# Part IB Complex Methods

## Lecture Notes


**Abstract**

The short lecture notes of the IB course *Complex Methods* contained in this document represent the material as displayed on the black board in the lectures. They are mainly provided as a concise summary of the course and for consumption together with the lecture course.

These notes assume that readers are already familiar with complex numbers, calculus in multi-dimensional real space $\mathbb{R}^n$, and Fourier transforms. The most important features of these three areas are briefly summarized, but not at the level of a dedicated lecture. This course is primarily aimed at applications of complex methods; readers interested in a more rigorous treatment of proofs are refered to the IB Complex Analysis lecture. Some more extensive discussions may also be found in the following books, though readers are not required to have studied them.



- M. J. Ablowitz and A. S. Fokas, *Complex variables: introduction and applications.* Cambridge University Press (2003)
- G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists.* Elsevier (2013)
- G. J. O. Jameson, *A First Course in Complex Functions.* Chapman and Hall (1970)
- T. Needham, *Visual complex analysis*, Clarendon (1998)
- H. A. Priestley, *Introduction to Complex Analysis.* Clarendon (1990)
- K. F. Riley, M. P. Hobson, and S. J. Bence, *Mathematical Methods for Physics and Engineering: a comprehensive guide.* Cambridge University Press (2002)


Example sheets will be on Moodle and on

`http://www.damtp.cam.ac.uk/user/examples`

Lectures Webpage:

`http://www.damtp.cam.ac.uk/user/us248/Lectures/lectures.html`

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# Contents

# A Background Material

## A.1 Complex numbers

Complex numbers: $z \in \mathbb{C}$: $\boxed{z = x + \mathrm{i}y = re^{\mathrm{i}\phi} = r\cos\phi + \mathrm{i}\, r\sin\phi}$ , where

$$\begin{aligned}
&\mathrm{i} = \sqrt{-1} = \text{“Imaginary unit”}\\
&x = \mathsf{Re}(z) = r\cos\phi = \text{“Real part”}\\
&y = \mathsf{Im}(z) = r\sin\phi = \text{“Imaginary part”}\\
&r = |z| = \sqrt{x^2+y^2} = \text{“Modulus”}\\
&\phi = \arg z = \text{“Argument”}\\
&\bar{z} \coloneqq x - \mathrm{i}y = \text{“Complex conjugate” of } z
\end{aligned}$$

**Comments:**

- $\arg z$ only defined up to adding $2n\pi$, $n \in \mathbb{Z}$
- *Principal argument* $\coloneqq$ the value $\phi = \arg z$ that falls in $(-\pi, \pi]$
- "$\phi = \arctan \frac{y}{x}$" does not in general work, since $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$
  E.g. $\arg\big(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\big) = \frac{2\pi}{3}$, but $\arctan\frac{y}{x} = -\frac{\pi}{3}$
- We have: $|z|^2 = r^2 = z\bar{z}\,, \qquad \mathsf{Re}(z) = \dfrac{z+\bar{z}}{2}\,, \qquad \mathsf{Im}(z) = \dfrac{z-\bar{z}}{2\mathrm{i}}\,.$

Triangle inequality: $\boxed{|z_1 + z_2| \le |z_1| + |z_2|}$ for all $z_1,\ z_2 \in \mathbb{C}\,.$

Setting $z_1 = \zeta_1 + \zeta_2$ and either $z_2 = -\zeta_2$ or $z_2 = -\zeta_1$, we also find:

$\boxed{\big||\zeta_1| - |\zeta_2|\big| \le |\zeta_1 + \zeta_2|}$ for all $\zeta_1,\ \zeta_2 \in \mathbb{C}\,.$

<u>**Geometric series**</u>

For $z \in \mathbb{C},\ z \neq 1$ and $n \in \mathbb{N}_0$: $\boxed{\displaystyle\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}}$

- Proof by induction
- For $|z| < 1$, the series converges: $\boxed{\displaystyle\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}}$
- This is the Taylor series of $f(z) = \dfrac{1}{1-z}$ around $z = 0$.

**Def.:** A set $\mathcal{D} \subset \mathbb{C}$ is an *open set* if:

$\forall z_0 \in \mathcal{D} \quad \exists\, \epsilon > 0 \;:\;$ the $\epsilon$ sphere $|z - z_0| < \epsilon$ lies in $\mathcal{D}$.

A *neighbourhood* of $z \in \mathbb{C}$ is an open set $\mathcal{D}$ that contains $z$.

## A.2 Trigonometric and hyperbolic functions

Euler's formula: $\boxed{e^{\mathrm{i}\phi} = \cos\phi + \mathrm{i}\sin\phi} \quad \Rightarrow \quad e^{-\mathrm{i}\phi} = \cos\phi - \mathrm{i}\sin\phi$

$$\Rightarrow \quad \cos\phi = \frac{e^{\mathrm{i}\phi} + e^{-\mathrm{i}\phi}}{2} \quad \wedge \quad \sin\phi = \frac{e^{\mathrm{i}\phi} - e^{-\mathrm{i}\phi}}{2\mathrm{i}}$$

Hyperbolic functions: $\boxed{e^{\phi} = \cosh\phi + \sinh\phi} \quad \Rightarrow \quad e^{-\phi} = \cosh\phi - \sinh\phi$

$$\Rightarrow \quad \cosh\phi = \frac{e^{\phi} + e^{-\phi}}{2} \quad \wedge \quad \sinh\phi = \frac{e^{\phi} - e^{-\phi}}{2}$$

We have: $\cos(\mathrm{i}x) = \dfrac{e^{-x} + e^{x}}{2} = \cosh x \quad \Leftrightarrow \quad \cosh(\mathrm{i}x) = \cos x\,,$

$$\sin(\mathrm{i}x) = \frac{e^{-x} - e^{x}}{2\mathrm{i}} = \mathrm{i}\sinh x \quad \Leftrightarrow \quad \sinh(\mathrm{i}x) = \mathrm{i}\sin x$$

Addition theorems:

$$\cos(\alpha + \beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta\,,$$
$$\sin(\alpha + \beta) = \sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta\,,$$

$$\cosh(x + y) = \cosh x\,\cosh y + \sinh x\,\sinh y\,,$$
$$\sinh(x + y) = \sinh x\,\cosh y + \cosh x\sinh y\,.$$

## A.3 Calculus of real functions in $\geq 1$ variables

Sometimes, we regard a complex function as 2 real functions on $\mathbb{R}^2$: $\boxed{f(z) = u(x, y) + \mathrm{i}v(x, y)}$

**Def.:** $C^m(\Omega)$ := set of functions $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ whose partial derivatives up to order $m$ exist and are continuous.

Note: Existence of partial derivatives does not mean much! E.g. :

$$f(x,y) = \begin{cases} x & \text{for } y = 0 \\ y & \text{for } x = 0 \\ \text{arbitrary} & \text{for } x \neq 0 \text{ and } y \neq 0 \end{cases}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}(0,0) \;=\; 1 \;=\; \frac{\partial f}{\partial y}(0,0)\,, \quad \text{but } f \text{ is not even continuous at } (0,0)!$$

**Def.:** $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is *differentiable* at $\boldsymbol{x} \in \Omega$ if: $\exists$ a linear function $A : \mathbb{R}^n \to \mathbb{R}$ with

$$f(\boldsymbol{x} + \Delta\boldsymbol{x}) - f(\boldsymbol{x}) = A(\Delta\boldsymbol{x}) + r(\Delta\boldsymbol{x}) \quad \text{with} \quad \lim_{\Delta\boldsymbol{x}\to 0} \frac{r(\Delta\boldsymbol{x})}{||\Delta\boldsymbol{x}||} = 0\,.$$

$f$ is *continuously differentiable* if furthermore the partial derivatives are continuous.

This generalizes to vector-valued $f : \Omega \to \mathbb{R}^m$ by considering each component $f_i$ separately.

One can show that:

$$f \text{ is continuously differentiable} \quad \Leftrightarrow \quad \text{all partial derivatives } \frac{\partial f}{\partial x_j} \text{ are continuous}$$

$$\Rightarrow \quad f \text{ is differentiable}$$

$$\Rightarrow \quad f \text{ is continuous and all partial derivatives } \frac{\partial f}{\partial x_j} \text{ exist}\,,$$

**Def.:** A sequence of functions $f_k : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is *uniformly convergent* with limit $f$

$$:\Leftrightarrow \quad \forall_{\epsilon>0} \;\; \exists_{N\in\mathbb{N}} \;\; \forall_{k\geq N,\; x\in\Omega} \;\; |f_k(x) - f(x)| < \epsilon$$

This allows: $\lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$

**Example:** The geometric series $\sum_{n=0}^{\infty} x^n$ converges uniformly for $|x| < 1$.

# B Analytic functions

## B.1 The Extended Complex Plane and the Riemann Sphere

We can identify $\mathbb{C}$ with $\mathbb{R}^2$: $z \leftrightarrow (x, y)$ is bijective with $z = x + \mathrm{i}y$

Addition: $z = z_1 + z_2 \quad \Leftrightarrow \quad (x,\ y) = (x_1 + x_2,\ y_1 + y_2)$

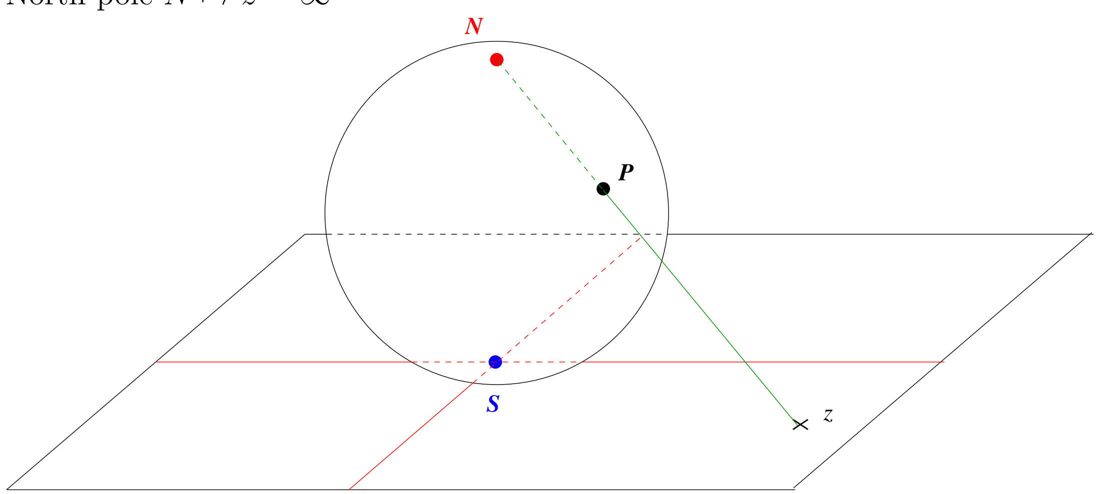
Multiplication: $z = z_1\, z_2 \quad \Leftrightarrow \quad (x,\ y) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$

Easier to see with $\mathrm{i}^2 = -1$: $z_1\, z_2 = (x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) = x_1 x_2 - y_1 y_2 + \mathrm{i}(x_1 y_2 + x_2 y_1)$

__Def.:__ The *extended complex domain* is $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$

**Comments**:

- $z = \infty$ is a single point!
- "$z = -\infty$" means we approach this point along the negative real axis
- This is best seen in the Riemann sphere:
  $P \leftrightarrow z$ via the line $\overline{NP}$
  South pole $P \mapsto z = 0$
  North pole $N \mapsto z = \infty$




- In practice: $f$ has a property at $z = \infty$ if $f(1/\zeta)$ has this property at $\zeta = 0$

## B.2 Complex differentiation and analytic functions

__Def.:__ $f : \mathbb{C} \to \mathbb{C}$ is *differentiable* at $z \in \mathbb{C}$

$:\Leftrightarrow \quad f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$ exists and is independent of the direction of approach.

Direction independence is a strong requirement!

In $\mathbb{R}$, we only have 2 directions. E.g. $f(x) = |x|$ is not differentiable at $x = 0$.

In $\mathbb{C}$, we have $\infty$ directions.

**Def.:** A complex function $f$ is *analytic* at $z \in \mathbb{C}$

:⇔ $\exists$ a neighbourhood $\mathcal{D}$ of $z$ where $f$ is differentiable.

**Comments:**

- We will see that analyticity implies a lot!
- E.g. an analytic function can be differentiated $\infty$ many times. This is not true for real functions: $\int |x| \mathrm{d}x$ can be differentiated exactly once.
- Good news: Many rules for differentiation of real functions hold for complex ones, too.

Let us consider 2 directions for the derivative of $f(z) = u(x, y) + \mathrm{i}v(x, y)$,

**(1)** Real direction: $\delta z = \delta x$

$$\begin{aligned} f'(z) &= \lim_{\delta x \to 0} \frac{f(z + \delta x) - f(z)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{u(x + \delta x, y) + \mathrm{i}v(x + \delta x, y) - u(x, y) - \mathrm{i}v(x, y)}{\delta x} \\ &= \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} \,. \end{aligned} \tag{B.1}$$

**(2)** Imaginary direction: $\delta z = \mathrm{i}\delta y$

$$\begin{aligned} f'(z) &= \lim_{\delta y \to 0} \frac{f(z + \mathrm{i}\delta y) - f(z)}{\mathrm{i}\delta y} \\ &= \lim_{\delta y \to 0} \frac{u(x, y + \delta y) + \mathrm{i}v(x, y + \delta y) - u(x, y) - \mathrm{i}v(x, y)}{\mathrm{i}\delta y} \\ &= \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y} \,. \end{aligned} \tag{B.2}$$

For a differentiable function, these must be equal!

**Proposition:** A differentiable function $f(z) = u(x, y) + \mathrm{i}v(x, y)$ satisfies the *Cauchy Riemann conditions*

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,.}$$

The reverse does not hold; we need that $u$ and $v$ are differentiable.

In practice we often use:

**Proposition:** If $f(z) = u(x,y) + \mathrm{i}v(x,y)$ satisfies $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}\,, \qquad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}\,,$
at $z = z_0$ and the partial derivatives are continuous in a neighbourhood of $z$, then $f$ is differentiable at $z_0$.

The proof (and others) are discussed in IB Complex Analysis.

Alternative viewpoint: Using $x = \dfrac{z+\bar{z}}{2}\,, \qquad y = \dfrac{z-\bar{z}}{2\mathrm{i}} = -\mathrm{i}\dfrac{z-\bar{z}}{2}\,,$

we can write any complex function as $g(z,\bar{z})$.

Then $g$ is differentiable if $g(z,\bar{z}) = g(z)$; cf. example sheet 1.

**Product rule**: The product of two analytic functions $f$, $g$ is analytic with

$$\boxed{(f\,g)'(z) = f'(z)g(z) + f(z)g'(z)}\,.$$

*Proof.* We can directly compute the derivative. Let

$$\begin{aligned}\varpi &:= \frac{f(z+h)-f(z)}{h} - f'(z)\,,\\ w &:= \frac{g(z+h)-g(z)}{h} - g'(z)\,,\end{aligned}$$

so both $\varpi \to 0$ and $w \to 0$ as $h \to 0$, and

$$\begin{aligned}(g\,f)' &= \lim_{h\to 0} \frac{g(z+h)f(z+h) - g(z)f(z)}{h}\\ &= \lim_{h\to 0} \frac{\{g(z) + [g'(z)+w]h\}\,\{f(z) + [f'(z)+\varpi]h\} - g(z)f(z)}{h}\\ &= \lim_{h\to 0} \frac{[g'(z)+w]h\,f(z) + [f'(z)+\varpi]h\,g(z) + [g'(z)+w]h\,[f'(z)+\varpi]h}{h}\\ &= g'(z)\,f(z) + f'(z)\,g(z)\,.\end{aligned}$$

□

**Chain rule**: The composition of two analytic functions $f$, $g$ is analytic with

$$\boxed{(f\circ g)'(z) = f'\big(g(z)\big)g'(z)}\,.$$

The proof works analogous to that for product rule; cf. long script.

**Examples**

**(1)** $f(z) = z = x + \mathrm{i}y$ is *entire* $:=$ analytic in all $\mathbb{C}$:

$\partial_x u = 1 = \partial_y v\,,\qquad \partial_y u = 0 = -\partial_x v\,,\qquad$ and they are continuous.

The definition of $f'(z)$ gives us immediately $f'(z) = 1$.

**(2)** $e^z = e^x\, e^{\mathrm{i}y} = e^x(\cos y + \mathrm{i}\sin y)$ is also entire:

$\partial_x u = e^x \cos y = \partial_y v\,,\qquad \partial_y u = -e^x \sin y = -\partial_x v\,,\qquad$ and they are continuous.

Computing $f'$ along the $x$ direction gives:

$f'(z) = \partial_x u + \mathrm{i}\partial_x v = e^x \cos y + \mathrm{i}e^x \sin y = e^z$

**(3)** $f(z) = z^n$, $n \in \mathbb{N}$ is also entire. This follows by induction using product rule and example (1), which also give us $f'(z) = nz^{n-1}$.

A linear combination $\alpha f + \beta g\,,\ \ \alpha,\, \beta \in \mathbb{C}$ of two analytic functions is also analytic.

$\Rightarrow$ Polynomials are analytic.

**(4)** $f(z) = \dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{x - \mathrm{i}y}{x^2 + y^2}$ is analytic everywhere except $z = 0$:

$$\partial_x u = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \partial_y v\,,\qquad \partial_y u = \frac{-2xy}{(x^2 + y^2)^2} = -\partial_x v\,.$$

Evaluating $f'$ along the $x$ direction, we get:

$$\frac{\partial}{\partial x}\frac{x - \mathrm{i}y}{x^2 + y^2} = \frac{-x^2 + y^2 + 2\mathrm{i}xy}{(x^2 + y^2)^2} = \frac{-(x - \mathrm{i}y)^2}{(x^2 + y^2)^2} = -\frac{\bar{z}^2}{z^2\bar{z}^2} = -\frac{1}{z^2}\,.$$

With product and chain rule, this also gives us the **Quotient rule**: $\boxed{\left(\dfrac{f}{g}\right)' = \dfrac{f'\,g - g'\,f}{g^2}}$

Also: If $P(z)$, $Q(z)$ are polynomials, then $\dfrac{P(z)}{Q(z)}$ is analytic except where $Q(z) = 0$.

**(5)** $\cos z = \dfrac{e^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2}\,,\quad$ and $\quad \sin z = \dfrac{e^{\mathrm{i}z} - e^{-\mathrm{i}z}}{2\mathrm{i}}\quad$ are analytic everywhere with

$(\sin)'(z) = \cos(z)\,,\qquad (\cos)'(z) = -\sin(z)\,.$

We likewise find $\ \cosh' = \sinh\,,\ \ \sinh' = \cosh$.

**(6)** $(\tan)'(z) = \dfrac{1}{\cos^2(z)}$ by quotient rule; is analytic except where $\cos(z) = 0$.

**(7)** One can show that $\log' z = \dfrac{1}{z}$ where $\log z$ is defined. This is more subtle; cf. below.

**Examples of non-analytic functions**

**(1)** $f(z) = \mathsf{Re}(z) \quad\Rightarrow\quad \partial_x u = 1 \neq \partial_y v,\quad$ so $\mathsf{Re}(z)$ is nowhere analytic.

**(2)** $f(z) = |z| \quad\Rightarrow\quad u = \sqrt{x^2 + y^2}\,,\ \ v = 0$

$$\Rightarrow \quad \partial_x u = \frac{x}{\sqrt{x^2+y^2}}\,, \quad \partial_y u = \frac{y}{\sqrt{x^2+y^2}}\,, \quad \partial_x v = \partial_y v = 0$$

$\Rightarrow$ The Cauchy-Riemann Eqs. are nowhere satisfied.

**(3)** $f(z) = \bar{z} \quad \Rightarrow \quad u = x\,,\ v = -y \quad \Rightarrow \quad \partial_x u = 1 \neq \partial_y v \quad \Rightarrow \quad \bar{z}$ is nowhere analytic.

**(4)** $f(z) = |z|^2 = x^2 + y^2$

$$\Rightarrow \quad \partial_x u = 2x\,, \quad \partial_y v = 0\,, \quad \partial_y u = 2y\,, \quad \partial_x v = 0$$

$\Rightarrow$ The Cauchy Riemann Eqs. are satisfied only at $z = 0$.

Analyticity requires differentiability in a neighbourhood, so $|z|^2$ is nowhere analytic.

## B.3 Harmonic functions

**Def.:** A function $f(x,y)$ is *harmonic* if it satisfies the Laplace equation $\triangle f = \partial_x^2 f + \partial_y^2 f = 0$

**Def.:** Two functions $u$, $v$ satisfying the Cauchy-Riemann Eqs. are *harmonic conjugates*.

The Cauchy-Riemann equations relate $u$ and $v$. If we know one, we can construct the other up to a constant and, thus, the analytic function $f$.

**Example.:** Let $u(x,y) = x^2 - y^2$

$\Rightarrow \quad \partial_y v = \partial_x u = 2x$

$\Rightarrow \quad v(x,y) = 2xy + g(x)$

Also

$\partial_y u = -2y \overset{!}{=} -\partial_x v = -2y - g'(x)$

$\Rightarrow \quad g'(x) = 0 \quad \Rightarrow \quad g(x) = c_0 = \text{const}$

$\Rightarrow \quad f(z) = u + \mathrm{i}v = x^2 - y^2 + 2\mathrm{i}xy + \mathrm{i}c_0 = (x + \mathrm{i}y)^2 + \mathrm{i}c_0 = z^2 + \mathrm{i}c_0\,.$

Note: You should compute $f(z)$, not merely $u(x,y)$, $v(x,y)$.

**Proposition:** The real and imaginary parts of any analytic complex function are harmonic.

*Proof.* Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$ be analytic

$\Rightarrow \quad \partial_x(\partial_x u) = \partial_x(\partial_y v) = \partial_y(\partial_x v) = \partial_y(-\partial_y u)$

$\Rightarrow \quad \partial_x^2 u + \partial_y^2 u = 0$

We likewise find $\triangle v = 0$. □

## B.4 Multi-valued functions and branch cuts

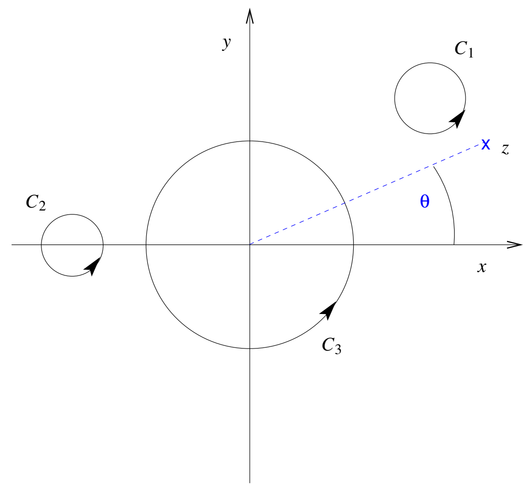
### B.4.1 Single branch cuts

Recall: $\log z = \log\big(re^{\mathrm{i}\theta}\big) = \log r + \mathrm{i}\theta$

Problem: $\theta = \arg z$ defined only up to adding $2n\pi, \quad n \in \mathbb{Z}$

E.g. $\log \mathrm{i} = \mathrm{i}\dfrac{\pi}{2} \quad \text{or} \quad \mathrm{i}\dfrac{5\pi}{2} \quad \text{or} \quad -\mathrm{i}\dfrac{3\pi}{2} \quad \text{or} \ \ldots$

Consider the curves $C_1$, $C_2$, $C_3$




- On $C_1$ : $\theta = \arg z \in \left(0, \frac{\pi}{2}\right)$ ; fine!
- On $C_2$ : $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ ; fine!
- On $C_3$ : $\theta$ increases by $2\pi$ everytime we go around the circle.

  $\Rightarrow \quad \theta$ is not single valued.

  We could require $\theta \in [0, 2\pi)$, but then $\theta$ is not continuous!

**Def.:** A *branch point* of a function $f(z)$ is a point $z_0$ that cannot be encircled by a curve $C$ such that $f$ is single-valued *and* continuous along $C$. $z_0$ is a *branch point singularity* of $f$.

**Examples**

**(1)** $f(z) = \log(z-a)\,, \quad a = \text{const} \in \mathbb{C}$ has a branch point at $z = a$

**(2)** $f(z) = \log\left(\frac{z-1}{z+1}\right) = \log(z-1) - \log(z+1)$ has two branch points at $\pm 1$

**(3)** Consider $f(z) = z^\alpha = r^\alpha e^{\mathrm{i}\alpha\theta}$ along a circle of radius $r_0$ around $z = 0$.

At $\theta = 0$ : $\quad f = r_0^\alpha$

At $\theta = 2\pi$ : $\quad f = r_0^\alpha e^{\mathrm{i}\alpha 2\pi}$

Equal only if $e^{\mathrm{i}\alpha 2\pi} = 1 \quad \Leftrightarrow \quad \alpha 2\pi = 2n\pi \quad \Leftrightarrow \quad \alpha \in \mathbb{Z}$.

$\Rightarrow$ For non-integer $\alpha$, $f(z)$ has a branch point at $z = 0$.

**(4)** $f(z) = \log z$ has a branchpoint at $z = \infty$ because $\log \frac{1}{\zeta} = -\log \zeta$ has one at $\zeta = 0$.

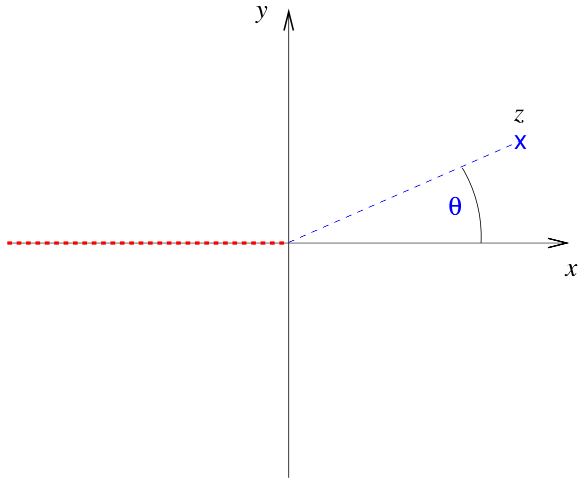
Likewise, $z^\alpha$ has a branchpoint at $z = \infty$ if $\alpha \notin \mathbb{Z}$.

**(5)** $f(z) = \log \frac{z-1}{z+1}$ does *not* have a branch point at $z = \infty$:

$f(z = 1/\zeta) = \log \frac{z-1}{z+1} = \log \frac{1-\zeta}{1+\zeta}$ stays near $\log 1 = 0$ for all $\zeta \approx 0$.

We handle branch points with *branch cuts*: "red" lines in $\mathbb{C}$ which no curve $C$ is allowed to cross.

**Example.:** Consider $\log z = \log|z| + \mathrm{i}\arg z$ with a branch cut along the negative real axis.

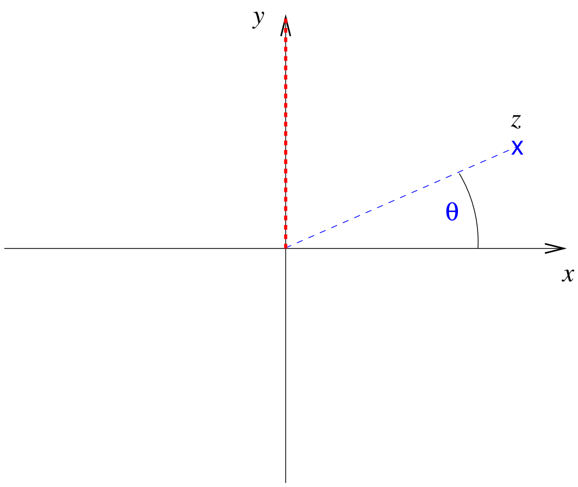


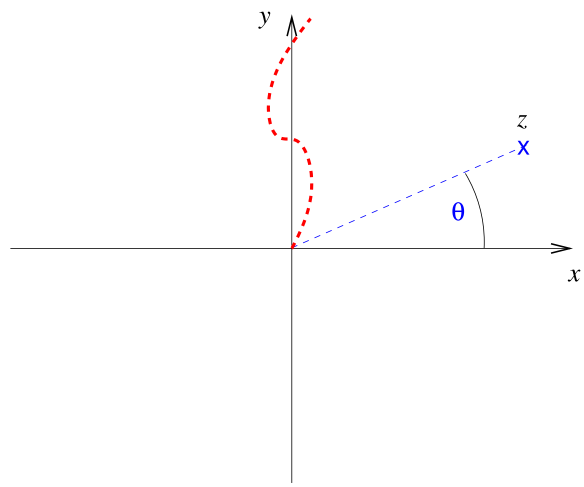

Then $\log z$ is continuous with derivative $\frac{\mathrm{d}}{\mathrm{d}z}\log z = \frac{1}{z}$ along any curve that does not cross the cut!

E.g. $\theta \in (-\pi, \pi]$ or $\theta \in (\pi, 3\pi]$

Either way, $\theta$ jumps by $2\pi$ across the cut!

We could also choose other branch cuts:



**Summary:**

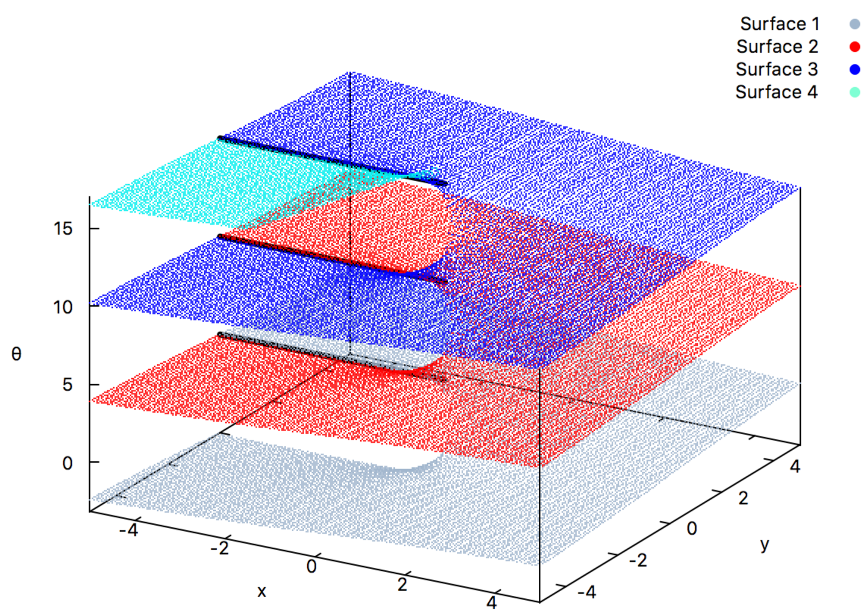
- We have 3 branch "thingies":
  1. *branch point*: A point we cannot encircle.
  2. *branch cut*: a red line we are not allowed to cross.
  3. *branch*: the choice of values $f(z)$ is allowed to take on.
- We have freedom in choosing branch cuts and branches, but not branch points.
- We can specify the branch of a function in two ways.
  1. Specify the function and range of values.
     E.g. $f(z) = \log|z| + \mathrm{i}\arg z\,, \quad \arg z \in (-\pi, \pi]$.
  2. Specify the function, the branch cut and $f(z)$ at one point.
     E.g. $f(z) = \log z$ with a cut on $\mathbb{R}^{\leq 0}$ and $\log 1 = 0$.

### B.4.2 Riemann surfaces*

Are branch cuts quite satisfactory?

Riemann suggested: Regard the branches of $f(z)$ as copies of $\mathbb{C}$ stacked on each other.

E.g. for $\log z = \log|z| + \mathrm{i}\theta$




Crossing a branch cut now carries us from one sheet to the next.

### B.4.3 Multiple branch cuts

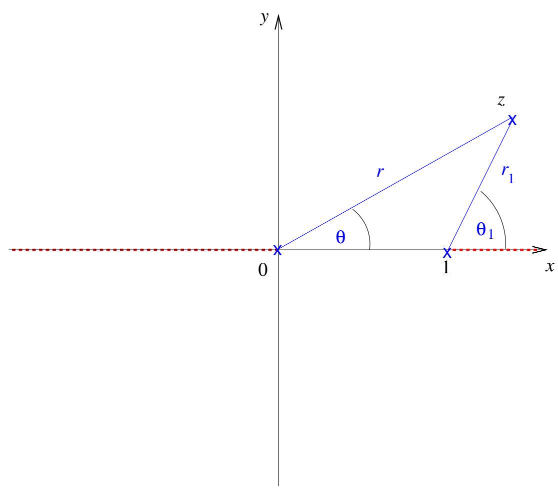
How to handle multiple branch points?

**Examples**

**(1)** $g(z) = [z(z-1)]^{1/3}$ has branch points: $z = 0, \quad z = 1$

Let $z = re^{i\theta}, \quad z - 1 = r_1 e^{i\theta_1}$

$\Rightarrow \quad g(z) = \sqrt[3]{r r_1} e^{i(\theta+\theta_1)/3}$



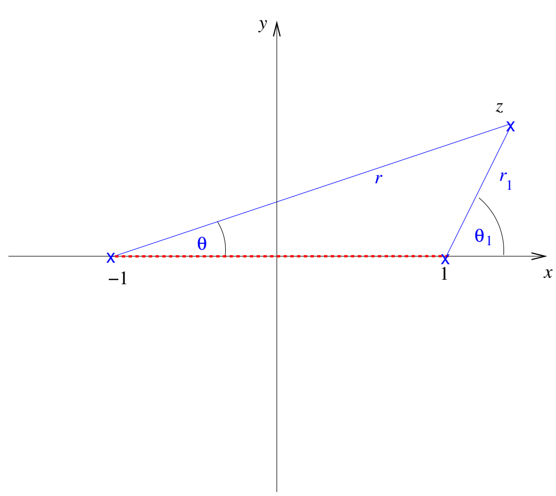

We must avoid either $\theta$ or $\theta_1$ completing a full circle.

$\rightarrow$ 2-segement branch cut $(-\infty, 0] \cup [1, \infty)$

**(2)** $f(z) = \log \dfrac{z-1}{z+1} = \log(z-1) - \log(z+1)$ has 2 branchpoints: $z = \pm 1$

Let $z + 1 = re^{i\theta}, \quad z - 1 = r_1 e^{i\theta_1}$, and avoid full circles in $\theta$, $\theta_1$

2-segment branch cut: $(-\infty, -1] \cup [1, \infty)$. But we can also use $[-1, 1]$!



Which is better? Depends...

| | |
|---|---|
| $(-\infty,-1]\cup[1,\infty)$ | does not handle legitimate curves encircling both $z=\pm 1$ |
| $[-1,1]$ | does not handle legitimate "little" curves between $z=1$ and $z=-1$ |

Why are curves encircling both $z=\pm 1$ ok?

Let $\theta,\ \theta_1\in[0,2\pi)$. Then, as we cross the real axis ...

(i) to the left of $-1$: All ok, since $\theta,\ \theta_1$ vary smoothly across $\pi$.

(ii) to the right of $+1$: both $\theta$ and $\theta_1$ jump by $2\pi$, but

$f(z)=\log|z-1|-\log|z+1|+\mathrm{i}(\theta_1-\theta)$ does not jump!

**(3)** Could we have used the branch cut $[0,1]$ in example 1?

No! So what's the difference between $f$ and $g$?

Answer: $g$ also has a branchpoint at $z=\infty$, $f$ doesn't.

$[0,1]$ would still allow us to encircle the branch point $z=\infty$ of $g(z)$.

**Proposition:** Let $f(z)$ have branch points $z_1,\ z_2,\ldots$. A complete branch cut of $f$ is a set of cuts with: (i) Every branchpoint has a cut ending on it. (ii) Both ends of each cut end on a branch point. (iii) Any curve in $\mathbb{C}$ that does not intersect the branch cut either encloses all or none of the branchpoints.

**Comments:**

- Regard $z=\infty$ as a single point!
- The branch cut $(-\infty,-1]\cup[1,\infty)$ in example 2 is really a single curve across the North Pole of the Riemann sphere! The cut $[-1,1]$ is a cut across the South Pole of the Riemann sphere!
- $z=\infty$ may also be a branch point, as in example 1.

## B.5 Möbius maps

**Def.:** *Möbius map*: a map $\mathcal{M}:\mathbb{C}\to\mathbb{C}\,,\quad z\mapsto w=\dfrac{az+b}{cz+d}$

where $a,\ b,\ c,\ d\in\mathbb{C}$ with $ad\neq bc$.

**Comments:**

- If $ad=bc$, $\mathcal{M}$ maps all $\mathbb{C}$ to a single point
- $\mathcal{M}$ is analytic everywhere except $z=-\frac{d}{c}$
- Regarded as $\mathcal{M}:\mathbb{C}^*\to\mathbb{C}^*$, $\mathcal{M}$ is bijective with

  $\mathcal{M}^{-1}:\mathbb{C}^*\to\mathbb{C}^*\,,\quad w\mapsto z=\dfrac{-dw+b}{cw-a}$, also a Möbius map! **(†)**

**Def.:** A *circline* is either a circle or a line.

**Proposition:** Any circline in $\mathbb{C}$ is given by the points $z$ with

$$|z - z_1| = \lambda |z - z_2| \quad \text{with} \quad z_1 \neq z_2 \in \mathbb{C}, \;\; \lambda \in \mathbb{R}^+$$

Proof: long script.

**Proposition:** A Möbius map maps a circline to a circline.

*Proof.* Plug (†) into a circline $|z - z_1| = \lambda |z - z_2|$.

$$\left| -\frac{dw-b}{cw-a} - z_1 \right| = \lambda \left| -\frac{dw-b}{cw-a} - z_2 \right| \qquad \Bigg| \cdot |cw-a|$$

$$\Rightarrow \quad |-dw + b - z_1(cw-a)| = |dw - b + z_1(cw-a)| = \lambda |dw - b + z_2(cw-a)|$$

$$\Rightarrow \quad |w(cz_1 + d) - (az_1 + b)| = \lambda |w(cz_2 + d) - (az_2 + b)| \,. \tag{B.3}$$

If $cz_1 + d = 0$ or $cz_2 + d = 0$, this trivially gives a circle. Otherwise,

$$\left| w - \frac{az_1 + b}{cz_1 + d} \right| = \lambda \left| w - \frac{az_2 + b}{cz_2 + d} \right| \,. \tag{B.4}$$

□

A circline is determined by 3 points. This suggests:

**Proposition:** Let $\alpha \neq \beta \neq \gamma \neq \alpha \in \mathbb{C}$ and $\tilde{\alpha} \neq \tilde{\beta} \neq \tilde{\gamma} \neq \tilde{\alpha} \in \mathbb{C}$.
$\Rightarrow$ There exists a Möbius map that sends $\alpha \mapsto \tilde{\alpha}$, $\beta \mapsto \tilde{\beta}$, $\gamma \mapsto \tilde{\gamma}$.

*Proof.* $\mathcal{M}_1(z) = \dfrac{\beta - \gamma}{\beta - \alpha} \dfrac{z - \alpha}{z - \gamma}$ sends $\alpha \mapsto 0$, $\beta \mapsto 1$, $\gamma \mapsto \infty$.

$\mathcal{M}_2(z) = \dfrac{\tilde{\beta} - \tilde{\gamma}}{\tilde{\beta} - \tilde{\alpha}} \dfrac{z - \tilde{\alpha}}{z - \tilde{\gamma}}$ sends $\tilde{\alpha} \mapsto 0$, $\tilde{\beta} \mapsto 1$, $\tilde{\gamma} \mapsto \infty$

$\mathcal{M}_2^{-1} \circ \mathcal{M}_1$ is the required map.
It is also a Möbius map (who form a group!) □

## B.6 The circle of Apollonius*

see long notes.

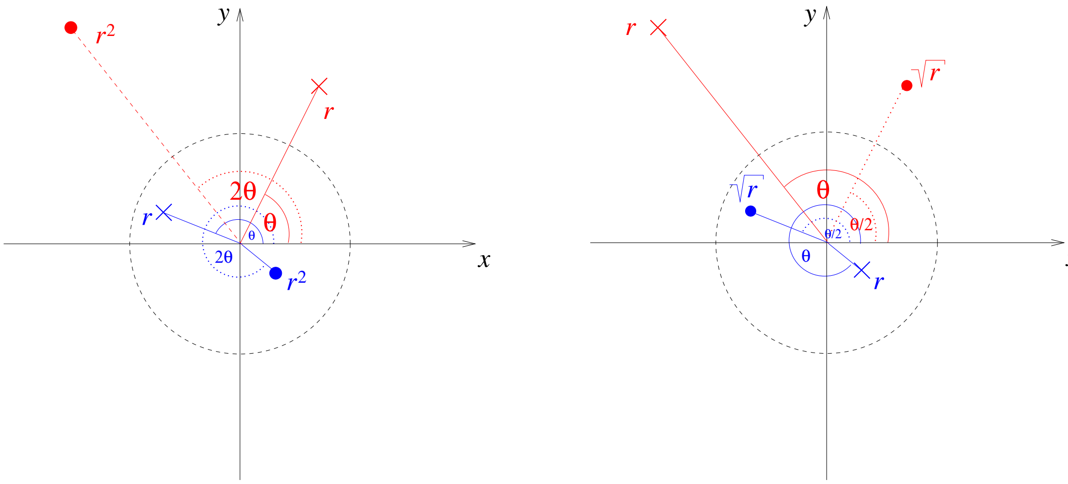
## B.7 Conformal mappings

### B.7.1 Simple operations in the complex plane

Let's first get some intuition about operations in the complex plane.

**Examples**

**(1)** Square function: $z = re^{\mathrm{i}\theta} \mapsto z^2 = r^2 e^{\mathrm{i}2\theta}$




2 effects:

- Rotation from $\theta$ to $2\theta$
- Points are pushed away from the unit circle (towards $\infty$ or 0)

Likewise for $z \mapsto z^\alpha$ with $\alpha > 1$, but we need to choose a branch if $\alpha \notin \mathbb{Z}$.

**(2)** Square root: $z = re^{\mathrm{i}\theta} \mapsto \sqrt{z} = \sqrt{r} e^{\mathrm{i}\frac{1}{2}\theta}$

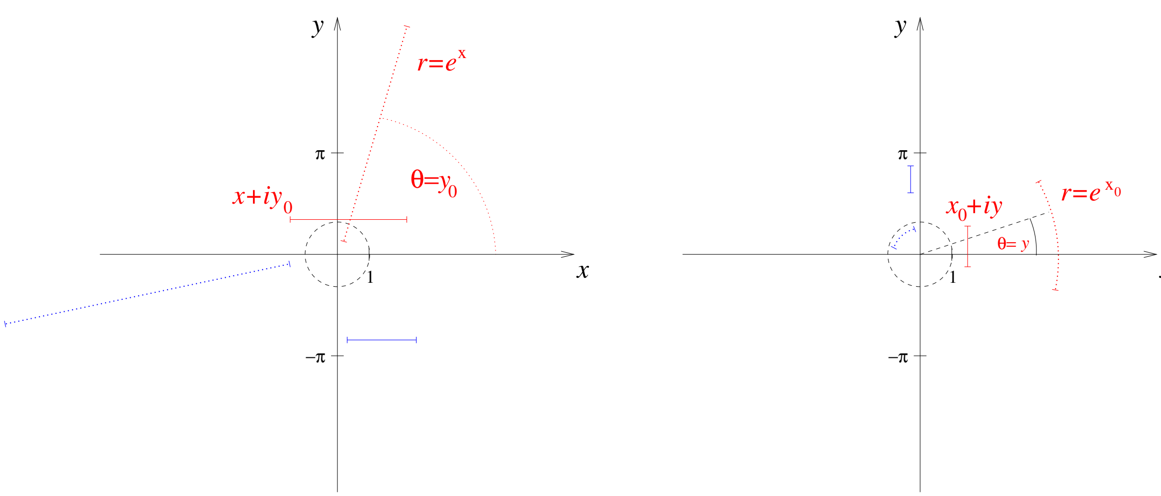
2 effects:

- Points rotate to half their original angle.
- Points are dragged towards the unit circle.

Likewise for any $z \mapsto z^\alpha$ with $0 < \alpha < 1$. We always need to chose a branch!

**(3)** Exponential function: $z = x + \mathrm{i}y \mapsto e^z = e^{x+\mathrm{i}y} = re^{\mathrm{i}y}$ with $r = e^x$.

- real part $x$ determines radius, imaginary part $y$ the angle.
- Horizontal lines $y = \text{const}$ $\rightarrow$ constant angle $\theta$, i.e. radial rays
- Vertical lines $x = \text{const}$ $\rightarrow$ constant $r$, i.e. circle segments



$\Rightarrow$ Rectangles map to sectors of annuli!

**(4)** log map: $z \mapsto \ln z$ is the inverse, provided we choose a branch.

sectors of annuli $\to$ rectangles

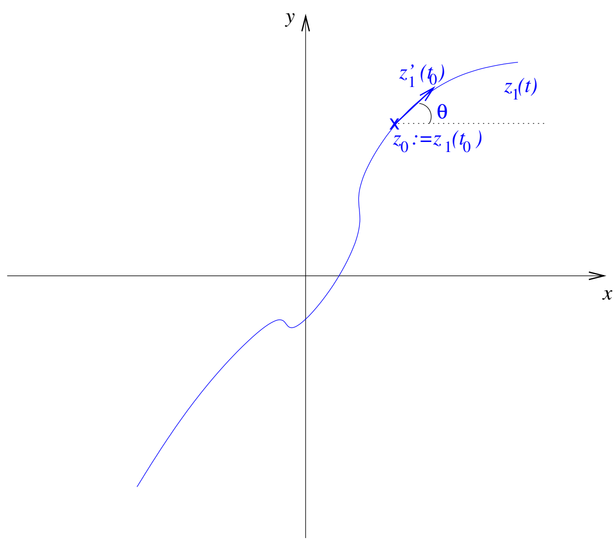
### B.7.2 Conformal maps

**Def.:** $f : U \to W$, $U,\ W \subseteq \mathbb{C}$ open, is a *conformal map* :⇔ f is analytic with $f'(z) \neq 0$ throughout $U$. If $f$ is also bijective, it is called a *conformal equivalence*.

**Proposition:** A conformal map preserves the angle in magnitude and direction between intersecting curves.

*Proof.* Let $z_1(t)$ be a curve in $\mathbb{C}$ with $z_0 = z_1(t_0)$.

$\Rightarrow \quad \theta = \arg z_1'(t_0) =$ angle of the curve with the $x$ direction




Let $f$ be a conformal map $\Rightarrow \quad \zeta_1(t) = f\big(z_1(t)\big)$ is a new curve with tangent

$$\zeta_1'(t_0) = \frac{\mathrm{d}f}{\mathrm{d}z_1}\bigg|_{t=t_0} \frac{\mathrm{d}z_1}{\mathrm{d}t}\bigg|_{t=t_0} = f'(z_0) z_1'(t_0)\,.$$

Angle with $x$ direction: $\vartheta = \arg\big(\zeta_1'(t_0)\big) = \arg\big(z_1'(t_0) f'(z_0)\big) = \theta + \arg f'(z_0)\,.$

The rotation angle $\arg f'(z_0)$ is well defined since $f'(z_0) \neq 0$.

Let $z_2(t)$ be a second curve through $z_0$. It also gets rotated by $\arg f'(z_0)$.

$\Rightarrow$ the angle between $z_1$ and $z_2$ is unchanged. □

**Comments:**

- Without proof: The reverse of this proposition is also true!

  $\Rightarrow$ Conformal maps and angle preserving maps are the same thing.

- We often determine the image $V := f(U)$ by taking the image of the boundary $\partial V = f(\partial U)$. But which side of $\partial V$ is $V$? Take one example point to find out.

**Examples**

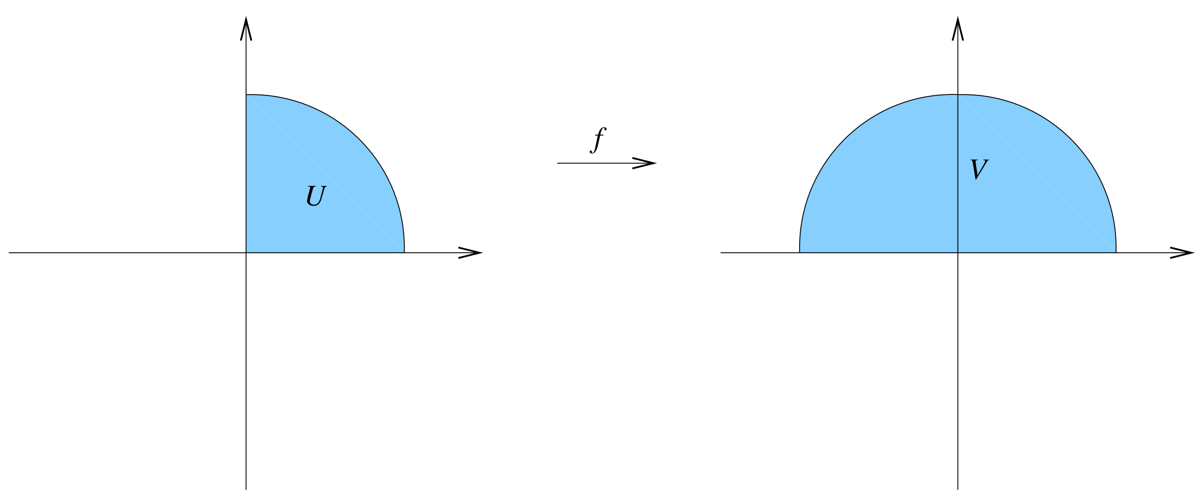
**(1)** $f(z) = az + b$ with $a\ b \in \mathbb{C}$, $a \neq 0$ is conformal everywhere:

it rotates by $\arg a$, translates by $b$ and rescales the radius by $|a|$

**(2)** $f(z) = z^2$ is a conformal map except at $z = 0$. Consider

$$U = \left\{ z \in \mathbb{C} \;\middle|\; 0 < |z| < 1 \quad \wedge \quad 0 < \arg z < \frac{\pi}{2} \right\}$$

$$\Rightarrow \quad V = f(U) = \left\{ w \in \mathbb{C} \;\middle|\; 0 < |w| < 1 \quad \wedge \quad 0 < \arg w < \pi \right\}$$

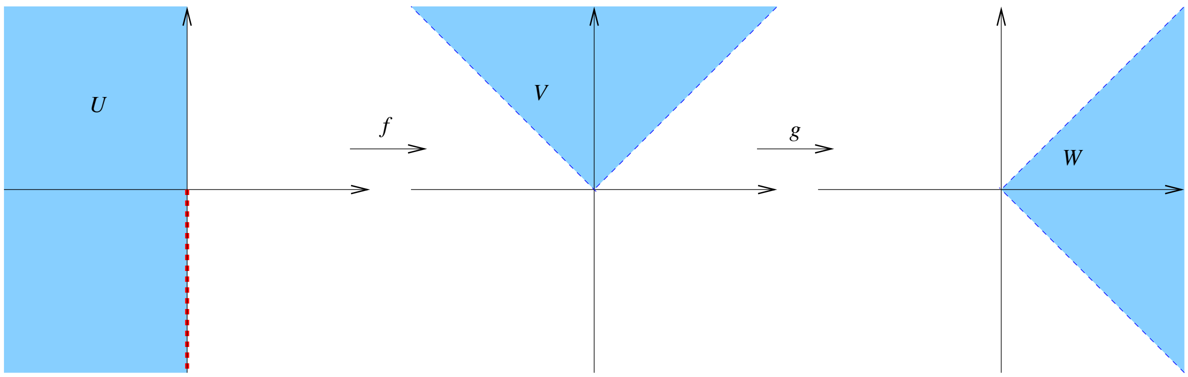



Note: The right angles of $\partial U$ at $z = 1,\ \mathrm{i}$ are preserved at $w = 1,\ -1$

The right angle at $z = 0$ is not! Because $f'(0) = 0$.

**(3)** Often, we know $U$, $W$ and want to find $f : U \to W$.

E.g. $U = \left\{ z \in \mathbb{C} \;\middle|\; \mathsf{Re}(z) < 0 \right\}, \qquad W = \left\{ w \in \mathbb{C} \;\middle|\; -\frac{\pi}{4} < \arg w \leq \frac{\pi}{4} \right\}$




1. Half the angular range: $f(z) = z^{1/2}$

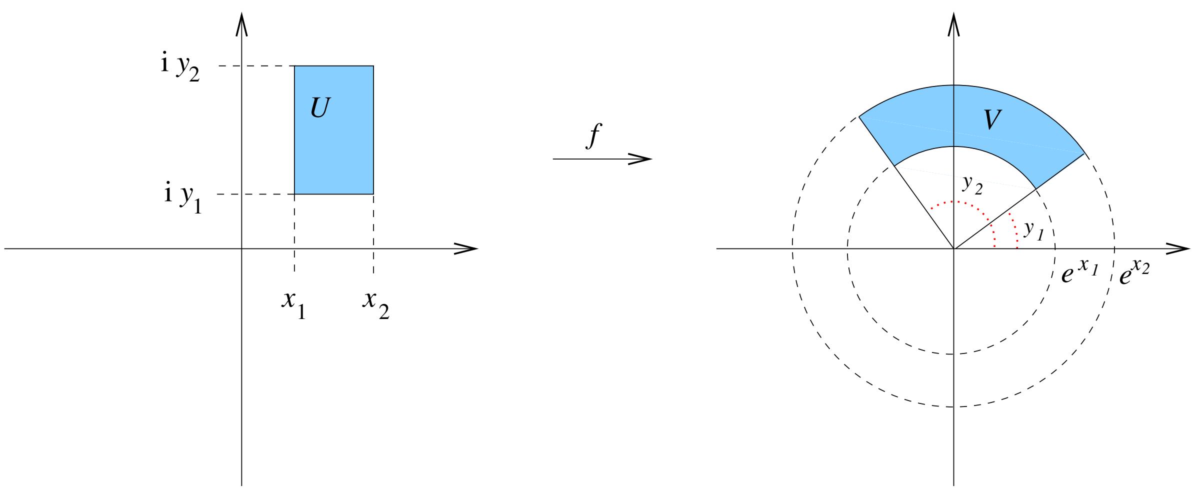
   We need a branch cut. This must not intersect $U$, so $f$ is analytic.

   E.g. $\arg z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$

2. We need to rotate $f(U)$ by $-\pi/2$: $g(\zeta) = e^{-\mathrm{i}\pi/2}\zeta = -\mathrm{i}\zeta$.

$\Rightarrow g \circ f : U \to W\,, \quad g \circ f(z) = -\mathrm{i} z^{\frac{1}{2}}$

**(4)** $f(z) = e^z$ is conformal throughout $\mathbb{C}$. It maps rectangles to sectors of annuli




With an appropriate branch, $\log z$ does the reverse.

**(5)** Möbius maps $x \mapsto \dfrac{az+b}{cz+d}$ are conformal on $\mathbb{C} \setminus \{-\frac{d}{c}\}$

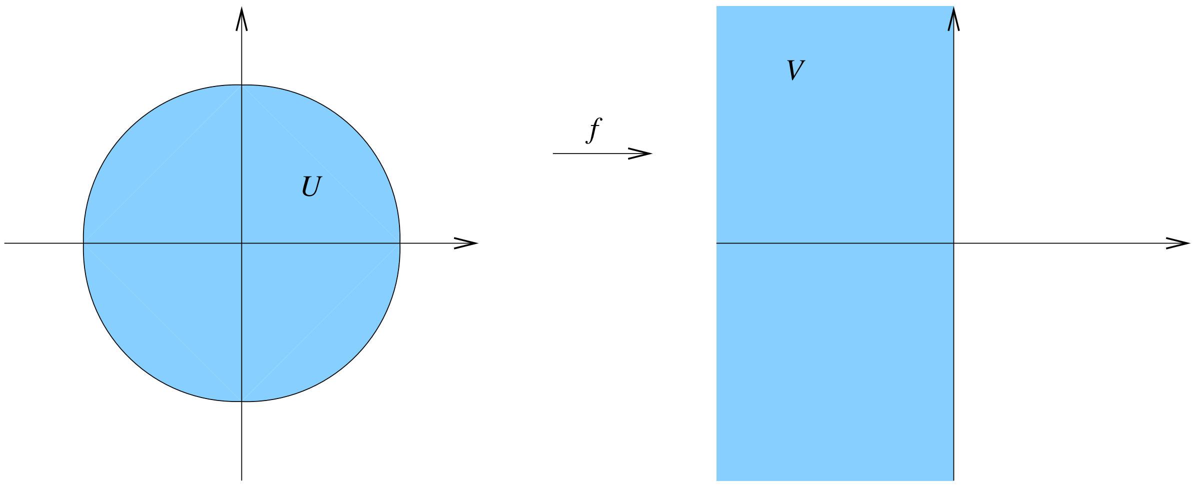
Recall: they map circlines to circlines.

Consider $\quad f(z) = \dfrac{z-1}{z+1}\,, \quad$ on $\quad U = \{z \in \mathbb{C} \;:\; |z| < 1\}$ "unit disk"

Let's find $\partial V = f(\partial U)$: $-1,\ \mathrm{i},\ 1 \in \partial U$:

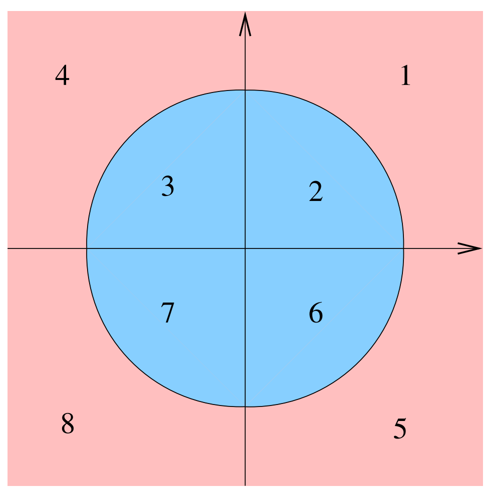
$f(-1) = \infty\,, \qquad f(\mathrm{i}) = \mathrm{i}\,, \qquad f(1) = 0 \qquad \Rightarrow \qquad \partial V =$ imaginary axis

Also: $f(z=0) = -1 \qquad \Rightarrow \qquad V =$ left half plane $\mathrm{Re}(w) < 0$




We can compute more points, e.g. $f\left(z = \frac{1+\mathrm{i}}{\sqrt{2}}\right) = \ldots = \dfrac{\mathrm{i}}{1+\sqrt{2}}$

One can show that $f$ maps regions 1-8 (1, 4, 5, 8 exterior to unit disk!) of




according to the pattern: $1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 1 \mapsto \ldots$

$5 \mapsto 6 \mapsto 7 \mapsto 8 \mapsto 5 \mapsto \ldots$

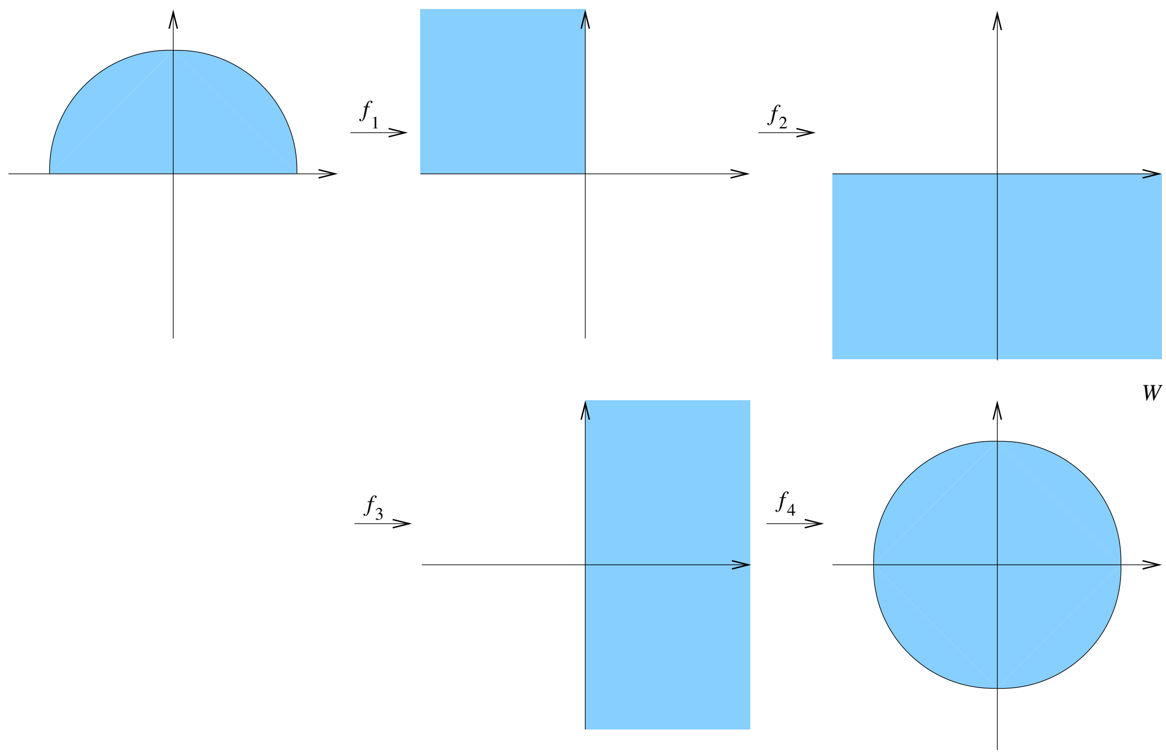
Agrees with unit disk $\xrightarrow{f}$ left half plane

**(6)** $f(z) = \frac{1}{z}$ is another Möbius map; cf. example sheet.

**(7)** Let's map the upper half disk $|z| < 1$, $\mathsf{Im}(z) > 0$ to the full disk $|z| < 1$

$f(z) = z^2$ doesn't work: no point gets mapped to $\mathbb{R}^+$ (e.g. to $\frac{1}{2}$) since $z \propto e^{\mathrm{i}\pi} \notin U$

Use Möbius maps...




- $f_1(z) = \dfrac{z-1}{z+1}$ takes the upper half disk to the 2nd quadrant
- $f_2(z) = z^2$ takes the 2nd quadrant to the lower half plane
- Rotate by $\frac{\pi}{2}$: $f_3(z) = \mathrm{i}z$
- $f_4(z) = f_1(z) = \dfrac{z-1}{z+1}$ maps the right half plane to the full circle

Looks like magic but works: E.g.

$$\frac{1}{2} \xrightarrow{f_4^{-1}} 3 \xrightarrow{f_3^{-1}} -3\mathrm{i} \xrightarrow{f_2^{-1}} \sqrt{3}e^{\mathrm{i}3\pi/4} \xrightarrow{f_1^{-1}} \text{somewhere in region 3}$$

### B.7.3 Laplace's equation and conformal maps

Let $U \subseteq \mathbb{R}^2$ be a "tricky" domain, $V \subseteq \mathbb{R}^2$ a "nice" domain.

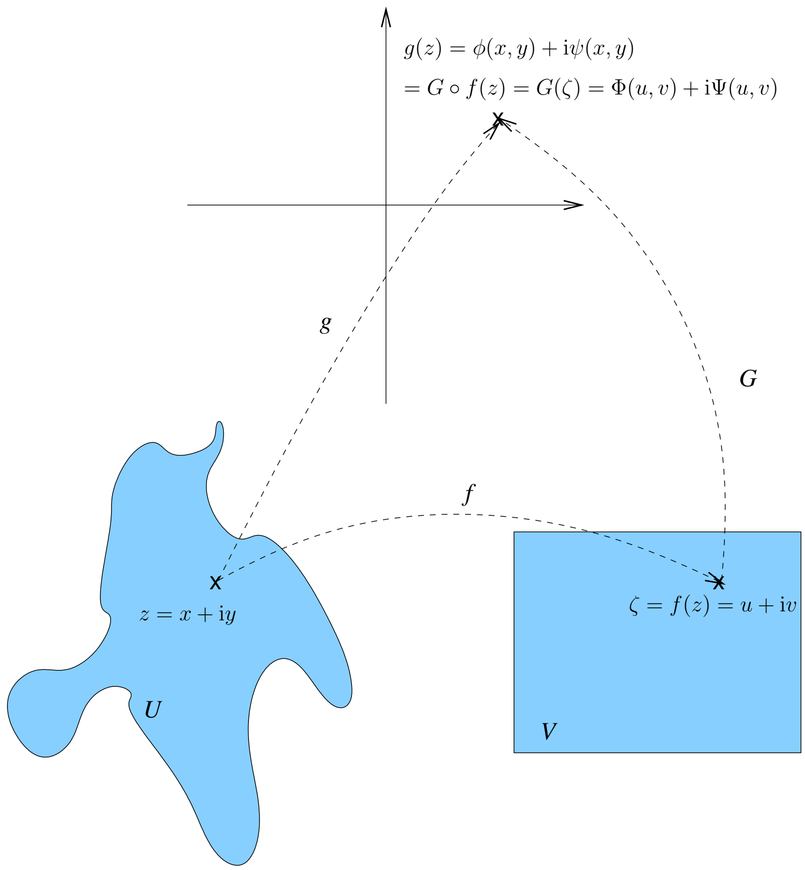
We write: $z = x + \mathrm{i}y \in U$, $\zeta = u + \mathrm{i}v \in V$.

Let $f : U \to V$ conformal: $\zeta := f(z) = u(x,y) + \mathrm{i}v(x,y)$

Recall: $G(\zeta) = \Phi(u,v) + \mathrm{i}\Psi(u,v)$ is analytic $\Rightarrow$ $\Phi$, $\Psi$ are harmonic: $\triangle\Phi = \triangle\Psi = 0$.

Clearly: $g := G \circ f$ is also analytic: $g(z) = G\big(f(z)\big) = \phi(x,y) + \mathrm{i}\psi(x,y)$

$\Rightarrow$ $\phi$, $\psi$ are also harmonic



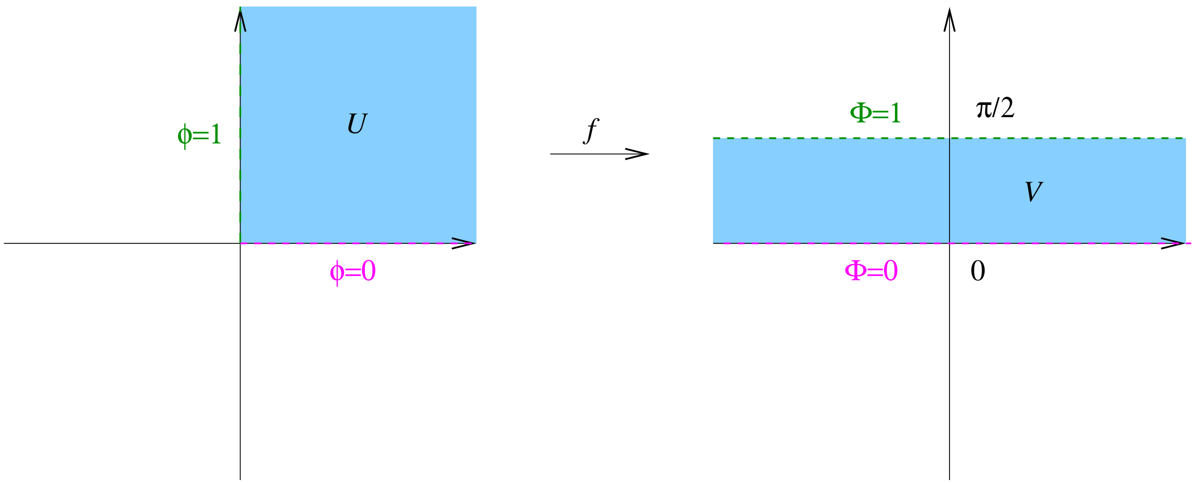

We can use this to solve the Laplace equations on complicated domains!

Goal: find solution of $\triangle\phi = \partial_x^2\phi + \partial_y^2\phi = 0$ on $U$ with Dirichlet boundary conditions on $\partial U$.

1. Find simple domain $V$ and conformal $f : U \to V$, $\quad z = x + \mathrm{i}y \mapsto \zeta = u + \mathrm{i}v$
2. Translate boundary conditions $\phi = \phi_0(x,y)$ on $\partial U$ into conditions $\Phi = \Phi_0(u,v)$ on $\partial V$
3. Solve $\triangle\Phi = 0$ on the nice domain $V$
4. $\phi(x,y) = \Phi\big(u(x,y), v(x,y)\big)$ solves $\quad \triangle\phi = 0$ on $U$

**Example**

Solve $\triangle\phi = 0$ on $U =$ the 1st quadrant of $\mathbb{R}^2$ with $\phi(x,0) = 0$, $\phi(0,y) = 1$




- $U$ is part of an annulus with: $r \in (0, \infty)$, $\theta \in \left(0, \frac{\pi}{2}\right)$.
- $f(z) = \log z$ maps $U$ to the "strip" $0 < \mathsf{Im}(z) < \frac{\pi}{2}$:

  $u(x,y) = \mathsf{Re}(\log z) = \log|z| \in (-\infty, \infty)$, $\quad v(x,y) = \mathsf{Im}(\log z) = \arg z\,.$
- So we need to solve $\partial_u^2\Phi + \partial_v^2\Phi = 0$ with $\Phi(u,0) = 0$, $\quad \Phi\left(u, \frac{\pi}{2}\right) = 1$
- Easy: $\Phi(u,v) = \dfrac{2}{\pi} v\,.$

  $\Rightarrow \phi(x,y) = \Phi(u,v) = \dfrac{2}{\pi} \arg z = \dfrac{2}{\pi} \arctan \dfrac{y}{x}$

Note: $\arg z = \arctan \frac{y}{x}$ is ok here, since $\arg z \in \left(0, \frac{\pi}{2}\right)$.

# C Contour integration and Cauchy's theorem

## C.1 Contours and integrals

Complex differentiation: We had $\infty$ directions.

Complex integration: We have $\infty$ paths from $a$ to $b$; unlike in $\mathbb{R}$!

In contrast to differentiation, we do *not* demand path independence of integration!

Let's define integration in $\mathbb{C}$...

**Def.:** A *curve* $\gamma$ is a continuous map $\gamma : [0, 1] \to \mathbb{C}$.

**Comments:**

- Without loss of generality, we use a parameter range $I = [0, 1]$. Use $\lambda(x) = a + (b - a)x$ to switch to $I = [a, b]$.
- We sometimes also denote by $\gamma$ the image $\gamma(I)$.
- A curve has a *direction*: from $\gamma(0)$ to $\gamma(1)$.

**Def.:** A *closed curve* is a curve $\gamma$ with $\gamma(0) = \gamma(1)$.

**Def.:** A *simple curve* is a curve $\gamma$ that does not intersect itself except at the end points $\gamma(0)$, $\gamma(1)$.

**Def.:** A *contour* is a piecewise differentiable curve.

**Notation:**

- $-\gamma :=$ reversed $\gamma$: $\quad t \mapsto (-\gamma)(t) = \gamma(1 - t)$.
- We can join two curves $\gamma_1$, $\gamma_2$ if $\gamma_1(1) = \gamma_2(0)$,

$$t \mapsto (\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{for } t < \frac{1}{2} \\ \gamma_2(2t - 1) & \text{for } t \geq \frac{1}{2} \end{cases}$$

**Def.:** The *contour integral* of a function $f$ along the contour $\gamma$ is:

$$\int_\gamma f(z)\mathrm{d}z := \int_0^1 f\big(\gamma(t)\big)\gamma'(t)\mathrm{d}t\,.$$

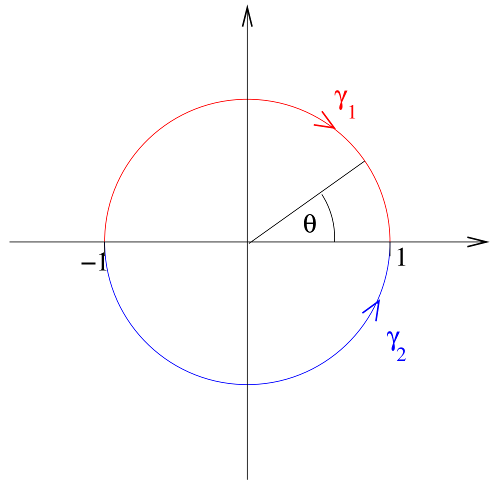
Cf. the integral of a vector field in $\mathbb{R}^n$: $\displaystyle\int_{\mathcal{C}} \boldsymbol{F}(\boldsymbol{r})\mathrm{d}\boldsymbol{r} = \int_A^B \boldsymbol{F}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)\mathrm{d}t\,.$

Alternatively: Dissect $[0, 1]$ into $0 = t_0 < t_1 < \ldots < t_n = 1$

Let $\delta t_k = t_{k+1} - t_k$, $\quad \delta z_k = z_{k+1} - z_k$, where $z_k = \gamma(t_k)$. Then

$$\int_\gamma f(z)\mathrm{d}z := \lim_{\Delta\to 0} \sum_{k=0}^{n-1} f(z_n)\delta z_n\,, \qquad \text{where} \quad \Delta = \max_{k=0,\ldots,n-1} \delta t_k\,.$$

**Example.:** Let $f(z) = \frac{1}{z}$ and $\gamma_1$, $\gamma_2$ be unit half circles, $z = \gamma(\theta) = e^{\mathrm{i}\theta}$, $\gamma'(\theta) = \mathrm{i}e^{\mathrm{i}\theta}$,




$$\Rightarrow \qquad I_1 = \int_{\pi}^{0} \frac{\mathrm{i}e^{\mathrm{i}\theta}\mathrm{d}\theta}{e^{\mathrm{i}\theta}} = -\mathrm{i}\pi\,, \qquad I_2 = \int_{-\pi}^{0} \frac{\mathrm{i}e^{\mathrm{i}\theta}\mathrm{d}\theta}{e^{\mathrm{i}\theta}} = \mathrm{i}\pi\,.$$

**Rules of integration:** (without proof)

**(1)** Joint contour: $\displaystyle\int_{\gamma_1+\gamma_2} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z\,;$ same as in $\mathbb{R}$ : $\int_a^c = \int_a^b + \int_b^c$

**(2)** Reversed contour: $\displaystyle\int_{-\gamma} f(z)\mathrm{d}z = -\int_{\gamma} f(z)\mathrm{d}z\,;$ as in $\mathbb{R}$ : $\int_a^b = -\int_b^a$

**(3)** If $f$ is differentiable along a contour $\gamma$ from $a$ to $b$, then

$$\int_{\gamma} f'(z)\mathrm{d}z = f(b) - f(a)\,.$$

Note: This does not contradict the above path dependence of $\int \frac{1}{z}\mathrm{d}z$ (Why?)

**(4)** Integration by parts and substitution work as in $\mathbb{R}$.

**(5)** Length of a curve: $\displaystyle L = \int_{\gamma} |\mathrm{d}z| = \int_0^1 |\gamma'(t)|\mathrm{d}t\,.$

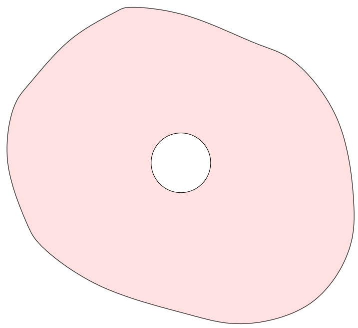
If $|f(z)| \le f_0$ along $\gamma$, then: $\displaystyle\left|\int_{\gamma} f(z)\mathrm{d}z\right| \le f_0 L\,.$

**Closed contours:**

- We denote integrals over closed contours by $\oint$.
- $\oint_{\gamma} f(z)\mathrm{d}z$ depends on the direction, but not on the strating point
- Convention: traverse $\gamma$ counter clockwise

  $\Leftrightarrow$ interior of $\gamma$ is on the left

**Def.:** An open set $\mathcal{D} \subseteq \mathbb{C}$ is a *connected domain* if each pair $z_1,\ z_2 \in \mathbb{C}$ can be connected by a curve whose image is in $\mathcal{D}$. $\mathcal{D}$ is *simply connected* if if is connected and every curve in $\mathcal{D}$ encloses only points in $\mathcal{D}$ ("no holes"!).



is not simply connected

Note: A single point is enough to make a hole.

## C.2 Cauchy's theorem

**Theorem:** If $f(z)$ is analytic in a simply connected domain $\mathcal{D}$ and $\gamma$ is a closed contour in $\mathcal{D}$,

$$\boxed{\oint_\gamma f(z)\mathrm{d}z = 0}$$

*Proof.* (slightly simplified)

Green's theorem for functions $P$, $Q$ with continuous partial derivatives on $\mathcal{D} \supseteq \mathcal{M}$, where $\mathcal{M}$ is the interior of a simple closed contour $\gamma$ in $\mathbb{R}^2$:

$$\oint_\gamma (P\mathrm{d}x + Q\mathrm{d}y) = \int\int_\mathcal{M} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y\,. \qquad (\dagger)$$

Write the complex $f(z) = u(x,y) + \mathrm{i}v(x,y)$, so:

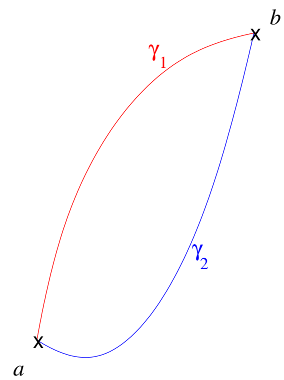
$$\begin{aligned}\oint_\gamma f(z)\mathrm{d}z &= \oint_\gamma (u+\mathrm{i}v)(\mathrm{d}x+\mathrm{i}\mathrm{d}y) = \oint_\gamma (u\mathrm{d}x - v\mathrm{d}y) + \mathrm{i}\oint_\gamma (v\mathrm{d}x + u\mathrm{d}y) \\ &\stackrel{(\dagger)}{=} \int\int_\mathcal{M} \underbrace{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{=0} \mathrm{d}x\,\mathrm{d}y + \mathrm{i}\int\int_\mathcal{M} \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{=0} \mathrm{d}x\,\mathrm{d}y = 0\,,\end{aligned}$$

thanks to the Cauchy-Riemann conditons. $\square$

## C.3 Deforming contours

**Proposition:** Let $\gamma_1$, $\gamma_2$ be contours from $a$ to $b$ in $\mathbb{C}$, and $f(z)$ be analytic on both contours and the region bounded by the contours. Then

$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z\,.$$




*Proof.* Let $\gamma_1$, $\gamma_2$ not intersect each other except at $a$, $b$.

$\Rightarrow \quad \gamma_1 - \gamma_2 := \gamma_1 + (-\gamma_2)$ is a simple closed contour

$$\Rightarrow \quad \oint_{\gamma_1-\gamma_2} f(z)\mathrm{d}z = 0\,.$$

If $\gamma_1$, $\gamma_2$ intersect each other, dissect the curve at each crossing point and apply the proof to each individual closed curve. □

**Comments**:

- Compare with exact differentials in $\mathbb{R}^2$: Write

  $$\mathrm{d}f = f(z)\mathrm{d}z = (u+\mathrm{i}v)(\mathrm{d}x+\mathrm{i}\mathrm{d}y) = \underbrace{(u+\mathrm{i}v)}_{=:P}\mathrm{d}x + \underbrace{(-v+\mathrm{i}u)}_{=:Q}\mathrm{d}y$$

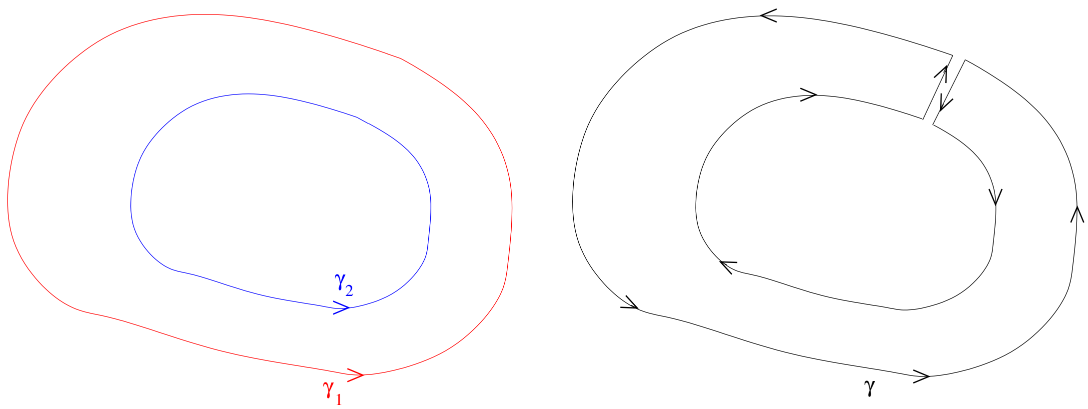
  $\Rightarrow \quad \partial_y P = \partial_y(u+\mathrm{i}v) = \partial_x(-v+\mathrm{i}u) = \partial_x Q$ by C.R. $\Rightarrow \quad \mathrm{d}f$ is exact!

  $\Rightarrow \quad$ The integral of $f$ is path independent.

- Cauchy's theorem lets us deform contours:

  Let $\gamma_1$, $\gamma_2$ be closed contours that can be continuously deformed into each other.

  Let $f(z)$ be analytic on and between $\gamma_1$, $\gamma_2$




Cut out a tiny piece from $\gamma_1$, $\gamma_2$ to get a single closed contour $\gamma$.

Use Cauchy's theorem on $\gamma$ and let the gap shrink to zero width,

$$\oint_\gamma f(z)\mathrm{d}z = 0 \qquad \Rightarrow \qquad \int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z.$$

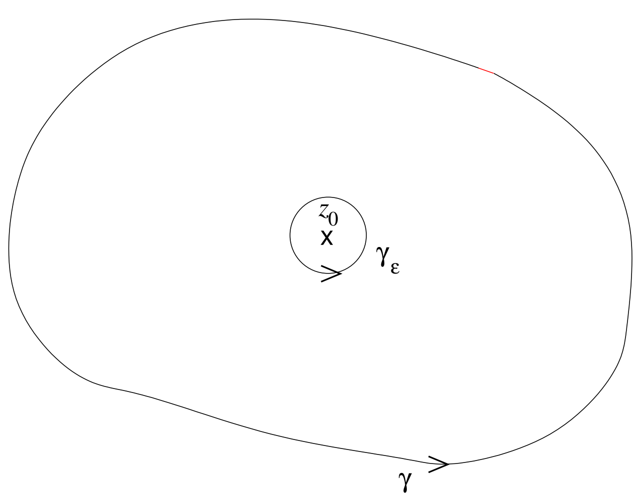
## C.4 Cauchy's integral formula

Integration along closed contours: Functions with singularities inside are more interesting!

**Theorem:** *Cauchy's integral formula*: Let $f(z)$ be analytic on an open domain $\mathcal{D}$, $z_0 \in \mathcal{D}$, and $\gamma$ a simple closed contour inside $\mathcal{D}$ that encircles $z_0$ counter clockwise.

$$\Rightarrow \quad \boxed{f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{z - z_0}\mathrm{d}z\,, \qquad \text{and} \qquad f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint \frac{f(z)}{(z - z_0)^{n+1}}\mathrm{d}z} \qquad (\ddagger)$$

*Proof.* Let $\gamma_\epsilon$ be a counter clockwise circular contour of radius $\epsilon$ inside $\gamma$




$$\Rightarrow \quad \frac{f(z)}{z - z_0}$$ is analytic between $\gamma$, $\gamma_\epsilon$. Write $z = z_0 + \epsilon e^{\mathrm{i}\theta}$ and let $\epsilon \to 0$, so:

$$\oint_\gamma \frac{f(z)}{z-z_0}\mathrm{d}z = \oint_{\gamma_\epsilon} \frac{f(z)}{z-z_0}\mathrm{d}z = \int_0^{2\pi} \frac{f(z_0+\epsilon e^{\mathrm{i}\theta})}{\epsilon e^{\mathrm{i}\theta}}\mathrm{i}\epsilon e^{\mathrm{i}\theta}\mathrm{d}\theta = \lim_{\epsilon\to 0}\ \mathrm{i}\int_0^{2\pi} f(z_0+\epsilon e^{\mathrm{i}\theta})\mathrm{d}\theta = 2\pi\mathrm{i}f(z_0)$$

That's the first Eq. **(‡)**. For the second, take $\frac{\mathrm{d}}{\mathrm{d}z_0}$ $n$ times

$$\Rightarrow \quad f'(z_0) = \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{f(z)}{(z-z_0)^2}\mathrm{d}z\,, \text{ etc.} \qquad \square$$

**Comments:**

- Knowing $f$ on $\gamma$ gives us $f(z)$ for every point inside. How?
  $f(z) = u(x,y) + \mathrm{i}v(x,y)$ is analytic
  $\Rightarrow$ $u$, $v$ are uniquely determined as solutions to the Laplace equation with Dirichlet boundary conditions on $\gamma$
  This does not work for $z_0$ outside $\gamma$: Then $\frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{f(z)}{z-z_0}\mathrm{d}z = 0$
- If $f$ is analytic at $z_0$, by Eq. **(‡)** it is differentiable $\infty$ times at $z_0$

**Liouville's theorem:** If $f$ is analytic on all $\mathbb{C}$ and bounded, it is constant.

*Proof.* $\exists c_0 \in \mathbb{R} \quad \forall z \in \mathbb{C} \quad |f(z)| \le c_0$

Let $\gamma_r$ be a counter clockwise circular contour of radius $r$ around $z_0$.

$$|f'(z_0)| \overset{(‡)}{=} \left|\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_r} \frac{f(z)}{(z-z_0)^2}\mathrm{d}z\right| \le \frac{1}{2\pi}\int_{\gamma_r}\frac{c_0}{r^2}\mathrm{d}z = \frac{c_0}{r} \overset{r\to\infty}{\longrightarrow} 0$$

We can use any $r$. $\square$

# D Laurent series and singularities

## D.1 Taylor series and Laurent series

Recall Taylor series in $\mathbb{R}$: $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$

On $\mathbb{C}$ we have the more general:

**Proposition:** Let $f(z)$ be analytic in an annulus $R_1 < |z - z_0| < R_2$.

Then $f$ has the Laurent series $\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n}$

If $f(z)$ is analytic at $z_0$, it has the Taylor series

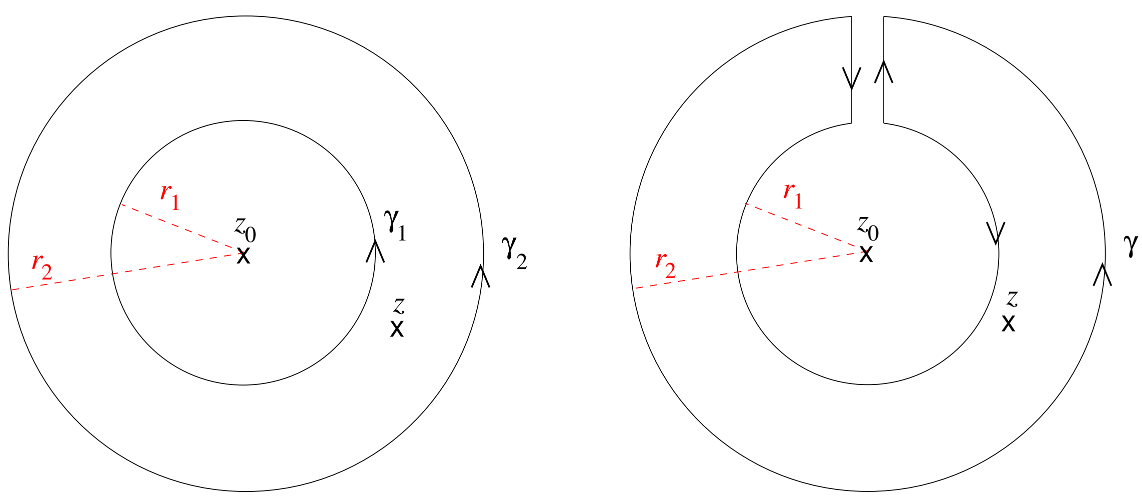
$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{with} \qquad a_n = \frac{f^{(n)}(z_0)}{n!}}$$

*Proof.* Without loss of generality, $z_0 = 0$.

Let $z \in \mathbb{C}$ with $R_1 < r_1 < |z| < r_2 < R_2$.

Let $\gamma_1$, $\gamma_2$ be counter clockwise circular contours of radius $r_1$, $r_2$.

Approximate with a closed contour $\gamma$:




Use Cauchy's integral formula (with $z \mapsto \zeta$, $z_0 \mapsto z$) and infinitesimal gap

$$\Rightarrow f(z) = \frac{1}{2\pi\mathrm{i}} \oint_{\gamma} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}} \oint_{\gamma_2} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta - \frac{1}{2\pi\mathrm{i}} \oint_{\gamma_1} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \,.$$

For $\gamma_1$: $\left|\frac{\zeta}{z}\right| < 1$. Use geometric series in $\frac{\zeta}{z}$, so

$$\begin{aligned} I_1 := -\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_1}\frac{f(\zeta)}{\zeta - z} &= \frac{1}{2\pi\mathrm{i}z}\oint_{\gamma_1}\frac{f(\zeta)}{1-\frac{\zeta}{z}}\mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}z}\oint_{\gamma_1} f(\zeta)\sum_{m=0}^{\infty}\left(\frac{\zeta}{z}\right)^m \mathrm{d}\zeta \\ &= \frac{1}{2\pi\mathrm{i}}\sum_{m=0}^{\infty}\left[z^{-m-1}\oint_{\gamma_1} f(\zeta)\zeta^m \mathrm{d}\zeta\right] \\ &= \sum_{n=-\infty}^{-1} a_n z^n\,, \qquad \text{with} \qquad a_n = \frac{1}{2\pi\mathrm{i}}\oint_{\gamma_1} f(\zeta)\zeta^{-n-1}\mathrm{d}\zeta\,. \end{aligned}$$

For $\gamma_2$: $\left|\frac{z}{\zeta}\right| < 1 \quad \Rightarrow \quad \ldots \quad \Rightarrow$

$$I_2 := \frac{1}{2\pi\mathrm{i}}\oint_{\gamma_2}\frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = \sum_{n=0}^{\infty} a_n z^n \qquad \text{with} \qquad a_n = \frac{1}{2\pi\mathrm{i}}\oint_{\gamma_2} f(\zeta)\zeta^{-n-1}\mathrm{d}\zeta$$

That's the Laurent series.

If $f$ is analytic at $z_0$, then it is also analytic inside $\gamma_1$ for small enough $r_1$.

For $n \leq -1$ (but not for $n \geq 0$ !), $\zeta^{-n-1}$ is also analytic inside $\gamma_1$

$\Rightarrow \quad I_1 = 0 \qquad$ by Cauchy's theorem!

$\Rightarrow \quad f(z) = I_2 \quad$ with $\quad a_n = \frac{1}{2\pi\mathrm{i}}\oint_{\gamma_2}\frac{f(\zeta)}{\zeta^{n+1}}\mathrm{d}\zeta \overset{(\ddagger)}{=} \frac{1}{n!}f^{(n)}(0) \quad$ Cauchy's integral formula! $\square$

**Comments:**
- One can show that the Laurent series is unique.
- The Taylor series is the same as for real functions.

**Examples**

**(1)** $f(z) = \frac{e^z}{z^3} = \sum_{n=0}^{\infty}\frac{z^{(n-3)}}{n!} = \sum_{n=-3}^{\infty}\frac{z^n}{(n+3)!}$

**(2)** $f(z) = e^{\frac{1}{z}}$ about 0: $\quad e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \ldots = \sum_{n=-\infty}^{0} a_n z^n\,, \qquad a_n = \frac{1}{(-n)!}$

**(3)** $f(z) = \frac{1}{z-a}$, $a \in \mathbb{C}$, about 0:

$$|z| < |a| \quad \Rightarrow \quad \frac{1}{z-a} = -\frac{1}{a}\frac{1}{1-\frac{z}{a}} = -\frac{1}{a}\sum_{m=0}^{\infty}\left(\frac{z}{a}\right)^m = -\sum_{n=0}^{\infty}\frac{1}{a^{n+1}}z^n$$

$$|z| > |a| \quad \Rightarrow \quad \frac{1}{z-a} = \frac{1}{z}\frac{1}{1-\frac{a}{z}} = \frac{1}{z}\sum_{m=0}^{\infty}\left(\frac{a}{z}\right)^m = \sum_{n=-\infty}^{-1} a^{-n-1}z^n\,.$$

**(4)** $f(z) = \dfrac{e^z}{z^2-1}$ is singular at $z = \pm 1$. What is its Laurent series about $z_0 = 1$?

Common trick: $\zeta = z - z_0 = z - 1$

$$\begin{aligned} f(z) &= \frac{e^\zeta\, e}{\zeta(\zeta+2)} = e\frac{e^\zeta}{2\zeta}\frac{1}{1+\frac{\zeta}{2}} \\ &= \frac{e}{2\zeta}\left(1+\zeta+\frac{1}{2!}\zeta^2+\ldots\right)\left[1-\frac{\zeta}{2}+\left(\frac{\zeta}{2}\right)^2\mp\ldots\right] \\ &= \frac{e}{2\zeta}\left(1+\frac{1}{2}\zeta+\ldots\right) = \frac{e}{2}\left(\frac{1}{z-1}+\frac{1}{2}+\ldots\right). \\ \Rightarrow a_{-1} &= \frac{e}{2}, \qquad a_0 = \frac{e}{4}. \end{aligned}$$

Often, we only need $a_{-1}$...

**(5)** $f(z) = z^{-1/2}$ about 0 does not work: $f$ has branchpoints at 0 and $\infty$

$\Rightarrow$ Every annulus crosses the branch cut, so $\nexists$ annulus where $f$ is analytic.

## D.2 Zeros and singularities

**Theorem:** Any polynomial $P(z)$ of degree $n \geq 1$ can be factorized as

$P(z) = a(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_k)^{m_k}$,

where $m_1 + m_2 + \ldots m_k = n$, $a \in \mathbb{C}$.

Use the Taylor expansion to generalize this to other functions:

**Def.:** The *zeros* of a function $f(z)$ are the points $z_0$ where $f(z_0) = 0$. A zero $z_0$ is of *order* $n$ if in its Taylor expansion $\sum_{k=0}^{\infty} a_k(z-z_0)^k$, the first non-zero coefficient is $a_k$ or, equivalently, if

$f(z_0) = f'(z_0) = \ldots = f^{(n-1)}(z_0) = 0$, but $f^{(n)}(z_0) \neq 0$.

A *simple zero* is a zero of order $n = 1$.

**Examples**

**(1)** $f(z) = z^3 + \mathrm{i}z^2 + z + \mathrm{i} = (z-\mathrm{i})(z+\mathrm{i})^2$:

a simple zero $z = \mathrm{i}$ and a zero of order 2 at $z = -\mathrm{i}$.

**(2)** $\sinh z = \frac{1}{2}(e^z - e^{-z}) = 0 \quad\Leftrightarrow\quad e^{2z} = 1 \quad\Leftrightarrow\quad z = \mathrm{i}n\pi$ with $n \in \mathbb{Z}$.

$\cosh(\mathrm{i}n\pi) = \cos(n\pi) = \pm 1 \quad\Rightarrow\quad$ all zeros are simple.

**(3)** $\sinh z$ has a simple zero at $\mathrm{i}\pi$, so:

$\sinh^3 z = [a_1(z-\mathrm{i}\pi)+\ldots]^3 = a_1^3(z-\mathrm{i}\pi)^3 + \ldots$ has a zero of order 3 at $\mathrm{i}\pi$.

We get its Taylor series using $\zeta = z - \mathrm{i}\pi$:

$$\sinh^3 z = [\sinh(\zeta + \mathrm{i}\pi)]^3 = (-\sinh\zeta)^3 = -\left(\zeta + \frac{1}{3!}\zeta^3 + \dots\right)^3 = -\zeta^3 - \frac{1}{2}\zeta^5$$
$$= -(z - \mathrm{i}\pi)^3 - \frac{1}{2}(z - \mathrm{i}\pi)^5 + \dots$$

singularities are "the inverse of zeros"

<u>**Def.:**</u> A *singularity* of $f(z)$ is a point $z_0$ where $f$ is not analytic.

*isolated singularity*: $f$ is analytic in a neighbourhood of $z_0$ (but not at $z_0$).

*non-isolated singularity*: $f$ is not analytic at $z_0$ nor in any neighbourhood of $z_0$

**Examples**

**(1)** $f(z) = \dfrac{1}{\sinh z}$ has isolated singularities at $z = \mathrm{i}n\pi$, $n \in \mathbb{Z}$, since sinh is zero there.

**(2)** $f(z) = \dfrac{1}{\sinh \frac{1}{z}}$ has isolated singularities at $z = \dfrac{1}{\mathrm{i}n\pi}$ for $n \neq 0$.

$f$ has a non-isolated singularity at $z = 0$, since for large enough $n$, another singularity $\dfrac{1}{\mathrm{i}n\pi}$ is arbitrarily close by

**(3)** $\dfrac{1}{\sinh z}$ has a non-isolated singularity at $z = \infty$, since $\dfrac{1}{\sinh \frac{1}{z}}$ has one at $z = 0$.

**(4)** $f(z) = \log z$ has a non-isolated singularity at $z = 0$: $z = 0$ must be connected to a branch cut! This is also called a *branch point singularity*.

For isolated singularities, $f$ is analytic in an annulus around it, so $f$ has a Laurent series.

Singularity checklist:

1. Is $z_0$ a branch-point singularity?
2. Is it a non-isolated singularity?
3. If neither, find the Laurent series and check:
   (a) If $a_n = 0 \ \ \forall\, n < 0$, then $f(z) = a_0 + a_1(z - z_0) + \dots$

   $\Rightarrow$ the singularity is *removable* by redefining $f(z_0) = a_0$.

   (b) If $\exists N > 0 \ \ \forall n < -N - 1 : a_n = 0$ and $a_{-N} \neq 0$

   $\Rightarrow$ $f$ has a pole of order $N$ at $z_0$.

   For $N = 1,\ 2,\ 3$ we say *Simple*, *Double* or *Triple* pole

   (c) If there is no such $N$, $f$ has an *essential isolated singularity* at $z_0$

**Examples**

**(1)** $f(z) = \frac{1}{z - \mathrm{i}}$ has a simple pole at $z = \mathrm{i}$.

**(2)** $f(z) = \dfrac{\cos z}{z}$ has a simple pole at $z = 0$: $\dfrac{\cos z}{z} = z^{-1} - \dfrac{1}{2}z + \dfrac{1}{24}z^3 \mp \ldots$

**(3)** $g(z) = \dfrac{z^2}{(z-1)^3(z-\mathrm{i})^2}$ has a double pole at $z = \mathrm{i}$:

Let $G(z) = \dfrac{z^2}{(z-1)^3} \quad \Rightarrow \quad g(z) = \dfrac{G(z)}{(z-\mathrm{i})^2}$

$G(z)$ is analytic at $z = \mathrm{i}$ with Taylor series $G(z) = b_0 + b_1(z-\mathrm{i}) + b_2(z-\mathrm{i})^2 + \ldots, \; b_0 \neq 0$

$\Rightarrow \quad g(z) = \dfrac{b_0}{(z-\mathrm{i})^2} + \dfrac{b_1}{z-\mathrm{i}} + b_2 + \ldots.$

We likewise show that $g$ has a triple pole at $z = 1$.

**(4)** More generally: Let $f(z)$ have a zero of order $n$ at $z_0$

$\Leftrightarrow$ $\frac{1}{f(z)}$ has a pole of order $n$ at $z_0$

**(5)** $f(z) = z^2$ has a double pole at $\infty$, since $\frac{1}{\zeta^2}$ has one at $\zeta = 0$

**(6)** $e^{\frac{1}{z}} = \displaystyle\sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n$ has an essential singularity at $z = 0$

**(7)** One likewise shows that $\sin \frac{1}{z}$ has an essential singularity at $z = 0$

**(8)** $f(z) = \dfrac{e^z - 1}{z} = 1 + \dfrac{1}{2!}z + \dfrac{1}{3!}z^2 + \ldots$ has a removable singularity at $z = 0$:

redefine $f(0) = 1$

**(9)** $f(z) = \frac{\sin z}{z}$ also has a removable singularity at $z = 0$: redefine $f(0) = 1$

**(10)** Let $P$, $Q$ be polynomials with zero of order $m$, $n$, respectively, at $z_0$ and $m \geq n$.

$\Rightarrow f(z) = \dfrac{P(z)}{Q(z)}$ has a removable singularity at $z_0$ with $f(z_0) = \dfrac{P^{(n)}(z_0)}{Q^{(n)}(z_0)}$

This follows from l'Hôpital's rule.

**Proposition:** Let $f(z)$ have an essential singularity at $z_0$

$\Rightarrow$ In any neighbourhood $\mathcal{D}$ of $z_0$, $f(z)$ takes on all possible complex values except at most one.

E.g. $e^{\frac{1}{z}}$ takes on any value except 0 around $z = 0$

## D.3 Residues

**Def.:** The *residue* $\operatorname*{Res}_{z=z_0} f$ of a function $f$ with isolated singularity at $z = z_0$ is the coefficient $a_{-1}$ in the Laurent expansion of $f$ about $z_0$.

**Proposition:** Let $f$ have a pole of order $n$ at $z = z_0$.

$$\Rightarrow \quad \boxed{\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \frac{1}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}[(z-z_0)^n f(z)]} \qquad (\star\star)$$

For $n = 1$
$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}[(z-z_0)f(z)]} \qquad (\star)$$

*Proof.* $n = 1$: The Laurent series starts $f(z) = a_{-1}\dfrac{1}{z-z_0} + a_0 + \dots$

$$\Rightarrow \quad \lim_{z\to z_0} [(z-z_0)f(z)] = a_{-1}$$

Use induction for $n > 1$ (example sheet) □

There are many tools to compute residuals...

**Examples**

**(1)** $f(z) = \dfrac{e^z}{z^3} = z^{-3} + z^{-2} + \dfrac{1}{2!}z^{-1} + \dfrac{1}{3!} + \dots \qquad \Rightarrow \qquad \operatorname*{Res}_{z=0} f(z) = \dfrac{1}{2}$

**(2)** $f(z) = \dfrac{e^z}{z^2-1} \quad \Rightarrow \quad \operatorname*{Res}_{z=1} \dfrac{e^z}{z^2-1} \overset{(\star)}{=} \lim_{z\to 1}(z-1)\dfrac{e^z}{z^2-1} = \lim_{z\to 1}\dfrac{e^z}{1+z} = \dfrac{e}{2}$

**(3)** Brute force does not always work:

$z^8 = w^8 \quad \Rightarrow \quad z = we^{\mathrm{i}n\pi/4}$ with $n = 0, \dots 7$

$$\operatorname*{Res}_{z=w} \frac{1}{z^8-w^8} = \lim_{z\to w}\frac{z-w}{z^8-w^8} = \frac{1}{(w-we^{\mathrm{i}\pi/4})\cdots(w-we^{\mathrm{i}7\pi/4})} = \text{hmm...}$$

Better use l'Hôpital: $\operatorname*{Res}_{z=w} \dfrac{1}{z^8-w^8} = \lim_{z\to w}\dfrac{z-w}{z^8-w^8} = \lim_{z\to w}\dfrac{1}{8z^7} = \dfrac{1}{8w^7}$

**(4)** $\sinh(\pi x)$ has simple zeros at $z = n\mathrm{i}$, $n \in \mathbb{Z}$

$\Rightarrow \dfrac{1}{\sinh(\pi z)}$ has simple poles at $z = n\mathrm{i}$.

Eq. $(\star)$ with l'Hôpital: $\operatorname*{Res}_{z=n\mathrm{i}} \dfrac{1}{\sinh(\pi z)} = \lim_{z\to n\mathrm{i}}\dfrac{z-n\mathrm{i}}{\sinh(\pi z)} = \lim_{z\to n\mathrm{i}}\dfrac{1}{\pi\cosh(\pi z)} = \dfrac{(-1)^n}{\pi}$

**(5)** Recall $\sinh^3 z = -(z-\mathrm{i}\pi)^3 - \dfrac{1}{2}(z-\mathrm{i}\pi)^5 + \dots$

$$\Rightarrow \quad \frac{1}{\sinh^3 z} = -(z-\mathrm{i}\pi)^{-3}\left[1+\frac{1}{2}(z-\mathrm{i}\pi)^2+\dots\right]^{-1} = -(z-\mathrm{i}\pi)^{-3}\left[1-\frac{1}{2}(z-\mathrm{i}\pi)^2+\dots\right]$$

$$= -(z-\mathrm{i}\pi)^{-3} + \frac{1}{2}(z-\mathrm{i}\pi)^{-1} + \dots$$

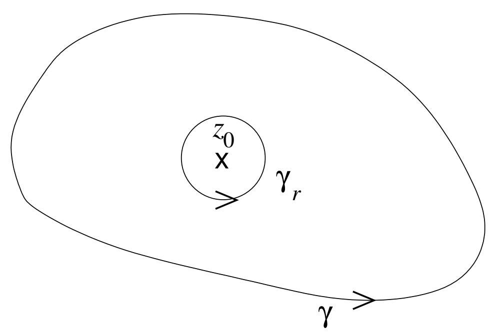
$$\Rightarrow \quad \operatorname*{Res}_{z=\mathrm{i}\pi} \frac{1}{\sinh^3 z} = \frac{1}{2}$$

What's special about $a_{-1}$?

**Theorem:** Let $\gamma$ be a simple closed contour in counter-clockwise direction, and $f(z)$ be analytic inside $\gamma$ except for an isolated singularity $z_0$. Then

$$\boxed{\oint_\gamma f(z)\mathrm{d}z = \mathrm{i}2\pi a_{-1} = \mathrm{i}2\pi \operatorname*{Res}_{z=z_0} f(z)}$$

*Proof.* $f(z)$ is analytic except for $z_0$. Deform the contour $\gamma$ into a circle $\gamma_r$ inside $\gamma$:




$$\Rightarrow \oint_\gamma f(z)\mathrm{d}z = \oint_{\gamma_r} f(z)\mathrm{d}z = \oint_{\gamma_r} \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n \oint_{\gamma_r} (z-z_0)^n \mathrm{d}z$$

The key point is

$$\begin{aligned}
\oint_{\gamma_r} (z-z_0)^n \mathrm{d}z &= \int_0^{2\pi} r^n e^{\mathrm{i}n\theta} \mathrm{i}re^{\mathrm{i}\theta}\mathrm{d}\theta = \mathrm{i}r^{n+1}\int_0^{2\pi} e^{\mathrm{i}(n+1)\theta}\mathrm{d}\theta \\
&= \begin{cases} \mathrm{i}2\pi\,, & \text{for } n=-1 \\ \frac{r^{n+1}}{n+1}\left[e^{\mathrm{i}(n+1)\theta}\right]_0^{2\pi} = 0 & \text{for } n\neq -1 \end{cases} .
\end{aligned} \tag{D.1}$$

□

Only the $a_{-1}$ term survives.

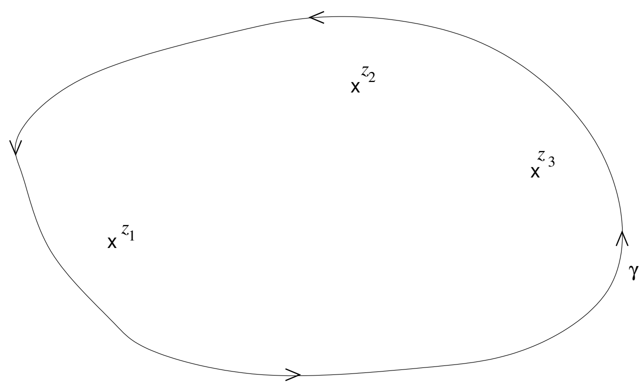
# E The calculus of residues

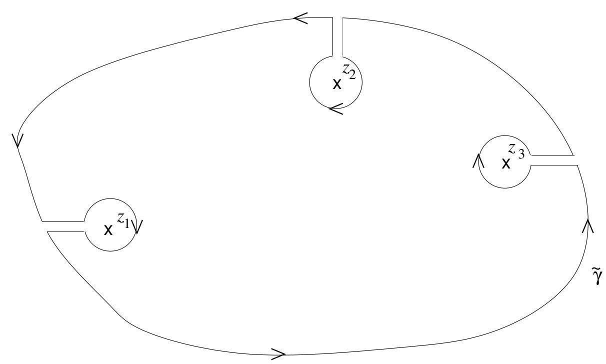
## E.1 The residue theorem

**Theorem:** Let $f(z)$ be analytic in a simply connected domain $\mathcal{D}$ except a finite number of isolated singularities $z_1, \ldots, z_n$. Let $\gamma$ be a simple closed counter-clockwise contour in $\mathcal{D}$ that encircles all $z_1, \ldots, z_n$.

$$\Rightarrow \quad \boxed{\oint_\gamma f(z)\mathrm{d}z = \mathrm{i}2\pi \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)}$$

*Proof.* Approximate $\gamma$ with $\tilde{\gamma}$ with the $z_k$ cut out:




□

$\Rightarrow \quad \tilde{\gamma}$ encloses no singularities!

$$\Rightarrow \quad 0 \stackrel{!}{=} \oint_{\tilde{\gamma}} f(z)\mathrm{d}z = \oint_\gamma f(z)\mathrm{d}z + \sum_{k=1}^{n} \oint_{\gamma_k} f(z)\mathrm{d}z$$

$$\Rightarrow \quad \oint_\gamma f(z)\mathrm{d}z = -\sum_{k=1}^{n} \oint_{\gamma_k} f(z)\mathrm{d}z = \sum_{k=1}^{n} \mathrm{i}2\pi \operatorname*{Res}_{z=z_k} f(z)\,, \qquad \text{since } \gamma_k \text{ are clockwise!}$$

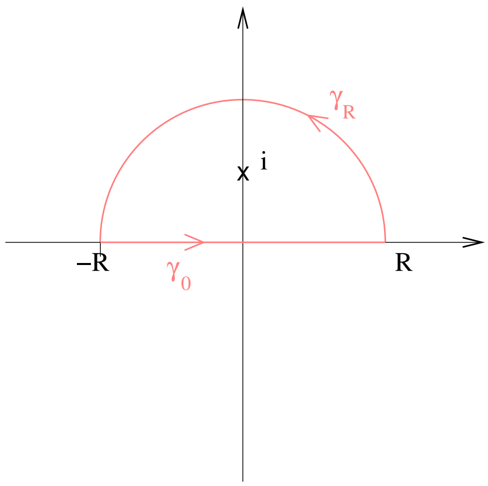
## E.2 Integrals along the real axis

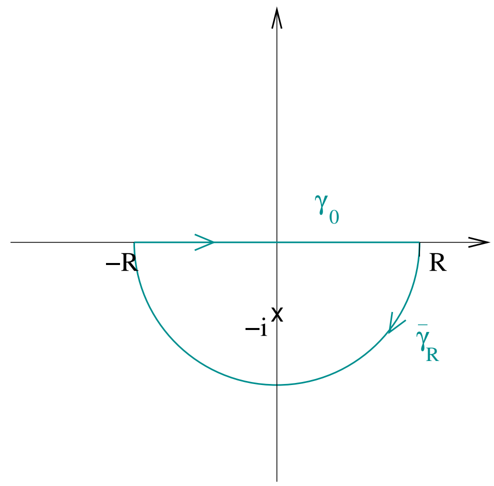
We can compute many integrals along parts or all of $\mathbb{R}$:

Complete with a circular segment and let radius $\to \infty$

### Examples

**(1)** $I = \displaystyle\int_0^\infty \frac{\mathrm{d}x}{1+x^2}$. Use the closed contour $\gamma_0 + \gamma_R$:




with $z = \gamma_0(t) = t,\ \gamma_0'(t) = 1$: $\displaystyle\lim_{R\to\infty}\int_{\gamma_0}\frac{\mathrm{d}z}{1+z^2} = \int_{-\infty}^{\infty}\frac{\mathrm{d}t}{1+t^2} = 2I$

The only singularity inside $\gamma_0 + \gamma_R$ is the simple pole $z = \mathrm{i}$:

$$\operatorname*{Res}_{z=\mathrm{i}}\frac{1}{1+z^2} = \lim_{z\to\mathrm{i}}\frac{1}{z+\mathrm{i}} = \frac{1}{2\mathrm{i}}$$

$$\Rightarrow \quad \oint_{\gamma_0+\gamma_R}\frac{\mathrm{d}z}{1+z^2} = \mathrm{i}2\pi\frac{1}{2\mathrm{i}} = \pi$$

Finally: $\left|1-|z|^2\right| \le |1+z^2|$

$$\Rightarrow \quad \lim_{R\to\infty}\left|\int_{\gamma_R}\frac{\mathrm{d}z}{1+z^2}\right| \le \lim_{R\to\infty}\int_{\gamma_R}\frac{|\mathrm{d}z|}{\left|1-|z|^2\right|} = \lim_{R\to\infty}\int_{\gamma_R}\frac{|\mathrm{d}z|}{R^2-1} = \lim_{R\to\infty}\frac{\pi R}{\mathcal{O}(R^2)} = 0$$

$$\Rightarrow \quad 2I = \pi \quad \Rightarrow \quad \int_0^{\infty}\frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{2}$$

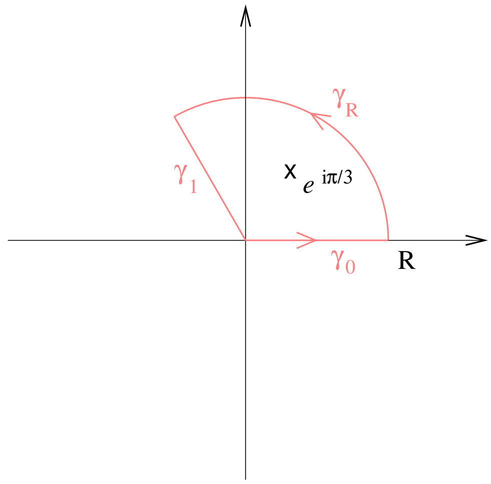
We could also close the contour with the lower half circle $\bar\gamma_R$. Then:

$$\operatorname*{Res}_{z=-\mathrm{i}}\frac{1}{z^2+1} = \lim_{z\to-\mathrm{i}}\frac{1}{z-\mathrm{i}} = -\frac{1}{2\mathrm{i}},$$

but $\gamma_0 + \bar\gamma_R$ is now clockwise giving another minus, so $I = \pi/2$.

**(2)** $\dfrac{1}{1+x^2}$ was conveniently symmetric. How about $f(z) = \dfrac{1}{1+z^3}$?

Still works! $f(z)$ is symmetric under rotations by 120°, so use




Using $\gamma_0(t) = t$, $\gamma_1(t) = e^{\mathrm{i}2\pi/3}t$, we get for $R \to \infty$

$$\int_{\gamma_0} \frac{\mathrm{d}z}{1+z^3} = \int_0^\infty \frac{1}{1+t^3}\gamma_0'(t)\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}t}{1+t^3} = I$$

$$\int_{\gamma_1} \frac{\mathrm{d}z}{1+z^3} = \int_\infty^0 \frac{1}{1+(e^{\mathrm{i}2\pi/3}t)^3} e^{\mathrm{i}2\pi/3}\mathrm{d}t = \int_\infty^0 \frac{1}{1+t^3} e^{\mathrm{i}2\pi/3}\mathrm{d}t = -e^{\mathrm{i}2\pi/3} I$$

$$\lim_{R\to\infty} \left| \int_{\gamma_R} \frac{\mathrm{d}z}{1+z^3} \right| \le \frac{2}{3}\pi R \sup_{z\in\gamma_R} \left| \frac{1}{1+z^3} \right| = \mathcal{O}(R^{-2}) = 0$$

$f(z)$ has 3 singularities, $e^{\mathrm{i}n\pi/3}$, $n = 1,\ 3,\ 5$: only $e^{\mathrm{i}\pi/3}$ is inside the contour

$$\text{l'Hôpital:}\ \operatorname*{Res}_{z=e^{\mathrm{i}\pi/3}} \frac{1}{1+z^3} = \lim_{z\to e^{\mathrm{i}\pi/3}} \frac{z-e^{\mathrm{i}\pi/3}}{1+z^3} = \lim_{z\to e^{\mathrm{i}\pi/3}} \frac{1}{3z^2} = \frac{1}{3}e^{-\mathrm{i}2\pi/3}$$

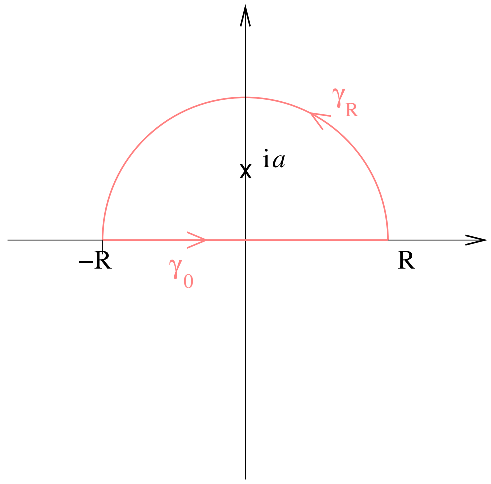
$$\Rightarrow \quad \int_{\gamma_1+\gamma_0+\gamma_R} \frac{\mathrm{d}z}{1+z^3} = -e^{\mathrm{i}2\pi/3}I + I + 0 = \mathrm{i}2\pi \operatorname*{Res}_{z=e^{\mathrm{i}\pi/3}} \frac{1}{1+z^3} = \mathrm{i}\frac{2\pi}{3}e^{-\mathrm{i}2\pi/3}$$

$$\Rightarrow I = \mathrm{i}\frac{2\pi}{3}\frac{e^{-\mathrm{i}2\pi/3}}{1-e^{\mathrm{i}2\pi/3}} = \ldots = \frac{2\pi}{3\sqrt{3}}$$

**(3)** Consider $I = \displaystyle\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}$, with $a > 0 \in \mathbb{R}$.

$f(z) = \dfrac{1}{(z^2+a^2)^2}$ has 2 double poles at $z = \pm\mathrm{i}a$. We need

$$\operatorname*{Res}_{z=\mathrm{i}a} \frac{1}{(z^2+a^2)^2} = \lim_{z\to\mathrm{i}a} \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{(z+\mathrm{i}a)^2} = \lim_{z\to\mathrm{i}a} \frac{-2}{(z+\mathrm{i}a)^3} = \frac{1}{\mathrm{i}4a^3}.$$ We use:




$$\left|\int_{\gamma_R} \frac{\mathrm{d}z}{(z^2+a^2)^2}\right| \leq \pi R \mathcal{O}(R^{-4}) \to 0 \quad \text{as} \quad R \to \infty$$

Symmetry of the integrand implies

$$2I = \oint_{\gamma_0+\gamma_R} \frac{\mathrm{d}z}{(z^2+a^2)^2} = \mathrm{i}2\pi \frac{1}{\mathrm{i}4a^3} = \frac{\pi}{2a^3} \qquad \Rightarrow \qquad I = \frac{\pi}{4a^3}$$

## E.3 Integrals of trigonometric functions

Consider integrals of the form $\displaystyle\int_0^{2\pi} f(\cos\theta, \sin\theta)\mathrm{d}\theta$

Using $\quad z = e^{\mathrm{i}\theta}\,, \qquad \cos\theta = \frac{1}{2}(z+z^{-1})\,, \qquad \sin\theta = \frac{1}{2\mathrm{i}}(z-z^{-1})\,.$

We get a contour integral along the unit circle:

$$\gamma : \theta \mapsto z = e^{\mathrm{i}\theta} \quad \Rightarrow \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = \mathrm{i}e^{\mathrm{i}\theta} = \mathrm{i}z \quad \Rightarrow \quad \mathrm{d}\theta = -\mathrm{i}\frac{\mathrm{d}z}{z}$$

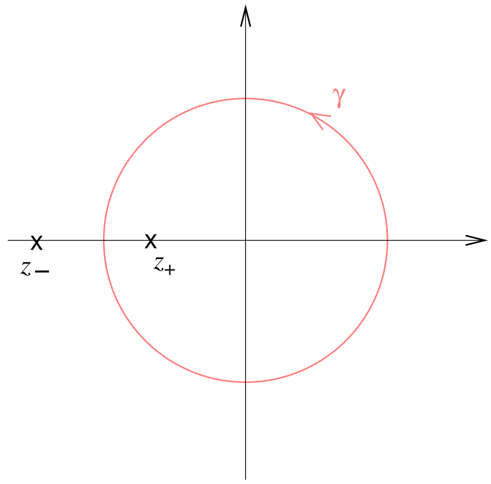
**Example**

**(1)** Let $a > 1 \in \mathbb{R}$, $\quad I = \displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta} = \oint_\gamma \frac{-\mathrm{i}\mathrm{d}z}{z\big[a+\frac{1}{2}(z+z^{-1})\big]} = -2\mathrm{i}\oint_\gamma \frac{\mathrm{d}z}{z^2+2az+1}$

$z^2+2az+1 = 0 \quad \Rightarrow \quad z_\pm = -a \pm \sqrt{a^2-1}$

One can show: $\quad z_- < -a < -1 \quad$ and $\quad -1 < z_+ < 0$

$\Rightarrow$ only $z_+$ is inside the unit circle




$$\operatorname*{Res}_{z=z_+} \frac{1}{z^2+2az+1} = \operatorname*{Res}_{z=z_+} \frac{1}{(z-z_+)(z-z_-)} = \frac{1}{z_+ - z_-} = \frac{1}{2\sqrt{a^2-1}}$$

$$\Rightarrow \quad I = -2\mathrm{i}\frac{\mathrm{i}2\pi}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}$$

## E.4 Branch cuts and keyhole contours

Functions with a branch cut need contours that do not cross the cut! This often results in *keyhole contours*

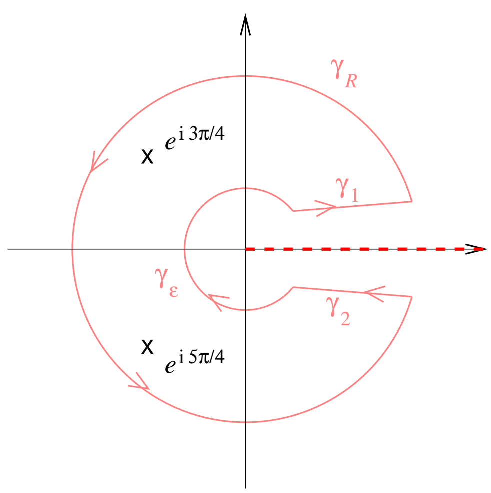
### Examples

**(1)** Consider $I = \displaystyle\int_0^\infty \frac{x^\alpha}{1+\sqrt{2}\,x+x^2}\mathrm{d}x\,, \quad 0 \neq \alpha \in (-1,1)$

Branch point: $z=0$. Branch cut: we take $\mathbb{R}^+$.

Branch: $0 \le \theta < 2\pi$, so: $\quad z = re^{\mathrm{i}\theta} \quad \Rightarrow \quad z^\alpha = r^\alpha e^{\mathrm{i}\alpha\theta}$

Take the contour




take the limit $R \to \infty$, $\epsilon \to 0$ such that the circles traverse $(0, 2\pi)$

4 contributions:

$\gamma_R$: $\displaystyle\int_{\gamma_R} \frac{z^\alpha}{1+\sqrt{2}z+z^2}\mathrm{d}z = 2\pi R\,\mathcal{O}(R^{\alpha-2}) = \mathcal{O}(R^{\alpha-1}) \to 0 \quad \text{as} \quad R\to\infty$

$\gamma_\epsilon$: Set $z = \epsilon e^{\mathrm{i}\theta}$

$$\Rightarrow \int_{\gamma_\epsilon} \frac{z^\alpha}{1+\sqrt{2}z+z^2}\mathrm{d}z = \int_{2\pi}^{0} \frac{\epsilon^\alpha e^{\mathrm{i}\alpha\theta}}{1+\sqrt{2}\epsilon e^{\mathrm{i}\theta}+\epsilon^2 e^{\mathrm{i}2\theta}}\mathrm{i}\epsilon e^{\mathrm{i}\theta}\mathrm{d}\theta = \mathcal{O}(\epsilon^{\alpha+1}) \quad \overset{\epsilon\to 0}{\longrightarrow} \quad 0\,.$$

$\gamma_1$: Let $\gamma_1(t) = te^{\mathrm{i}\delta\theta}$ with the limit $\delta\theta\to 0$

$$\Rightarrow \int_{\gamma_1} \frac{z^\alpha}{1+\sqrt{2}z+z^2}\mathrm{d}z = \lim_{\delta\theta\to 0}\int_0^\infty \frac{t^\alpha e^{\mathrm{i}\alpha\delta\theta}}{1+\sqrt{2}te^{\mathrm{i}\delta\theta}+t^2e^{\mathrm{i}2\delta\theta}}e^{\mathrm{i}\delta\theta}\mathrm{d}t = \int_0^\infty \frac{t^\alpha}{1+\sqrt{2}t+t^2}\mathrm{d}t = I$$

$\gamma_2$: $\gamma_2(t) = te^{\mathrm{i}\delta\theta}$ with the limit $\delta\theta\to 2\pi$

$$\int_{\gamma_2} \frac{z^\alpha}{1+\sqrt{2}z+z^2}\mathrm{d}z = \int_\infty^0 \frac{t^\alpha e^{\mathrm{i}2\alpha\pi}}{1+\sqrt{2}t+t^2}\mathrm{d}t = -e^{\mathrm{i}2\alpha\pi}I$$

In summary: $$\oint_{\gamma_1+\gamma_R+\gamma_2+\gamma_\epsilon} \frac{z^\alpha}{1+\sqrt{2}z+z^2}\mathrm{d}z = \big(1-e^{\mathrm{i}2\alpha\pi}\big)I\,. \qquad (\dagger)$$

One can show: $z^2+1+\sqrt{2}z = (z-e^{\mathrm{i}3\pi/4})(z-e^{\mathrm{i}5\pi/4})$

$e^{\mathrm{i}3\pi/4} - e^{\mathrm{i}5\pi/4} = \sqrt{2}\mathrm{i}$

$$\Rightarrow \quad \operatorname*{Res}_{z=e^{\mathrm{i}3\pi/4}} \frac{z^\alpha}{1+\sqrt{2}z+z^2} = \frac{e^{\mathrm{i}3\alpha\pi/4}}{e^{\mathrm{i}3\pi/4}-e^{\mathrm{i}5\pi/4}} = \frac{e^{\mathrm{i}3\alpha\pi/4}}{\sqrt{2}\mathrm{i}}$$

$$\operatorname*{Res}_{z=e^{\mathrm{i}5\pi/4}} \frac{z^\alpha}{1+\sqrt{2}z+z^2} = \frac{e^{\mathrm{i}5\alpha\pi/4}}{e^{\mathrm{i}5\pi/4}-e^{\mathrm{i}3\pi/4}} = -\frac{e^{\mathrm{i}5\alpha\pi/4}}{\sqrt{2}\mathrm{i}}$$ With $(\dagger)$, we get

$$\mathrm{i}2\pi\left(\frac{e^{\mathrm{i}3\alpha\pi/4}}{\sqrt{2}\mathrm{i}} - \frac{e^{\mathrm{i}5\alpha\pi/4}}{\sqrt{2}\mathrm{i}}\right) = \sqrt{2}\pi e^{\mathrm{i}3\alpha\pi/4}(1-e^{\mathrm{i}\alpha\pi/2}) \overset{!}{=} (1-e^{\mathrm{i}2\pi\alpha})I$$

$$\Rightarrow Ie^{\mathrm{i}\alpha\pi}(e^{-\mathrm{i}\alpha\pi}-e^{\mathrm{i}\alpha\pi}) = \sqrt{2}\pi e^{\mathrm{i}\alpha\pi}(e^{-\mathrm{i}\alpha\pi/4}-e^{\mathrm{i}\alpha\pi/4})$$

$$\Rightarrow \quad I = \sqrt{2}\pi\frac{\sin\left(\frac{\alpha\pi}{4}\right)}{\sin(\alpha\pi)}$$

Note: The two poles $e^{\mathrm{i}3\pi/4}$ and $e^{\mathrm{i}5\pi/4}$ are in our branch $\theta\in[0,2\pi)$.

They must be! $e^{\mathrm{i}5\pi/4} = e^{-\mathrm{i}3\pi/4}$, but $e^{-\mathrm{i}3\pi\theta/4}$ would have given us a different residue and a wrong $I$. Stay by your branch!

## E.5 Rectangular contours

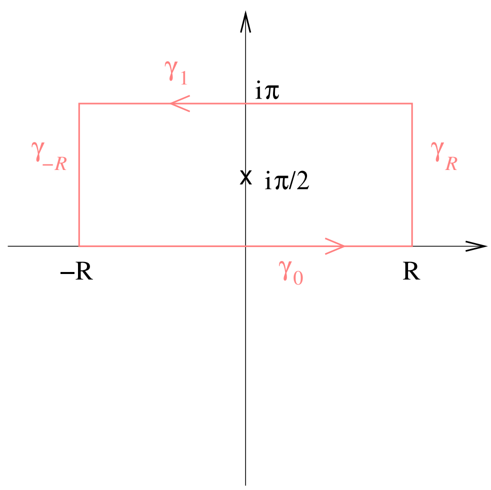
Advantage: Rectangular contours can strecth to $\infty$ in selected directions.

### Examples

**(1)** Consider $I = \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{\cosh x} \mathrm{d}x\,, \quad -1 < \alpha < 1$

$$\cosh(\mathrm{i}z) = \cos z \quad \Rightarrow \quad \frac{e^{\alpha z}}{\cosh z} \quad \text{has poles at} \;\; z = \mathrm{i}\left(n + \frac{1}{2}\right)\pi\,, \quad n \in \mathbb{Z}$$

The contour with $R \to \infty$




encircles only one pole, $z = \mathrm{i}\frac{\pi}{2}$. We have 4 contributions to the integral:

$$\gamma_0\!: \;\; z(t) = t \quad \Rightarrow \quad \int_{\gamma_0} \frac{e^{\alpha z}}{\cosh z}\mathrm{d}z = \int_{-\infty}^{\infty} \frac{e^{\alpha t}}{\cosh t}\mathrm{d}t = I$$

$\gamma_1$: $\quad z(t) = t + \mathrm{i}\pi\,, \quad \cosh(z + \mathrm{i}\pi) = -\cosh z$, so

$$\int_{\gamma_1} \frac{e^{\alpha z}}{\cosh z}\mathrm{d}z = \int_{\infty}^{-\infty} \frac{e^{\alpha(t+\mathrm{i}\pi)}}{\cosh(t+\mathrm{i}\pi)}\mathrm{d}t = -e^{\alpha\mathrm{i}\pi}\int_{-\infty}^{\infty} \frac{e^{\alpha t}}{-\cosh t}\mathrm{d}t = e^{\mathrm{i}\alpha\pi} I$$

$\gamma_R$: Parametrize $z(t) = R + \mathrm{i}t$

One can show: $|\cosh(R + \mathrm{i}t)| = \sqrt{\cos^2 t + \sinh^2 R} \geq \sinh R\,.$

$$\Rightarrow \quad \left|\int_{\gamma_R} \frac{e^{\alpha z}}{\cosh z}\mathrm{d}z\right| \leq \int_0^{\pi} \frac{|e^{\alpha R}e^{\mathrm{i}\alpha t}|}{|\sinh R|}\mathrm{d}t = e^{\alpha R}\int_0^{\pi} \frac{1}{\sinh R}\mathrm{d}t = \frac{\pi e^{\alpha R}}{\sinh R} = \mathcal{O}\big(e^{(\alpha-1)R}\big) \overset{R\to\infty}{\longrightarrow} 0$$

since $\alpha < 1$.

$$\gamma_{-R}\!: \quad \text{Likewise, } \alpha > -1 \quad \Rightarrow \quad \int_{\gamma_{-R}} \frac{e^{\alpha z}}{\cosh z}\mathrm{d}z \overset{R\to-\infty}{\longrightarrow} 0$$

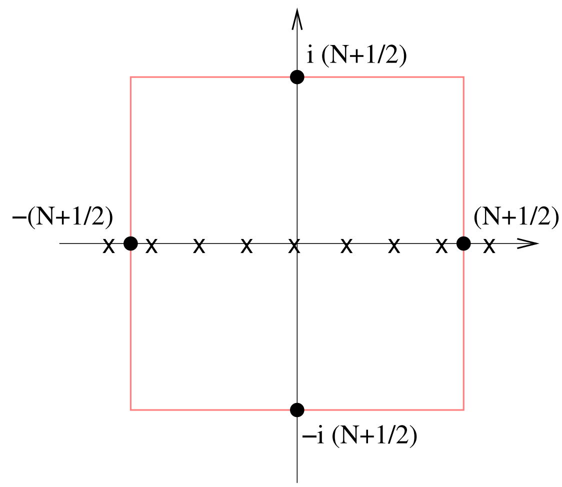
For $\gamma = \gamma_0 + \gamma_R + \gamma_1 + \gamma_{-R}$ we have: $\displaystyle\oint_\gamma \frac{e^{\alpha z}}{\cosh z} \mathrm{d}z = (1 + e^{\mathrm{i}\alpha\pi})I.$

One simple pole at $z = \mathrm{i}\frac{\pi}{2}$. With l'Hôpital:

$$\operatorname*{Res}_{z=\mathrm{i}\frac{\pi}{2}} \frac{e^{\alpha z}}{\cosh z} = \lim_{z\to\mathrm{i}\frac{\pi}{2}} \frac{(z-\mathrm{i}\frac{\pi}{2})e^{\alpha z}}{\cosh z} = \lim_{z\to\mathrm{i}\frac{\pi}{2}} \frac{(z-\mathrm{i}\frac{\pi}{2})\alpha e^{\alpha z} + e^{\alpha z}}{\sinh z} = \frac{e^{\mathrm{i}\alpha\pi/2}}{\sinh \mathrm{i}\frac{\pi}{2}} = -\mathrm{i}e^{\mathrm{i}\alpha\pi/2}$$

$$\Rightarrow \quad I = \frac{1}{1+e^{\mathrm{i}\alpha\pi}} 2\pi\mathrm{i}(-\mathrm{i}e^{\mathrm{i}\alpha\pi/2}) = \frac{2\pi}{e^{-\mathrm{i}\alpha\pi/2} + e^{\mathrm{i}\alpha\pi/2}} = \frac{\pi}{\cos\left(\frac{\alpha\pi}{2}\right)}.$$

**(2)** Consider $I = \displaystyle\oint_\gamma f(z)\mathrm{d}z$ with $f(z) = \dfrac{1}{z^2\tan(\pi z)}$ and the contour




$f$ has poles at $z = n \in \mathbb{Z}$. $z = 0$ is a triple pole, the others are simple.

The Taylor series of tan gives us $\operatorname*{Res}_{z=0} f(z)$:

$$\tan z = z + \frac{1}{3}z^3 + \dots$$

$$\Rightarrow \quad z^2\tan(\pi z) = \pi z^3 \left(1 + \frac{\pi^2}{3}z^2 + \dots\right)$$

$$\frac{1}{z^2\tan(\pi z)} = \frac{1}{\pi z^3}\left(1 - \frac{\pi^2}{3}z^2 - \dots\right) = \frac{1}{\pi}z^{-3} \underbrace{-\frac{\pi}{3}}_{=a_{-1}} z^{-1} - \dots$$

For $n \neq 0$: $\displaystyle\operatorname*{Res}_{z=n} f(z) = \lim_{z\to n} \frac{z-n}{z^2\tan(\pi z)} = \lim_{z\to n} \frac{1}{2z\tan(\pi z) + \frac{z^2\pi}{\cos^2(\pi z)}} = \frac{1}{n^2\pi}$

Along the right edge: $z(t) = N + \frac{1}{2} + \mathrm{i}t$.

One can show: $\left|\tan\left[(N+\frac{1}{2})\pi + \mathrm{i}\pi t\right]\right| \geq 1$

$$\Rightarrow \quad \left|\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{\mathrm{i}\mathrm{d}t}{z(t)^2\tan[\pi z(t)]}\right| \leq \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \left|\frac{1}{z(t)^2}\right| \mathrm{d}t = \mathcal{O}(N^{-1}) \overset{N\to\infty}{\longrightarrow} 0$$

Along the upper edge: $z(t) = \mathrm{i}(N + \frac{1}{2}) + t$

One can show: $\left|\tan[t\pi + \mathrm{i}(N+\frac{1}{2})\pi]\right| \geq \tanh\frac{\pi}{2}$

$$\Rightarrow \quad \left|\int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{\mathrm{i}\mathrm{d}t}{z(t)^2\tan[\pi z(t)]}\right| \leq \int_{-N-\frac{1}{2}}^{N+\frac{1}{2}} \frac{\mathrm{d}t}{\left|z(t)^2 \underbrace{\tan[(t\pi + \mathrm{i}(N+\frac{1}{2})\pi]}_{\geq \tanh\frac{\pi}{2}}\right|} = \mathcal{O}(N^{-1}) \overset{N\to\infty}{\longrightarrow} 0$$

Likewise, the integrals along the lower and left edge vanish for $N \to \infty$. So:

$$\oint_\gamma \frac{\mathrm{d}z}{z^2\tan(\pi z)} = \mathrm{i}2\pi\left(-\frac{\pi}{3} + 2\sum_{n=1}^{N}\frac{1}{n^2\pi}\right) \overset{N\to\infty}{\longrightarrow} 0$$

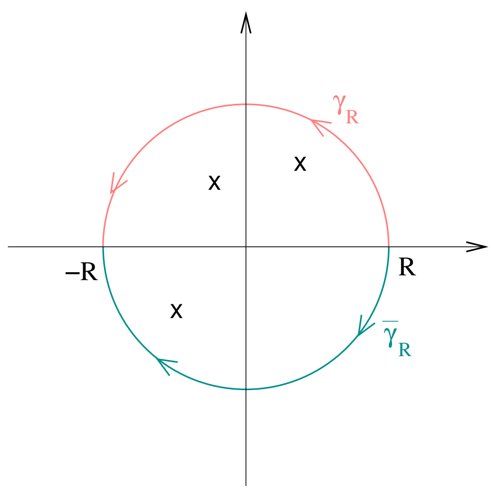
$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

## E.6 Jordan's Lemma

We have eliminated some integrals through falloff of the integrand.

E.g. $\displaystyle\oint_{\gamma_R} \mathcal{O}(R^{-2})\mathrm{d}z \quad \overset{R\to\infty}{\longrightarrow} \quad 0$

An even stronger tool is *Jordan's lemma* for contours $\gamma_R$, $\bar{\gamma}_R$




**Lemma:** Let $f(z)$ be analytic in $\mathbb{C}$, except for a finite number of singular points, with $f(z) \to 0$ as $|z| \to \infty$. Let $\lambda,\ \mu \in \mathbb{R},\ \lambda > 0,\ \mu < 0$. Then

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)e^{\mathrm{i}\lambda z}\mathrm{d}z = 0\,, \qquad \lim_{R\to\infty}\int_{\bar{\gamma}_R} f(z)e^{\mathrm{i}\mu z}\mathrm{d}z = 0\,.$$

*Proof.* One can show that: $\sin x \geq \frac{2}{\pi}x \quad$ for $\quad x \in [0, \frac{\pi}{2}] \qquad$ (long script)

Parametrize $\gamma_R : \theta \mapsto Re^{\mathrm{i}\theta}$, so:

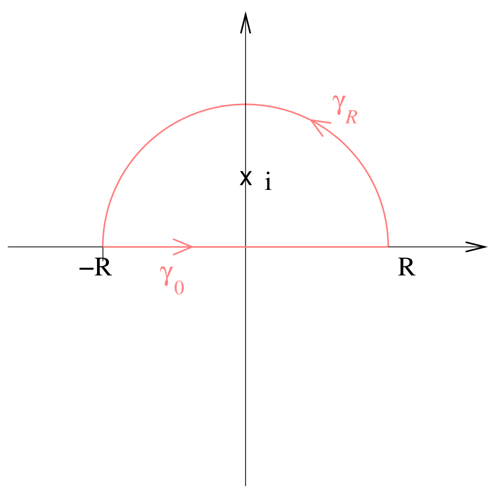
$$\begin{aligned}
\left|\int_{\gamma_R} f(z)e^{\mathrm{i}\lambda z}\mathrm{d}z\right| &= \left|\int_0^\pi f(Re^{\mathrm{i}\theta})e^{\mathrm{i}\lambda Re^{\mathrm{i}\theta}}\mathrm{i}Re^{\mathrm{i}\theta}d\theta\right| \leq R\int_0^\pi \left|f(Re^{\mathrm{i}\theta})\right| \left|e^{\mathrm{i}\lambda Re^{\mathrm{i}\theta}}\right| \mathrm{d}\theta \\
&= R\int_0^\pi \left|f(Re^{\mathrm{i}\theta})\right| \underbrace{e^{-\lambda R\sin\theta}}_{>0}\mathrm{d}\theta \leq R\sup_{z\in\gamma_R}|f(z)| \; 2\int_0^{\pi/2} e^{-\lambda R\sin\theta}\mathrm{d}\theta \\
&\leq 2R\sup_{z\in\gamma_R}|f(z)| \int_0^{\pi/2} e^{-2\lambda R\theta/\pi}\mathrm{d}\theta = \frac{\pi}{\lambda}\left(1-e^{-\lambda R}\right)\sup_{z\in\gamma_R}|f(z)| \\
&\xrightarrow{R\to\infty} 0\,.
\end{aligned}$$

Likewise for $\mu < 0$ and the contour $\bar{\gamma}_R$. □

**Examples**

**(1)** Consider $I = \displaystyle\int_0^\infty \frac{\cos(\alpha x)}{1+x^2}\mathrm{d}x\,, \quad \alpha > 0$

Use contour $\gamma = \gamma_0 + \gamma_R$




Trick: Use $\mathsf{Re}\displaystyle\int_\gamma \frac{e^{\mathrm{i}\alpha z}}{1+z^2}\mathrm{d}z$

Along $\gamma_0$: this gives us $2I$

Along $\gamma_R$: we get 0 by Jordan's lemma

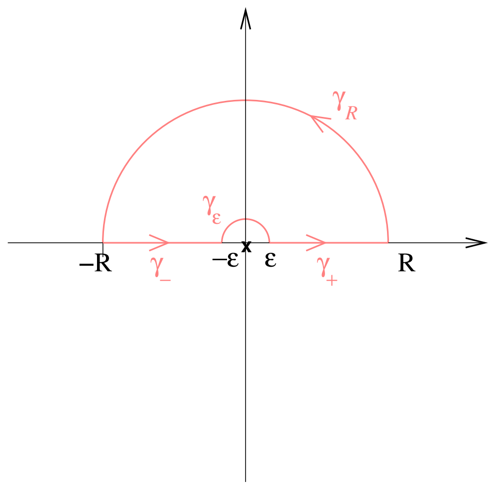
We have one simple pole $z = \mathrm{i}$ inside $\gamma$

$$\Rightarrow \quad I = \frac{1}{2}\mathsf{Re}\left(\mathrm{i}2\pi\operatorname*{Res}_{z=\mathrm{i}}\frac{e^{\mathrm{i}\alpha z}}{1+z^2}\right) = \frac{1}{2}\mathsf{Re}\left(\mathrm{i}2\pi\frac{e^{-\alpha}}{2\mathrm{i}}\right) = \frac{\pi}{2}e^{-\alpha}.$$

**(2)** Consider $I = \int_{-\infty}^{\infty} \frac{\sin x}{x} \mathrm{d}x$

For Jordan's lemma we want $\frac{e^{\mathrm{i}z}}{z}$, but that's not regular at $z = 0$.

Solution: Cut out $z = 0$ with




$$\Rightarrow \quad I = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left( \int_{-R}^{-\epsilon} \frac{\sin x}{x} \mathrm{d}x + \int_{\epsilon}^{R} \frac{\sin x}{x} \mathrm{d}x \right) = \mathsf{Im} \left[ \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left( \int_{-R}^{-\epsilon} \frac{e^{\mathrm{i}x}}{x} \mathrm{d}x + \int_{\epsilon}^{R} \frac{e^{\mathrm{i}x}}{x} \mathrm{d}x \right) \right]$$

The closed contour encircles no singularity, so $\oint_\gamma \ldots \mathrm{d}z = 0$ and

$$I_{\epsilon,R} := \int_{\gamma_-} \frac{e^{\mathrm{i}z}}{z} \mathrm{d}z + \int_{\gamma_+} \frac{e^{\mathrm{i}z}}{z} \mathrm{d}z = -\int_{\gamma_\epsilon} \frac{e^{\mathrm{i}z}}{z} \mathrm{d}z - \int_{\gamma_R} \frac{e^{\mathrm{i}z}}{z} \mathrm{d}z$$

$\gamma_\epsilon$: parametrize $z(\theta) = \epsilon e^{\mathrm{i}\theta}$

$$\Rightarrow \quad \int_{\gamma_\epsilon} \frac{e^{\mathrm{i}z}}{z} \mathrm{d}z = \int_{\pi}^{0} \frac{1 + \mathcal{O}(\epsilon)}{\epsilon e^{\mathrm{i}\theta}} \mathrm{i}\epsilon e^{\mathrm{i}\theta} \mathrm{d}\theta \quad \xrightarrow{\epsilon\to 0} \quad -\mathrm{i}\pi$$

$\gamma_R$: We get 0 by Jordan's lemma

$$\Rightarrow \quad I = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \mathsf{Im}(I_{\epsilon,R}) = -\mathsf{Im}(-\mathrm{i}\pi) = \pi.$$

# F Transform theory

## F.1 Fourier transforms

**Def.:** Let $f : \mathbb{R} \to \mathbb{C}$ be absolutely integrable with bounded variation and a finite number of discontinuities. The *Fourier transform* and its inverse are

$$\tilde{f}(k) = \mathcal{F}\big[f(x)\big](k) = \int_{-\infty}^{\infty} f(x)e^{-\mathrm{i}kx}\mathrm{d}x \qquad (\dagger)$$

$$f(x) = \mathcal{F}^{-1}\big[\tilde{f}(k)\big](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{\mathrm{i}kx}\mathrm{d}k \qquad (\ddagger)$$

**Comments:**

- There are many conventions: shifting minus signs, factors of $2\pi$ etc.
- We often use variable pairs $(x, k)$ for space and $(t, \omega)$ for time.

  $k = \dfrac{2\pi}{\lambda}$ = wave number. $\omega = \dfrac{2\pi}{T}$ = angular frequency.
- Some non-square integrable functions can be handled with *distributions*.

  In particular: $f(x) = 1 \quad \Rightarrow \quad \tilde{f}(k) = 2\pi\delta(k) = \int_{-\infty}^{\infty} 1e^{-\mathrm{i}kx}\mathrm{d}x$
- At discontinuities, the Fourier transform returns the average:
  $\dfrac{f(x^+) + f(x^-)}{2}$.
- We don't quite need $\int_{-\infty}^{\infty}$.

  **Def.:** The *Cauchy principal value* of an integral $\int_{-\infty}^{\infty} g(x)\mathrm{d}x$ is

  $$⨍_{-\infty}^{\infty} g(x)\mathrm{d}x := \lim_{R\to\infty}\int_{-R}^{R} g(x)\mathrm{d}x$$

  Other notations: PV $\int$, p.v. $\int$, P $\int$

  $⨍$ may exist even when $\int$ does not. E.g.:

  $$⨍_{-\infty}^{\infty} \frac{x}{1+x^2}\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R}\frac{x}{1+x^2}\mathrm{d}x = 0$$

  $$\lim_{R\to\infty}\int_{-R^2}^{R}\frac{x}{1+x^2}\mathrm{d}x = \ldots = -\infty\,.$$
- Eqs. $(\dagger)$, $(\ddagger)$ stand for:

  $$\frac{1}{2}[\tilde{f}(k^+) + \tilde{f}(k^-)] = ⨍_{-\infty}^{\infty} f(x)e^{-\mathrm{i}kx}\mathrm{d}x\,,$$

  $$\frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2\pi}⨍_{-\infty}^{\infty} \tilde{f}(k)e^{\mathrm{i}kx}\mathrm{d}k\,,$$

  But we keep using $(\dagger)$, $(\ddagger)$

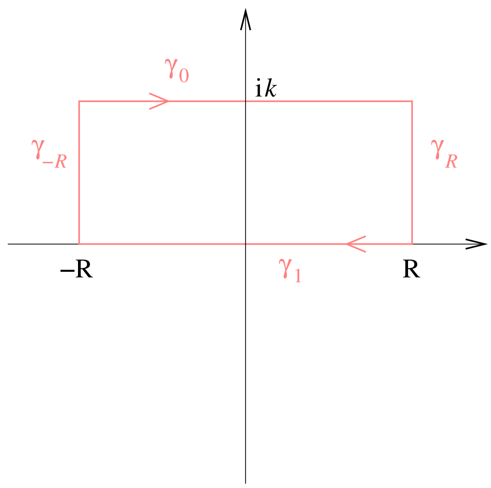
**Examples**

**(1)** Consider $f(x) = e^{-x^2/2}$.

$$\Rightarrow \quad \tilde{f}(k) = \int_{-\infty}^{\infty} e^{-x^2/2} e^{-\mathrm{i}kx} \mathrm{d}x = \int_{-\infty}^{\infty} e^{-(x+\mathrm{i}k)^2/2} e^{-k^2/2} \mathrm{d}x \qquad \Big| \; z = x + \mathrm{i}k$$

$$= e^{-k^2/2} \int_{-\infty+\mathrm{i}k}^{\infty+\mathrm{i}k} e^{-z^2/2} \mathrm{d}z$$

This is the contribution along $\gamma_0$ for $R \to \infty$ in the contour




Along $\gamma_R,\ \gamma_{-R}$: We get 0 as $R \to \infty$ since $e^{-R^2} \to 0$

We have no singularity

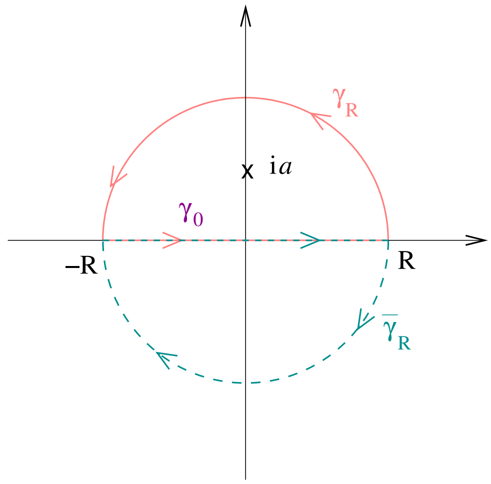
$$\Rightarrow \int_{\gamma_0} e^{-z^2/2} \mathrm{d}z = -\int_{\gamma_1} e^{-z^2/2} \mathrm{d}z = +\int_{-\infty}^{\infty} e^{-t^2/2} \mathrm{d}t = \sqrt{2\pi} \qquad \text{(Gauss integral)}$$

$$\Rightarrow \tilde{f}(k) = \sqrt{2\pi} e^{-k^2/2}$$

**(2)** Consider $\tilde{f}(k) = \dfrac{1}{a + \mathrm{i}k}, \quad a > 0.$

$$\Rightarrow f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a + \mathrm{i}k} e^{\mathrm{i}kx} \mathrm{d}k$$

Reminiscent of Jordan's lemma! We use contours $\gamma_R$ and $\bar{\gamma}_R$




Case 1, $x>0$: Use $\gamma_R$ with $\displaystyle\operatorname*{Res}_{k=\mathrm{i}a}\frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}=\lim_{k\to\mathrm{i}a}\frac{k-\mathrm{i}a}{a+\mathrm{i}k}e^{\mathrm{i}kx}=-\mathrm{i}e^{-xa}$

Use Jordan's lemma with $\lambda=x$

$$\lim_{R\to\infty}\oint_{\gamma_R+\gamma_0}\frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\mathrm{d}k=\mathrm{i}2\pi(-\mathrm{i}e^{-ax})\stackrel{!}{=}\underbrace{\lim_{R\to\infty}\int_{\gamma_R}\frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\mathrm{d}k}_{=0}+\lim_{R\to\infty}\int_{\gamma_0}\frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\mathrm{d}k$$

$$\Rightarrow f(x)=\frac{1}{2\pi}2\pi e^{-ax}=e^{-ax}\,.$$

Case 2, $x<0$: Use $\bar\gamma_R$. Now $\bar\gamma_R+\gamma_0$ encircles no singularity

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\mathrm{d}k=\lim_{R\to\infty}\left[\int_{\gamma_0}\frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\mathrm{d}k+\underbrace{\int_{\bar\gamma_R}\frac{e^{\mathrm{i}kx}}{a+\mathrm{i}k}\mathrm{d}k}_{\to 0}\right]=0$$

So $f(x)=\begin{cases}0 & \text{for } x<0\\ e^{-ax} & \text{for } x>0\end{cases}$

## F.2 Laplace transforms

Motivation: Handle growing functions, e.g. $e^t$, and initial conditions.

### F.2.1 Definition of the Laplace transform

**Def.:** Let $f(t)$ be defined for all $t\geq 0$. Its *Laplace transform* is

$$F(s)=\mathcal{L}\{f(t)\}(s)=\int_0^\infty f(t)e^{-st}\mathrm{d}t\,,\quad s\in\mathbb{C}\,,$$

provided the integral exists.

**Comments:**

- This works for functions that grow no more than exponential
- Some people use $p$ instead of $s$
- If $f(t) = 0$ for $t < 0$, we have: $F(s) = \tilde{f}(-\mathrm{i}s)$ (Fourier transform)

**Examples**

**(1)** $f(t) = 1 \quad \Rightarrow \quad \mathcal{L}\{1\}(s) = \int_0^\infty e^{-st}\mathrm{d}t = \frac{1}{s}$

**Note:** The integral only exists for $\mathsf{Re}(s) > 0$.

But we may still take the result for all $s$ where it is defined! Here $s \in \mathbb{C} \setminus \{0\}$.

This is called *analytic continuation* and can be useful.

**(2)** Use integration by parts for $f(t) = t$.

$$\Rightarrow \quad F(s) = \int_0^\infty t e^{-st}\mathrm{d}t = \left[t\frac{-1}{s}e^{-st}\right]_{t=0}^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\mathrm{d}t = 0 - \frac{1}{s^2}\left[e^{-st}\right]_{t=0}^\infty = \frac{1}{s^2}$$

**(3)** $\mathcal{L}\{e^{\lambda t}\}(s) = \int_0^\infty e^{(\lambda - s)t}\mathrm{d}t = \frac{1}{s-\lambda}$ for $\mathsf{Re}(\lambda) > \mathsf{Re}(s)$.

But we can use again analytic continuation: $F(s) = \frac{1}{s-\lambda}$ for $s \in \mathbb{C} \setminus \{\lambda\}$

**(4)** For $\lambda = \pm\mathrm{i}$, we get:

$$\mathcal{L}\{\sin t\}(s) = \mathcal{L}\left\{\frac{1}{2\mathrm{i}}(e^{\mathrm{i}t} - e^{-\mathrm{i}t})\right\}(s) = \frac{1}{2\mathrm{i}}\left(\frac{1}{s-\mathrm{i}} - \frac{1}{s+\mathrm{i}}\right) = \frac{1}{s^2+1}$$

### F.2.2 Properties of the Laplace transform

Let $f(t)$, $g(t)$ be functions with Laplace transforms $F(s)$, $G(s)$. Then:

**(A)** *Linearity.* For $\alpha,\ \beta \in \mathbb{C}$ : $\quad \mathcal{L}\{\alpha f + \beta g\} = \alpha\mathcal{L}\{f\} + \beta\mathcal{L}\{g\}$ .

*Proof.* Directly from definition □

**(B)** *Translation.* For $t_0 \in \mathbb{R}$ : $\quad \mathcal{L}\{f(t-t_0)\, H(t-t_0)\}(s) = e^{-st_0}F(s)$ ,

where $H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$ "Heaviside function"

*Proof.* With $\tilde{t} = t - t_0$,

$$\int_0^\infty f(t-t_0)H(t-t_0)e^{-st}\mathrm{d}t = e^{-st_0}\int_{-t_0}^\infty f(\tilde{t})H(\tilde{t})e^{-s\tilde{t}}\mathrm{d}\tilde{t} = e^{-st_0}\int_0^\infty f(\tilde{t})e^{-s\tilde{t}}\mathrm{d}\tilde{t} = e^{-st_0}F(s)\,.$$

□

**(C)** *Scaling.* For $\lambda > 0$: $\quad \mathcal{L}\{f(\lambda t)\}(s) = \frac{1}{\lambda}F\left(\frac{s}{\lambda}\right)$

*Proof.* With $\tilde{t} = \lambda t$ : $\displaystyle\int_0^\infty f(\lambda t)e^{-st}\mathrm{d}t = \int_0^\infty f(\tilde{t})e^{-\frac{s}{\lambda}\tilde{t}}\,\frac{\mathrm{d}\tilde{t}}{\lambda} = \frac{1}{\lambda}F\left(\tfrac{s}{\lambda}\right)$ □

**(D)** *Shifting.* For $s_0 \in \mathbb{C}$: $\mathcal{L}\{e^{s_0 t}f(t)\}(s) = F(s - s_0)$

*Proof.* $\displaystyle\int_0^\infty e^{s_0 t}f(t)e^{-st}\mathrm{d}t = F(s - s_0)$ □

**(E)** *Transform of derivatives.* $\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0)$
$\mathcal{L}\{f''(t)\}(s) = s\mathcal{L}\{f'(t)\}(s) - f'(0) = s^2F(s) - sf(0) - f'(0)$ etc.

*Proof.* $\displaystyle\int_0^\infty f'(t)e^{-st}\mathrm{d}t = \left[f(t)\,e^{-st}\right]_{t=0}^\infty - \int_0^\infty -sf(t)e^{-st}\mathrm{d}t = sF(s) - f(0)$ □

**(F)** *Derivative of transform.* $F'(s) = \mathcal{L}\{-tf(t)\}(s)\,,\qquad F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}(s)$

We often use this right-to-left.

*Proof.* $\displaystyle F'(s) = \int_0^\infty -tf(t)e^{-st}\mathrm{d}t = \mathcal{L}\{-tf(t)\}(s)$ □

**(G)** *Asymptotic limits.* If $\lim_{t\to\infty} f(t)$ exists: $\displaystyle\lim_{s\to\infty} sF(s) = f(0)$

$\displaystyle\lim_{s\to 0} sF(s) = f(\infty)$

*Proof.* $\displaystyle sF(s) \overset{(E)}{=} f(0) + \int_0^\infty f'(t)e^{-st}\mathrm{d}t$

For $s = 0$, this gives us $f(t)$ with $t \to \infty$.

For $s \to \infty$, we use that $f,\ f'$ grow at most exponential, so the integral vanishes. □

**Examples**

**(1)** $\displaystyle\mathcal{L}\{t\sin t\}(s) \overset{(F)}{=} -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\sin t\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}$

**(2)** We already know $\displaystyle\mathcal{L}\{1\}(s) = \frac{1}{s} \quad\overset{(F)}{\Rightarrow}\quad \mathcal{L}\{t^n\}(s) = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}s^n}\frac{1}{s} = \frac{n!}{s^{n+1}}$

Euler's *Gamma function*: $\displaystyle\Gamma(n) := \int_0^\infty e^{-t}t^{n-1}\mathrm{d}t$

$\Rightarrow\quad \Gamma(n) = \mathcal{L}\{t^{n-1}\}(s=1) = (n-1)!$

**(3)** For $a > 0$: $\displaystyle\mathcal{L}\{\sin(at)\}(s) \overset{(C)}{=} \frac{1}{a}\mathcal{L}\{\sin t\}\left(\tfrac{s}{a}\right) = \frac{1}{a}\frac{1}{\frac{s^2}{a^2}+1} = \frac{a}{s^2+a^2}$

$$\Rightarrow\quad \mathcal{L}\left\{\frac{\sin(at)}{a}\right\}(s) = \frac{1}{s^2+a^2}$$

**(4)** $\mathcal{L}\{e^{\mathrm{i}at}\}(s) = \dfrac{1}{s - \mathrm{i}a} = \dfrac{s + \mathrm{i}a}{s^2 + a^2} \overset{(A)}{=} \mathcal{L}\{\cos(at) + \mathrm{i}\sin(at)\}(s)$

$$\Rightarrow \quad \mathcal{L}\{\cos(at)\}(s) = \frac{s + \mathrm{i}a}{s^2 + a^2} - \mathrm{i}\mathcal{L}\{\sin(at)\}(s) = \frac{s + \mathrm{i}a}{s^2 + a^2} - \mathrm{i}\frac{a}{s^2 + a^2}$$

$$\Rightarrow \quad \mathcal{L}\{\cos(at)\}(s) = \frac{s}{s^2 + a^2}\,.$$

### F.2.3 The inverse Laplace transform

**Proposition:** We can compute the inverse Laplace transform $f(t)$ of $F(s)$ from the *Bromwich inversion formula* $f(t) = \dfrac{1}{\mathrm{i}2\pi}\displaystyle\int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} F(s)e^{st}\mathrm{d}s\,,$

where $\alpha \in \mathbb{R}$ is chosen greater than the real part of all singular points of $F(s)$.

*Proof.* By assumption, $f(t)$ has a Laplace transform

$\Rightarrow \quad \exists \alpha \in \mathbb{R}: \quad g(t) = f(t)e^{-\alpha t}$ decays exponentially as $t \to \infty$

$\Rightarrow \quad g(t)$ has a Fourier transform: $\quad \tilde{g}(\omega) = \displaystyle\int_{-\infty}^{\infty} f(t)e^{-\alpha t}e^{-\mathrm{i}\omega t}\mathrm{d}t = F(\alpha + \mathrm{i}\omega)$

$$\Rightarrow g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha + \mathrm{i}\omega)e^{\mathrm{i}\omega t}\mathrm{d}\omega \qquad \Big| \quad s := \alpha + \mathrm{i}\omega\,, \quad \mathrm{d}\omega = \frac{\mathrm{d}s}{\mathrm{i}}$$

$$\Rightarrow \quad f(t)e^{-\alpha t} = \frac{1}{2\pi\mathrm{i}}\int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} F(s)e^{(s-\alpha)t}\mathrm{d}s$$

$$\Rightarrow \quad f(t) = \frac{1}{\mathrm{i}2\pi}\int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} F(s)e^{st}\mathrm{d}s\,.$$

□

Why $\alpha > \mathsf{Re}(s_0)$ for all singularities $s_0$ of F? This ensures $f(t) = 0$ for $t < 0$; cf. below.

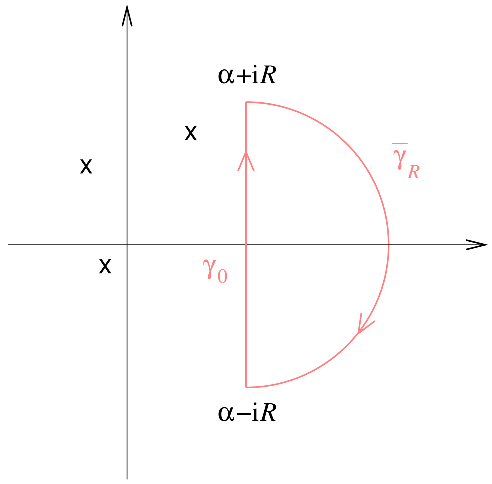
In practice, we often have the case:

**Proposition:** Let $F(s)$ be the Laplace transform of $f(t)$ and have only a finite number of isolated singularities $s_k \in \mathbb{C},\ k = 1, \ldots, n$. If $\lim\limits_{|s| \to \infty} F(s) = 0$, then $f(t) = 0$ for $t < 0$ and for $t > 0$,

$$f(t) = \sum_{k=1}^{n} \operatorname*{Res}_{s = s_k}\left(F(s)e^{st}\right)$$

*Proof.* (i) $t < 0$:

Consider the contour $\gamma = \gamma_0 + \bar{\gamma}_R$




If $F(s) = o(s^{-1})$, we get $\left|\int_{\bar\gamma_R} F(s)e^{st}\mathrm{d}s\right| \leq \pi R e^{\alpha t} \sup_{s\in\bar\gamma_R} |F(s)| \quad \xrightarrow{R\to\infty} \quad 0\,,$

since $\alpha \leq \mathrm{Re}(s)$ along $\bar\gamma_R$, so that for $t < 0$ we have $\alpha t \geq \mathrm{Re}(s)t \quad \Rightarrow \quad |e^{\alpha t}| \geq |e^{st}|$.

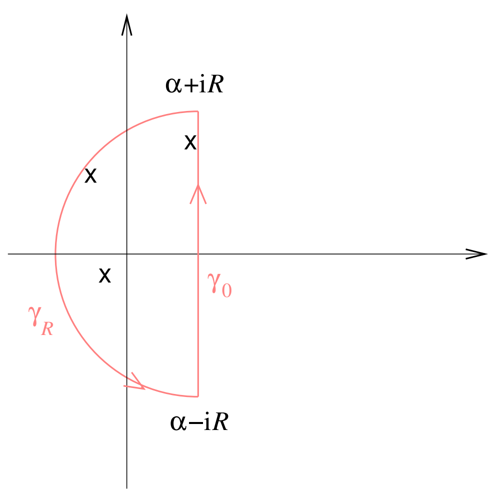
If $F(s) \to 0$ more slowly than $o(s^{-1})$, one can show the same with Jordan's lemma.

In any case: $\lim_{R\to\infty} \int_{\bar\gamma_R} F(s)e^{st}\mathrm{d}s = 0 \quad \Rightarrow \quad \int_{\gamma_0} F(s)e^{st}\mathrm{d}s = \int_{\gamma_0+\bar\gamma_R} F(s)e^{st}\mathrm{d}s \overset{!}{=} 0,$

since $\gamma_0 + \bar\gamma_R$ encloses no singularities.

(ii) $t > 0$:

Use the contour $\gamma = \gamma_0 + \gamma_R$




For $R \to \infty$, $\gamma$ encircles all singularities!

As before, $\int_{\gamma_R} \ldots \to 0$ as $R \to \infty$. The Bromwich formula and residual theorem give us

$$f(t) = \frac{1}{\mathrm{i}2\pi} \lim_{R\to\infty} \int_{\gamma_0} F(s)e^{st}\mathrm{d}s = \frac{1}{\mathrm{i}2\pi} \lim_{R\to\infty} \oint_{\gamma_0+\gamma_R} F(s)e^{st}\mathrm{d}s = \sum_{k=1}^{n} \operatorname*{Res}_{s=s_k} \left(F(s)e^{st}\right). \qquad \square$$

**Examples**

**(1)** $F(s) = \dfrac{1}{s-1}$ has a simple pole at $s = 1$ and $F(s) \to 0$ as $|s| \to \infty$. Let $\alpha > 1$.

$$\Rightarrow \quad f(t) = \operatorname*{Res}_{s=1}\left(\frac{e^{st}}{s-1}\right) = e^t\,.$$

**(2)** $F(s) = s^{-n}$, $n \in \mathbb{N}$ has a pole of order $n$ at $s = 0$, and $F(s) \to 0$ as $|s| \to \infty$.

$$\Rightarrow \quad f(t) = \operatorname*{Res}_{s=0}\left(\frac{e^{st}}{s^n}\right) = \lim_{s\to 0}\left[\frac{1}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}s^{n-1}}e^{st}\right] = \frac{t^{n-1}}{(n-1)!}$$

### F.2.4 Solving differential equations with the Laplace transform

**Examples**

**(1)** Solve $t\ddot{f}(t) - t\dot{f}(t) + f(t) = 2\,,\qquad f(0) = 2\,,\quad \dot{f}(0) = -1\,.$

$$\mathcal{L}\{t\dot{f}(t)\}(s) \overset{(F)}{=} -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\dot{f}(t)\}(s) \overset{(E)}{=} -\frac{\mathrm{d}}{\mathrm{d}s}[sF(s) - f(0)] = -sF'(s) - F(s)$$

$$\begin{aligned}\mathcal{L}\{t\ddot{f}(t)\}(s) &\overset{(F)}{=} -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{\ddot{f}(t)\}(s) \overset{(E)}{=} -\frac{\mathrm{d}}{\mathrm{d}s}\left[s^2F(s) - sf(0) - \dot{f}(0)\right] \\ &= -s^2F'(s) - 2sF(s) + f(0)\end{aligned}$$

$$\mathcal{L}\{2\}(s) = \frac{2}{s}$$

Transform the ODE

$$\Rightarrow \quad -s^2F'(s) - 2sF(s) + f(0) + sF'(s) + F(s) + F(s) = \frac{2}{s}$$

$$\Rightarrow \quad -s(s-1)F'(s) - 2(s-1)F(s) = \frac{2}{s} - 2 = \frac{2-2s}{s} = -(s-1)\frac{2}{s}$$

$$\Rightarrow \quad sF'(s) + 2F(s) \overset{!}{=} \frac{1}{s}(s^2F)' = \frac{2}{s}$$

$$\Rightarrow \quad s^2F = 2s + A \quad \Rightarrow \quad F(s) = \frac{2}{s} + \frac{A}{s^2}\,, \quad \text{where } A = \text{const}$$

$\Rightarrow f(t) = 2 + At\,.$ With $\dot{f}(0) = -1$ we get: $A = -1$

**(2)** Solve the PDE $\dfrac{\partial}{\partial t}f(t,x) = \dfrac{\partial^2}{\partial x^2}f(t,x)$ on $0 \le x \le 2$, $t \ge 0$ with

$f(t,0) = 0\,,\qquad f(2,t) = 0\,,\qquad f(0,x) = 3\sin(2\pi x)$

Laplace transform in $t$ with rule (E); $x$ unaffected!

$$\Rightarrow \quad sF(s,x) - f(0,x) = \partial_x^2 F(s,x)$$

$$\Rightarrow \quad \partial_x^2 F(s,x) - sF(s,x) = -3\sin(2\pi x)\,. \qquad \text{This is an ODE!}$$

Homogeneous part: $F_h(s,x) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x}\,,\qquad c_1,\, c_2 = \text{const}$

For the particular solution, we guess

$$F_p(s,x) = A\cos(2\pi x) + B\sin(2\pi x)$$

$$\Rightarrow \quad -(2\pi)^2 A\cos(2\pi x) - (2\pi)^2 B\sin(2\pi x) - s[A\cos(2\pi x) + B\sin(2\pi x)] \overset{!}{=} -3\sin(2\pi x)$$

$$\Rightarrow \quad -[(2\pi)^2 + s]A = 0 \qquad \wedge \qquad -[(2\pi)^2 + s]B = -3$$

$$\Rightarrow A = 0 \qquad \wedge \qquad B = \frac{3}{4\pi^2 + s}$$

$$\Rightarrow \quad F(s,x) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{3}{4\pi^2 + s}\sin(2\pi x)$$

Laplace transform the boundary conditions:

$f(t,0) = 0 \qquad \Rightarrow \qquad F(s,0) = c_1 + c_2 = 0\,,$

$f(t,2) = 0 \qquad \Rightarrow \qquad F(s,2) = c_1 e^{2\sqrt{s}} + c_2 e^{-2\sqrt{s}} = 0\,,$

So $c_1 = c_2 = 0$ and: $\quad F(s,x) = \dfrac{3}{s + 4\pi^2}\sin(2\pi x)$

$$\Rightarrow \quad f(t) = 3e^{-4\pi^2 t}\sin(2\pi x)\,.$$

### F.2.5 The convolution theorem for Laplace transforms

**<u>Def.:</u>** The *convolution* $f * g$ of two functions $f,\ g : \mathbb{R} \to \mathbb{R}$ is

$$(f * g)(t) = (g * f)(t) = \int_{-\infty}^{\infty} f(t-u)g(u)\mathrm{d}u$$

If $f(t) = g(t) = 0$ for $t < 0$, this becomes

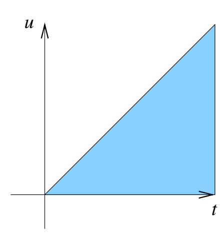
$$(f * g)(t) = (g * f)(t) = \int_0^t f(t-u)g(u)\mathrm{d}u$$

For Fourier transforms: $\mathcal{F}[f * g] = \mathcal{F}[f]\,\mathcal{F}[g]$

**<u>Theorem:</u>** The Laplace transform of a convolution is

$$\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f\}(s)\,\mathcal{L}\{g\}(s) = F(s)\,G(s)$$

*Proof.* $\displaystyle \mathcal{L}\{f * g\}(s) = \int_0^\infty \left[\int_0^t f(t-u)g(u)\mathrm{d}u\right] e^{-st}\mathrm{d}t = \int_0^\infty \left[\int_u^\infty f(t-u)g(u)e^{-st}\mathrm{d}t\right]\mathrm{d}u$




With $x = t - u$, $\mathrm{d}t = \mathrm{d}x$,

$$\mathcal{L}\{f*g\}(s) = \int_0^\infty \left[\int_0^\infty f(x)g(u)e^{-sx}e^{-su}\mathrm{d}x\right]\mathrm{d}u = \int_0^\infty \left[\int_0^\infty f(x)e^{-sx}\mathrm{d}x\right] g(u)e^{-su}\mathrm{d}u = F(s)G(s)$$

□

**Examples**

**(1)** Find the inverse of $H(s) = \dfrac{1}{s(s^2+1)} \overset{!}{=} F(s)\,G(s)$ with $F(s) = \dfrac{1}{s}$, $G(s) = \dfrac{1}{s^2+1}$.

$$\Rightarrow \quad f(t) = 1, \quad g(t) = \sin t \quad \Rightarrow \quad h(t) = 1 * \sin t = \int_0^t \sin u \mathrm{d}u = 1 - \cos t$$

**(2)** Consider the ODE $4\ddot{f}(t) + f(t) = h(t)\,, \quad f(0) = 3\,, \quad \dot{f}(0) = -7\,,$

with an unspecified forcing term $h(t)$. Laplace transform the ODE using rule (E):

$$4\left[s^2 F(s) - s f(0) - \dot{f}(0)\right] + F(s) = H(s)$$

$$\Rightarrow \quad (4s^2+1)F(s) - 12s + 28 = H(s)$$

$$\Rightarrow \quad F(s) \;=\; \frac{12s-28}{4(s^2+\frac{1}{4})} + \frac{H(s)}{4(s^2+\frac{1}{4})} \;=\; \frac{3s}{s^2+\frac{1}{4}} - \frac{7}{s^2+\frac{1}{4}} + \frac{H(s)}{4}\frac{1}{s^2+\frac{1}{4}}$$

The first two terms are inverted with our results for $\sin(at)$, and $\cos(at)$, the third with convolution:

$$f(t) = 3\cos\frac{t}{2} - 14\sin\frac{t}{2} + \frac{1}{4}h(t) * \left(2\sin\frac{t}{2}\right) = 3\cos\frac{t}{2} - 14\sin\frac{t}{2} + \frac{1}{2}\int_0^t \sin\frac{u}{2}\,h(t-u)\mathrm{d}u\,.$$

Remarkably complete given that we have no information about $h(t)$!