}{2}\right)\pi$ for all $n \in \mathbb{Z}$. Whether we choose a full circle or a half circle, we would end

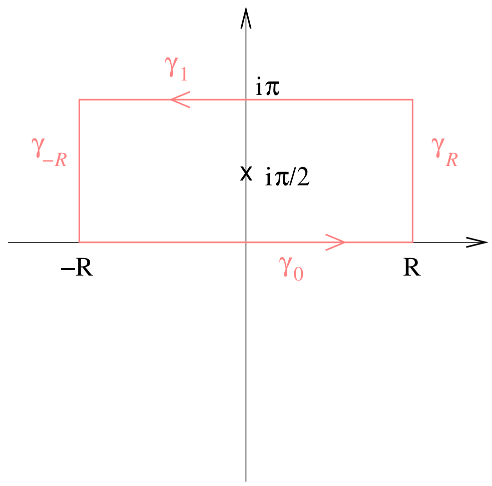

Figure 37: A rectangular contour with two horizontal lines through $z = 0$ and $z = \mathrm{i}\pi$, bounded by $\mathsf{Re}(z) = \mp R$ on the left and right, respectively. Only one of the zeros of $\cosh z$ lies inside this contour: $z = \mathrm{i}\frac{\pi}{2}$.

up with an infinite number of poles inside. All poles are located on the imaginary axis, however, so we can bypass this problem by using a rectangle stretching infinitely far in the real direction only. Furthermore, we exploit the periodicity

$$\cosh(z + \mathrm{i}\pi) = \cosh z \underbrace{\cosh(\mathrm{i}\pi)}_{=\cos\pi=-1} + \sinh z \underbrace{\sinh(\mathrm{i}\pi)}_{=\mathrm{i}\sin\pi=0} = -\cosh z\,,$$

and construct a rectangle with two horizontal edges γ_0 and γ_1 with $\mathsf{Im}(z) = 0$ and $\mathsf{Im}(z) = \pi$ as shown in Fig. 37. The rectangle is completed by two vertical segments γ_{-R} and γ_R with fixed $\mathsf{Re}(z) = \mp R$. Letting $R \to \infty$, we have four contribution to the integral over the closed contour. Along γ_0, we parametrize $t \mapsto z = t$ and find

$$\int_{\gamma_0} \frac{e^{\alpha z}}{\cosh z}\mathrm{d}z = \int_{-\infty}^{\infty} \frac{e^{\alpha t}}{\cosh t}\mathrm{d}t = I\,.$$

Likewise, along $\gamma_1 : t \mapsto z = t + \mathrm{i}\pi$, we find

$$\int_{\gamma_1} \frac{e^{\alpha z}}{\cosh z}\mathrm{d}z = \int_{\infty}^{-\infty} \frac{e^{\alpha(t+\mathrm{i}\pi)}}{\cosh(t+\mathrm{i}\pi)}\mathrm{d}t = -e^{\alpha\mathrm{i}\pi}\int_{-\infty}^{\infty} \frac{e^{\alpha t}}{-\cosh t}\mathrm{d}t = e^{\mathrm{i}\alpha\pi}I\,.$$

Along γ_R, we have

$$\begin{aligned}
\cosh z &= \cosh(R+\mathrm{i}y) = \cosh R\,\cosh(\mathrm{i}y) + \sinh R\,\sinh(\mathrm{i}y) \\
&= \cosh R\,\cos y + \mathrm{i}\cosh R\,\sin y \\
\Rightarrow \big|\cosh z\big| &= \sqrt{\cosh^2 R\,\cos^2 y + \sinh^2 R\,\sin^2 y} \\
&= \sqrt{(1+\sinh^2 R)\cos^2 y + \sinh^2 R\,\sin^2 y} \\
&= \sqrt{\cos^2 y + \sinh^2 R} \quad \geq \quad \sinh R\,.
\end{aligned} \tag{E.23}$$

This gives us with $\gamma_R : t \mapsto z = R + \mathrm{i}t$,

$$\left|\int_{\gamma_R} \frac{e^{\alpha z}}{\cosh z}\mathrm{d}z\right| \leq \int_0^\pi \frac{|e^{\alpha R}e^{\mathrm{i}\alpha t}|}{|\sinh R|}\mathrm{d}t = e^{\alpha R}\int_0^\pi \frac{1}{\sinh R}\mathrm{d}t = \frac{\pi e^{\alpha R}}{\sinh R} = \mathcal{O}\big(e^{(\alpha-1)R}\big) \quad \overset{R\to\infty}{\longrightarrow} \quad 0\,,$$

since $\alpha < 1$. One likewise uses $\alpha > -1$ to show that

$$\int_{\gamma_{-R}} \frac{e^{\alpha z}}{\cosh z}\mathrm{d}z \quad \overset{R\to-\infty}{\longrightarrow} \quad 0\,.$$

In summary, with $\gamma = \gamma_0 + \gamma_R + \gamma_1 + \gamma_{-R}$, we obtain

$$\oint_\gamma \frac{e^{\alpha z}}{\cosh z}\mathrm{d}z = (1 + e^{\mathrm{i}\alpha\pi})I\,.$$

The only singular point inside γ is $z = \mathrm{i}\pi/2$ where, using l'Hôpital,

$$\operatorname*{Res}_{z=\mathrm{i}\frac{\pi}{2}} \frac{e^{\alpha z}}{\cosh z} = \lim_{z\to\mathrm{i}\frac{\pi}{2}} \frac{\big(z-\mathrm{i}\frac{\pi}{2}\big)e^{\alpha z}}{\cosh z} = \lim_{z\to\mathrm{i}\frac{\pi}{2}} \frac{\big(z-\mathrm{i}\frac{\pi}{2}\big)\alpha e^{\alpha z} + e^{\alpha z}}{\sinh z} = \frac{e^{\mathrm{i}\alpha\pi/2}}{\sinh\frac{\mathrm{i}\pi}{2}} = -\mathrm{i}e^{\mathrm{i}\alpha\pi/2}\,.$$

This gives us

$$I = \frac{1}{1+e^{\mathrm{i}\alpha\pi}}2\pi\mathrm{i}\left(-\mathrm{i}e^{\mathrm{i}\alpha\pi/2}\right) = \frac{2\pi}{e^{-\mathrm{i}\alpha\pi/2} + e^{\mathrm{i}\alpha\pi/2}} = \frac{\pi}{\cos\left(\frac{\alpha\pi}{2}\right)}\,. \tag{E.24}$$

(2) Sometimes, we do not mind having an infinite number of singular points inside our contour. This allows us, for example, to compute the limits of some complicated series. Let us illustrate this for the contour integral

$$I = \oint_\gamma f(z)\mathrm{d}z\,, \qquad \text{with} \qquad f(z) = \frac{1}{z^2\tan(\pi z)}\,, \tag{E.25}$$

where γ is the square contour of radius $N + \frac{1}{2}$, $N \in \mathbb{N}$, centered on the origin as shown in Fig. 38. The singularities of $f(z)$ are $z = n$, $n \in \mathbb{Z}$ with $z = 0$ being a triple pole

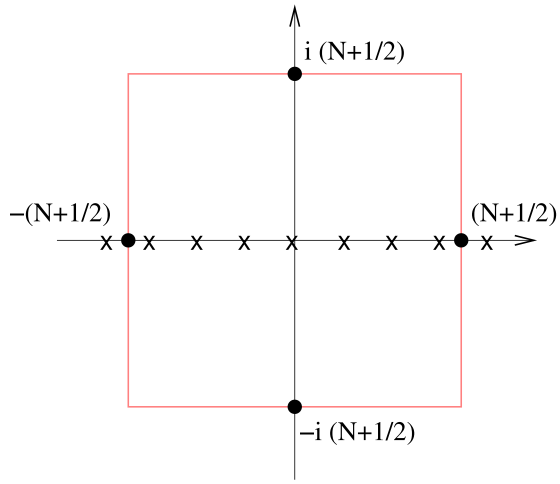

Figure 38: A square contour centered on the origin with radius $N+\frac{1}{2}$, $N\in\mathbb{N}$, encloses singularities of $f(z)=1/[z^2\tan(\pi z)]$ located at $n\in\mathbb{Z}$.

and all the others simple. The residue at the triple pole is obtained from the Taylor series of $\tan z$,

$$\begin{aligned} &\tan z = z + \frac{1}{3}z^3 + \dots \\ \Rightarrow\quad & z^2\tan\pi z = z^2\left(\pi z + \frac{1}{3}\pi^3 z^3 + \dots\right) = \pi z^3\left(1+\frac{\pi^2}{3}z^2+\dots\right) \\ \Rightarrow\quad & \frac{1}{z^2\tan\pi z} = \frac{1}{\pi z^3}\left(1-\frac{\pi^2}{3}z^2-\dots\right) = \frac{1}{\pi}z^{-3}-\frac{\pi}{3}z^{-1}-\dots\,, \end{aligned} \tag{E.26}$$

i.e. $a_{-1}=-\frac{\pi}{3}$. For the simple poles we find

$$\operatorname*{Res}_{z=n}\frac{1}{z^2\tan\pi z} = \lim_{z\to n}\frac{z-n}{z^2\tan\pi z} = \lim_{z\to n}\frac{1}{2z\tan\pi z+\frac{z^2\pi}{\cos^2\pi z}} = \frac{1}{n^2\pi}\,.$$

We will show in a moment that the integral contributions along all four edges of the square vanish in the limit $N\to\infty$, so that

$$\oint_\gamma \frac{\mathrm{d}z}{z^2\tan\pi z} = \mathrm{i}2\pi\left(-\frac{\pi}{3}+2\sum_{n=1}^{N}\frac{1}{n^2\pi}\right) \xrightarrow{N\to\infty} 0\,,$$

and therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}\,. \tag{E.27}$$

Of course, we might have computed this sum with less exhaustion by other means; our method, however, has the benefit that it can be easily modified to computer alternative sums such as $\sum\frac{1}{n^4}$ or $\sum\frac{1}{n^{28}}$ should you feel any desire to do so.

So how about the four edges of the square contour? Let's start with the right edge where we parametrize $t \mapsto z(t) = N + \frac{1}{2} + \mathrm{i}t$. The trick consists in bounding the integrand; abbreviating $\nu := N + \frac{1}{2}$, we find

$$\begin{aligned}
|\tan(\nu\pi + \mathrm{i}\pi t)| &= \left|\frac{\sin(\nu\pi + \mathrm{i}\pi t)}{\cos(\nu\pi + \mathrm{i}\pi t)}\right| = \left|\frac{\overbrace{\sin\nu\pi}^{=(-1)^N}\,\cos\mathrm{i}\pi t + \overbrace{\cos\nu\pi}^{=0}\,\sin\mathrm{i}\pi t}{\cos\nu\pi\,\cos\mathrm{i}\pi t - \sin\nu\pi\,\sin\mathrm{i}\pi t}\right| \\
&= \left|\frac{(-1)^N\cosh\pi t}{\mathrm{i}(-1)^{N+1}\sinh\pi t}\right| = \left|\mathrm{i}\frac{\cosh\pi t}{\sinh\pi t}\right| = \frac{|\cosh\pi t|}{|\sinh\pi t|} \geq 1
\end{aligned}$$

$$\Rightarrow \left|\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{\mathrm{i}\mathrm{d}t}{z(t)^2\tan\pi z(t)}\right| \leq \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \left|\frac{1}{z(t)^2}\right|\mathrm{d}t = \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \mathcal{O}(N^{-2})\mathrm{d}t = \mathcal{O}(N^{-1}) \stackrel{N\to\infty}{\longrightarrow} 0\,.$$

At the upper edge of the square, we parametrize $z(t) = \mathrm{i}\big(N + \frac{1}{2}\big) + t =: \mathrm{i}\nu + t$ and find

$$\begin{aligned}
|\tan(t\pi + \mathrm{i}\nu\pi)| &= \frac{|\sin(t\pi + \mathrm{i}\nu\pi)|}{|\cos(t\pi + \mathrm{i}\nu\pi)|} = \frac{|\sin t\pi\,\cosh\nu\pi + \mathrm{i}\cos t\pi\,\sinh\nu\pi|}{|\cos t\pi\,\cosh\nu\pi - \mathrm{i}\sin t\pi\,\sinh\nu\pi|} \\
&= \frac{\sqrt{\sin^2 t\pi\,\cosh^2\nu\pi + \cos^2 t\pi\,\sinh^2\nu\pi}}{\sqrt{\cos^2 t\pi\,\cosh^2\nu\pi + \sin^2 t\pi\,\sinh^2\nu\pi}} \\
&= \frac{\sqrt{\sin^2 t\pi\,\cosh^2\nu\pi + \sinh^2\nu\pi - \sin^2 t\pi\,\sinh^2\nu\pi}}{\sqrt{\cosh^2\nu\pi - \cosh^2\nu\pi\,\sin^2 t\pi + \sinh^2\nu\pi\,\sin^2 t\pi}} \\
&= \frac{\sqrt{\sinh^2\nu\pi + \sin^2 t\pi}}{\sqrt{\cosh^2\nu\pi - \sin^2 t\pi}} \geq \left|\frac{\sinh\nu\pi}{\cosh\nu\pi}\right| = |\tanh\nu\pi| \geq \tanh\frac{\pi}{2}\,,
\end{aligned}$$

where the last inequality arises from the monotonicity of the tanh function; cf. Fig. 39. For the integral along the upper edge, we therefore find

$$\left|\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{\mathrm{i}\mathrm{d}t}{z(t)^2\tan\pi z(t)}\right| \leq \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{\mathrm{d}t}{|z(t)^2\tan(t\pi + \mathrm{i}\nu\pi)|} \leq \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{\mathrm{d}t}{|z(t)|^2\tanh\frac{\pi}{2}} = \mathcal{O}(N^{-1}) \stackrel{N\to\infty}{\longrightarrow} 0\,.$$

We likewise find that the integral contributions along the left and lower edge of the square vanish in the limit $N \to \infty$.

E.6 Jordan's Lemma

In some of our applications in this section, we have eliminated integrals along half circles by using a sufficient falloff of the integrand and then taking the limit $R \to \infty$; in Eqs. (E.7), (E.8), for example, the $1/R^2$ falloff of the integrand more than compensates the $\propto R$ increase

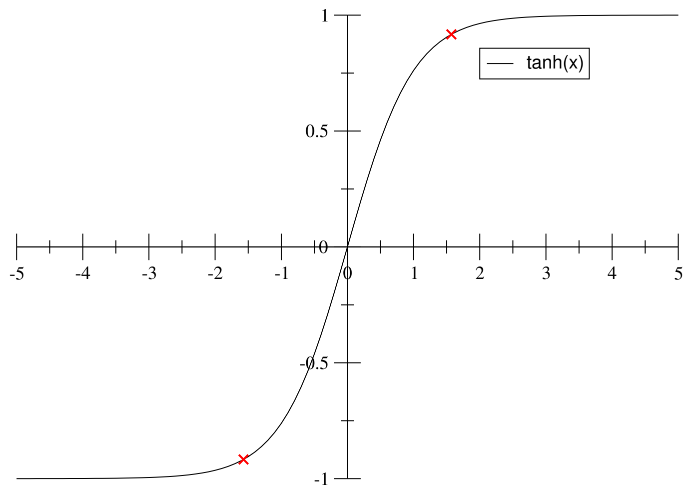

Figure 39: The tanh function. The values $\tanh(\pm\pi/2)$ are marked by the red '$\times$' symbols.

in length of the path, resulting in the vanishing of the integral as $R \to \infty$. This is all fine, but sometimes integrals of this type vanish even without such a rapid falloff. We can then evoke *Jordan's lemma.*

Lemma : Let $f(z)$ be a function that is analytic in $\mathbb{C}$ except for a finite number of singular points with $f(z) \to 0$ as $|z| \to \infty$. Let γ_R denote the contour traversed counter clockwise along the half circle of radius R in the upper half plane, and $\bar{\gamma}_R$ the clockwise contour along the half circle of radius R in the lower half plan as illustrated in Fig. 40. Then for any positive real constant $\lambda > 0$

$$\lim_{R\to\infty} \int_{\gamma_R} f(z) e^{\mathrm{i}\lambda z} \mathrm{d}z = 0\,, \tag{E.28}$$

and for any negative real constant $\mu < 0$

$$\lim_{R\to\infty} \int_{\bar{\gamma}_R} f(z) e^{\mathrm{i}\mu z} \mathrm{d}z = 0\,, \tag{E.29}$$

This result is evident for functions that fall off sufficiently fast with $|z|$, i.e. behave like $f(z) = \mathcal{O}(|z|^{-1-\epsilon})$ for $|z| \to \infty$ and some $\epsilon > 0$. Then

$$\left| \int_{\gamma_R} f(Z) e^{\mathrm{i}\lambda z} \mathrm{d}z \right| \leq \pi R\, \mathcal{O}(R^{-1-\epsilon}) \stackrel{R\to\infty}{\longrightarrow} 0\,. \tag{E.30}$$

Jordan's lemma, however, tells us that the integral even vanishes for functions with a weaker falloff, say $O(|z|^{-\epsilon})$; any function that tends to zero at infinity will do.

Proof. For this proof we first need to show that

$$\sin x \geq \frac{2}{\pi} x \qquad \text{for} \quad x \in \left[0, \tfrac{\pi}{2}\right].$$

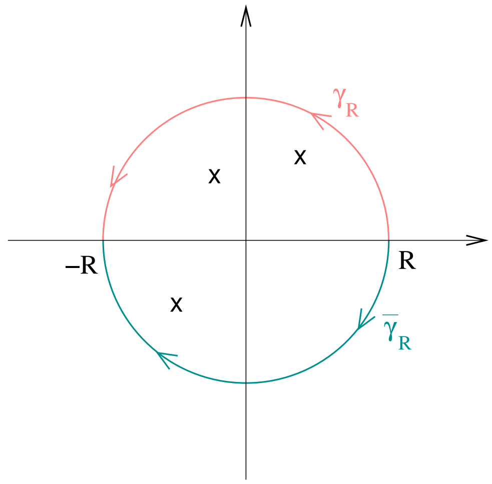

Figure 40: The curves γ_R and $\bar{\gamma}_R$ are semi-circles as required for Jordan's lemma.

This looks obvious when plotting the functions $\sin x$ and x, but is not trivial to derive from first principles. We start with

$$h(x) := \frac{\sin x}{x} \qquad \Rightarrow \qquad h'(x) = \frac{x\cos x - \sin x}{x^2}\,, \quad \lim_{x\to 0} h(x) = 1\,, \quad h\left(\tfrac{\pi}{2}\right) = \frac{2}{\pi}\,.$$

Next, we define

$$\begin{aligned}
& g(x) := x\cos x - \sin x \\
\Rightarrow\quad & g(0) = 0 \qquad \wedge \qquad g'(x) = \cos x - x\sin x - \cos x = -x\sin x \le 0 \;\text{ for }\; x \in \left[0, \tfrac{\pi}{2}\right] \\
\Rightarrow\quad & g(x) \le 0 \;\text{ for }\; x \in \left[0, \tfrac{\pi}{2}\right] \\
\Rightarrow\quad & h'(x) \overset{!}{=} \frac{g(x)}{x^2} \le 0 \;\text{ for }\; x \in \left[0, \tfrac{\pi}{2}\right] \\
\Rightarrow\quad & h(x) \ge h\left(\tfrac{\pi}{2}\right) = \frac{2}{\pi} \;\text{ for }\; x \in \left[0, \tfrac{\pi}{2}\right] \\
\Rightarrow\quad & \sin x \ge \frac{2}{\pi}x \;\text{ for }\; x \in \left[0, \tfrac{\pi}{2}\right]\,.
\end{aligned} \tag{E.31}$$

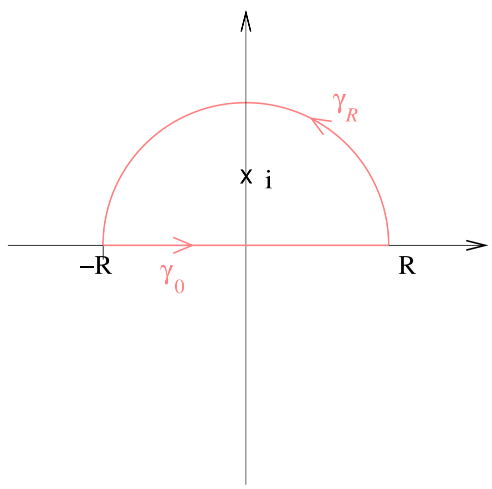

Figure 41: The contour $\gamma = \gamma_0 + \gamma_R$ consists of the real interval $[-R, R]$ and the upper half circle of radius R.

Returning to the Jordan lemma, we parametrize the half circle with $\theta \mapsto Re^{\mathrm{i}\theta}$ and find

$$\begin{aligned}
\left|\int_{\gamma_R} f(z)e^{\mathrm{i}\lambda z}\mathrm{d}z\right| &= \left|\int_0^\pi f(Re^{\mathrm{i}\theta})e^{\mathrm{i}\lambda Re^{\mathrm{i}\theta}}\mathrm{i}Re^{\mathrm{i}\theta}d\theta\right| \leq R\int_0^\pi \left|f(Re^{\mathrm{i}\theta})\right| \left|e^{\mathrm{i}\lambda Re^{\mathrm{i}\theta}}\right|\mathrm{d}\theta \\
&= R\int_0^\pi \left|f(Re^{\mathrm{i}\theta})\right| \underbrace{e^{-\lambda R\sin\theta}}_{>0}\mathrm{d}\theta \leq R\sup_{z\in\gamma_R}|f(z)|\; 2\int_0^{\pi/2} e^{-\lambda R\sin\theta}\mathrm{d}\theta \\
&\overset{(E.31)}{\leq} 2R\sup_{z\in\gamma_R}|f(z)|\int_0^{\pi/2} e^{-2\lambda R\theta/\pi}\mathrm{d}\theta = \frac{\pi}{\lambda}\left(1-e^{-\lambda R}\right)\sup_{z\in\gamma_R}|f(z)| \\
&\overset{R\to\infty}{\longrightarrow} 0\,.
\end{aligned}$$

One likewise shows that the integral along $\bar{\gamma}_R$ in the lower half plane vanishes with $\mu < 0$. □

Example applications of Jordan's lemma

(1) Let α be a positive real constant and

$$I = \int_0^\infty \frac{\cos(\alpha x)}{1+x^2}\mathrm{d}x\,. \tag{E.32}$$

We compute I from the contour integral

$$\mathsf{Re}\int_\gamma \frac{e^{\mathrm{i}\alpha z}}{1+z^2}\mathrm{d}z \tag{E.33}$$

where $\gamma = \gamma_0 + \gamma_R$ is the contour shown in Fig. 41. The contribution along the baseline γ_0 gives us $2I$ as $R \to \infty$ whereas the contribution along the half circle γ_R vanishes by

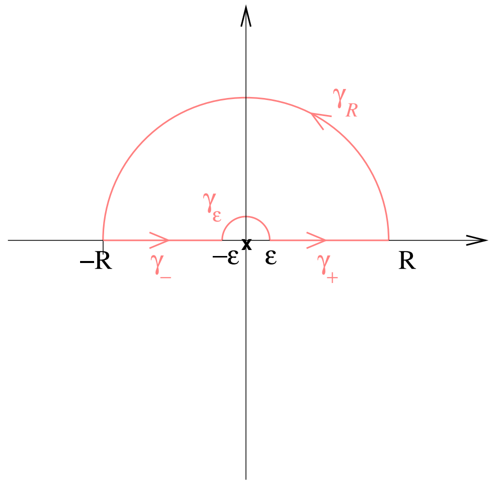

Figure 42: A closed semicircular contour with the origin cut out with a second semi-circle of small radius ϵ.

Jordan's lemma; we could derive the latter result without the lemma by simply noting the $\mathcal{O}(R^{-2})$ falloff of the integrand, but now that we have the lemma, we might just as well use it to save time.

The integrand of Eq. (E.33) has only one singular point inside γ, the simple pole at $z = \mathrm{i}$, so that

$$I = \frac{1}{2}\mathsf{Re}\left(\mathrm{i}2\pi \operatorname*{Res}_{z=\mathrm{i}} \frac{e^{\mathrm{i}\alpha z}}{1+z^2}\right) = \frac{1}{2}\mathsf{Re}\left(\mathrm{i}2\pi\frac{e^{-\alpha}}{2\mathrm{i}}\right) = \frac{\pi}{2}e^{-\alpha}\,.$$

Note the importance of using $e^{\mathrm{i}\alpha}$ in the integrand in Eq. (E.33) and taking the real part after evaluating the integral. If we had attempted to calculate $\int_\gamma \frac{\cos\alpha z}{1+z^2}\mathrm{d}z$ instead, the contribution along γ_R would not have vanished, because $\cos(\alpha z)$ is unbounded as $|z| \to \infty$; recall Eq. (A.17).

(2) This example is a little trickier. We wish to calculate

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\mathrm{d}x\,. \tag{E.34}$$

Here we really need Jordan's lemma, since the $1/R$ falloff of the integrand is not sufficient to conclude the vanishing of the integral along γ_R simply by power counting. Furthermore, closing the contour with the baseline γ_0 as in Fig. 41, would run across the origin $z = 0$. Even though the integrand $\frac{\sin x}{x}$ is regular at $z = 0$, we actually need to integrate $\frac{e^{\mathrm{i}z}}{z}$ in order to use Jordan's lemma and that has a non-removable singularity at $z = 0$. We bypass this problem by modifying the contour as shown in Fig. 42. We cut out a small piece $(-\epsilon, \epsilon)$ from the real axis and replace it with a small half circle. This enables us to write the integral I as

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}\mathrm{d}x = \lim_{\substack{\epsilon\to 0\\ R\to\infty}}\left(\int_{-R}^{-\epsilon}\frac{\sin x}{x}\mathrm{d}x + \int_{\epsilon}^{R}\frac{\sin x}{x}\mathrm{d}x\right) = \mathsf{Im}\left[\lim_{\substack{\epsilon\to 0\\ R\to\infty}}\left(\int_{-R}^{-\epsilon}\frac{e^{\mathrm{i}x}}{x}\mathrm{d}x + \int_{\epsilon}^{R}\frac{e^{\mathrm{i}x}}{x}\mathrm{d}x\right)\right]\,.$$

The resulting closed contour $\gamma = \gamma_- + \gamma_\epsilon + \gamma_+ + \gamma_R$ in Fig. 42 now encloses no singularity of the integrand and the round trip integral is zero and, therefore,

$$I_{\epsilon,R} := \int_{\gamma_-} \frac{e^{\mathrm{i}z}}{z}\mathrm{d}z + \int_{\gamma_+} \frac{e^{\mathrm{i}z}}{z}\mathrm{d}z = -\int_{\gamma_\epsilon} \frac{e^{\mathrm{i}z}}{z}\mathrm{d}z - \int_{\gamma_R} \frac{e^{\mathrm{i}z}}{z}\mathrm{d}z\,. \tag{E.35}$$

The last term on the right-hand side vanishes in the limit $R \to \infty$ by Jordan's lemma. Along γ_ϵ, we parametrize $\theta \mapsto z(\theta) = \epsilon e^{\mathrm{i}\theta}$ and find

$$\int_{\gamma_\epsilon} \frac{e^{\mathrm{i}z}}{z}\mathrm{d}z = \int_\pi^0 \frac{1+\mathcal{O}(\epsilon)}{\epsilon e^{\mathrm{i}\theta}} \mathrm{i}\epsilon e^{\mathrm{i}\theta}\mathrm{d}\theta \xrightarrow{\epsilon\to 0} -\mathrm{i}\pi\,.$$

In the simultaneous limit $\epsilon \to 0$, $R \to 0$, Eq. (E.35) therefore gives us

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\mathrm{d}x = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \mathsf{Im}(I_{\epsilon,r}) = -\mathsf{Im}(-\mathrm{i}\pi) = \pi\,. \tag{E.36}$$

One can likewise calculate

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\mathrm{d}x = \pi\,, \tag{E.37}$$

but this result is more easily derived with the following trick,

$$\begin{aligned}
& I(a) := \int_{-\infty}^{\infty} \frac{\sin^2(ax)}{x^2}\mathrm{d}x \\
\Rightarrow\quad & \frac{\mathrm{d}I}{\mathrm{d}a} = \int_{-\infty}^{\infty} \frac{2x\sin(ax)\,\cos(ax)}{x^2}\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\sin(2ax)}{x}\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\sin y}{y}\mathrm{d}y = \pi \\
\Rightarrow\quad & I(a) = \pi a\,, \qquad \text{since} \quad I(0) = 0\,.
\end{aligned} \tag{E.38}$$

F Transform theory

In this section, we will discuss the *Fourier transform* and the *Laplace transform*. Both methods find wide ranging applications in physics, engineering, numerical methods, image compression and other areas. The Fourier transform has already been dealt with in the IB Methods lecture, but our new tool of contour integration will enable us to compute more Fourier transforms than before. The Laplace transform is a purely real operation and you may have wondered why it is not covered in IB Methods. The reason is the *inverse* Laplace transform which requires the computation of a comparatively complicated contour integral.

F.1 Fourier transforms

It turns out rather complicated to state the conditions for the existence of a Fourier transform in all generality [3]; a common requirement used in practice is absolute integrability, though we will rapidly note an important exception.

Def. : Let $f:\mathbb{R}\to\mathbb{C}$ be absolutely integrable, i.e. $\int_{-\infty}^{\infty}|f(x)|\,\mathrm{d}x$ exists, have bounded variation and a finite number of discontinuities. Then its *Fourier transform* and its inverse are defined as

$$\tilde{f}(k)=\mathcal{F}\big[f(x)\big](k) = \int_{-\infty}^{\infty} f(x)e^{-\mathrm{i}kx}\mathrm{d}x\,, \tag{F.1}$$

$$f(x)=\mathcal{F}^{-1}\big[\tilde{f}(k)\big](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{\mathrm{i}kx}\mathrm{d}k\,. \tag{F.2}$$

A few comments are in order.

1. We will employ, whenever possible, the notation $\tilde{f}(k)$ and $f(x)$, but sometimes the use of the operator $\mathcal{F}$ turns out convenient if not necessary; this operator notation may then be abbreviated to $\tilde{f}=\mathcal{F}[f]$ or $\tilde{f}(k)=\mathcal{F}[f](k)$. In the literature, you may also find the version $\tilde{f}(k)=\mathcal{F}[f(x)]$, but this is misleading as the left- and right-hand sides suggest dependency on k and x, respectively, which doesn't match. We will not use this latter notation.

2. There are many ways to shift around $\pm$ signs and factors of 2π in the definitions (F.1) and (F.2). For instance, the factors $e^{-\mathrm{i}kx}$ and $e^{\mathrm{i}kx}$ in the integrands of (F.1) and (F.2) may be swapped. We may also use a factor $1/\sqrt{2\pi}$ in front of *both* integrals rather than $1/2\pi$ in front of one of them. All of these variations provide legitimate Fourier transforms and, needless to say, every single variation is in use somewhere in the literature.

3. The variable pair (x,k) as used in Eqs. (F.1), (F.2) is commonly used for Fourier transforms on spatial domains; the *wavenumber* $k=2\pi/\lambda$ is a measure for the inverse wavelength λ. For functions of time, it is common to use the variable pair (t,ω), where the angular frequency is related to the period by $\omega=2\pi/T$. Sometimes, and especially in numerical applications [5], one works with the frequency $\nu=\omega/(2\pi)$ instead. This has the added

benefit of eliminating all factors of 2π in Eqs. (F.1) and (F.2). Here, we will stick to ω, however, as there is already enough trouble with conventions.

4. Sometimes we are interested in taking the Fourier transform of functions that do not satisfy the requirement of absolute integrability, e.g. $f(x) = 1$. These can be handled with a generalized concept of functions, the so-called *distributions*. The most common example is the δ distribution

$$f(x) = 1 \qquad \Rightarrow \qquad \tilde{f}(k) = 2\pi\delta(k) = \int_{-\infty}^{\infty} 1\, e^{-\mathrm{i}kx}\mathrm{d}x\,. \tag{F.3}$$

5. You may recall from IB Methods that the Fourier transform returns the average of the left and right limits of a function at points of discontinuity. Furthermore, we have been unnecessarily restrictive in Eqs. (F.1), (F.2): instead of $\int_{-\infty}^{\infty}$, we only need the so-called Cauchy principal value of the integrals. Let us briefly define and discuss this subtlety.

Def. : The *Cauchy principal value* of an integral $\int_{-\infty}^{\infty} g(x)\mathrm{d}x$ is defined as

$$⨍_{-\infty}^{\infty} g(x)\mathrm{d}x := \lim_{R\to\infty}\int_{-R}^{R} g(x)\mathrm{d}x\,. \tag{F.4}$$

Alternative notations include PV $\int$, p.v. $\int$, P $\int$.

The key point is that both limits are approached in the same manner, which ensures that many functions have Cauchy principle value integrals even if they do not have "normal" ones. For example,

$$⨍_{-\infty}^{\infty} \frac{x}{1+x^2}\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R}\frac{x}{1+x^2}\mathrm{d}x = 0\,, \tag{F.5}$$

by symmetry of the integrand, whereas

$$\lim_{R\to\infty}\int_{-R^2}^{R}\frac{x}{1+x^2}\mathrm{d}x = \lim_{R\to\infty}\left[\frac{1}{2}\ln\!\big(1+x^2\big)\right]_{x=-R^2}^{R} = \frac{1}{2}\lim_{R\to\infty}\frac{\ln(1+R^2)}{\ln(1+R^4)} = -\infty\,. \tag{F.6}$$

The correct version of Eqs. (F.1), (F.2) is therefore given by

$$\frac{1}{2}\big[\tilde{f}(k^+) + \tilde{f}(k^-)\big] \;=\; ⨍_{-\infty}^{\infty} f(x)e^{-\mathrm{i}kx}\mathrm{d}x\,, \tag{F.7}$$

$$\frac{1}{2}\big[f(x^+) + f(x^-)\big] \;=\; \frac{1}{2\pi}⨍_{-\infty}^{\infty}\tilde{f}(k)e^{\mathrm{i}kx}\mathrm{d}k\,. \tag{F.8}$$

Here, the left-hand side expressions explicitly state that the Fourier transform and its inverse return the average of the respective left and right limits $f(x^+)$, $f(x^-)$ or $\tilde{f}(k^+)$, $\tilde{f}(k^-)$ at points of discontinuity, and the right-hand side shows that we only need the Cauchy principal value. Of course, we have always taken into account both effects when computing Fourier transforms

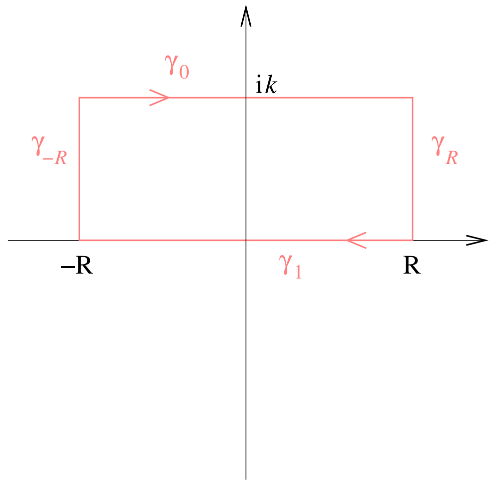

Figure 43: A closed rectangular contour consisting of four segments, γ_1 along the real axis, the parallel γ_0 through $\mathrm{i}k$ and the vertical segmants at $x = \pm R$.

in practice. Now that we are fully aware of what we are doing and why, we can safely return to the simple notation of Eqs. (F.1) and (F.2).

Let us calculate some Fourier transforms then, using contour integration.

Examples of Fourier integrals

(1) The Fourier transform of $f(x) = e^{-x^2/2}$ is given by

$$\begin{aligned}\tilde{f}(k) &= \int_{-\infty}^{\infty} e^{-x^2/2} e^{-\mathrm{i}kx}\mathrm{d}x = \int_{-\infty}^{\infty} e^{-(x+\mathrm{i}k)^2/2} e^{-k^2/2}\mathrm{d}x \qquad \Big|\; z = x + \mathrm{i}k \\ &= e^{-k^2/2} \int_{-\infty+\mathrm{i}k}^{\infty+\mathrm{i}k} e^{-z^2/2}\mathrm{d}z\,.\end{aligned}$$

We evaluate this integral using the rectangular contour shown in Fig. 43; we need the contribution along γ_0. In the limit $R \to \infty$, the integrals along γ_R and γ_{-R} vanish since $e^{-R^2} \to 0$. Furthermore, e^{-z^2} has no singularities inside the contour (nor anywhere else), so that $\int_{\gamma_0+\gamma_R+\gamma_1+\gamma_{-R}} e^{-z^2/2}\mathrm{d}z = 0$ and, therefore,

$$\int_{\gamma_0} e^{-z^2/2}\mathrm{d}z = -\int_{\gamma_1} e^{-z^2/2}\mathrm{d}z = +\int_{-\infty}^{\infty} e^{-t^2/2}\mathrm{d}t = \sqrt{2\pi}$$

$$\Rightarrow \quad \tilde{f}(k) = \sqrt{2\pi}e^{-k^2/2}\,, \tag{F.9}$$

where we have used the Gauss integral $\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$.

(2) For the calculation of inverse Fourier transforms, we often use semi-circular contours and employ Jordan's lemma in either of its versions (E.28) or (E.29). Let us assume

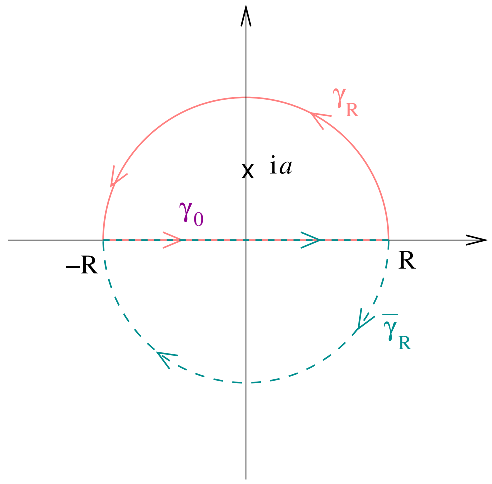

Figure 44: γ_R and $\bar{\gamma}_R$ denote the semi-circles in the upper and lower half plane, respectively. A closed contour is obtained in either case by adding the straight segement γ_0.

we know the Fourier transform

$$\tilde{f}(k) = \frac{1}{a + \mathrm{i}k} \qquad \text{with} \;\; a > 0 \in \mathbb{R}\,,$$

and wish to reconstruct its original $f(x)$ from

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{\mathrm{i}kx} \mathrm{d}k = \int_{-\infty}^{\infty} \frac{1}{a+\mathrm{i}k} e^{\mathrm{i}kx} \mathrm{d}k\,.$$

Recalling Jordan's lemma, we suspect that we need to distinguish the two cases $x > 0$ and $x < 0$ and will require Eqs. (E.28) and (E.29), respectively. Let us start with $x > 0$, where Jordan's lemma tells us to use the upper semi-circle γ_R in Fig. 44. γ_R encloses the only singularity of the integrand at $k = \mathrm{i}a$, which is a simple pole with residue

$$\operatorname*{Res}_{k=\mathrm{i}a} \frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k} = \lim_{k\to\mathrm{i}a} \frac{k - \mathrm{i}a}{a+\mathrm{i}k} e^{\mathrm{i}kx} = -\mathrm{i}e^{-xa}\,.$$

We thus obtain for $x > 0$, using Jordan's lemma (with $\lambda = x$),

$$\lim_{R\to\infty} \oint_{\gamma_R+\gamma_0} \frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\mathrm{d}k = \mathrm{i}2\pi(-\mathrm{i}e^{-ax}) \stackrel{!}{=} \underbrace{\lim_{R\to\infty}\int_{\gamma_R} \frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\mathrm{d}k}_{=0} + \lim_{R\to\infty}\int_{\gamma_0} \frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\mathrm{d}k$$

$$\Rightarrow \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\mathrm{d}k = \frac{1}{2\pi}\lim_{R\to\infty}\int_{\gamma_0} \frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\mathrm{d}k = e^{-ax}\,.$$

For $x < 0$, we employ Jordan's lemma using the semi-circle $\bar{\gamma}_R$ in the lower half-plane in Fig. 44. The resulting closed trajectory $\bar{\gamma}_R + \gamma_0$ encircles no singularity of $\tilde{f}(k)$ and we simply obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\mathrm{d}k = \lim_{R\to\infty}\int_{\gamma_0} \tilde{f}(k)e^{\mathrm{i}kx}\mathrm{d}k = \lim_{R\to\infty}\int_{\gamma_0} \tilde{f}(k)e^{\mathrm{i}kx}\mathrm{d}k + \underbrace{\lim_{R\to\infty}\int_{\bar{\gamma}_R} \tilde{f}(k)e^{\mathrm{i}kx}\mathrm{d}k}_{=0} = 0\,.$$

The original function is therefore

$$f(x) = \begin{cases} 0 & \text{for } \ x<0 \\ e^{-ax} & \text{for } \ x>0 \end{cases} . \tag{F.10}$$

Just to check, the Fourier transform of this function is

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-\mathrm{i}kx}\mathrm{d}x = \int_0^{\infty} e^{-ax-\mathrm{i}kx}\mathrm{d}x = \frac{1}{-a-\mathrm{i}k}\left[e^{-ax-\mathrm{i}kx}\right]_{x=0}^{\infty} = \frac{1}{a+\mathrm{i}k}\,,$$

as expected.

F.2 Laplace transforms

The main shortcoming of Fourier transforms is the restriction to absolutely integrable functions. Many systems encountered in the real world involve growing functions, such as e^t, which we cannot manage with Fourier transforms, not even by resorting to distributional theory. The Laplace transform provides a handle to treat such functions in a manner analogous to the Fourier domain. Furthermore, we will see that Laplace transforms have a natural way to incorporate boundary conditions, which makes them very suitable for solving ordinary differential equations. These features of the Laplace transform come at a price, but a relatively small one: They are restricted to functions $f(t)$ that vanish for $t<0$.

F.2.1 Definition of the Laplace transform

Def. : Let $f(t)$ be a function defined for all $t \geq 0$. The *Laplace transform* of $f(t)$ is given by the integral

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^{\infty} f(t)e^{-st}\mathrm{d}t\,, \qquad s\in\mathbb{C}\,, \tag{F.11}$$

provided that the integral exists.

Again, we shall not dwell too long on a general formulation of necessary or sufficient conditions for the existence of a Laplace transform; for our practical purposes it is sufficient that the integral (F.11) exists for functions $f(t)$ that grow no more than exponential. The range of notational conventions is less confusing than for Fourier transforms, but you may come across literature where the Laplace domain coordinate is called p instead of s and some authors denote the Laplace transform by $\hat{f}$; we use the capital F here to have as distinct a symbol from the Fourier transform as sensibly possible. Finally, note the relation

$$F(s) = \tilde{f}(-\mathrm{i}s)\,, \tag{F.12}$$

between the two transforms for functions f with $f(t) = 0$ for $t < 0$.

In contrast to the Fourier transform, we have not dealt with Laplace transforms in IB Methods, so we should start with a few examples to acquire some intuition about this animal.

Examples of Laplace transforms

(1) By integration, we directly find

$$\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st}\mathrm{d}t = \frac{1}{s}\,. \tag{F.13}$$

This looks innocent and trivial enough, but examplifies a subtlety we will encounter more frequenyly in our discussion of Laplace transforms. Clearly, the integral is only defined for $\mathsf{Re}(s) > 0$. In practice, however, we may at times use the resulting function $F(s)$ over the entire range where it is defined; in the present case, this would be for all $s \in \mathbb{C} \setminus \{0\}$. This process is called *analytic continuation* [4] and finds applications in many areas of physics and mathematics. In general relativity, for example, it enables us to extend the famous Schwarzschild solution of a static black hole to *Kruskal-Szekeres* coordinates which turn out vital to comprehensively understand the Schwarzschild spacetime; see e.g. [6]. A thorough treatment of analytic continuation is beyond the scope of these notes, but we will bear it in mind in the discussion of some of the examples in this section.

(2) We integrate by parts to find

$$\mathcal{L}\{t\}(s) = \int_0^\infty t e^{-st}\mathrm{d}t = \left[t\frac{-1}{s}e^{-ts}\right]_{t=0}^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\mathrm{d}t = 0 - \frac{1}{s^2}\left[e^{-st}\right]_{t=0}^\infty = \frac{1}{s^2}\,. \tag{F.14}$$

(3) For a constant λ, we can directly integrate

$$\mathcal{L}\{e^{\lambda t}\}(s) = \int_0^\infty e^{(\lambda-s)t}\mathrm{d}t = \frac{1}{\lambda - s}\left[e^{(\lambda-s)t}\right]_{t=0}^\infty = \frac{1}{s-\lambda}\,. \tag{F.15}$$

Once again, the integral only exists if $\mathsf{Re}(s) > \mathsf{Re}(\lambda)$, but we can analytically continue the function $1/(s-\lambda)$ for all $s \in \mathbb{C}$ except $s = \lambda$.

(4) Using the previous example, we find in particular that

$$\mathcal{L}\{\sin t\}(s) = \mathcal{L}\left\{\frac{1}{2\mathrm{i}}(e^{\mathrm{i}t} - e^{-\mathrm{i}t})\right\}(s) = \frac{1}{2\mathrm{i}}\left(\frac{1}{s-\mathrm{i}} - \frac{1}{s+\mathrm{i}}\right) = \frac{1}{s^2+1}\,. \tag{F.16}$$

F.2.2 Properties of the Laplace transform

The Laplace transform obeys a number of rules analogous to the Fourier transform.

Proposition:

1. *Linearity*: For constants $\alpha,\ \beta \in \mathbb{C}$,

$$\mathcal{L}\{\alpha f + \beta g\} = \alpha\mathcal{L}\{f\} + \beta\mathcal{L}\{g\}\,. \tag{F.17}$$

2. *Translation*: For a constant $t_0 \in \mathbb{R}$,

$$\mathcal{L}\{f(t-t_0)H(t-t_0)\}(s) = e^{-st_0}F(s)\,, \tag{F.18}$$

   where H denotes the Heaviside stepfunction

$$H(x) = \begin{cases} 0 & \text{for} \;\; x<0 \\ 1 & \text{for} \;\; x>0 \end{cases}\,,$$

3. *Scaling*: For a constant $\lambda > 0$,

$$\mathcal{L}\{f(\lambda t)\}(s) = \frac{1}{\lambda}F\left(\tfrac{s}{\lambda}\right)\,. \tag{F.19}$$

4. *Shifting*: For a constant $s_0 \in \mathbb{C}$,

$$\mathcal{L}\{e^{s_0 t}f(t)\}(s) = F(s-s_0)\,. \tag{F.20}$$

5. *Transform of a derivative*:

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0)\,. \tag{F.21}$$

   By repeated application of this formula, we find

$$\mathcal{L}\{f''(t)\}(s) = s\mathcal{L}\{f'(t)\}(s) - f'(0) = s^2F(s) - sf(0) - f'(0)\,, \tag{F.22}$$

   and so forth.

(to be continued)

(continued)

6. *Derivative of a transform*:

$$F'(s) = \mathcal{L}\{-tf(t)\}(s)\,. \tag{F.23}$$

We usually use this equation in its inverse form; suppose we know a pair $f(t) \leftrightarrow F(s)$, then we easily evaluate $F'(s)$ and thus obtain the Laplace transform of the function $-tf(t)$. E.g $\mathcal{L}\{t^2\}(s) = 2/s^3$. More generally, we have

$$F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}(s)\,. \tag{F.24}$$

7. *Asymptotic limits*:

$$\lim_{s\to\infty} sF(s) = f(0)\,, \tag{F.25}$$

$$\lim_{s\to 0} sF(s) = f(\infty)\,, \tag{F.26}$$

provided the limit $\lim_{t\to\infty} f(t)$ exists.

Proof.

1. Linearity of the Laplace transform directly follows from linearity of the integral operation.
2. Setting $\tilde{t} := t - t_0$, we obtain

$$\int_0^\infty f(t-t_0)e^{-st}\mathrm{d}t = \int_{-t_0}^\infty f(\tilde{t})e^{-s(\tilde{t}+t_0)}\mathrm{d}\tilde{t} = e^{-st_0}\int_{-t_0}^\infty f(\tilde{t})e^{-s\tilde{t}}\mathrm{d}\tilde{t}$$
$$\Rightarrow \int_0^\infty f(t-t_0)\,H(t-t_0)e^{-st}\mathrm{d}t = e^{-st_0}\int_{-t_0}^\infty f(\tilde{t})\,H(\tilde{t})e^{-s\tilde{t}}\mathrm{d}\tilde{t} = \int_0^\infty f(\tilde{t})e^{-s\tilde{t}}\mathrm{d}\tilde{t} = e^{-st_0}F(s)\,.$$

3. Defining $\tilde{t} = \lambda t$, $\mathrm{d}\tilde{t} = \lambda \mathrm{d}t$, we find

$$\int_0^\infty f(\lambda t)e^{-st}\mathrm{d}t = \int_0^\infty f(\tilde{t})e^{-\frac{s}{\lambda}\tilde{t}}\frac{\mathrm{d}\tilde{t}}{\lambda} = \frac{1}{\lambda}F\left(\tfrac{s}{\lambda}\right).$$

4. We directly see

$$\int_0^\infty e^{s_0 t}f(t)e^{-st}\mathrm{d}t = F(s-s_0)\,.$$

5. Integrating by parts, we obtain

$$\int_0^\infty f'(t)e^{-st}\mathrm{d}t = \left[f(t)e^{-st}\right]_{t=0}^\infty - \int_0^\infty -sf(t)e^{-st}\mathrm{d}t = sF(s) - f(0)\,.$$

Once again, we encounter the subtlety of analytic continutation. The proof breaks down if the integral $\int_0^\infty f(t)e^{-st}\mathrm{d}t$ does not exist. The relation still holds for analytically continued Laplace transforms, though. Consider, for example,

$$\begin{aligned} & f(t) = e^{2t} && \Rightarrow \quad F(s) = \frac{1}{s-2} \quad \text{for} \quad \mathsf{Re}(s) > 2 \\ \Rightarrow \quad & f'(t) = 2e^{2t} && \Rightarrow \quad \mathcal{L}\{f'(t)\}(s) = \frac{2}{s-2} \overset{!}{=} \frac{s}{s-2} - 1 = sF(s) - f(0)\,, \end{aligned}$$

so Eq. (F.21) holds for all $s \neq 2$.

Note how Eqs. (F.21), (F.22) relate the Laplace transform of derivatives to the boundary values $f(0)$, $f'(0)$ etc. This is the key property of the Laplace transform in solving ordinary differential equations.

6. Differentiating Eq. (F.11) with respect to s yields

$$F'(s) = \int_0^\infty -tf(t)e^{-st}\mathrm{d}t = \mathcal{L}\{-tf(t)\}\,, \tag{F.27}$$

and Eq. (F.24) follows from multiple application of $\mathrm{d}/\mathrm{d}s$.

7. From Eq. (F.21), we obtain

$$sF(s) = f(0) + \int_0^\infty f'(t)e^{-st}\mathrm{d}t\,.$$

By requirement, the limit $\lim_{t\to\infty} f(t)$ exists, so $f(t)$ and $f'(t)$ do not grow faster than exponential. For $s \to \infty$, the integral on the right-hand side therefore vanishes and we obtain Eq. (F.25). In the limit $s \to 0$, on the other hand, we have $e^{-st} \to 1$, the inegral just becomes $\int_0^\infty f'(t)\mathrm{d}t$, and we recover Eq. (F.26).

□

Example applications of the properties of the Laplace transform

(1) Let us exploit Eq. (F.23) to convert a known pair of Laplace transforms, namely Eq. (F.16), into a new one,

$$\mathcal{L}\{t\sin t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\sin t\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}\,.$$

(2) From Eq. (F.13), we know $\mathcal{L}\{1\}(s) = 1/s$. Applying Eq. (F.24) to this result, we find

$$\mathcal{L}\{t^n\}(s) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\frac{1}{s} = \frac{n!}{s^{n+1}}\,.$$

More generally, *Euler's Gamma function* is defined for complex n with $\mathsf{Re}(n) > 0$ as

$$\Gamma(n) := \int_0^\infty e^{-t}t^{n-1}\mathrm{d}t\,. \tag{F.28}$$

This function generalizes factorials to non-integer values since for $n \in \mathbb{Z},\ n > 0$,

$$\Gamma(n) = \mathcal{L}\{t^{n-1}\}(s=1) = (n-1)!\,.$$

(3) We can extend the first example using Eq. (F.19). For $a > 0$, we find

$$\mathcal{L}\{\sin(at)\}(s) \overset{(F.19)}{=} \frac{1}{a}\mathcal{L}\{\sin t\}\left(\tfrac{s}{a}\right) = \frac{1}{a}\frac{1}{\frac{s^2}{a^2}+1} = a\frac{1}{s^2+a^2}$$

$$\Rightarrow \quad \mathcal{L}\left\{\frac{\sin(at)}{a}\right\}(s) = \frac{1}{s^2+a^2}\,. \tag{F.29}$$

(4) With the Laplace transform of the sine function, it is quite easy to get that of the cosine,

$$\mathcal{L}\{e^{\mathrm{i}at}\}(s) \overset{(F.15)}{=} \frac{1}{s-\mathrm{i}a} = \frac{s+\mathrm{i}a}{s^2+a^2} \overset{!}{=} \mathcal{L}\{\cos(at)+\mathrm{i}\sin(at)\}(s)$$

$$\Rightarrow \quad \mathcal{L}\{\cos(at)\}(s) = \mathcal{L}\{e^{\mathrm{i}at}\}(s) - \mathrm{i}\mathcal{L}\{\sin(at)\}(s) = \frac{s+\mathrm{i}a}{s^2+a^2} - \mathrm{i}\frac{a}{s^2+a^2}$$

$$\Rightarrow \quad \mathcal{L}\{\cos(at)\}(s) = \frac{s}{s^2+a^2}\,. \tag{F.30}$$

F.2.3 The inverse Laplace transform

So far, our discussion of the Laplace transform hardly merits its place in a Complex Methods course; this is going to change, however, now that we discuss its inverse.

Proposition: Given a function $F(s)$, we can compute its inverse Laplace transform $f(t)$ from the *Bromwich inversion formula*,

$$f(t) = \frac{1}{\mathrm{i}2\pi}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} F(s)e^{st}\mathrm{d}s\,, \tag{F.31}$$

where α is a real constant chosen such that the Bromwich inversion contour γ consisting of the points $\{s \in \mathbb{C} \;:\; \mathrm{Re}(s) = \alpha\}$ lies to the *right* of all singularities of $F(s)$.

Proof. Since $f(t)$ has a Laplace transform, we have $f(t) = 0$ for $t < 0$ and f does not grow faster than exponential. We can therefore choose an $\alpha \in \mathbb{R}$ such that

$$g(t) = f(t)e^{-\alpha t}\,,$$

decays exponentially as $t \to \infty$ and therefore has a Fourier transform,

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\alpha t}e^{-\mathrm{i}\omega t}\mathrm{d}t = F(\alpha+\mathrm{i}\omega)\,.$$

This enables us to apply the inverse Fourier transform, whence

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha+\mathrm{i}\omega)e^{\mathrm{i}\omega t}\mathrm{d}\omega\,.$$

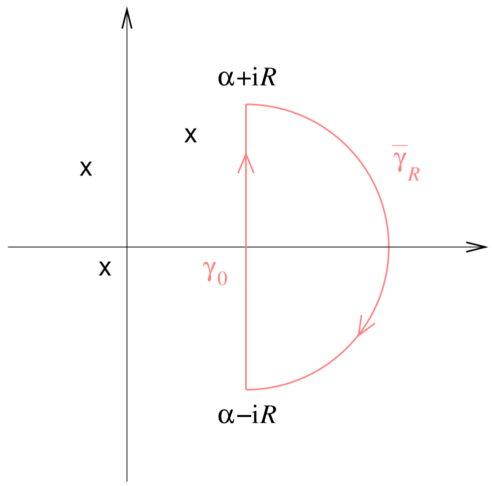

Figure 45: The closed semi-circular contour $\gamma_0 + \bar{\gamma}_R$ is located to the right of all singularities of the function $F(s)$ marked by $\times$ symbols.

Finally, we substitute $s = \alpha + \mathrm{i}\omega \quad \Rightarrow \quad \mathrm{d}\omega = \mathrm{d}s/\mathrm{i}$, and obtain

$$f(t)e^{-\alpha t} = \frac{1}{2\pi \mathrm{i}} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} F(s) e^{(s-\alpha)t} \mathrm{d}s \qquad \Big| \cdot e^{\alpha t}$$

$$\Rightarrow \quad f(t) = \frac{1}{\mathrm{i}2\pi} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} F(s) e^{st} \mathrm{d}s \,.$$

The additional requirement that the contour of points with $\mathrm{Re}(s) > \alpha$ lies to the right of all singularities of $F(s)$ fixes a constant of integration and thus ensures that $f(t) = 0$ for $t < 0$; we will see how this works in the next proof. □

In practice, the Laplace transform and its inverse are often applied to functions with a finite number of singularities. This simplifies the inverse Laplace transform considerably.

Proposition: Let $F(s)$ be the Laplace transform of a function $f(t)$ and have only a finite number of isolated singularities at points $s_k \in \mathbb{C}$, $k = 1, \ldots, n$. Let furthermore $F(s) \to 0$ as $|s| \to \infty$. Then $f(t) = 0$ for $t < 0$ and for $t > 0$,

$$f(t) = \sum_{k=1}^{n} \operatorname*{Res}_{s=s_k} \left(F(s) e^{st} \right) . \tag{F.32}$$

Proof. Let us first consider the case $t < 0$ and construct the contour $\gamma = \gamma_0 + \bar{\gamma}_R$ as shown in Fig. 45. If $F(s) = o(s^{-1})$, we directly obtain

$$\left| \int_{\bar{\gamma}_R} F(s) e^{st} \mathrm{d}s \right| \leq \pi R e^{\alpha t} \sup_{s \in \bar{\gamma}_R} \left| F(s) \right| \quad \xrightarrow{R \to \infty} \quad 0 \,.$$

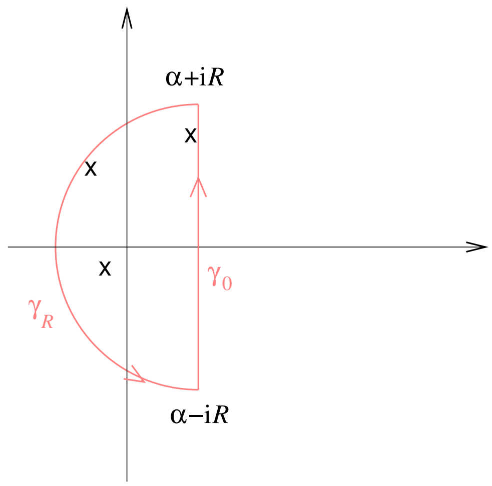

Figure 46: The closed semi-circular contour $\gamma_R + \gamma_0$ closes γ_0 on the left and encircles all singularities of $F(s)$.

Note that we have used here that $\alpha \leq \mathsf{Re}(s)$ along $\bar{\gamma}_R$ and for $t < 0$ we therefore get $\alpha t \geq \mathsf{Re}(s)t$ and $|e^{\alpha t}| \geq |e^{st}|$ along $\bar{\gamma}_R$. One can show, using a modified version of Jordan's lemma, that this result still holds if $F(s)$ approaches 0 less rapidly than $o(s^{-1})$. We skip the details and merely note that in any case,

$$\lim_{R\to\infty} \int_{\bar{\gamma}_R} F(s)e^{st}\mathrm{d}s = 0\,,$$

and therefore $\int_{\gamma_0} F(s)e^{st}\mathrm{d}s = \int_{\gamma_0+\bar{\gamma}_R} F(s)e^{st}\mathrm{d}s$ in the limit $R \to \infty$. But our contour does not enclose any singularitities of the integrand, so that by Cauchy's Theorem (C.15), we get $f(t) = 0$ for $t < 0$ as expected.

For $t > 0$, we close the contour to the left of $\mathsf{Re}(s) = \alpha$ as shown in Fig. 46. Since $F(s)$ only has a finite number of isolated singularities, we will encircle the whole lot as $R \to \infty$. As before, one shows that the contribution along γ_R vanishes as $R \to \infty$. We can then use the residue theorem to compute $f(t)$ according to the Bromwich inversion formula,

$$f(t) = \frac{1}{\mathrm{i}2\pi} \lim_{R\to\infty} \int_{\gamma_0} F(s)e^{st}\mathrm{d}s = \frac{1}{\mathrm{i}2\pi} \lim_{R\to\infty} \int_{\gamma_0+\gamma_R} F(s)e^{st}\mathrm{d}s = \sum_{k=1}^{n} \operatorname*{Res}_{s=s_k} \left(F(s)e^{st}\right),$$

□

Let us gather some practical experience with this method.

Examples of inverting the Laplace transform

(1) Consider

$$F(s) = \frac{1}{s-1}\,,$$

which has a simple pole at $s = 1$, so that we choose $\alpha > 1$ for compatibility with the Bromwich inversion formula. Clearly, $F(s) \to 0$ as $|s| \to \infty$ and we can use Eq. (F.32) to find

$$f(t) = \operatorname*{Res}_{s=1} \left(\frac{e^{st}}{s-1} \right) = e^t \,, \tag{F.33}$$

in agreement with our previous result in Eq. (F.15).

(2) The function $F(s) = s^{-n}$, $n \in \mathbb{N}$, has a pole of order n at $s = 0$. Again, the requirements for using Eq. (F.32) are met. We therefore choose $\alpha > 0$ and obtain

$$f(t) = \operatorname*{Res}_{s=0} \left(\frac{e^{st}}{s^n} \right) \overset{(D.19)}{=} \lim_{s\to 0} \left[\frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}s^{n-1}} e^{st} \right] = \frac{t^{n-1}}{(n-1)!} \,. \tag{F.34}$$

The cases $n = 1$, $n = 2$ correspond to the transform pairs we have already calculated in Eqs. (F.13), (F.14).

F.2.4 Solving differential equations with the Laplace transform

The Laplace transform can greatly simplify differential equations, and sometimes even convert them into algebraic equations. This is best seen in practice.

Examples of solving differential equations

(1) Consider the ordinary differential equation (ODE)

$$\ddot{f}(t) - 3\dot{f}(t) + 2f(t) = 4e^{2t} \,, \qquad f(0) = -3 \,, \quad \dot{f}(0) = 5 \,. \tag{F.35}$$

Using Eqs. (F.21), (F.22) on the left and our result (F.15) on the right-hand side, we convert this ODE into an algebraic equation for $F(s)$,

$$\begin{aligned}
& s^2F(s) - sf(0) - \dot{f}(0) - 3[sF(s) - f(0)] + 2F(s) = \frac{4}{s-2} \\
\Rightarrow\quad & (s^2 - 3s + 2)F(s) = \frac{4}{s-2} + sf(0) + \dot{f}(0) - 3f(0) = \frac{4}{s-2} - 3s + 14 \\
\Rightarrow\quad & (s-2)(s-1)F(s) = \frac{4}{s-2} - 7s + 14 + 4s - 4 + 4\frac{s-2}{s-2} = -7(s-2) + 4(s-1) + 4\frac{s-1}{s-2} \\
\Rightarrow\quad & F(s) = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2} \\
\Rightarrow\quad & f(t) = -7e^t + 4e^{2t} + 4te^{2t} \,,
\end{aligned}$$

where we have used Eq. (F.15) and, for the last term, Eq. (F.23) in the form

$$\mathcal{L}\{te^{2t}\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{e^{2t}\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s-2} = \frac{1}{(s-2)^2} \,.$$

(2) Consider the ODE

$$t\ddot{f}(t) - t\dot{f}(t) + f(t) = 2\,, \qquad f(0) = 2\,, \quad \dot{f}(0) = -1\,.$$

We transform into the Laplace domain using

$$\begin{aligned}
\mathcal{L}\{t\dot{f}(t)\}(s) &\overset{(F.23)}{=} -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\dot{f}(t)\}(s) \overset{(F.21)}{=} -\frac{\mathrm{d}}{\mathrm{d}s}[sF(s) - f(0)] = -sF'(s) - F(s) \\
\mathcal{L}\{t\ddot{f}(t)\}(s) &\overset{(F.23)}{=} -\frac{\mathrm{d}}{\mathrm{d}s}\left[\mathcal{L}\{\ddot{f}(t)\}(s)\right] \overset{(F.22)}{=} -\frac{\mathrm{d}}{\mathrm{d}s}\left[s^2F(s) - sf(0) - \dot{f}(0)\right] \\
&= -s^2F'(s) - 2sF(s) + f(0) \\
\mathcal{L}\{2\}(s) &\overset{(F.13)}{=} \frac{2}{s}\,.
\end{aligned}$$

The transform of the entire ODE therefore gives

$$\begin{aligned}
& -s^2F'(s) - 2sF(s) + f(0) + sF'(s) + F(s) + F(s) = \frac{2}{s} \\
\Rightarrow\quad & -s(s-1)F'(s) - 2(s-1)F(s) = \frac{2}{s} - 2 = \frac{2-2s}{s} = -(s-1)\frac{2}{s} \\
\Rightarrow\quad & sF'(s) + 2F(s) = \frac{2}{s}\,.
\end{aligned}$$

We can solve this comparatively simple ODE with an integrating factor; we could formally compute this factor I from $(2I) = (sI)'$, but here we can see directly that

$$\begin{aligned}
& sF' + 2F = \frac{1}{s}(s^2F)' \overset{!}{=} \frac{2}{s} \\
\Rightarrow\quad & s^2F = 2s + A\,, \qquad A = \text{const} \\
\Rightarrow\quad & F = \frac{2}{s} + \frac{A}{s^2}\,,
\end{aligned}$$

and using Eqs. (F.13), (F.14) for the inversion,

$$f(t) = 2 + At\,. \tag{F.36}$$

Finally, the boundary condition determines $A = -1$.

(3) We can also use the Laplace transform to solve partial differential equations (PDEs). Consider the heat equation

$$\frac{\partial}{\partial t}f(t,x) = \frac{\partial^2}{\partial x^2}f(t,x)\,,$$

on the domain $0 \le x \le 2$, $t \ge 0$ with initial and boundary conditions

$$f(t,0) = 0\,, \qquad f(t,2) = 0\,, \qquad f(0,x) = 3\sin(2\pi x)\,.$$

Physically, this can be regarded as a one-dimensional cross section of your room with some initial (admittedly weird) sinusoidal temperature profile bounded by two walls which are connected to a freezing exterior. Intuitively, we would expect the temperature profile to decrease towards zero at late times, and this is indeed what our calculation will give.

When dealing with partial differential equations, we typically apply the Laplace transform to one direction only, time in our case. In consequence, the x dependence will not be affected at all by our transformation witchcraft. The heat equation in the Laplace domain then becomes using Eq. (F.21)

$$\begin{aligned} sF(s,x) - f(0,x) &= \frac{\partial^2}{\partial x^2} F(s,x) \\ \Rightarrow \quad \frac{\partial^2}{\partial x^2} F(s,x) - sF(s,x) &= -3\sin(2\pi x)\,. \end{aligned}$$

Note how the Laplace transform has converted a PDE into a much simpler ODE and also naturally incorporated the initial conditions into our solution procedure. We solve this linear, inhomogeneous ODE using the standard method of deriving the general homogeneous solution F_h plus one particular solution F_p and then using the boundary conditions to eliminate free coefficients. The homogenuous part is evidently given by

$$F_h(s,x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}\,,$$

with constants c_1, c_2. In contructing a particular solution, we note that s acts merely as a parameter in our ODE, so that a rather obvious guess is

$$\begin{aligned} & F_p(s,x) = A\cos(2\pi x) + B\sin(2\pi x) \\ \Rightarrow \quad & -(2\pi)^2 A\cos(2\pi x) - (2\pi)^2 B\sin(2\pi x) - s\left[A\cos(2\pi x) + B\sin(2\pi x)\right] \stackrel{!}{=} -3\sin(2\pi x) \\ \Rightarrow \quad & -\left[(2\pi)^2 + s\right] A = 0 \qquad \wedge \qquad -\left[(2\pi)^2 + s\right] B = -3 \\ \Rightarrow \quad & A = 0 \qquad \wedge \qquad B = \frac{3}{4\pi^2 + s}\,, \end{aligned}$$

so that the general solution is

$$F(s,x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{3}{4\pi^2 + s}\sin(2\pi x)\,.$$

Laplace transforming the boundary conditions gives us

$$\begin{aligned} f(t,0) = 0 \quad &\Rightarrow \quad F(s,0) = c_1 + c_2 = 0\,, \\ f(t,2) = 0 \quad &\Rightarrow \quad F(s,2) = c_1 e^{2\sqrt{s}} + c_2 e^{-2\sqrt{s}} = 0\,, \end{aligned}$$

so that $c_1 = c_2 = 0$, so that our solution is

$$F(s,x) = \frac{3}{s + 4\pi^2}\sin(2\pi x)\,.$$

We transform this back into the time domain using Eq. (F.15),

$$f(t) = 3e^{-4\pi^2 t}\sin(2\pi x)\,.$$

Note that the $\sin(2\pi x)$ factor is just a constant in this process!

F.2.5 The convolution theorem for Laplace transforms

The convolution of two functions appears in many applications such as signal processing or probability theory. The Fourier transform provides a neat way to manage convolutions since they are transformed into products. In this section, we will see that Laplace transformations also have this property.

Def. : The *convolution* $f * g$ of two functions $f,\ g : \mathbb{R} \to \mathbb{R}$ is defined by the integral transform

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-u)g(u)\mathrm{d}u\,. \tag{F.37}$$

If the functions $f(t)$ and $g(t)$ vanish for negative values of the argument $t < 0$, this becomes

$$(f * g)(t) = \int_{0}^{t} f(t-u)g(u)\mathrm{d}u\,. \tag{F.38}$$

The convolution operator is commutative, so that we may interchange f and g on the right-hand side of Eqs. (F.37), (F.38). From IB Methods, we know that the Fourier transform of a convolution is given by the product of the individual Fourier transforms,

$$\mathcal{F}[f * g] = \mathcal{F}[f]\cdot\mathcal{F}[g]\,.$$

This also holds for Laplace transforms.

Theorem : The Laplace transform of a convolution is given by

$$\mathcal{L}\{f * g\}(s) \quad = \quad \mathcal{L}\{f\}(s)\cdot\mathcal{L}\{g\}(s) \quad = \quad F(s)\,G(s)\,. \tag{F.39}$$

Proof.

$$\begin{aligned}
\mathcal{L}\{f * g\}(s) &= \int_0^\infty \left[\int_0^t f(t-u)g(u)\mathrm{d}u\right] e^{-st}\mathrm{d}t \\
&= \int_0^\infty \left[\int_u^\infty f(t-u)g(u)\,e^{-st}\mathrm{d}t\right]\mathrm{d}u\,,
\end{aligned}$$

where we have swaped the order of integration and adjusted the limits such that the integration

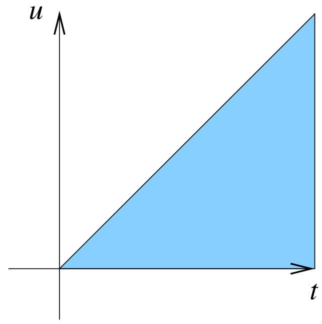

Figure 47: The integral regions $\int_0^\infty (\int_0^t \ldots \mathrm{d}u)\mathrm{d}t$ and $\int_0^\infty (\int_u^\infty \ldots \mathrm{d}t)\mathrm{d}u$ are the same: the blue shaded area.

area remains the same; cf. Fig. 47. Defining $x = t - u$, we obtain

$$\begin{aligned} \mathcal{L}\{f * g\}(s) &= \int_0^\infty \left[\int_0^\infty f(x) g(u) e^{-sx} e^{-su} \mathrm{d}x \right] \mathrm{d}u \\ &= \int_0^\infty \left[\int_0^\infty f(x) e^{-sx} \mathrm{d}x \right] g(u) e^{-su} \mathrm{d}u = F(s)\, G(s)\,. \end{aligned} \tag{F.40}$$

$\square$

Example applications of the convolution theorem

(1) The convolution theorem can help us in finding new pairs of Laplace transforms. Say, we wish to find the inverse of

$$H(s) = \frac{1}{s(s^2+1)}\,.$$

Then we set $F(s) = s^{-1}$ and $G(s) = (s^2+1)^{-1}$, so that $f(t) = 1$ and $g(t) = \sin t$ and

$$\mathcal{L}^{-1}\left\{ \frac{1}{s(s^2+1)} \right\}(t) = 1 * \sin t = \int_0^t \sin u \, \mathrm{d}u = 1 - \cos t\,.$$

(2) We can also express solutions to differential equations in more general terms. Suppose, we are given the ODE

$$4\ddot{f}(t) + f(t) = h(t)\,, \qquad f(0) = 3\,, \qquad \dot{f}(0) = -7\,, \tag{F.41}$$

for an unspecified forcing term $h(t)$. Using Eq. (F.22), we transform the ODE into

$$\begin{aligned} & 4\left[s^2 F(s) - s f(0) - \dot{f}(0) \right] + F(s) = H(s) \\ \Rightarrow\ & (4s^2+1) F(s) - 12s + 28 = H(s) \\ \Rightarrow\ & F(s) = \frac{12s - 28}{4\left(s^2 + \frac{1}{4}\right)} + \frac{H(s)}{4\left(s^2+\frac{1}{4}\right)} = \frac{3s}{s^2+\frac{1}{4}} - \frac{7}{s^2+\frac{1}{4}} + \frac{H(s)}{4} \frac{1}{s^2+\frac{1}{4}}\,. \end{aligned} \tag{F.42}$$

The first two terms are readily inverted using Eqs. (F.29), (F.30), but the third term requires the convolution theorem in the form $\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = f * g$,

$$\begin{aligned} f(t) &= 3\cos\frac{t}{2} - 14\sin\frac{t}{2} + \frac{1}{4}h(t) * \left(2\sin\frac{t}{2}\right) \\ &= 3\cos\frac{t}{2} - 14\sin\frac{t}{2} + \frac{1}{2}\int_0^t \sin\frac{u}{2}\, h(t-u)\mathrm{d}u\,. \end{aligned} \tag{F.43}$$

This expression is remarkably complete given that we had no knowledge about $h(t)$.

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