Part II General Relativity

Lecture Notes

Abstract

These notes represent the material covered in the Part II lecture *General Relativity* (GR). While the course is largely self-contained and some aspects of Newtonian Gravity and Special Relativity will be reviewed, it is assumed that readers will already be familiar with these topics. Also, calculus in N dimensions and Linear Algebra will be used extensively without being introduced.

This set of notes differs from the long version by representing in almost verbatim style how the material is presented in the lecture room. It is primarily designed to dispense with the necessity to take notes during the lectures.

A more in-depth discussion of books is given in the long set of notes. Here we merely discuss a few books seemingly most suitable for an introduction to general relativity.

- S. M. Carroll: "Spacetime and Geometry: An Introduction to General Relativity" [2] ; cf. also [1] .
- R. d'Inverno: "Introducing Einstein's Relativity" [3] .
- J. B. Hartle: "Gravity, An Introduction to Einstein's General Relativity" [4] .
- L. Ryder: "Introduction to General Relativity" [5] .
- B. Schutz, "A first course in general relativity" [6] .

I would not set any of them apart over the others, but recommend each reader to have a look at them and find where the best chemistry is found.

Example sheets will be pointed to at some later stage, probably on

`http://www.damtp.cam.ac.uk/user/examples`

Lectures Webpage:

`http://www.damtp.cam.ac.uk/user/us248/Lectures/lectures.html`

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Contents

A Preliminaries

A.1 Units

SI units:
- metres, second, kilogram etc.
- Adapted to "ourselves" $\rightarrow$ numbers $\mathcal{O}(1)$

Natural units:
- Take into account constants of nature.

  $\rightarrow$ 1) unifies physical dimensions (space, time etc.)

  2) indicate possible breakdown of theories
- Better adapted to *extreme* physics.

Speed of light

$c = 299\,792\,458$ m/s $\approx 3 \times 10^8$ m/s $\overset{!}{=}$ const

Suggests to measure all speeds in units of c.

$$\boxed{\Rightarrow \quad c = 3.00 \times 10^8 \text{ m/s} \overset{!}{=} 1}$$

$\Rightarrow \quad 1 \text{ s} = 3.00 \times 10^8 \text{ m}$

Familiar from the *light year*: $1 \text{ yr} = 9.4607 \times 10^{15}$ m

$v \ll c = 1 \quad \Rightarrow$ Galileo trafo, Newtonian kinematics accurate

$v \lesssim 1 \quad \Rightarrow$ both break down $\rightarrow$ need special relativity (SR)

Gravitational constant

$$\boxed{G = 6.67408 \times 10^{-11} \frac{\text{m}^3}{\text{kg}\,\text{s}^2} \overset{!}{=} \text{const}}$$

Set $c = 1,\ G = 1 \quad \Rightarrow \quad 1 \text{ m} = 1.3466 \times 10^{27} \text{ kg}$ or $1 \text{ s} = 4.0370 \times 10^{35} \text{ kg}$

Example: Solar mass $M_\odot = 1.4771 \text{ km} = 4.9269\ \mu\text{s} \quad \rightarrow \quad \sim$ Schwarzschild radius of sun

$\dfrac{M}{R} \ll \dfrac{c^2}{G} = 1 \quad \Rightarrow$ Newtonian gravity (NG) accurate

$\dfrac{M}{R} \approx 1 \quad \Rightarrow$ NG breaks down $\rightarrow$ need general relativity (GR)

Comment: If velocities determined by gravity, the regimes $v \approx 1$ and $M/R \approx 1$ overlap.

E.g.: (1) Spherical orbit around mass M: $\frac{v^2}{c^2} = \frac{G}{c^2}\frac{M}{R} \quad \Rightarrow \quad v^2 = \frac{M}{R}$

(2) Escape velocity from sphere of mass M: $\frac{v_e^2}{c^2} = \frac{2G}{c^2}\frac{M}{R} \quad \Rightarrow \quad v_e^2 = \frac{2M}{R}$

Planck's constant

$$\boxed{\hbar := \frac{h}{2\pi} = 1.0545718 \times 10^{-34}\ \frac{\mathrm{kg\,m^2}}{\mathrm{s}}}$$

Set $c = 1,\ \hbar = 1 \quad \Rightarrow 1\ \mathrm{kg} = 8.5223 \times 10^{50}\ \mathrm{Hz} \qquad$ or $\qquad 1\ \mathrm{m} = \frac{1}{3.51767288 \times 10^{-43}\ \mathrm{kg}}$

Compton wavelength: $\lambdabar = \frac{\hbar}{mc} = \frac{1}{m}$

Compare Compton wavelength of a body with its size or available volume.

E.g. : Sun: $\lambdabar_\odot = \frac{\hbar}{M_\odot c} = 0.177 \times 10^{-72}\ \mathrm{m} \quad \Rightarrow \quad \frac{\lambdabar_\odot}{R_\odot} \ll 1$

Proton: $\lambdabar_p = \frac{\hbar}{m_p c} = 0.210268\ \mathrm{fm} \sim$ radius of atomic nuclei $\sim 1 \ldots 10$ fm

$\frac{\lambdabar}{R} \ll 1 \quad \Rightarrow \quad$ Classical physics accurate

$\frac{\lambdabar}{R} \approx 1 \quad \Rightarrow \quad$ Need quantum mechanics (QM)

Planck mass

Consider a system with $\frac{G}{c^2}\frac{M}{R} = 1$ (GR!) and $\frac{\hbar}{McR} = 1$ (QM!).

$$\Rightarrow \quad M = \sqrt{\frac{\hbar c}{G}} = 2.18 \times 10^{-8}\ \mathrm{kg} = 1.22 \times 10^{19}\ \mathrm{GeV}$$

In this regime, quantum and GR effects are important.

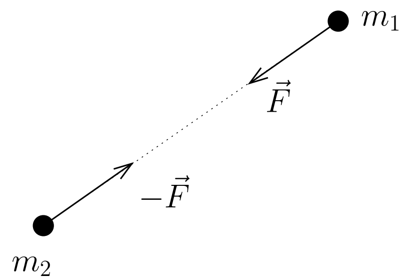
The *theory of quantum gravity* for this regime remains unknown.

A.2 Newtonian Gravity

A tale of 3 masses

Newtonian 2-body problem

$$\vec{F}_{1\text{on}2} = G m_{1a} m_{2p} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \stackrel{!}{=} m_{2i} \ddot{\vec{r}}_2$$



$m_a =$ active mass: source of grav. field ,

$m_p =$ passive mass: sensitivity to grav. fields ,

$m_i =$ inertial mass: resistance to change motion .

"Actio = Reactio" $\Rightarrow$ $\vec{F}_{2\text{on}1} = G m_{1p} m_{2a} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} \stackrel{!}{=} -\vec{F}_{1\text{on}2} = G m_{1a} m_{2p} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3}$

$\vec{r}_1$, $\vec{r}_2$ are arbitrary, so: $m_{1p} m_{2a} = m_{1a} m_{2p}$

$$\Rightarrow \frac{m_{1p}}{m_{1a}} = \frac{m_{2p}}{m_{2a}}$$

Ratio of passive to active mass is the same for all bodys. Without loss of generality (WLoG),

$$\boxed{m_a = m_p}$$

Note: Same holds for electric charge: $q_a = q_p$.

How about inertial mass?

$\sim$ **1590: Galileo**: Balls of different mass need same time to roll down a slope.

1922: Eötvös: Torque from Sun's gravity on torsion balance is $< 5 \times 10^{-9}$

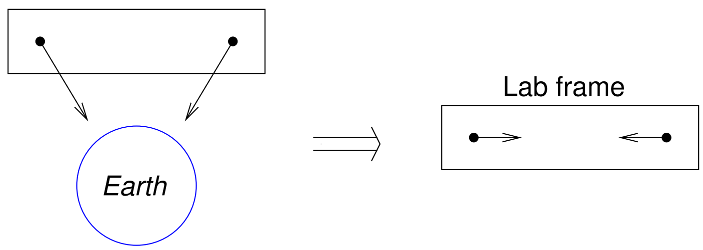
+ many more, all compatible with $m_a = m_p = m_i$. This leads to the...

Equivalence principles

Weak Equivalence Principle (WEP): Freely falling bodies with negligible gravitational self interaction follow the same path if they have the same initial velocity and position.

Einstein promotes this to a more general level.

Def. : "local inertial frame" $:=$ coordinate frame (t, x, y, z) defined by a freely falling observer in the same way as an inertial frame is defined in Minkowski spacetime. "local" means $\ll$ length scale of variations in the gravitational field $\vec{g}$ (cf. figure).



Einstein Equivalence Principle (EEP): In a local inertial frame, the results of all non-gravitational experiments are indistinguishable from those of the same experiment performed in an inertial frame in Minkowski spacetime.

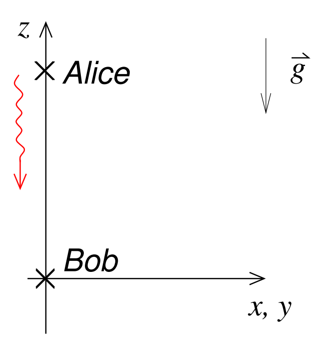
Strong equivalence principle (SEP): The gravitational motion of a small test body (that may have gravitational self interaction) depends only on its initial velocity and position but not on its constitution.

Comments:

- SEP $\Rightarrow$ WEP ; in general WEP $\not\Rightarrow$ SEP
- Need "small" objects to avoid tidal effects. E.g. Moon drifting away from Earth.
- SEP is related to equality of active and passive mass. Say Earth and Moon have different m_a/m_p. They'd fall differently in the Sun's field $\Rightarrow$ "Nortvedt effect"
- SEP implies $G =$ const everywhere.
- GR satisfies all three EPs. Gravity is a feature of spacetime!

Gravitational redshift

Consider: $\vec{g} = (0,\, 0,\, -g)$, Alice at $z = h$, Bob at $z = 0$
Alice sends light to Bob.



EEP $\Rightarrow$ equivalent to frame accelerated with $(0,\, 0,\, +g)$ in Minkowski spacetime

Assumption: v of Bob, Alice $\ll c$

$\Rightarrow$ ignore $\dfrac{v^2}{c^2}$ and higher-order SR terms

$$\Rightarrow \; z_A(t) = h + \frac{1}{2}gt^2 \,, \qquad z_B(t) = \frac{1}{2}gt^2 \,, \qquad v_A = v_B = gt \overset{!}{\ll} c \,.$$

- Alice emits first signal at t_1

  $\Rightarrow z_1(t) = z_A(t_1) - c(t - t_1) = h + \frac{1}{2}gt_1^2 - c(t - t_1)$

- This reaches Bob at T_1, i.e. $\quad h + \frac{1}{2}gt_1^2 - c(T_1 - t_1) = \frac{1}{2}gT_1^2 \qquad$ **(∗∗)**
- Alice emits second signal at $t_2 = t_1 + \Delta\tau_A$.

  This reaches Bob at $T_2 = T_1 + \Delta\tau_B$.

  $\Rightarrow h + \frac{1}{2}g(t_1 + \Delta\tau_A)^2 - c(T_1 + \Delta\tau_B - t_1 - \Delta\tau_A) = \frac{1}{2}g(T_1 + \Delta\tau_B)^2 \quad \Big| \;$ subtract **(∗∗)**

  $\Rightarrow c(\Delta\tau_A - \Delta\tau_B) + \frac{1}{2}g\,\Delta\tau_A\,(2t_1 + \Delta\tau_A) = \frac{1}{2}g\,\Delta\tau_B\,(2T_1 + \Delta\tau_B)$

- Assumption: $\Delta\tau_A \ll t_1\,, \quad \Delta\tau_B \ll T_1\,,$ e.g. period in light waves

  $\Rightarrow c(\Delta\tau_A - \Delta\tau_B) + g\,\Delta\tau_A\,t_1 = g\,\Delta\tau_B\,T_1$

  $\Rightarrow \Delta\tau_B\,(gT_1 + c) = \Delta\tau_A\,(gt_1 + c)$

  $\Rightarrow \Delta\tau_B = \left(1 + \frac{gT_1}{c}\right)^{-1}\left(1 + \frac{gt_1}{c}\right)\Delta\tau_A \approx \left[1 - \frac{g(T_1 - t_1)}{c}\right]\Delta\tau_A \quad \Big| \;$ we used $\frac{gt}{c} \ll 1$

- **(∗∗)** $\;\Rightarrow\; \frac{h}{c} - (T_1 - t_1) = \frac{1}{2}\underbrace{\frac{g}{c}(T_1 + t_1)}_{\ll 1}(T_1 - t_1) \approx 0 \quad \Big| \;$ we used $\frac{gt}{c} \ll 1$

  $\Rightarrow T_1 - t_1 = \frac{h}{c}$ to leading order.

- $\Rightarrow \boxed{\Delta\tau_B \approx \left(1 - \frac{gh}{c^2}\right)\Delta\tau_A \overset{!}{<} \Delta\tau_A}$

  $\Rightarrow$ Signal appears blue shifted to Bob: $c\,\Delta\tau_B = \lambda_B \approx \left(1 - \frac{gh}{c^2}\right)\lambda_A$

Confirmed in Pound-Rebka experiment (1960): light falling in tower.
Light climbing out of a gravity well is red shifted.

Redshift in curved spacetime

Recall: Invariant interval in SR: $\quad c^2\Delta\tau^2 = c^2\,\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$

For weak, static gravitational field, this generalizes to (cf. later):

$$c^2\,d\tau^2 = \left[1 + \frac{2\phi(x,y,z)}{c^2}\right]\,c^2dt^2 - \left[1 - \frac{2\phi(x,y,z)}{c^2}\right](dx^2+dy^2+dz^2)\,; \qquad \frac{\phi}{c^2} \ll 1$$

- Alice: $\vec{x}_A$, Bob: $\vec{x}_B$, at fixed positions!
- Alice emits signals at t_A, $t_A + \Delta t$

  Bob receives the first at t_B . When does he see the second?
- The spacetime is static: ϕ does not depend on t

  $\Rightarrow$ The two signals travel on identical trajectories, just shifted in time

  $\Rightarrow$ Bob receives the second signal at $t_B + \Delta t$.
- But what proper times do Alice's and Bob's clocks measure?

$$\Delta\tau_A^2 = \left(1 + \frac{2\phi_A}{c^2}\right)\Delta t^2\,, \qquad \Delta\tau_B^2 = \left(1 + \frac{2\phi_B}{c^2}\right)\Delta t^2$$

$$\Rightarrow \Delta\tau_A \approx \left(1 + \frac{\phi_A}{c^2}\right)\Delta t\,, \qquad \Rightarrow \Delta\tau_B \approx \left(1 + \frac{\phi_B}{c^2}\right)\Delta t\,,$$

$$\Rightarrow \Delta\tau_B \approx \left(1 + \frac{\phi_B}{c^2}\right)\left(1 + \frac{\phi_A}{c^2}\right)^{-1}\Delta\tau_A \approx \left(1 + \frac{\phi_B - \phi_A}{c^2}\right)\Delta\tau_A$$

Newtonian gravity for matter fields

Index notation:

- Write vectors, matrices as components: $x_i = (x_1, x_2, x_3) = (x, y, z)$; $v_i = (v_1, v_2, v_3)$ etc.
- Repeated indices in a product are summed over: $A_{ij}v_j := \sum_{j=1}^{3} A_{ij}v_j$
- No index may appear more than twice: $A_{ii}v_i$ is not defined.
- We may rename indices summed over: $A_{ij}v_j = A_{ik}v_k$
- In an equation or a sum, free (not repeated) indices must match on both sides:

  $w_i + A_{ik}v_k = 0$ is correct; $w_j = A_{ik}v_k$ is not.
- We denote partial derivatives by $\partial_i = \dfrac{\partial}{\partial x_i}$. Sometimes, we also use a comma:

  $v_{k,i} := \partial_i v_k = \dfrac{\partial v_k}{\partial x_i}$

Example: Motion of point particle in gravitational field $\vec{g}$.

$m\ddot{\vec{x}} = m\vec{g}(\vec{x}, t) \quad \Rightarrow \quad \ddot{x}_i = g_i(x_k, t).$

Let $\tilde{x}_i$ be a non-inertial coordinate system: $\tilde{x}_i = x_i - b_i(t)$.

$\Rightarrow \qquad \ddot{\tilde{x}}_i = \tilde{g}_(\tilde{x}_k, t) = g_i(\tilde{x}_k, t) - \ddot{b}_i(t)$

Comments: 1) If g_i is uniform (x_k independent) $\Rightarrow$ $\exists\, b_i$ such that $\tilde{g}_i = 0$.

2) g_i not uniform $\Rightarrow$ we can only get $\tilde{g}_i = 0$ locally $\rightarrow$ freely falling frame

Index version of Newtonian gravity

Tidal forces on two particles at x_i, $x_i + \delta x_i$:

$$\frac{d^2}{dt^2} x_i = g_i(x_j, t)\,, \qquad \frac{d^2}{dt^2}(x_i + \delta x_i) = g_i(x_j + \delta x_j, t)$$

$$\Rightarrow \quad \frac{d^2}{dt^2}\delta x_i = \delta x_k \partial_k g_i + \mathcal{O}(\delta x_j^2)\,,$$

$$\Rightarrow \quad \boxed{\frac{d^2}{\partial t^2}\delta x_i + E_{ij}\delta x_j = 0}\,, \qquad E_{ij} := -\partial_j g_i\,.$$

$\vec{g}$ is curl free $\Rightarrow \boxed{\vec{g} = -\vec{\nabla}\phi \quad \Leftrightarrow \quad g_i = -\partial_i \phi}$

It follows: $\boxed{E_{ij} = E_{ji}}$.

Poisson equation: $\vec{\nabla}\cdot\vec{g} = -4\pi G\rho \quad \Rightarrow \quad \boxed{\vec{\nabla}^2\phi = \partial_i\partial_i\phi = 4\pi G\rho} \quad \Rightarrow \quad E_{ii} = 4\pi G\rho\,.$

The definition $E_{ij} = -\partial_j g_i$ implies

$$\partial_k E_{ij} = -\partial_k\partial_j g_i = \partial_j E_{ki} \qquad \Rightarrow \qquad \boxed{E_{i[j,k]} := \frac{1}{2}(E_{ij,k} - E_{ik,j}) = 0}$$

The need for GR: not so much from experiment, but the incompatibility of SR with Newtonian space and time.

A.3 Special Relativity

Extend index notation

- Distinguish upstairs and downstairs indices: $v^i \neq v_i$.
- Summation only over one up and one downstairs index: $\quad v^j u_j := \sum_{j=1}^{3} v^j u_j$.
- Latin indices $i, j, \ldots = 1, 2, 3$. Greek indices $\alpha, \beta, \ldots = 0 \ldots 3$.

Metric

Pythagoras as matrix equation: $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = \delta_{ij}\Delta x^i \,\Delta x^j \qquad (\dagger)$

$\delta_{ij} = \text{diag}(1,\,1,\,1) =$ flat Euclidean metric in Cartesian coords.

In polar coordinates:
$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \\ &= g_{ij} d\tilde{x}^i d\tilde{x}^j \,, \qquad g_{ij} = \text{diag}(1,\, r^2,\, r^2\sin^2\theta) \,. \end{aligned}$$

Note: Unlike ($\dagger$), this only works for infinitesimal distances!

Lorentz transformations

Consider inertial (non-accelerated) frames with Cartesian coords.

Proper distance between spacetime events $(t,\, x,\, y,\, z)$ and $(t + \Delta t,\, x + \Delta x,\, y + \Delta y,\, z + \Delta z)$:
$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$

In SR, no inertial frame is prefered over another $\Rightarrow$ same proper distance in $\tilde{x}^\alpha$:
$\Delta s^2 = -\Delta \tilde{t}^2 + \Delta \tilde{x}^2 + \Delta \tilde{y}^2 + \Delta \tilde{z}^2$

Note: For $\Delta s = 0$, the events are connected by a light ray.
$\Rightarrow$ All inertial frames measure the same speed of light.

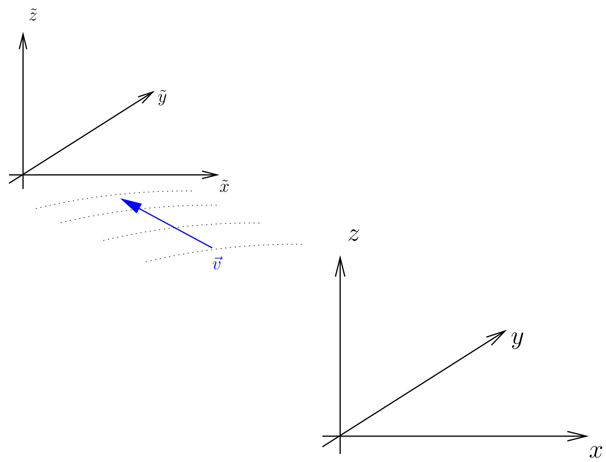
Index notation: $\Delta s^2 = \eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta = \eta_{\tilde{\alpha}\tilde{\beta}}\Delta \tilde{x}^{\tilde{\alpha}} \Delta \tilde{x}^{\tilde{\beta}}$

$$\eta_{\alpha\beta} = \eta_{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Leftrightarrow \qquad \eta^{\alpha\beta} = \eta^{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\eta^{\alpha\beta}$ is the *inverse* of $\eta_{\alpha\beta}$.

Inertial frames are related by $\quad \tilde{x}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_{\mu} x^{\mu} + x_0^{\mu}\,, \qquad \Lambda^{\tilde{\alpha}}{}_{\mu} = \text{const}$

WLoG: $x_0^{\mu} = 0$.



We want: $\quad \eta_{\tilde{\alpha}\tilde{\beta}} \Delta\tilde{x}^{\tilde{\alpha}} \Delta\tilde{x}^{\tilde{\beta}} = \eta_{\tilde{\alpha}\tilde{\beta}} \Lambda^{\tilde{\alpha}}{}_{\mu} \Delta x^{\mu}\, \Lambda^{\tilde{\beta}}{}_{\nu} \Delta x^{\nu} \overset{!}{=} \eta_{\mu\nu} \Delta x^{\mu}\, \Delta x^{\nu}$

$$\Rightarrow \eta_{\mu\nu} = \Lambda^{\tilde{\alpha}}{}_{\mu} \Lambda^{\tilde{\beta}}{}_{\nu} \eta_{\tilde{\alpha}\tilde{\beta}}$$

One can show that this is satisfied by the *Lorentz transformations*

$$\Lambda^{\tilde{\alpha}}{}_{\mu} = \left(\begin{array}{c|ccc} \gamma & -\gamma v & 0 & 0 \\ \hline -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad \Leftrightarrow \quad \Lambda^{\mu}{}_{\tilde{\alpha}} = \left(\begin{array}{c|ccc} \gamma & \gamma v & 0 & 0 \\ \hline \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \qquad \gamma = \frac{1}{\sqrt{1 - v^2}}$$

Comments
- WLoG, coordinates oriented such that the velocity v points in the x direction
- One can show $\Lambda^{\tilde{\alpha}}{}_{\mu} \Lambda^{\mu}{}_{\tilde{\beta}} = \delta^{\tilde{\alpha}}{}_{\tilde{\beta}}\,, \qquad \Lambda^{\mu}{}_{\tilde{\alpha}} \Lambda^{\tilde{\alpha}}{}_{\nu} = \delta^{\mu}{}_{\nu}$

World lines and 4-velocity

Def.: The interval between two spacetime events x^{α} and $x^{\alpha} + \Delta x^{\alpha}$ is called

timelike $\quad :\Leftrightarrow \quad \eta_{\mu\nu} \Delta x^{\mu}\, \Delta x^{\nu} < 0$

null $\quad :\Leftrightarrow \quad \eta_{\mu\nu} \Delta x^{\mu}\, \Delta x^{\nu} = 0$

spacelike $\quad :\Leftrightarrow \quad \eta_{\mu\nu} \Delta x^{\mu}\, \Delta x^{\nu} > 0\,.$

Def.: Proper time $\quad \Delta\tau^2 := -\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$

Postulate: A clock moving on a world line $x^\alpha(\lambda)\,,\ \lambda \in \mathbb{R}$, that is in every point timelike or null, measures the proper time along this world line

$$\tau := \int_{\lambda_1}^{\lambda_2} \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\mu}{d\lambda}}d\lambda\,. \tag{‡}$$

Comments:

- τ is invariant under reparametrizing $\lambda \to \mu(\lambda)$.
- We often parametrize timelike curves with τ
- (‡) $\quad\Rightarrow\quad d\tau = \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau \quad\Rightarrow\quad \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu := \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$

Def.: The four velocity along a timelike curve is $\quad u^\alpha := \dfrac{dx^\alpha}{d\tau}$

By def. $\quad \eta_{\mu\nu}u^\mu u^\nu = -1$

Geodesics

Consider the action $\quad \mathcal{S}[x^\alpha(\lambda)] = \int \underbrace{\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}_{=:\mathcal{L}}d\lambda$

Geodesics are curves that extremize this action.

They follow from the Euler-Lagrange (EL) equations

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = \frac{\partial\mathcal{L}}{\partial x^\mu} \quad\Rightarrow\quad \dots \quad\Rightarrow \frac{d^2x^\alpha}{d\tau^2} = 0\,.$$

One can derive the same equation for null and spacelike geodesics (cf. GR case below).

Postulate: Free massive (massless) particles in special relativity move on straight timelike (null) curves,

$$\frac{d^2x^\alpha}{d\tau^2} = 0\,. \tag{A.1}$$

Time dilation

Let $\mathcal{O}$, $\tilde{\mathcal{O}}$ be 2 observers with coordinates x^μ, $\tilde{x}^{\tilde{\alpha}}$.

Let $\tilde{\mathcal{O}}$ move with v in the x direction relative to $\mathcal{O}$.

Clock at rest in $\tilde{\mathcal{O}}$: $\tilde{u}^{\tilde{\alpha}} = \left(\dfrac{d\tilde{t}}{d\tau}, 0, 0, 0\right)$

Viewed from $\mathcal{O}$: $u^\mu = \left(\dfrac{dt}{d\tau}, \dfrac{dx^i}{d\tau}\right) \overset{!}{=} \Lambda^\mu{}_{\tilde{\alpha}} \tilde{u}^{\tilde{\alpha}} = \left(\gamma\dfrac{d\tilde{t}}{d\tau}, \gamma v \dfrac{d\tilde{t}}{d\tau}, 0, 0\right)$ (†)

u^t component: $\dfrac{dt}{d\tau} = \gamma \dfrac{d\tilde{t}}{d\tau} \quad \Rightarrow \quad \dfrac{dt}{d\tilde{t}} = \gamma$

$\Rightarrow \quad dt = \dfrac{d\tilde{t}}{\sqrt{1-v^2}}$: $\mathcal{O}$ sees the moving $\tilde{\mathcal{O}}$ age more slowly.

Lorentz contraction

Def.: Length of a rod in $\mathcal{O}$:= proper distance Δs between two events $\mathcal{A}$ and $\mathcal{B}$, where $x^i_{\mathcal{A}}$ is the position of the rod's tail at a specified time $t_{\mathcal{A}} = t_0$ and $x^i_{\mathcal{B}}$ is the position of the rod's head at the same time $t_{\mathcal{B}} = t_0$.

$$\Delta s = \sqrt{\eta_{\alpha\beta} \Delta x^\alpha \Delta x^\beta} = \sqrt{\delta_{ij} \Delta x^i \Delta x^j}\,, \qquad \Delta x^i = x^i_{\mathcal{B}} - x^i_{\mathcal{A}} \tag{A.2}$$

Let the rod be at rest in $\mathcal{O}$.

World lines of head and tail: $x^\mu = (t_{\text{tail}}, x^i_0)\,, \qquad y^\mu = (t_{\text{head}}, x^i_0 + \Delta x^i)$

$\mathcal{O}$ will pick events with $t_{\text{tail}} = t_{\text{head}} \quad \Rightarrow \quad \ell^2 = \Delta s^2 = \delta_{ij} \Delta x^i \Delta x^j$

World lines in $\tilde{\mathcal{O}}$: $(\tilde{t}_{\text{tail}}, \tilde{x}^i) = \tilde{x}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_\mu x^\mu$

$$(\tilde{t}_{\text{head}}, \tilde{y}^i) = \tilde{y}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_\mu y^\mu = \Lambda^{\tilde{\alpha}}{}_\mu (x^\mu + \Delta x^\mu)$$

$\tilde{\mathcal{O}}$ will pick events $\tilde{A}$, $\tilde{B}$ with $\tilde{t}_{\text{tail}} = \tilde{t}_{\text{head}}$.

$\Rightarrow \ldots \Rightarrow \quad t_{\text{tail}} = t_{\text{head}} + \dfrac{\Lambda^{\tilde{0}}{}_i \,\Delta x^i}{\Lambda^{\tilde{0}}{}_0} = t_{\text{head}} - v_i \Delta x^i$

Proper distance $\Delta s^2_{\tilde{A}\tilde{B}}$: $x^\mu_{\tilde{\mathcal{A}}} = (t_{\text{head}} - v_i \Delta x^i, x^i_0)\,, \qquad x^\mu_{\tilde{\mathcal{B}}} = (t_{\text{head}}, x^i_0 + \Delta x^i)$

$$\begin{aligned} \Rightarrow \tilde{\ell}^2 = \Delta s^2_{\tilde{\mathcal{A}}\tilde{\mathcal{B}}} &= \eta_{\mu\nu}(x^\mu_{\tilde{\mathcal{B}}} - x^\mu_{\tilde{\mathcal{A}}})(x^\nu_{\tilde{\mathcal{B}}} - x^\nu_{\tilde{\mathcal{A}}}) \\ &= -(v_i \Delta x^i)^2 + \delta_{ij} \Delta x^i \,\Delta x^j \end{aligned}$$

Orient the rod along the x axis $\Rightarrow \quad \ell = \Delta x\,, \qquad \tilde{\ell} = \sqrt{1-v_x^2}\Delta x$

Comments:
- The sign of v does not matter.
- Velocity perpendicular to the rod causes no contraction.

Four momentum and Doppler shift

Def.: Four momentum of a particle with rest mass m and 4-velocity u^μ: $p^\alpha = m u^\alpha$

$\eta_{\mu\nu} u^\mu u^\nu = -1 \quad \Rightarrow \quad \eta_{\mu\nu} p^\mu p^\nu = -m^2$

Let $\tilde{\mathcal{O}}$ move with v in x direction relative to $\mathcal{O}$; consider particle at rest in $\tilde{\mathcal{O}}$.

$\Rightarrow \quad \tilde{p}^{\tilde{\alpha}} = (m,\,0,\,0,\,0)$

$p^\mu = \Lambda^\mu{}_{\tilde{\alpha}} \tilde{p}^{\tilde{\alpha}} = \gamma m(1,\,v,\,0,\,0)$

$\gamma m =$ total relativistic energy of particle in $\mathcal{O}$

$\gamma m v =$ linear momentum of particle in $\mathcal{O}$

$\Rightarrow \quad p^\mu = (E,\,p,\,0,\,0)$

$\eta_{\mu\nu} p^\mu p^\nu = -E^2 + p^2 = -m^2 \quad \Rightarrow \quad E^2 = m^2 c^4 + p^2 c^2$

Comment: Null curves do not have a four velocity (unit tangent vector), but they have a four momentum.

Recall for massless particles: $E = h\nu$, $p = h/\lambda$.

$\Rightarrow \quad p^\alpha = h\nu(1,\,1,\,0,\,0)$ e.g. photon moving in x direction

Now consider such a photon and let $\tilde{\mathcal{O}}$ move with v in x-direction relative to $\mathcal{O}$.

$\Rightarrow$ in $\mathcal{O}$: $p^\alpha = (E,\,E,\,0,\,0)\,, \quad E = h\nu$

$\Rightarrow$ in $\tilde{\mathcal{O}}$: $\tilde{p}^{\tilde{\alpha}} = \Lambda^{\tilde{\alpha}}{}_\mu p^\mu = \big(\gamma E - \gamma v E,\; -\gamma v E + \gamma E,\; 0,\; 0\big) =: \big(\tilde{E},\; \tilde{E},\; 0,\; 0\big)\,, \quad \tilde{E} = h\tilde{\nu}$

Redshift $\dfrac{\tilde{\nu}}{\nu} = \dfrac{\tilde{E}}{E} = \gamma(1-v) = \sqrt{\dfrac{1-v}{1+v}}$

Note:
- Redshift if $\tilde{\mathcal{O}}$ moves in same direction as photon. Blueshift if $v < 0$.
- *Transverse Doppler shift* if v perpendicular to propagation of photon.
  More complicated to calculate!

B Differential geometry

Goal: Extend Euclidean geometry to curved spaces.

Motivation: GR generalizes SR like Riemannian geometry generalizes Euclidean geometry.

Conventions:

- An upstairs index in a denominator counts as a downstairs index. E.g.: $\partial_i \coloneqq \dfrac{\partial}{\partial x^i}$
- *Contravariant* indices: upstairs. *Covariant* indices: downstairs.

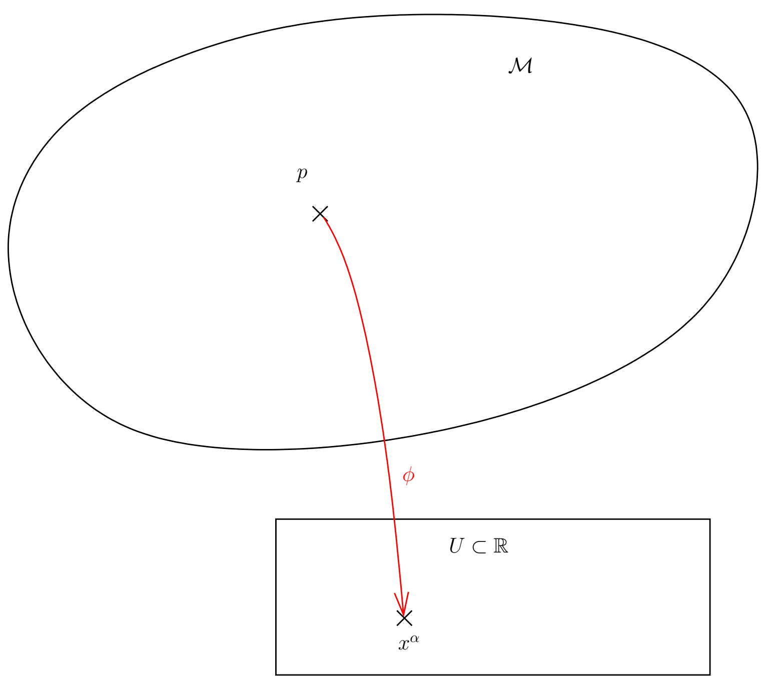
B.1 Manifolds and tensors

Strategy: Start with manifold $\mathcal{M}$. Establish structure on $\mathcal{M}$ step by step.

Def.: n dimensional manifold $\mathcal{M} \coloneqq$ set of points that locally resembles Euclidean space $\mathbb{R}^n$ at each point. For our purposes, this means that there exists a one-to-one and onto map

$$\phi : \mathcal{M} \to U \subset \mathbb{R}^n \,, \quad p \in \mathcal{M} \mapsto x^\alpha \in U \subset \mathbb{R}^n \,, \quad \alpha = 0, \ldots, n-1 \,, \tag{B.1}$$

where U is an open subset of $\mathbb{R}^n$. x^α are the coordinates on $\mathcal{M}$.



Comments:

- It is sufficient if we can chop up $\mathcal{M}$ and map each piece separately to $\mathbb{R}^n$. Everything we will develop also holds for such subdivisions of $\mathcal{M}$.
- Think of coordinates like house numbers in a street. Houses don't change if we change their numbers. Likewise, we will find objects on $\mathcal{M}$ to be invariant under coordinate changes.
- Curves, vectors etc. live on $\mathcal{M}$, not in the coordinate space. But ϕ is one-to-one, so this distinction often blurred.

Functions and curves

Def.: Function on $\mathcal{M}$: $\quad f : \mathcal{M} \to \mathbb{R}$

f is smooth $:\Leftrightarrow$ $\forall$ coordinate systems x^α: $f(x^\alpha)$ is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$. If f is invariant under a change of coordinates, it is a *scalar*.

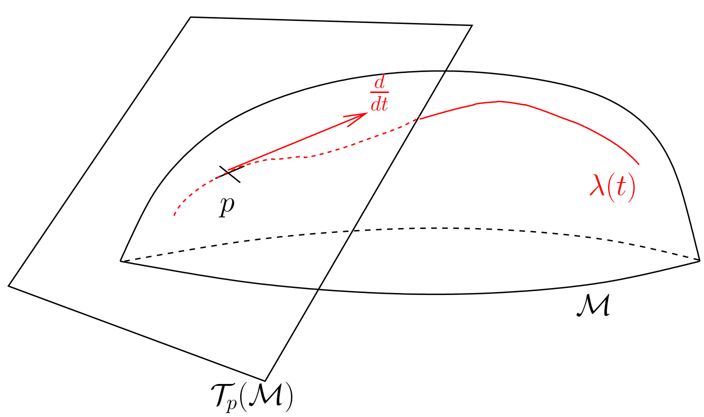
Def.: Curve $:=$ a map $\lambda : I \subset \mathbb{R} \to \mathcal{M}$, I open. λ is smooth $:\Leftrightarrow \forall$ coordinate systems x^α: the map $x^\alpha \circ \lambda : I \to \mathbb{R}^n$ is smooth

Vectors

Def.: Let $\mathcal{C}^\infty$ be the space of all smooth functions $f : \mathcal{M} \to \mathbb{R}$, λ be a smooth curve and $p \equiv \lambda(0) \in \mathcal{M}$. The tangent vector to the curve λ at $p \in \mathcal{M}$ is the map

$$\boldsymbol{V} : \mathcal{C}^\infty \to \mathbb{R}\,, \qquad f \mapsto \boldsymbol{V}(f) = \frac{d}{dt} f\big(\lambda(t)\big)\bigg|_{t=0}\,.$$

$\mathcal{T}_p(\mathcal{M}) \quad := \quad$ space of all vectors at p.



A vector is a derivative operator! It obeys:

(i) Linearity: $\alpha,\, \beta \in \mathbb{R}, \quad f,\ g \in \mathcal{C}^\infty \quad \Rightarrow \quad \boldsymbol{V}(\alpha f + \beta g) = \alpha \boldsymbol{V}(f) + \beta \boldsymbol{V}(g)$

(ii) Leibniz rule: $f,\ g \in \mathcal{C}^\infty \quad \Rightarrow \quad \boldsymbol{V}(f\, g) = \boldsymbol{V}(f)\, g(p) + f(p)\, \boldsymbol{V}(g)$

Consider coordinates $x^\alpha \quad \Rightarrow \quad \boldsymbol{V}(f) = \frac{d}{dt} f\big(x^\mu\big(\lambda(t)\big)\big) = \left.\frac{dx^\mu}{dt}\right|_\lambda \frac{\partial}{\partial x^\mu}\ f(x^\alpha)$

vector components basis vectors

One can show: $\mathcal{T}_p(\mathcal{M})$ has dimension n and $\mathbf{e}_\mu \coloneqq \partial_\mu \coloneqq \frac{\partial}{\partial x^\mu}$ form a basis.

Components: $V^\mu = \frac{dx^\mu}{dt} \quad \Rightarrow \quad \boxed{\boldsymbol{V} = V^\mu \mathbf{e}_\mu = V^\mu \partial_\mu = \frac{dx^\mu}{dt}\frac{\partial}{\partial x^\mu} = \frac{d}{dt}}$

Coordinate change $x^\mu \ \to \tilde{x}^\alpha$: $\mathbf{e}_\mu = \frac{\partial}{\partial x^\mu} \quad \to \quad \tilde{\mathbf{e}}_\alpha = \frac{\partial}{\partial \tilde{x}^\alpha} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial}{\partial x^\mu} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\mathbf{e}_\mu$

$$V^\mu = \frac{dx^\mu}{dt} \quad \to \quad \tilde{V}^\alpha = \frac{d\tilde{x}^\alpha}{dt} = \frac{\partial \tilde{x}^\alpha}{\partial x^\nu}\frac{dx^\nu}{dt} = \frac{\partial \tilde{x}^\alpha}{\partial x^\nu} V^\nu$$

$\Rightarrow \quad \boldsymbol{V} = V^\mu \mathbf{e}_\mu = \tilde{V}^\alpha \tilde{\mathbf{e}}_\alpha$ is invariant!

∂_μ is a *coordinate basis.* Non-coordinate bases also exist but we do not consider them.

Covectors / one-forms

Def.: A covector or one-form is a linear map

$$\boldsymbol{\eta} : \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad \boldsymbol{V} \mapsto \boldsymbol{\eta}(\boldsymbol{V})$$

$\mathcal{T}_p^*(\mathcal{M}) \coloneqq$ *Cotangent space* of all covectors at $p \in \mathcal{M}$. $\mathcal{T}_p^*(\mathcal{M})$ is an n dimensional vector space. Let $\mathbf{e}_\mu$ be a basis of $\mathcal{T}_p(\mathcal{M})$. The components of a covector $\boldsymbol{\eta}$ are $\eta_\mu \coloneqq \boldsymbol{\eta}(\mathbf{e}_\mu)$.

Properties: Linearity: $\alpha,\, \beta \in \mathbb{R}, \quad \boldsymbol{V},\, \boldsymbol{W} \in \mathcal{T}_p(\mathcal{M}) \quad \Rightarrow \quad \boldsymbol{\eta}(\alpha \boldsymbol{V} + \beta \boldsymbol{W}) = \alpha \boldsymbol{\eta}(\boldsymbol{V}) + \beta \boldsymbol{\eta}(\boldsymbol{W})$

Components: $\boldsymbol{\eta}(\boldsymbol{V}) = \boldsymbol{\eta}(V^\mu \mathbf{e}_\mu) = V^\mu \boldsymbol{\eta}(\mathbf{e}_\mu) = V^\mu \eta_\mu$

Transformation rule: We require $\boldsymbol{\eta}(\boldsymbol{V})$ to be a scalar

$$\Rightarrow \boldsymbol{\eta}(\boldsymbol{V}) = \eta_\mu V^\mu \overset{!}{=} \tilde{\eta}_\alpha \tilde{V}^\alpha = \tilde{\eta}_\alpha \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} V^\mu \quad \Rightarrow \quad \boxed{\tilde{\eta}_\beta = \frac{\partial x^\mu}{\partial \tilde{x}^\beta}\eta_\mu}$$

Def.: Gradient $\mathbf{d}f$ of a smooth function f: $\quad \mathbf{d}f : \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad \dfrac{d}{dt} \mapsto \dfrac{df}{dt}$

$$\text{Let } \boldsymbol{V} = \frac{d}{dt} \in \mathcal{T}_p(\mathcal{M}) \quad \Rightarrow \quad \mathbf{d}f(\boldsymbol{V}) = \frac{df}{dt} = \boldsymbol{V}(f)$$

Basis: Let $f = x^\alpha\,,\ \alpha$ fixed $\quad \Rightarrow \quad \mathbf{d}x^\alpha(\mathbf{e}_\beta) = \mathbf{d}x^\alpha\left(\dfrac{\partial}{\partial x^\beta}\right) = \dfrac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta$

$$\Rightarrow \quad \eta_\alpha \mathbf{d}x^\alpha(\boldsymbol{V}) = \eta_\alpha \mathbf{d}x^\alpha(V^\beta \partial_\beta) = \eta_\alpha V^\beta \mathbf{d}x^\alpha(\partial_\beta) = \eta_\alpha V^\beta \delta^\alpha{}_\beta = \eta_\alpha V^\alpha = \boldsymbol{\eta}(\boldsymbol{V})$$

$$\Rightarrow \quad \boxed{\boldsymbol{\eta} = \eta_\alpha \mathbf{d}x^\alpha}$$

Tensors

Def. : A tensor $\boldsymbol{T}$ at $p \in \mathcal{M}$ of rank $\binom{r}{s}$, $r,\, s \in \mathbb{N}_0$, is a multilinear map

$$T : \underbrace{\mathcal{T}_p^*(\mathcal{M}) \times \ldots \times \mathcal{T}_p^*(\mathcal{M})}_{r \text{ factors}} \times \underbrace{\mathcal{T}_p(\mathcal{M}) \times \ldots \times \mathcal{T}_p(\mathcal{M})}_{s \text{ factors}} \to \mathbb{R}$$

Plug in r one-forms and s vectors and out pops a real number.

Examples:

- Covector $\boldsymbol{\eta}$ = tensor of rank $\binom{0}{1}$: input $\boldsymbol{V}$, output $\boldsymbol{\eta}(\boldsymbol{V})$
- Vector $\boldsymbol{V}$ can be viewed as: $\quad \boldsymbol{V} : \mathcal{T}_p^*(\mathcal{M}) \to \mathbb{R}\,, \qquad \boldsymbol{\eta} \mapsto \boldsymbol{\eta}(\boldsymbol{V})$

  $\Rightarrow \boldsymbol{V}$ is a $\binom{1}{0}$ tensor.

  Components of $\boldsymbol{V}$: $\boldsymbol{\eta}(\boldsymbol{V}) = \eta_\alpha \mathbf{d}x^\alpha(\boldsymbol{V}) = \eta_\alpha V^\alpha \quad \Rightarrow \quad V^\alpha = \mathbf{d}x^\alpha(\boldsymbol{V}) = \boldsymbol{V}(\mathbf{d}x^\alpha)$

  This holds for all tensors: $\boxed{T^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s} = \boldsymbol{T}(\mathbf{d}x^{\alpha_1}, \ldots, \mathbf{d}x^{\alpha_r}, \mathbf{e}_{\beta_1}, \ldots, \mathbf{e}_{\beta_s})}$
- $\boldsymbol{\delta} : \mathcal{T}_p^*(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad (\boldsymbol{\eta}, \boldsymbol{V}) \mapsto \boldsymbol{\eta}(\boldsymbol{V}) \quad \forall\ \eta \in \mathcal{T}_p^*(\mathcal{M})\,,\ \ \boldsymbol{V} \in \mathcal{T}_p(\mathcal{M})$

  is a $\binom{1}{1}$ tensor with components $\boldsymbol{\delta}(\mathbf{d}x^\alpha, \partial_\beta) = \mathbf{d}x^\alpha(\partial_\beta) = \dfrac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta$

One can show that tensors of rank $\binom{r}{s}$ form a vector space of dimension n^{r+s} and transform according to

$$\boxed{\tilde{T}^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s} = \frac{\partial \tilde{x}^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial \tilde{x}^{\alpha_r}}{\partial x^{\mu_r}} \frac{\partial x^{\nu_1}}{\partial \tilde{x}^{\beta_1}} \cdots \frac{\partial x^{\nu_s}}{\partial \tilde{x}^{\beta_s}} T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}}$$

Tensor operations

(1) Addition, scalar multiplication: Let $c_1,\ c_2 \in \mathbb{R}$, $\boldsymbol{S},\ \boldsymbol{T}$ $\binom{1}{1}$ tensors

$$\Rightarrow \quad c_1\boldsymbol{S} + c_2\boldsymbol{T} : \mathcal{T}_p^*(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad \boldsymbol{\eta},\, \boldsymbol{V} \mapsto c_1\boldsymbol{S}(\boldsymbol{\eta},\boldsymbol{V}) + c_2\boldsymbol{T}(\boldsymbol{\eta},\boldsymbol{V})$$

(2) (Anti-) symmetrization: E.g. for $\binom{0}{2}$ tensor $\boldsymbol{T}$

symmetric part: $S_{\alpha\beta} \coloneqq \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) =: T_{(\alpha\beta)}$

antisymm. part: $\frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) =: T_{[\alpha\beta]}$

Index subset: $T^{(\alpha\beta)\gamma}{}_{\delta} \coloneqq \frac{1}{2}(T^{\alpha\beta\gamma}{}_{\delta} + T^{\beta\alpha\gamma}{}_{\delta})$

non-adjacent indices: $T_{(\alpha|\beta\gamma|\delta)} \coloneqq \frac{1}{2}(T_{\alpha\beta\gamma\delta} - T_{\delta\beta\gamma\alpha})$

Over $n > 2$ indices:
- sum over all permutations
- apply sign of permutation for anti-symm.
- divide by $n!$

E.g.: $T^{\alpha}{}_{[\beta\gamma\delta]} = \frac{1}{3!}\big(T^{\alpha}{}_{\beta\gamma\delta} + T^{\alpha}{}_{\delta\beta\gamma} + T^{\alpha}{}_{\gamma\delta\beta} - T^{\alpha}{}_{\delta\gamma\beta} - T^{\alpha}{}_{\gamma\beta\delta} - T^{\alpha}{}_{\beta\delta\gamma}\big)$

(3) Contraction of $\binom{r}{s}$ tensor $\coloneqq$ Summation over 1 upper and 1 lower index

$\rightarrow$ $\binom{r-1}{s-1}$ tensor

Example: Let $\boldsymbol{T}$ be a $\binom{3}{2}$ tensor

$\Rightarrow$ $\binom{2}{1}$ tensor $\boldsymbol{S}(\boldsymbol{\omega},\boldsymbol{\eta},\boldsymbol{V}) \coloneqq \boldsymbol{T}(\mathbf{d}x^{\mu},\boldsymbol{\omega},\boldsymbol{\eta},\partial_{\mu},\boldsymbol{V})$

This is basis independent:

$$\boldsymbol{T}\Big(\mathbf{d}\tilde{x}^{\mu},\boldsymbol{\omega},\boldsymbol{\eta},\frac{\partial}{\partial\tilde{x}^{\mu}},\boldsymbol{V}\Big) \;=\; \underbrace{\frac{\partial\tilde{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial\tilde{x}^{\mu}}}_{=\delta^{\beta}{}_{\alpha}}\boldsymbol{T}\Big(\mathbf{d}x^{\alpha},\boldsymbol{\omega},\boldsymbol{\eta},\frac{\partial}{\partial x^{\beta}},\boldsymbol{V}\Big) \;=\; \boldsymbol{T}(\mathbf{d}x^{\alpha},\boldsymbol{\omega},\boldsymbol{\eta},\partial_{\alpha},\boldsymbol{V})$$

Components: $S^{\mu\nu}{}_{\rho} = T^{\alpha\mu\nu}{}_{\alpha\rho}$

(4) Let $\boldsymbol{S}$ be a $\binom{p}{q}$ tensor, $\boldsymbol{T}$ a $\binom{r}{s}$ tensor

"outer product" $\boldsymbol{S} \otimes \boldsymbol{T}$ is a $\binom{p+r}{q+s}$ tensor with

$$\big(\boldsymbol{S} \otimes \boldsymbol{T}\big)(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_p, \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_r, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_q, \boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_s)$$
$$:= \boldsymbol{S}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_p, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_q)\, \boldsymbol{T}(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_r, \boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_s)$$

One straightforwardly shows:

(i) $\big(\boldsymbol{S} \otimes \boldsymbol{T}\big)^{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_r}{}_{\mu_1 \ldots \mu_q \nu_1 \ldots \nu_s} = S^{\alpha_1 \ldots \alpha_p}{}_{\mu_1 \ldots \mu_q}\, T^{\beta_1 \ldots \beta_r}{}_{\nu_1 \ldots \nu_s}$

(ii) In a coord. basis, a $\binom{2}{1}$ tensor can be written as

$\boldsymbol{T} = T^{\mu\nu}{}_{\rho} \mathbf{e}_\mu \otimes \mathbf{e}_\nu \otimes \mathbf{d}x^\rho$

likewise (r, s) tensor

Tensor fields

So far: tensors at point $p \in \mathcal{M}$

Def.: Tensor field of rank $\binom{r}{s}$:= collection of $\binom{r}{s}$ tensors at each $p \in \mathcal{M}$. Like a map : $p \to \boldsymbol{T}_p$ of rank $\binom{r}{s}$. The tensor field is smooth :⇔ its components in a coordinate basis are smooth functions.

Sometimes we write $\boldsymbol{X}_p$ = vector, $\boldsymbol{X}$ = field. Often it's clear from context.

Example: Vector field (VF) $\boldsymbol{X} : \mathcal{M} \to \mathcal{T}_p(\mathcal{M})\,, \quad p \mapsto \boldsymbol{X}_p$

For function f: $\boldsymbol{X}(f) : \mathcal{M} \to \mathbb{R}\,, \quad p \mapsto \boldsymbol{X}_p(f)$ is a function

Henceforth we assume all tensor fields to be smooth.

Integral curves

Def.: Integral curve λ of a VF $\boldsymbol{V}$ through $p \in \mathcal{M}$:= curve through p whose tangent at every point q along the curve is $\boldsymbol{V}_q$.

In coords: $\left.\dfrac{d}{dt}\right|_\lambda = \boldsymbol{V} \quad \Rightarrow \quad \dfrac{dx^\mu\big(\lambda(t)\big)}{dt} = V^\mu(x^\alpha) \quad \text{with} \quad x^\mu\big(\lambda(t_0)\big) = x^\mu(p)$

Has a unique solution by ODE theory.

B.2 The metric tensor

Goal: Measure distances, volumes, etc. $\rightarrow$ need metric!

Def.: A metric at $p \in \mathcal{M}$ $:= \binom{0}{2}$ tensor that is:

(i) symmetric: $\boldsymbol{g}(\boldsymbol{V},\boldsymbol{W}) = \boldsymbol{g}(\boldsymbol{W},\boldsymbol{V}) \quad \forall\, \boldsymbol{V},\, \boldsymbol{W} \in \mathcal{T}_p(\mathcal{M}) \quad \Leftrightarrow \quad g_{\alpha\beta} = g_{\beta\alpha}$

(ii) non-degenerate: $\boldsymbol{g}(\boldsymbol{V},\boldsymbol{W}) = 0 \;\; \forall\, \boldsymbol{W} \in \mathcal{T}_p(\mathcal{M}) \quad \Leftrightarrow \quad \boldsymbol{V} = 0$

Components: $\boldsymbol{g} = g_{\alpha\beta}\mathbf{d}x^\alpha \otimes \mathbf{d}x^\beta\,, \qquad g_{\mu\nu} = \boldsymbol{g}(\partial_\mu, \partial_\nu)\,, \qquad ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$

A metric maps vectors to 1-forms:

$\boldsymbol{V} \mapsto \boldsymbol{g}(\boldsymbol{V},\,.\,) =: \underline{\boldsymbol{V}}\,,$ i.e. $\underline{\boldsymbol{V}} : \mathcal{T}_p(\mathcal{M}) \rightarrow \mathbb{R}\,, \quad \boldsymbol{W} \mapsto \underline{\boldsymbol{V}}(\boldsymbol{W}) := \boldsymbol{g}(\boldsymbol{V},\boldsymbol{W}) = \underline{V}_\mu W^\mu = g_{\mu\nu}V^\mu W^\nu$

Components: $V_\mu := \underline{V}_\mu = g_{\mu\nu}V^\nu$

$\boldsymbol{g}$ non-degenerate $\Rightarrow$ $\boldsymbol{g}$ invertible

Def.: inverse metric $\boldsymbol{g}^{-1} :=$ symmetric $\binom{2}{0}$ tensor $g^{\alpha\beta}$ with $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha{}_\gamma$

Example: Line element on the unit sphere, $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$: $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$,

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \qquad g^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

$\boldsymbol{g}^{-1}$ maps 1-forms to vectors: $\big(\boldsymbol{g}^{-1}(\boldsymbol{\eta},\,.\,)\big)(\boldsymbol{\omega}) := \boldsymbol{g}^{-1}(\boldsymbol{\eta},\boldsymbol{\omega})$

The metric mappings between vectors and 1-forms are inverses of each other:

$\boldsymbol{g}^{-1}\big(\boldsymbol{g}(\boldsymbol{V},\,.\,),\,.\,\big) = \boldsymbol{V}\,, \quad \boldsymbol{g}\big(\boldsymbol{g}^{-1}(\boldsymbol{\eta},\,.\,),\,.\,\big) = \boldsymbol{\eta}$

$\rightarrow$ natural isomorphism

Signature

$\boldsymbol{g}$ symmetric $\Rightarrow$ components of $\boldsymbol{g}$ at $p \in \mathcal{M}$ are a symmetric matrix

$\Rightarrow$ $\exists$ basis where $g_{\mu\nu}$ is diagonal

$\boldsymbol{g}$ non-degenerate $\Rightarrow$ all diagonal elements are $\neq 0$

$\Rightarrow$ we can rescale the basis such that the diagonal elements $= \pm 1$

"orthonormal basis" $\leftarrow$ basis non-unique!

"Sylvester's law" $\Rightarrow$ the number of $+1$ and -1 entries is independent of basis

Def.: Signature $:=$ sum $+1$, -1 over all diagonal elements

Riemannian metrics: signature $= + + \ldots +$ or $+n = \#$ of dims.

Lorentzian metrics: $- + + \ldots +$ or $n-2$. Some people use $+ - - \ldots -$

Note: Equivalence principle

$\Rightarrow$ in a local inertial frame, the laws of SR hold

$\Rightarrow$ $\exists$ coords: metric $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ "Lorentz invariant"

Only possible locally! At $q \neq p$, $g_{\mu\nu} \neq \eta_{\mu\nu}$ in general

Def.: A Riemannian (Lorentzian) manifold

$:= (\mathcal{M}, \boldsymbol{g})$ where $\mathcal{M}$ is a diff. manifold and $\boldsymbol{g}$ a Riemannian (Lorentzian) metric

spacetime $:=$ Lorentzian manifold

Example: Minkowski metric in $\mathbb{R}^4$ with Cartesian coords. x^0, x^1, x^2, x^3:

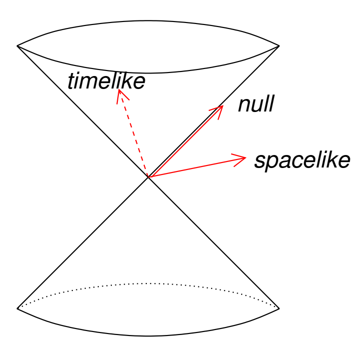
$$\boldsymbol{\eta} = -(\mathbf{d}x^0)^2 + (\mathbf{d}x^1)^2 + (\mathbf{d}x^2)^2 + (\mathbf{d}x^3)^2\,, \qquad (\mathbf{d}x^0)^2 \equiv \mathbf{d}x^0 \otimes \mathbf{d}x^0\,, \ldots$$

Def.: Let $(\mathcal{M}, \boldsymbol{g})$ be a Lorentzian manifold, $\boldsymbol{V} \in \mathcal{T}_p(\mathcal{M})$, $\boldsymbol{V} \neq 0$

$\boldsymbol{V}$ is timelike $:\Leftrightarrow$ $\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{V}) < 0$

null $:\Leftrightarrow$ $\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{V}) = 0$

spacelike $:\Leftrightarrow$ $\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{V}) > 0$



local inertial frame: $g_{\mu\nu} = \eta_{\mu\nu}$

$\Rightarrow$ locally we have the light cone structure of SR

Def.: Norm of spacelike vector $\boldsymbol{V} \in \mathcal{T}_p(\mathcal{M})$ is $|\boldsymbol{V}| := \sqrt{\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{V})}$.

Angle between spacelike $\boldsymbol{V}$, $\boldsymbol{W} \in \mathcal{T}_p(\mathcal{M})$ is $\theta := \arccos\left(\frac{\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{W})}{|\boldsymbol{V}|\,|\boldsymbol{W}|}\right)$.

B.3 Geodesics

Curves

Def.: A curve is timelike (null, spacelike) at a point $p \in \mathcal{M}$
:⇔ its tangent vector at that point is timelike (null, spacelike).

Note: This can change along a curve $\lambda(t)$.

Def.: Length along a spacelike curve

$$s := \int_{t_0}^{t_1} \sqrt{\boldsymbol{g}(\boldsymbol{V}, \boldsymbol{V})|_{\lambda(t)}} dt = \int_{t_0}^{t_1} \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}}\, dt\,, \quad \boldsymbol{V} = \frac{d}{dt}\,.$$

Proper time along timelike curve $\lambda(t)$

$$\tau(t_1) := \int_{t_0}^{t_1} \sqrt{-\, \boldsymbol{g}(\boldsymbol{V}, \boldsymbol{V})|_{\lambda(t)}}\, dt = \int_{t_0}^{t_1} \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}}\, dt\,.$$

Def.: Four velocity along timelike curves $u^\mu := \left. \dfrac{dx^\mu}{d\tau} \right|_{\lambda(\tau)}$

$\Rightarrow g_{\mu\nu} u^\mu u^\nu = -1$

Noether's theorem

Action $\mathcal{S} = \displaystyle\int \mathcal{L}(q_k, \dot{q}_k, \lambda)\, d\lambda$ is extremized by the curve satisfying the

Euler-Lagrange eqs.: $\quad \dfrac{d}{d\lambda} \left(\dfrac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) = \dfrac{\partial \mathcal{L}}{\partial q_k}$

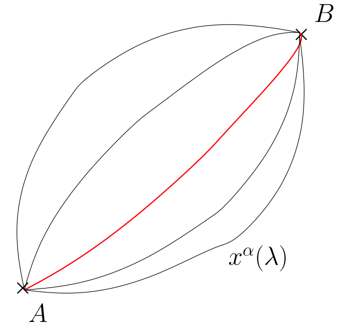
Noether's theorem: (i) $\mathcal{L}$ not explicitly dependend on q_k

$\Rightarrow p_k := \dfrac{\partial \mathcal{L}}{\partial \dot{q}_k}$ conserved along curve that extremizes $\mathcal{S}$.

(ii) $\mathcal{L}$ not explicitly dependend on parameter λ

$\Rightarrow I := \dot{q}_k \dfrac{\partial \mathcal{L}}{\partial \dot{q}_k} - \mathcal{L}$ conserved along curve.

Geodesics, variation 1



Consider timelike curves from A to B.

WLoG: $\lambda = 0$ (1) at A (B).

$$\mathcal{S} = \int_0^1 \mathcal{L} d\lambda\,, \qquad \mathcal{L} = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

Note: $\mathcal{S}$ invariant under reparametrization $\kappa(\lambda)$, $\dfrac{d\kappa}{d\lambda} > 0$:

$$\mathcal{S} = \int_0^1 \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} d\lambda = \int_{\kappa(0)}^{\kappa(1)} \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\kappa}\frac{dx^\nu}{d\kappa}}\, d\kappa$$

EL eqs.: $$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = \frac{1}{2\mathcal{L}}\left(-g_{\mu\nu}\delta^\mu{}_\alpha \dot{x}^\nu - g_{\mu\nu}\dot{x}^\mu \delta^\nu{}_\alpha\right) = -\frac{g_{\mu\alpha}\dot{x}^\mu}{\mathcal{L}}$$

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{1}{2\mathcal{L}}\left(-\dot{x}^\mu \dot{x}^\nu\, \partial_\alpha g_{\mu\nu}\right)$$

$$\Rightarrow \frac{d}{d\lambda}\left(-\frac{g_{\mu\alpha}\dot{x}^\mu}{\mathcal{L}}\right) + \frac{\dot{x}^\mu \dot{x}^\nu\, \partial_\alpha g_{\mu\nu}}{2\mathcal{L}} = 0$$

Change parameter: $$\tau(\lambda) = \int_0^\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tilde{\lambda}}\frac{dx^\nu}{d\tilde{\lambda}}}\, d\tilde{\lambda} \qquad \Rightarrow \qquad \frac{d\tau}{d\lambda} = \mathcal{L}$$

$$\Rightarrow -\mathcal{L}\frac{d}{d\tau}\left(g_{\mu\alpha}\frac{dx^\mu}{d\tau}\right) + \frac{\mathcal{L}}{2}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\partial_\alpha g_{\mu\nu} = 0$$

$$\Rightarrow \frac{d^2x^\mu}{d\tau^2}g_{\mu\alpha} + \partial_\nu g_{\mu\alpha}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad \Big| \cdot g^{\beta\alpha}$$

$$\Rightarrow \boxed{\frac{d^2x^\beta}{d\tau^2} + \left\{{\beta \atop \mu\ \nu}\right\}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0} \qquad (\dagger)$$

<u>Def.:</u> Christoffel symbols: $\boxed{\left\{{\beta \atop \mu\ \nu}\right\} := \frac{1}{2}g^{\beta\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)}$

symmetric in μ, ν !

For spacelike geodesics: $\quad \tilde{\mathcal{L}} = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\,, \quad \dfrac{ds}{d\lambda} = \tilde{\mathcal{L}} \quad \Rightarrow \quad$ Eq. ($\dagger$) with $\tau \to s$

Geodesics, variation 2

Alternatively: $\hat{\mathcal{S}} = \int_A^B \hat{\mathcal{L}} d\lambda \,, \qquad \hat{\mathcal{L}} = g_{\alpha\beta}\dfrac{dx^\alpha}{d\lambda}\dfrac{dx^\beta}{d\lambda}$

Differences: (1) No restriction to timelike geodesics.

(2) Not invariant under reparamterization.

Euler-Lagrange Eqs. $\Rightarrow \quad \ldots \quad \Rightarrow \quad \ddot{x}^\alpha + \left\{ {\alpha \atop \nu\ \beta} \right\} \dot{x}^\nu \dot{x}^\beta = 0 \,, \quad \dot{} = \dfrac{d}{d\lambda}$ $(\star)$

So far so good. But now take (†) and let

$$\tau = \tau(\lambda), \quad \frac{d\tau}{d\lambda} > 0 \quad \Rightarrow \quad \frac{d}{d\tau} = \frac{d\lambda}{d\tau}\frac{d}{d\lambda} \quad \Rightarrow \quad \frac{d^2}{d\tau^2} = \frac{d^2\lambda}{d\tau^2}\frac{d}{d\lambda} + \left(\frac{d\lambda}{d\tau}\right)^2 \frac{d^2}{d\lambda^2}$$

$$\Rightarrow \quad \frac{d^2x^\alpha}{d\lambda^2} + \left\{ {\alpha \atop \nu\ \beta} \right\} \frac{dx^\nu}{d\lambda}\frac{dx^\beta}{d\lambda} = -\left(\frac{d\lambda}{d\tau}\right)^{-2} \frac{d^2\lambda}{d\tau^2}\frac{dx^\alpha}{d\lambda} \propto \frac{dx^\alpha}{d\lambda} \qquad (\star\star)$$

That's not $(\star)$ above! What's going on?

Answer: $\hat{\mathcal{S}}$ not invariant under parameter change $\quad \Rightarrow \quad$ Variation gives different curve

Eqs. (†), $(\star)$ only agree if $\quad \dfrac{d^2\lambda}{d\tau^2} = 0 \quad \Leftrightarrow \quad \lambda = c_1\tau + c_2 \,, \quad c_1,\, c_2 = \text{const} \in \mathbb{R}$ (††)

Def.: The parameter λ along a timelike (spacelike) curve is affine $\quad \Leftrightarrow \quad$ it is related to proper time τ (proper distance s) by Eq. (††). For a non-affine parameter, the geodesic equation is $(\star\star)$.

Summary for all curves (incl. null):

Def.: If a curve $\quad \mathcal{C} : I \subset \mathbb{R} \to \mathcal{M} \,, \qquad \lambda \mapsto x^\alpha(\lambda)$

(1) satisfies $(\star) \quad \to \quad$ geodesic, λ is affine.

(2) satisfies Eq. $(\star\star)$ with non-zero right-hand side $\quad \to \quad$ geodesic; λ is non-affine.

(3) satisfies neither $(\star)$ nor $(\star\star) \quad \to \quad$ it is not a geodesic.

ODE theory $\quad \Rightarrow \quad$ solutions to $(\star)$, $(\star\star)$ unique if x^α, $\dot{x}^\alpha$ fixed at $\lambda = \lambda_0$

Geodesic postulate: Test particles with positive (zero) rest mass move on timelike (null) geodesics.

$\hat{\mathcal{L}}$ gives an easy way to calculate $\left\{ {}^{\ \alpha}_{\nu\ \beta} \right\}$:

Example: Schwarzschild metric in Schwarzschild coords.:

$$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\,, \quad f = 1 - \frac{2M}{r}\,, \quad M = \text{const}$$

$$\Rightarrow \quad \hat{\mathcal{L}} = f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2$$

$$\text{EL for } t(\tau): \ \frac{d}{d\tau}\big(2f\dot{t}\big) = 0 \quad \Rightarrow \quad \frac{d^2 t}{d\tau^2} + f^{-1}\frac{df}{dr}\,\dot{t}\,\dot{r} = 0$$

$$\Rightarrow \quad \left\{ {}^{\ t}_{t\ r} \right\} = \left\{ {}^{\ t}_{r\ t} \right\} = \frac{df/dr}{2f}\,, \quad \left\{ {}^{\ t}_{\mu\ \nu} \right\} = 0 \ \text{ otherwise}$$

B.4 Covariant derivative

Physical laws involve derivatives.

Problem: Cannot take difference between vectors at different points:

$$\boldsymbol{U} \in \mathcal{T}_p(\mathcal{M}), \quad \boldsymbol{V} \in \mathcal{T}_q(\mathcal{M})$$

$\rightarrow$ Covariant derivative ∇ on manifold $\mathcal{M}$

Def.: For functions: $\nabla f : \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad \boldsymbol{V} \mapsto \nabla_{\boldsymbol{V}} f := \boldsymbol{V}(f) = V^\alpha \partial_\alpha f$

∇f is $\binom{0}{1}$ tensor: $\nabla_\alpha f := (\nabla f)_\alpha = \partial_\alpha f$

Def.: For vectors: $\nabla \boldsymbol{V} : \mathcal{T}_p(\mathcal{M}) \to \mathcal{T}_p(\mathcal{M})\,, \qquad \boldsymbol{X} \mapsto \nabla_{\boldsymbol{X}} \boldsymbol{V}$ with

(1) $\nabla_{f\boldsymbol{X}+g\boldsymbol{Y}} \boldsymbol{V} = f\nabla_{\boldsymbol{X}}\boldsymbol{V} + g\nabla_{\boldsymbol{Y}}\boldsymbol{V}\,,$

(2) $\nabla_{\boldsymbol{X}}(\boldsymbol{V} + \boldsymbol{W}) = \nabla_{\boldsymbol{X}}\boldsymbol{V} + \nabla_{\boldsymbol{X}}\boldsymbol{W}$

(3) $\nabla_{\boldsymbol{X}}(f\boldsymbol{V}) = f\nabla_{\boldsymbol{X}}\boldsymbol{V} + (\nabla_{\boldsymbol{X}} f)\boldsymbol{V}$ (Leibnitz rule)

Equivalently: $\nabla \boldsymbol{V} : \mathcal{T}^*_p(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad (\boldsymbol{\eta}, \boldsymbol{X}) \mapsto \boldsymbol{\eta}(\nabla_{\boldsymbol{X}}\boldsymbol{V})$

$\nabla \boldsymbol{V}$ is a $\binom{1}{1}$ tensor: $V^\alpha{}_{;\beta} := \nabla_\beta V^\alpha := (\nabla \boldsymbol{V})^\alpha{}_\beta$

Def.: Let $\{\mathbf{e}^\mu\}$ be a basis of $\mathcal{T}_p(\mathcal{M})$.

Connection coefficients $\Gamma^\rho_{\mu\nu}$: $\quad \nabla_\nu \mathbf{e}_\mu := \nabla_{\mathbf{e}_\nu}\mathbf{e}_\mu = \Gamma^\rho_{\mu\nu}\mathbf{e}_\rho$

We get for $\boldsymbol{V} = V^\mu \mathbf{e}_\mu\,, \quad \boldsymbol{W} = W^\mu \mathbf{e}_\mu$

$$\begin{aligned}
\Rightarrow \quad \nabla_{\boldsymbol{V}} \boldsymbol{W} &= \nabla_{\boldsymbol{V}}(W^\mu \mathbf{e}_\mu) = \boldsymbol{V}(W^\mu)\mathbf{e}_\mu + W^\mu \nabla_{\boldsymbol{V}} \mathbf{e}_\mu \\
&= V^\nu \mathbf{e}_\nu (W^\mu)\, \mathbf{e}_\mu + W^\mu \nabla_{V^\nu \mathbf{e}_\nu} \mathbf{e}_\mu \\
&= V^\nu \mathbf{e}_\nu (W^\mu)\, \mathbf{e}_\mu + W^\mu V^\nu \underbrace{\nabla_\nu \mathbf{e}_\mu}_{=\Gamma^\rho_{\mu\nu}\mathbf{e}_\rho} \\
&= V^\nu \Big(\partial_\nu W^\rho + W^\mu \Gamma^\rho_{\mu\nu} \Big) \mathbf{e}_\rho
\end{aligned}$$

$$\Rightarrow \quad (\nabla_{\boldsymbol{V}} \boldsymbol{W})^\rho = V^\nu \partial_\nu W^\rho + V^\nu \Gamma^\rho_{\mu\nu} W^\mu \qquad \Big| \; \boldsymbol{V} \text{ arbitrary}$$

$$\Rightarrow \quad \boxed{W^\rho{}_{;\nu} := \nabla_\nu W^\rho := (\nabla \boldsymbol{W})^\rho{}_\nu = \partial_\nu W^\rho + \Gamma^\rho_{\mu\nu} W^\mu}$$

Coordinate transformation $x^\mu \to \tilde{x}^\alpha$

$$\Rightarrow \ \ldots \ \Rightarrow \quad \tilde{\Gamma}^\sigma_{\mu\nu} = \frac{\partial \tilde{x}^\sigma}{\partial x^\rho} \frac{\partial x^\alpha}{\partial \tilde{x}^\nu} \frac{\partial x^\beta}{\partial \tilde{x}^\mu} \Gamma^\rho_{\beta\alpha} + \frac{\partial \tilde{x}^\sigma}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial \tilde{x}^\nu \partial \tilde{x}^\mu} \tag{†}$$

$\Rightarrow \Gamma^\sigma_{\mu\nu}$ is not a tensor! But the difference of two connections is. E.g.:

Def.: Torsion tensor: $\quad T_{\mu\nu}{}^\lambda := \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\,.$

Γ is called torsion free $\quad :\Leftrightarrow T_{\mu\nu}{}^\lambda = 0.$

Comments:

- $\partial_\nu W^\mu = \dfrac{\partial W^\mu}{\partial x^\nu}$ is not a tensor either!

  Second term in (†) just cancels this to make $\nabla_\nu W^\mu$ a tensor.
- Example: The Christoffel symbols are a connection.
- Convention: Derivative index = 2nd downstairs index in $\Gamma^\rho_{\mu\nu}$.

  Some people use the first.

Covariant derivative of tensors

Obtained from Leibniz rule; $\binom{r}{s}$ tensor $\boldsymbol{T} \quad \mapsto \quad \nabla \boldsymbol{T}$ is $\binom{r}{s+1}$ tensor

E.g. 1-form: $\nabla_{\boldsymbol{V}}\big(\boldsymbol{\eta}(\boldsymbol{W})\big) := \big(\nabla_{\boldsymbol{V}}\boldsymbol{\eta}\big)(\boldsymbol{W}) + \boldsymbol{\eta}(\nabla_{\boldsymbol{V}}\boldsymbol{W})$

$$\Rightarrow \big(\nabla_{\boldsymbol{V}}\boldsymbol{\eta}\big)(\boldsymbol{W}) = \nabla_{\boldsymbol{V}}\big(\boldsymbol{\eta}(\boldsymbol{W})\big) - \boldsymbol{\eta}(\nabla_{\boldsymbol{V}}\boldsymbol{W})$$

$\nabla\boldsymbol{\eta}$ is a $\binom{0}{2}$ tensor since:

$$\begin{aligned}\big(\nabla\boldsymbol{\eta}\big)(\boldsymbol{V},\boldsymbol{W}) &= \big(\nabla_{\boldsymbol{V}}\boldsymbol{\eta}\big)(\boldsymbol{W}) = \nabla_{\boldsymbol{V}}(\eta_\mu W^\mu) - \eta_\mu(\nabla_{\boldsymbol{V}}\boldsymbol{W})^\mu \\ &= V^\rho\partial_\rho(\eta_\mu W^\mu) - \eta_\mu\big(V^\rho\partial_\rho W^\mu + V^\rho\Gamma^\mu_{\nu\rho}W^\nu\big) \\ &= V^\rho W^\mu\partial_\rho\eta_\mu - \Gamma^\mu_{\nu\rho}\eta_\mu V^\rho W^\nu \\ &= \big(\partial_\rho\eta_\mu - \Gamma^\nu_{\mu\rho}\eta_\nu\big)V^\rho W^\mu\end{aligned}$$

Components: $\boxed{\eta_{\mu;\rho} := \nabla_\rho\eta_\mu := (\nabla\boldsymbol{\eta})_{\rho\mu} = \partial_\rho\eta_\mu - \Gamma^\nu_{\mu\rho}\eta_\nu}$.

Covariant derivative of (r,s) tensor:

$$\boxed{\begin{aligned}\nabla_\rho T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_2} = \; &\partial_\rho T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} + \Gamma^{\mu_1}_{\sigma\rho}T^{\sigma\mu_2\ldots\mu_r}{}_{\nu_1\ldots\nu_2} + \ldots + \Gamma^{\mu_r}_{\sigma\rho}T^{\mu_1\ldots\mu_{r-1}\sigma}{}_{\nu_1\ldots\nu_s} \\ &- \Gamma^\sigma_{\nu_1\rho}T^{\mu_1\ldots\mu_r}{}_{\sigma\nu_2\ldots\nu_s} - \ldots - \Gamma^\sigma_{\nu_s\rho}T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_{s-1}\sigma}\end{aligned}}$$

B.5 The Levi-Civita connection

Note: We need no metric for the covariant derivative.

But: A metric singles out a special connection.

Theorem: On a manifold $\mathcal{M}$ with metric $\boldsymbol{g}$ $\exists$ unique connection with

(1) $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu} = \left\{{\alpha \atop \mu\ \nu}\right\}$ Christoffel symbols, torsion free

(2) $\nabla\boldsymbol{g} = 0$ "metric compatible"

This connection is called the Levi-Civita connection.

<u>Proof:</u> "⇒": Let $\Gamma^{\alpha}_{\beta\gamma}$ be metric compatible, symmetric

$$\Rightarrow \nabla_\alpha g_{\beta\gamma} = 0 \qquad \Rightarrow \qquad \partial_\alpha g_{\beta\gamma} = \Gamma^{\rho}_{\beta\alpha} g_{\rho\gamma} + \Gamma^{\rho}_{\gamma\alpha} g_{\beta\rho}$$

By definition the Christoffel symbols are

$$\begin{aligned}
\left\{ {}^{\ \mu}_{\beta\ \gamma} \right\} &= \frac{1}{2} g^{\mu\nu} (\partial_\beta g_{\gamma\nu} + \partial_\gamma g_{\nu\beta} - \partial_\nu g_{\beta\gamma}) \\
&= \frac{1}{2} g^{\mu\nu} \left(\Gamma^{\rho}_{\gamma\beta} g_{\rho\nu} + \underline{\Gamma^{\rho}_{\nu\beta} g_{\gamma\rho}} + \Gamma^{\rho}_{\nu\gamma} g_{\rho\beta} + \Gamma^{\rho}_{\beta\gamma} g_{\nu\rho} - \underline{\Gamma^{\rho}_{\beta\nu} g_{\rho\gamma}} - \Gamma^{\rho}_{\gamma\nu} g_{\beta\rho} \right) \\
&= \Gamma^{\mu}_{\beta\gamma}
\end{aligned}$$

"⇐": Likewise: $\Gamma^{\alpha}_{\beta\gamma} = \left\{ {}^{\ \alpha}_{\beta\ \gamma} \right\}$

$$\Rightarrow \ldots \Rightarrow \Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}\,, \qquad \nabla_\alpha g_{\beta\gamma} = 0$$

In GR we use the Levi-Civita connection.

B.6 Parallel transport

A connection gives us a notion of "a tensor that does not change along a curve"

<u>Def.:</u> Let $\boldsymbol{V}$ be a vector field and $\mathcal{C}$ an integral curve of $\boldsymbol{V}$. A tensor $\boldsymbol{T}$ is parallel transported along $\mathcal{C}$ $\;:\Leftrightarrow\;$ $\nabla_{\boldsymbol{V}} \boldsymbol{T} = 0$ along the curve.

Comments:

- The tangent of an affinely parametrized geodesic is parallel transported along itself:

  $$U^\mu \nabla_\mu U^\alpha = U^\mu \partial_\mu U^\alpha + U^\mu \Gamma^{\alpha}_{\nu\mu} U^\nu = \frac{dx^\mu}{d\lambda} \frac{\partial}{\partial x^\mu} \frac{dx^\alpha}{d\lambda} + \frac{dx^\mu}{d\lambda} \Gamma^{\alpha}_{\nu\mu} \frac{dx^\nu}{d\lambda} = 0$$

  $$\Rightarrow \frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^{\alpha}_{\nu\mu} \frac{dx^\nu}{d\lambda} \frac{dx^\mu}{d\lambda} = 0$$

  with non-affine parameter: $\nabla_{\boldsymbol{U}} \boldsymbol{U} = f \boldsymbol{U}$

- $\nabla_{\boldsymbol{V}}\boldsymbol{T} = 0$ determines $\boldsymbol{T}$ uniquely along the curve:

  in coords. (x^μ) the curve is $x^\mu(\lambda)$

  $$\Rightarrow \quad V^\sigma \nabla_\sigma T^\mu{}_\nu = V^\sigma \partial_\sigma T^\mu{}_\nu + \Gamma^\mu_{\rho\sigma} T^\rho{}_\nu V^\sigma - \Gamma^\rho_{\nu\sigma} T^\mu{}_\rho V^\sigma$$

  $$= \frac{d}{d\lambda} T^\mu{}_\nu + \Gamma^\mu_{\rho\sigma} T^\rho{}_\nu V^\sigma - \Gamma^\rho_{\nu\sigma} T^\mu{}_\rho V^\sigma = 0$$

  ODE theory $\Rightarrow$ unique solution for all $T^\mu{}_\nu$

- parallel transport $\boldsymbol{T}$ along a curve from p to q

  $\rightarrow$ isomorphism between tensors at p, q

  Unlike SR, this is path dependent in GR!

- Parallel transport preserves length of vectors:

  $$\frac{d}{d\lambda}(W_\alpha W^\alpha) = V^\mu \nabla_\mu (W_\alpha W^\alpha) = 2W_\alpha \underbrace{V^\mu \nabla_\mu W^\alpha}_{=0}$$

  $\Rightarrow$ Geodesics do not change their timelike, spacelike or null character.

Def.: Acceleration along timelike curve: $a^\mu := u^\rho \nabla_\rho u^\mu$

The curve is a geodesic if $a^\mu = 0$ or $a^\alpha = f\, u^\alpha$

B.7 Normal coordinates

Def.: Let $\mathcal{M}$ be a manifold, Γ a connection, $p \in \mathcal{M}$.

"exponential map" : $e : \mathcal{T}_p \rightarrow \mathcal{M}\,,\quad \boldsymbol{X}_p \mapsto q$ with

$q :=$ point a unit affine parameter distance along

geodesic through p with tangent $\boldsymbol{X}_p$

Comments: 1) e can be shown to be one-to-one and onto locally

2) The vector $\boldsymbol{X}_p$ fixes the parametrization of the geodesic:

One can show that $\lambda \boldsymbol{X}_p\,,\;\; 0 \le \lambda \le 1$ is mapped to point at

affine par. distance λ along the geodesic of $\boldsymbol{X}_p$. $(**)$

Def.: Let $\{\mathbf{e}_\mu\}$ be a basis of $\mathcal{T}_p(\mathcal{M})$. "Normal coords. in nbhd. of $p \in \mathcal{M}$":

coords. that assigns to $q = e(\boldsymbol{X}) \in \mathcal{M}$ the coordinates X^μ

Note: The coords. X^μ are not fixed by the vector $\boldsymbol{X}$.

We still have the freedom to choose a basis for $\mathcal{T}_p(\mathcal{M})$.

Lemma: In normal coordinates, $\Gamma^\mu_{(\nu\rho)} = 0$ at p.

If Γ is torsion free, then $\Gamma^\mu_{\nu\rho} = 0$.

Proof: From **(**)** $\Rightarrow$ affinely parametrized geodesic is given by

$$x^\mu(\lambda) = \lambda\, X^\mu_p \quad \text{in normal coords.}$$

$$\Rightarrow \text{ geodesic eq.: } 0 + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = \Gamma^\mu_{\nu\rho}X^\nu_p X^\rho_p = 0 \text{ at } p \;\; \forall \boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$$

$\Rightarrow \Gamma^\mu_{(\nu\rho)} = 0$

torsion free $\Rightarrow$ $\Gamma^\mu_{[\nu\rho]} = 0$ $\Rightarrow$ $\Gamma^\mu_{\nu\rho} = 0$

Note: in general $\Gamma^\mu_{\nu\rho} \neq 0$ away from p!

Lemma: With Levi-Civita connection $\Rightarrow$ in normal coords. at p: $\partial_\rho g_{\mu\nu} = 0$

Proof: $\Gamma^\rho_{\mu\nu} = 0 \;\Rightarrow\; 2g_{\sigma\rho}\Gamma^\rho_{\mu\nu} = \partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} = 0$

symmetrize on σ, μ, add $\Rightarrow \partial_\nu g_{\sigma\mu} = 0$

Lemma: Let $(\mathcal{M}, \boldsymbol{g})$ be a spacetime with LC connection

$\Rightarrow$ $\exists$ coordinates at p with: $\partial_\rho g_{\mu\nu} = 0\,,\quad g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$

Proof: Choose an orthonormal basis $\{\mathbf{e}_\mu\}$ for $\mathcal{T}_p(\mathcal{M})$

$\Rightarrow$ at p: $\boldsymbol{X} = X^0\mathbf{e}_0 + \ldots + X^3\mathbf{e}_3$ defines normal coords. $x^\mu = X^\mu$

(**) $\Rightarrow$ point an affine par.distance along geodesic through p with

tangent $\mathbf{e}_0$ has coords. $\lambda\,(\mathbf{e}_0)^\mu = (\lambda, 0, 0, 0)$

$\Rightarrow$ The geodesic is $x^\mu(\lambda) = (\lambda, 0, 0, 0)$

In any coordinate system, the tangent to the curve $(\lambda, 0, 0, 0)$ is $\partial_0 = \partial/\partial x^0$

Likewise $\partial_\mu = \mathbf{e}_\mu \;\Rightarrow\; \{\partial_\mu\}$ is orthonormal $\Rightarrow\; g_{\mu\nu} = \boldsymbol{g}(\partial_\mu, \partial_\nu) = \eta_{\mu\nu}$ at p.

Summary: Locally, we can choose coordinates such that the metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and its first derivatives vanish.

Def.: "local inertial frame at $p \in \mathcal{M}$" := normal coord. chart with these properties

B.8 The Riemann tensor

The commutator

Def.: Commutator of 2 vectorfields $\boldsymbol{V},\ \boldsymbol{W}$: $\quad [\boldsymbol{V}, \boldsymbol{W}]^\alpha := V^\mu \partial_\mu W^\alpha - W^\mu \partial_\mu V^\alpha$

$[\boldsymbol{X}, \boldsymbol{Y}]$ is indeed a vectorfield:

One shows $\quad \tilde{V}^\nu \tilde{\partial}_\nu \tilde{W}^\mu - \tilde{W}^\nu \tilde{\partial}_\nu \tilde{V}^\mu = \quad \ldots \quad = \dfrac{\partial \tilde{x}^\mu}{\partial x^\gamma} \left(V^\beta \partial_\beta W^\gamma - W^\beta \partial_\beta V^\gamma \right)$

Properties:

$$[\boldsymbol{V}, \boldsymbol{W}] = -[\boldsymbol{W}, \boldsymbol{V}]$$

$$[\boldsymbol{V}, \boldsymbol{W} + \boldsymbol{U}] = [\boldsymbol{V}, \boldsymbol{W}] + [\boldsymbol{V} + \boldsymbol{U}]$$

$$[\boldsymbol{V}, f\, \boldsymbol{W}] = f\, [\boldsymbol{V}, \boldsymbol{W}] + \boldsymbol{V}(f)\, \boldsymbol{W}$$

$$[\boldsymbol{U}, [\boldsymbol{V}, \boldsymbol{W}]] + [\boldsymbol{V}, [\boldsymbol{W}, \boldsymbol{U}]] + [\boldsymbol{W}, [\boldsymbol{U}, \boldsymbol{V}]] = 0 \quad \text{“Jacobi identity”}$$

Note: $\left[\dfrac{\partial}{\partial x^\mu}, \dfrac{\partial}{\partial x^\nu} \right] = 0 \quad$ (coord. basis $\Rightarrow$ commutators vanish)

Conversely, one can show:

If $\boldsymbol{V}_0, \ldots, \boldsymbol{V}_{m-1}$, $m \leq \dim(\mathcal{M})$ are vector fields which are

lin. indep. $\forall p \in \mathcal{M}$ with all $[\boldsymbol{V}_i, \boldsymbol{V}_j] = 0$

$\Rightarrow$ In a nbhd. of p $\exists$ coords. (x^μ): $\qquad \boldsymbol{V}_i = \dfrac{\partial}{\partial x^i}\ ,\quad i = 0, \ldots, m-1$

Second derivatives

For function f: $\nabla_\nu \nabla_\mu f = \quad \ldots \quad = \nabla_\mu \nabla_\nu f - 2\Gamma^\rho_{[\mu\nu]} \partial_\rho f = \nabla_\mu \nabla_\nu f \quad$ for torsion free Γ

For vector $\boldsymbol{V}$: $\quad \nabla_\alpha \nabla_\beta V^\gamma - \nabla_\beta \nabla_\alpha V^\gamma = \quad \ldots$

$$= \partial_\alpha \Gamma^\gamma_{\rho\beta} V^\rho + \Gamma^\gamma_{\rho\beta} \partial_\alpha V^\rho + \Gamma^\gamma_{\rho\alpha} \partial_\beta V^\rho + \Gamma^\gamma_{\rho\alpha} \Gamma^\rho_{\sigma\beta} V^\sigma - (\ \alpha \ \leftrightarrow \ \beta\ )$$

Def.: Riemann tensor: $\boxed{R^{\gamma}{}_{\rho\alpha\beta} := \partial_\alpha \Gamma^{\gamma}_{\rho\beta} - \partial_\beta \Gamma^{\gamma}_{\rho\alpha} + \Gamma^{\mu}_{\rho\beta}\Gamma^{\gamma}_{\mu\alpha} - \Gamma^{\mu}_{\rho\alpha}\Gamma^{\gamma}_{\mu\beta}}$.

$\Rightarrow$ "Ricci Identity": $\nabla_\alpha \nabla_\beta V^\gamma - \nabla_\beta \nabla_\alpha V^\gamma = R^{\gamma}{}_{\rho\alpha\beta} V^\rho$

Equivalently, we can define:

Def.: For 3 VFs $\boldsymbol{U},\ \boldsymbol{V},\ \boldsymbol{W}$: $\quad \big(\boldsymbol{R}(\boldsymbol{U},\boldsymbol{V})\big)(\boldsymbol{W}) = \nabla_{\boldsymbol{U}}\nabla_{\boldsymbol{V}}\boldsymbol{W} - \nabla_{\boldsymbol{V}}\nabla_{\boldsymbol{U}}\boldsymbol{W} - \nabla_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{W} \qquad (\ddagger)$

Proof: One shows for a function f:

$$\begin{aligned}
\boldsymbol{R}(f\boldsymbol{U},\boldsymbol{V})\boldsymbol{W} &= f\boldsymbol{R}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W}\\
\boldsymbol{R}(\boldsymbol{U},f\boldsymbol{V})\boldsymbol{W} &= f\boldsymbol{R}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W}\\
\boldsymbol{R}(\boldsymbol{U},\boldsymbol{V})f\boldsymbol{W} &= f\boldsymbol{R}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W}
\end{aligned}$$

rhs of ($\ddagger$) is a vector $\Rightarrow$ Contraction of $\boldsymbol{R}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W}$ with one-form is linear.

Components: $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = 0, \quad \nabla_\alpha \mathbf{e}_\beta = \Gamma^{\mu}_{\beta\alpha}\mathbf{e}_\mu$

$$\begin{aligned}
\Rightarrow \boldsymbol{R}(\mathbf{e}_\alpha, \mathbf{e}_\beta)\mathbf{e}_\rho &= \nabla_\alpha\nabla_\beta \mathbf{e}_\rho - \nabla_\beta\nabla_\alpha \mathbf{e}_\rho = \nabla_\alpha(\Gamma^{\mu}_{\rho\beta}\mathbf{e}_\mu) - \nabla_\beta(\Gamma^{\mu}_{\rho\alpha}\mathbf{e}_\mu)\\
&= \ldots = \big[\underbrace{\partial_\alpha\Gamma^{\nu}_{\rho\beta} - \partial_\beta\Gamma^{\nu}_{\rho\alpha} + \Gamma^{\mu}_{\rho\beta}\Gamma^{\nu}_{\mu\alpha} - \Gamma^{\mu}_{\rho\alpha}\Gamma^{\nu}_{\mu\beta}}_{=R^{\nu}{}_{\rho\alpha\beta}\ !}\big]\mathbf{e}_\nu
\end{aligned}$$

Symmetries of Riemann

(1) $\quad R^{\alpha}{}_{\beta\gamma\delta} = -R^{\alpha}{}_{\beta\delta\gamma} \quad \Leftrightarrow \quad \boxed{R^{\alpha}{}_{\beta(\gamma\delta)} = 0}$ by def.

Torsion $= 0$, let $p \in \mathcal{M}$, (x^μ) normal coords. Then:

(2) $\quad \Gamma^{\mu}_{\nu\rho} = 0$ at p, $\Gamma^{\mu}_{[\nu\rho]} = 0$ everywhere

$\Rightarrow R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho\Gamma^{\mu}_{\nu\sigma} - \partial_\sigma\Gamma^{\mu}_{\nu\rho}$ | antisymmetrize on $\nu\rho\sigma$

$\Rightarrow \boxed{R^{\mu}{}_{[\nu\rho\sigma]} = 0}$ tensorial equation!

(3) $\quad \nabla_\tau R^{\mu}{}_{\nu\rho\sigma} = \partial_\tau R^{\mu}{}_{\nu\rho\sigma}$ | "$\partial R = \partial\partial\Gamma - \Gamma\,\partial\Gamma = \partial\partial\Gamma$"

$= \partial_\tau\partial_\rho\Gamma^{\mu}_{\nu\sigma} - \partial_\tau\partial_\sigma\Gamma^{\mu}_{\nu\rho}$ | antisymmetrize on $\rho\sigma\tau$

$\boxed{\Rightarrow R^{\mu}{}_{\nu[\rho\sigma;\tau]} = 0}$ "Bianchi identity"

(4) Levi-Civita connection, $p \in \mathcal{M} \quad \Rightarrow \quad$ in normal coords.: $\partial_\mu g_{\nu\rho} = 0$

$$\Rightarrow 0 = \partial_\mu \delta^\nu{}_\rho = \partial_\mu (g^{\nu\sigma} g_{\sigma\rho}) = g_{\sigma\rho} \partial_\mu g^{\nu\sigma} \qquad \Big| \; \cdot g^{\rho\tau}$$

$$\Rightarrow \partial_\mu g^{\nu\tau} = 0$$

$$\Rightarrow \partial_\rho \Gamma^\tau_{\nu\sigma} = \frac{1}{2} g^{\tau\mu} \big(\partial_\rho \partial_\sigma g_{\mu\nu} + \partial_\rho \partial_\nu g_{\sigma\mu} - \partial_\rho \partial_\mu g_{\nu\sigma} \big)$$

$$\Rightarrow R_{\mu\nu\rho\sigma} = \frac{1}{2} \big(\partial_\rho \partial_\nu g_{\sigma\mu} + \partial_\sigma \partial_\mu g_{\nu\rho} - \partial_\sigma \partial_\nu g_{\rho\mu} - \partial_\rho \partial_\mu g_{\nu\sigma} \big) + \underbrace{\text{“}\Gamma\Gamma - \Gamma\Gamma\text{”}}_{=0}$$

$$\Rightarrow \boxed{R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}}$$

It follows: $\boxed{R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}}$

<u>Parallel transport and curvature</u>

Let $\boldsymbol{X}, \boldsymbol{Y}$ be VFs with: lin. indep. everywhere and $[\boldsymbol{X}, \boldsymbol{Y}] = 0$; let torsion $= 0$

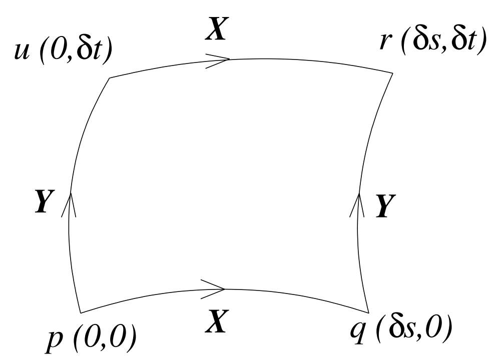
$\Rightarrow$ we can choose coords. $(s,\, t,\, \ldots)$ such that $\boldsymbol{X} = \dfrac{\partial}{\partial s}\,, \quad \boldsymbol{Y} = \dfrac{\partial}{\partial t}$

Let $p,\, q,\, r,\, u \in \mathcal{M}$ along integral curves of $\boldsymbol{X}, \boldsymbol{Y}$ with coords.

$(0,\, \ldots,\, 0), \;\; (\delta s,\, 0,\, \ldots), \;\; (\delta s,\, \delta t,\, 0,\, \ldots), \;\; (0,\, \delta t,\, 0,\, \ldots)$

Let $\boldsymbol{Z}_p \in \mathcal{T}_p(\mathcal{M})$, parallel trapo $\boldsymbol{Z}$ along $pqrup$

to get $\boldsymbol{Z}'_p \in \mathcal{T}_p(\mathcal{M})$



$$\Rightarrow \boxed{\lim_{\delta s, \delta t \to 0} \frac{(\boldsymbol{Z}'_p - \boldsymbol{Z}_p)^\alpha}{\delta s\, \delta t} = (R^\alpha{}_{\beta\gamma\delta} Z^\beta Y^\gamma X^\delta)_p}$$

<u>Proof:</u> long script.

Geodesic deviation

Goal: quantify relative acceleration of geodesics

Def.: Let $(\mathcal{M}, \Gamma)$ be a manifold with connection.

"1-parameter family of geodesics" := a map

$\gamma : I \times I' \to \mathcal{M}$ with $I, I' \subset \mathbb{R}$, open and

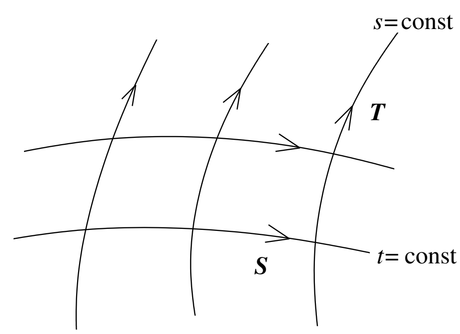
(i) for fixed s, $\gamma(s,t)$ is a geodesic with affine par. t

(ii) locally, $(s,t) \mapsto \gamma(s,t)$ is smooth, 1-to-1 has smooth inverse

$\Rightarrow$ the family of geodesics forms a 2-dim. surface $\Sigma \subset \mathcal{M}$

Let $\boldsymbol{T}$ be the tangent vector to $\gamma(s = \text{const}, t)$ and $\boldsymbol{S}$ to $\gamma(s, t = \text{const})$

In coords. (x^μ): $S^\mu = \dfrac{dx^\mu}{ds}$



$\Rightarrow x^\mu(s + \delta s, t) = x^\mu(s,t) + \delta s\, S^\mu(s,t) + \mathcal{O}(\delta s^2)$

$\Rightarrow \delta s\, \boldsymbol{S}$ points from one geodesic to a nearby one: "deviation vector"

$\Rightarrow$ "relative velocity" of nearby geodesics: $\nabla_{\boldsymbol{T}}(\delta s\, \boldsymbol{S}) = \delta s\, \nabla_{\boldsymbol{T}} \boldsymbol{S}$

$\Rightarrow$ "relative acceleration" of nearby geodesics: $\delta s\, \nabla_{\boldsymbol{T}} \nabla_{\boldsymbol{T}} \boldsymbol{S}$

Theorem: $\boxed{\nabla_{\boldsymbol{T}} \nabla_{\boldsymbol{T}} \boldsymbol{S} = \boldsymbol{R}(\boldsymbol{T}, \boldsymbol{S}) \boldsymbol{T}}$

$$\Leftrightarrow \quad T^\mu \nabla_\mu (T^\nu \nabla_\nu S^\alpha) = R^a{}_{\lambda\mu\nu} T^\lambda T^\mu S^\nu$$

Lemma: $V^\mu \nabla_\mu W^\alpha - W^\mu \nabla_\mu V^\alpha = V^\mu \partial_\mu W^\alpha + V^\mu \Gamma^\alpha_{\rho\mu} W^\rho - W^\mu \partial_\mu V^\alpha - W^\mu \Gamma^\alpha_{\rho\mu} V^\rho$

$$= V^\mu \partial_\mu W^\alpha - W^\mu \partial_\mu V^\alpha = [\boldsymbol{V}, \boldsymbol{W}]^\alpha$$

$$\Rightarrow \nabla_{\boldsymbol{V}} \boldsymbol{W} - \nabla_{\boldsymbol{W}} \boldsymbol{V} = [\boldsymbol{V}, \boldsymbol{W}]$$

<u>Proof</u>: Use coords. (s,t) on Σ and extend to $(s,t,\dots)$ in nbhd. of Σ

$$\Rightarrow \boldsymbol{S} = \frac{\partial}{\partial s}\,, \quad \boldsymbol{T} = \frac{\partial}{\partial t} \quad \Rightarrow \quad [\boldsymbol{S},\boldsymbol{T}] = 0$$

No torsion $\Rightarrow \nabla_{\boldsymbol{T}}\boldsymbol{S} - \nabla_{\boldsymbol{S}}\boldsymbol{T} = [\boldsymbol{T},\boldsymbol{S}] = 0$

$$\Rightarrow \nabla_{\boldsymbol{T}}\nabla_{\boldsymbol{T}}\boldsymbol{S} = \nabla_{\boldsymbol{T}}\nabla_{\boldsymbol{S}}\boldsymbol{T} = \nabla_{\boldsymbol{S}}\underbrace{\nabla_{\boldsymbol{T}}\boldsymbol{T}}_{=0 \ \text{geodesic!}} + \boldsymbol{R}(\boldsymbol{T},\boldsymbol{S})\boldsymbol{T} \qquad \square$$

Comments:
- $R^\alpha{}_{\beta\gamma\delta}$ measures geodesic deviation; manifestation of curvature.
- $R^\alpha{}_{\beta\gamma\delta} = 0 \Leftrightarrow$ relative acceleration $= 0$ for all families of geodesics.
- Tidal forces arise from geodesic deviation.

<u>The Ricci tensor</u>

<u>Def.:</u> The "Ricci tensor" is $\boxed{R_{\alpha\beta} := R^\mu{}_{\alpha\mu\beta}}$.

The "Ricci scalar" is $\boxed{R := g^{\mu\nu}R_{\mu\nu}}$.

The "Einstein tensor" is $\boxed{G_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R}$.

Recall Bianchi Identity: $R_{\alpha\beta[\gamma\delta;\mu]} = 0 \qquad \Big| \cdot g^{\alpha\gamma}g^{\beta\delta}$

$$\Rightarrow \frac{1}{6}g^{\alpha\gamma}g^{\beta\delta}\left[R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\delta\mu;\gamma} + R_{\alpha\beta\mu\gamma;\delta} - \underbrace{R_{\alpha\beta\delta\gamma;\mu}}_{=-R_{\alpha\beta\gamma\delta;\mu}} - R_{\alpha\beta\gamma\mu;\delta} - R_{\alpha\beta\mu\delta;\gamma}\right] = 0$$

$$\Rightarrow \frac{1}{3}g^{\alpha\gamma}g^{\beta\delta}\left[R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\delta\mu;\gamma} + R_{\alpha\beta\mu\gamma;\delta}\right] = 0$$

$$\Rightarrow R_{;\mu} - g^{\alpha\gamma}R_{\alpha\mu;\gamma} - g^{\beta\delta}R_{\beta\mu;\delta} = 0$$

$$\Rightarrow \nabla_\mu R - 2\nabla_\gamma R^\gamma{}_\mu = -2\nabla^\gamma\left(R_{\gamma\mu} - \frac{1}{2}g_{\gamma\mu}R\right)$$

$$\Rightarrow \boxed{\nabla^\mu G_{\mu\alpha} = 0} \qquad \text{"Contracted Bianchi Id."}$$

C Physical laws in curved spacetimes

Goal: How do we use differential geometry to describe physics?

C.1 The covariance principle

What we have so far:

- ***Equivalence principle:*** freely falling frame = inertial frame in SR
- ***Normal coordinates:*** $\exists$ coordinates: $g_{\alpha\beta} = \eta_{\alpha\beta}, \quad \Gamma^{\alpha}_{\beta\gamma} = 0$
- laws of SR invariant under Lorentz trafos

this motivates the

Covariance principle: The laws of GR are tensorial, i.e. coordinate invariant, and follow from SR laws by replacing

(1) the metric: $\eta_{\mu\nu} \quad \to \quad g_{\mu\nu}\,.$

(2) the derivative: $\partial \quad \to \quad \nabla\,.$

Example: Electromagnetic field in SR:

$$F_{\mu\nu} = F_{[\mu\nu]} \quad \text{with} \quad F_{0i} = -E_i\,, \quad F_{ij} = \epsilon_{ijk} B_k\,, \qquad (i,\, j,\, k = 1 \ldots 3)$$

$$\text{vacuum Maxwell eqs.:} \quad \eta^{\mu\nu}\partial_\mu F_{\nu\rho} = 0\,, \quad \partial_{[\mu} F_{\nu\rho]} = 0$$

$$\to \text{ in GR:} \quad g^{\alpha\beta}\nabla_\alpha F_{\beta\gamma} = 0\,, \quad \nabla_{[\alpha} F_{\beta\gamma]} = 0$$

Note: Step SR $\to$ GR not unique since $R_{\alpha\beta} = 0$ in SR

C.2 The energy momentum tensor

Postulate: Mass-energy, stress in GR is described by the "energy momentum tensor" with

(1) $T^{\alpha\beta} :=$ flux of α momentum across a surface of constant x^β

(2) $T_{\alpha\beta} = T_{\beta\alpha}\,, \qquad \nabla^\mu T_{\mu\nu} = 0$

Interpretation: $T^{\alpha\beta} = \boldsymbol{T}(\mathbf{d}x^\alpha, \mathbf{d}x^\beta)$

$T^{00} = 0$ momentum across $t = \text{const} =$ energy density

$T^{i0} = x^i$ momentum density

$T^{0i} =$ energy flux across surface $x^i = \text{const}$

$T^{ij} =$ flux of i momentum across $x^j = \text{const}$

$T_{\alpha\beta}$ often obtained from SR + covariance principle

Particles

1) in SR: Point like object with 4-momentum $P^\mu = mu^\mu = (E,\, P^i)$

4-velocity of observer in particle's rest frame: $w^\mu = (1,\, 0,\, 0,\, 0)$

particle energy measured by this observer: $E = -\eta_{\mu\nu} w^\mu P^\nu$

particle's rest mass: $\eta_{\mu\nu} P^\mu P^\nu = -E^2 + \vec{p}^{\,2} = -m^2$

2) in GR: Covariance $\Rightarrow$ $P^\alpha = mu^\alpha$ $\Rightarrow$ $g_{\alpha\beta} P^\alpha P^\beta = -m^2$

Particle energy measured by observer: $E = -g_{\alpha\beta}(p)\; w^\alpha(p)\; P^\beta(p)$

works only if both are at p!

electromagnetic field

1) pre-relativistic notation, Cartesian coordinates:

energy density: $\rho = \dfrac{1}{8\pi}(E_i E_i + B_i B_i)$

momentum density, energy flux: $j_i = \frac{1}{4\pi}\epsilon_{ijk} E_j B_k$ "Poynting vector"

stress tensor: $S_{ij} = \dfrac{1}{4\pi}\left[\dfrac{1}{2}(E_k E_k + B_k B_k)\delta_{ij} - E_i E_j - B_i B_j\right]$

Maxwell eqs. $\Rightarrow$ $\dfrac{\partial \rho}{\partial t} + \partial_i j_i = 0\,, \qquad \dfrac{\partial j_i}{\partial t} + \partial_j S_{ij} = 0$

2) Special relativity:

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}\eta_{\mu\nu}\right) = T_{\nu\mu}$$

$T_{00} = \rho\,, \quad T_{0i} = -j_i\,, \quad T_{ij} = S_{ij}\,;$ from 1)

Conservation: $\partial^\mu T_{\mu\nu} = \eta^{\mu\sigma}\partial_\sigma T_{\mu\nu} = 0$

3) GR: $T_{\alpha\beta} = \dfrac{1}{4\pi}\left(F_{\alpha\mu}F_\beta{}^\mu - \dfrac{1}{4}F^{\mu\nu}F_{\mu\nu}g_{\alpha\beta}\right)$ with $\nabla^\mu T_{\mu\alpha} = 0$

Dust

Continuum limit of non-interacting particles of rest mass m and number density n

$\Rightarrow$ Particles are freely falling!

In comoving frame: $u^\mu = [1,\, 0,\, 0,\, 0]$, locally $g_{\alpha\beta} = \eta_{\alpha\beta}$

$$\Rightarrow \rho = mn\,, \qquad T^{i0} = T^{0j} = 0\,, \qquad T^{ij} = 0$$

$$\Rightarrow \quad \boxed{T^{\alpha\beta} = \rho u^\alpha u^\beta = mnu^\alpha u^\beta}\,.$$

Perfect fluids

Def.: Continuous matter with (1) no heat conduction, (2) no viscosity in locally comoving frame.

(1) = energy can only flow if the particles flow.

(2) = Force between particles only has radial component.

$\Rightarrow$ For particles on x^i axis, only p^i momentum can flow.

$\Rightarrow T^{ij} \neq 0$ only for $i = j$.

No direction preferred $\Rightarrow T^{11} = T^{22} = T^{33} =: P$

In SR: In comoving frame $T^{\alpha\beta} = \mathrm{diag}(\rho,\, P,\, P,\, P)\,;\quad u^\alpha = (1,\, 0,\, 0,\, 0)\,,\quad g^{\alpha\beta} = \eta^{\alpha\beta}$

$$\Rightarrow \quad T^{\alpha\beta} = (\rho + P)u^{\alpha\beta} + P\eta^{\alpha\beta}$$

Covariance principle $\Rightarrow$ $\boxed{T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + Pg^{\alpha\beta}}$

Conservation $\nabla_\alpha T^{\alpha\beta} = 0$

$$\Rightarrow \quad \boxed{u^\alpha \nabla_\alpha \rho + (\rho + P)\nabla_\alpha u^\alpha = 0}$$

$$\boxed{(\rho + P)u^\alpha \nabla_\alpha u^\beta = -(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\alpha P}$$ GR's version of Euler equation

Comments:

- We still need an equation of state describing the matter. E.g. polytrope $P \propto \rho^\Gamma$
- $P = 0 \quad \Rightarrow \quad$ fluid moves on geodesics

C.3 The Einstein equations

Postulates of GR

(1) Spacetime is a 4-dim. Lorentzian manifold with metric and Levi-Civita connection.

(2) Free particles follow timelike or null geodesics.

(3) Energy, momentum and stress of matter is described by a symmetric, conserved tensor $T_{\alpha\beta}$: $\nabla^\alpha T_{\alpha\beta} = 0$.

(4) Curvature is related to matter by the Einstein eqs.

$$\boxed{G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta}} \; ; \quad G = \text{Newton's constant}$$

Comments:

- Factor $\frac{8\pi G}{c^4}$ from Newtonian limit (cf. below)
- Vacuum $\Rightarrow G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\, R = 0 \quad \Big| \cdot g^{\alpha\beta}$

  $\Rightarrow R = 0 \quad \Rightarrow \quad \boxed{R_{\alpha\beta} = 0}$
- $\boldsymbol{G} = 8\pi \boldsymbol{T}$ represents 10 coupled, non-linear PDEs. Hard to solve!

Theorem: (Lovelock 1972) Let $H_{\alpha\beta}$ be a symmetric tensor with

(i) in any coords. $H_{\mu\nu} = H_{\mu\nu}(g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\sigma \partial_\rho g_{\mu\nu})$ at every $p \in \mathcal{M}$

(ii) $\nabla^\alpha H_{\alpha\beta} = 0$

(iii) $H_{\mu\nu}$ linear in $\partial_\sigma \partial_\rho g_{\mu\nu}$

$\Rightarrow \exists_{A,B\in\mathbb{R}} \quad H_{\alpha\beta} = A\, G_{\alpha\beta} + B\, g_{\alpha\beta}$

$\Rightarrow$ we can modify Einstein's eqs.: $\boxed{G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}}$

Λ = Cosmological constant

$\Lambda g_{\alpha\beta}$ is equivalent to perfect fluid with $\rho = -P = \dfrac{\Lambda c^4}{8\pi G}$

D The Schwarzschild solution and classic tests of GR

Einstein was pessimistic about finding solutions to $\boldsymbol{G} = 8\pi\boldsymbol{T}$.

But 1915 Schwarzschild found his "black hole" solution. Key simplification: symmetry.

D.1 Schwarzschild's solution

Symmetric spacetimes

Def.: A spacetime $(\mathcal{M}, \boldsymbol{g})$ is "symmetric in a variable s" if $\exists$ coordinates x^α such that one of the $x^\alpha = s$ and $g_{\alpha\beta}$ are independent of s in this coordinate system.

Def.: A spacetime $(\mathcal{M}, \boldsymbol{g})$ is "stationary" if $\exists$ coordinates x^α such that x^0 is a timelike coordinate and $g_{\alpha\beta}$ do not depend on x^0.

Def.: A spacetime $(\mathcal{M}, \boldsymbol{g})$ is "static" if it is stationary and in that coordinate system $g_{0i} = 0$ for $i = 1, 2, 3$.

Note: Time reversal $t \to -t \quad \Rightarrow \quad g_{0i} dt\, dx^i \to -g_{0i} dt\, dx^i$

Invariance under time reversal implies $g_{0i} = 0$: static spacetimes.

Spherically symmetric spacetimes

$\exists$ point $\mathcal{O}$: spacetime is invariant under rotations about $\mathcal{O}$

$\Rightarrow \ldots \Rightarrow$ The angular part of the line element is $\propto$ 2-sphere metric: $d\Omega^2 \coloneqq d\theta^2 + \sin^2\theta\, d\phi^2$

We assume the metric is invariant under reflection $\theta \to \pi - \theta\,, \quad \phi \to -\phi$: no mixed ϕ, θ terms

$$\Rightarrow \; ds^2 = -\tilde{A} d\tilde{t}^2 + 2\tilde{B} d\tilde{t}\, d\tilde{r} + \tilde{C} d\tilde{r}^2 + \tilde{D} d\Omega^2$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ are functions of $(\tilde{t}, \tilde{r})$ and $\tilde{D} > 0$

(1) $r \coloneqq \sqrt{\tilde{D}} \quad \Rightarrow \quad ds^2 = -\hat{A}(\tilde{t}, r) d\tilde{t}^2 + 2\hat{B}(\tilde{t}, r) d\tilde{t}\, dr + \hat{C}(\tilde{t}, r) dr^2 + r^2 d\Omega^2$

(2) ODE theory $\quad \Rightarrow \quad \exists$ integrating factor $I(\tilde{t}, r)$ with

$$I(\tilde{t}, r)\big[-\hat{A}(\tilde{t}, r)\, d\tilde{t} + \hat{B}(\tilde{t}, r)\, dr\big] = d\hat{t}$$

$$\Rightarrow \ldots \Rightarrow \quad ds^2 = -j(\hat{t}, r) d\hat{t}^2 + k(\hat{t}, r) dr^2 + r^2 d\Omega^2 \quad \text{with} \quad j = \frac{1}{\hat{A} I^2}\,, \quad k = \hat{C} + \frac{\hat{B}^2}{\hat{A}}$$

(3) Now Einstein eqs.: $R^{\hat{t}}{}_{r} = \dfrac{\partial_{\hat{t}} k}{k^2 r} = 0 \quad \Rightarrow \quad k = k(r)$

$$R^{\hat{t}}{}_{\hat{t}} = \frac{r^2 \partial_r k + k^2 - k}{k^2 r^2} = 0 \quad \Rightarrow \ldots \Rightarrow \quad k = \frac{r}{r - 2M}\,, \quad M = \text{const}$$

$$R^{r}{}_{r} = \frac{-r\partial_r j + jk - j}{-jkr^2} = 0 \quad \Rightarrow \quad j = \frac{r-2M}{r} f(\hat{t})$$

(4) signature +2 $\Rightarrow$ $f(\hat{t}) > 0$

Rescale $\hat{t}$ such that $\sqrt{f(\hat{t})}d\hat{t} = dt$

$$\Rightarrow \boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)} \qquad (\dagger)$$

Comments:

- $(\dagger)$ = unique solution in spherical symmetry and vacuum. It is static!
- $\lim_{r\to\infty} g_{\alpha\beta} = \eta_{\alpha\beta}$: *asymptotic flatness*
- M can be shown to be the mass-energy of the spacetime.
- $(\dagger)$ also describes the spacetime exterior to spherically symmetric stars.

$\Rightarrow$ **Birkhoff's Theorem:** Any spherically symmetric solution of the vacuum Einstein equations is given by the Schwarzschild metric. It is static and asymptotically flat.

D.2 Geodesics in the Schwarzschild spacetime

$$\hat{\mathcal{L}} = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2$$

θ component: $\Rightarrow \dfrac{d}{d\lambda}\left(\dfrac{\partial \hat{\mathcal{L}}}{\partial \dot{\theta}}\right) - \dfrac{\partial \hat{\mathcal{L}}}{\partial \theta} = 2r^2\ddot{\theta} + 4r\dot{r}\dot{\theta} - 2r^2 \sin\theta\,\cos\theta\,\dot{\phi}^2 = 0$

$$\Rightarrow \quad \ddot{\theta} + 2\frac{\dot{r}\dot{\theta}}{r} - \sin\theta\,\cos\theta\,\dot{\phi}^2 = 0$$

Rotate coordinates such that $\theta = \pi/2\,,\ \dot{\theta} = 0$ at λ_0

$\Rightarrow$ $\theta = \pi/2$ always WLoG!

Noether $\Rightarrow$ (i) $\dfrac{\partial\hat{\mathcal{L}}}{\partial t} = 0 \quad \Rightarrow \quad E = -\dfrac{1}{2}\dfrac{\partial\hat{\mathcal{L}}}{\partial \dot{t}} = \left(1 - \dfrac{2M}{r}\right)\dot{t} = \text{const}$

(ii) $\dfrac{\partial\hat{\mathcal{L}}}{\partial \phi} = 0 \quad \Rightarrow \quad L = \dfrac{1}{2}\dfrac{\partial\hat{\mathcal{L}}}{\partial \dot{\phi}} = r^2 \sin^2\theta\,\dot{\phi} = r^2\dot{\phi} = \text{const}$

(iii) $\dfrac{\partial\hat{\mathcal{L}}}{\partial \lambda} = 0 \quad \Rightarrow \quad Q = -\left(1 - \dfrac{2M}{r}\right)\dot{t}^2 + \left(1 - \dfrac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \\ 1 & \text{spacelike} \end{cases}$

Meaning of E, L: Consider $r \gg M \quad \Rightarrow \quad g_{\alpha\beta} \to \eta_{\alpha\beta} \quad \Rightarrow \quad$ SR

E: for timelike geodesics with $\lambda = \tau$: $\quad E = \dfrac{dt}{d\tau} = \gamma$

$\Rightarrow \quad E\,m = m\,\gamma \quad \Rightarrow \quad E =$ energy per rest mass m

L: Likewise $L\,m = mr^2\dot{\phi} = m\gamma r^2\dfrac{d\phi}{dt} =$ angular momentum per rest mass

Plug E, L into Eq. for Q

$$\Rightarrow \quad -E^2 + \dot{r}^2 + \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)L^2 = \left(1 - \frac{2M}{r}\right)Q$$

$$\Rightarrow \boxed{\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2\,, \qquad V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} - Q\right)} \tag{‡}$$

Comparison with Newtonian equations

Energy balance for particle in Newtonian gravity

$$E_{\text{kin}} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 = \frac{1}{2}m\dot{r}^2 + \frac{m}{2}\frac{L^2}{r^2}$$

$E_{\text{pot}} = -G\dfrac{Mm}{r}\,, \qquad$ Let's set $G = 1$

$$\Rightarrow \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\frac{L^2}{r^2} - \frac{Mm}{r} = E_{\text{kin}} + E_{\text{pot}} = \text{const}$$

Eq. (‡) for $Q = -1$: $\quad (E^2 - 1)m/2 = E_{\text{kin}} + E_{\text{pot}} - m\frac{ML^2}{r^3}$

$\Rightarrow$ GR has extra term $-\dfrac{ML^2}{r^3}$

Summary: $\frac{1}{2}\dot{r}^2 + V_{\mathrm{N/GR}}(r) = \mathrm{const}\,.$ $\qquad V_{\mathrm{N}}(r) = \frac{1}{2}\frac{L^2}{r^2} - \frac{M}{r}$

$$V_{\mathrm{GR}}(r) = \frac{1}{2}\frac{L^2}{r^2} + Q\frac{M}{r} - \frac{ML^2}{r^3}$$

Effective potential V determines trajectories

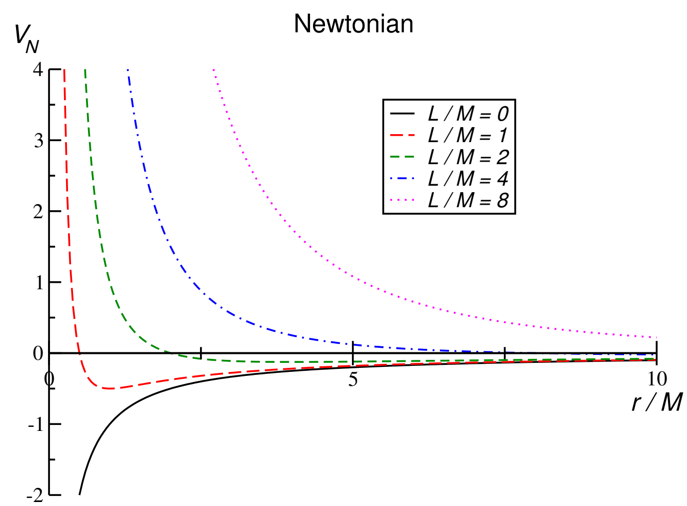
Newtonian: For $L \neq 0$: $\lim\limits_{r\to 0} V = \infty\,, \qquad \lim\limits_{r\to\infty} V = 0$

Extrema from: $V_{\mathrm{N}}'(r) = -\frac{L^2}{r^3} + \frac{M}{r^2} = 0 \quad \Rightarrow \quad r = \frac{L^2}{M}$

$$V_{\mathrm{N}}''(r) = \frac{3L^2}{r^4} - \frac{2M}{r^3} \qquad \Rightarrow \quad V_{\mathrm{N}}''(L^2/M) = M^4/L^6 > 0$$

$\Rightarrow$ circular orbit at $r = \frac{L^2}{M}$

V_{N} min. $\Rightarrow$ orbit stable



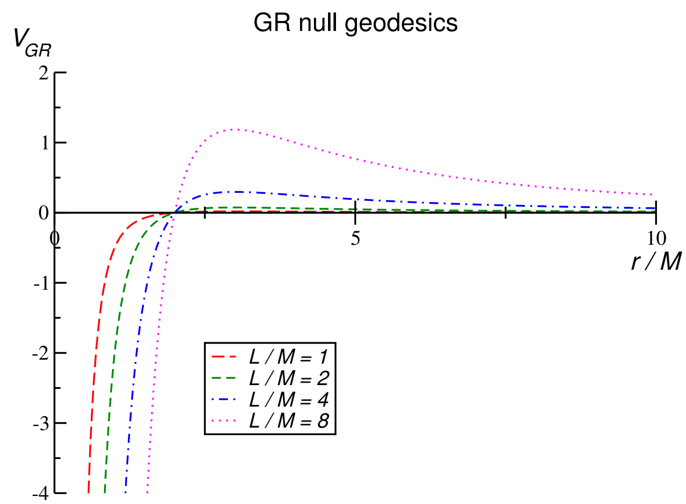
GR null: $\lim_{r\to 0} V = -\infty\,, \qquad \lim_{r\to\infty} V = 0$

$$\text{Extrema:}\quad V'_{\text{GR}}(r) = -\frac{L^2}{r^3} + 3\frac{ML^2}{r^4} = 0 \quad \Rightarrow \quad r = 3M$$

$$V''_{\text{GR}}(r) = \frac{3L^2}{r^4} - \frac{12ML^2}{r^5} \qquad \Rightarrow \quad V''_{\text{GR}}(3M) = -\frac{L^2}{81M^4} < 0$$

$\Rightarrow$ circular orbit at $r = 3M$

V_{GR} max. $\Rightarrow$ orbit unstable



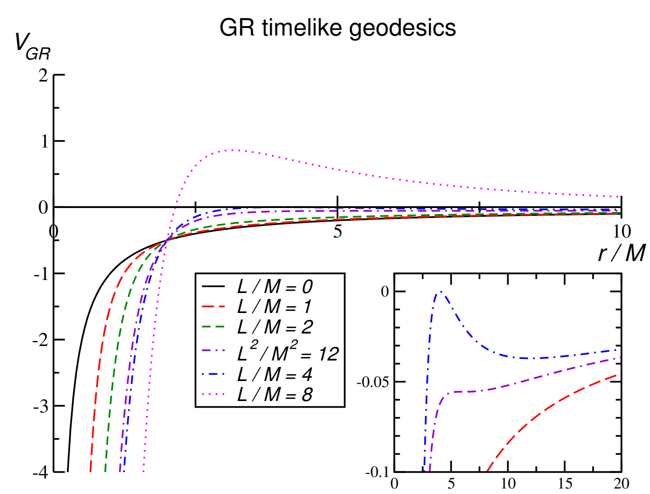
GR timelike: $\lim_{r\to 0} V = -\infty\,, \qquad \lim_{r\to\infty} V = 0$

$$\text{Extrema:}\quad r = r_\pm = \frac{L^2}{2M} \pm \sqrt{\frac{L^4}{4M^2} - 3L^2} > 0 \text{ for } L^2 > 12\ M^2$$

$$\text{For } L^2 > 12\ M^2: \quad V''_{\text{GR}}(r_+) > 0\,, \qquad V''_{\text{GR}}(r_-) < 0$$

$\Rightarrow$ stable circular orbit at r_+

$\Rightarrow$ unstable circular orbit at r_-



D.3 Classic tests of GR

Mercury's perihelion precession

Newtonian: $\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\frac{L^2}{r^2} - \frac{Mm}{r} =: \mathcal{E}$ (†)

$$\dot{\phi} = L/r^2 \quad \Rightarrow \quad \phi(t) \text{ monotonic} \quad \to \quad \text{parametrize orbit with } \phi$$

$$y := \frac{1}{r} \quad \Rightarrow \ldots \Rightarrow \quad \dot{r} := \frac{dr}{dt} = -L\frac{dy}{d\phi} =: -Ly'$$

$$(\dagger) \quad \Rightarrow \quad (y')^2L^2 + L^2y^2 - 2My = 2\mathcal{E} \quad \Big| \ \frac{d}{d\phi}$$

$$\Rightarrow \quad 2L^2y'y'' + 2L^2yy' - 2My' = 0$$

$$\Rightarrow \quad y' = 0 \quad \vee \quad y'' + y = \frac{M}{L^2}$$

$$\Rightarrow \quad y = \frac{M}{L^2}(1 + \epsilon\cos\phi) \quad \text{Keppler ellipse for } \epsilon < 1\text{: No perihelion precession}$$

GR: Use again $y(\phi)$, $Q = -1$

$$\Rightarrow \ldots \Rightarrow \quad L^2(y')^2 + L^2y^2 - 2My - 2ML^2y^3 = E^2 - 1 \quad \Big| \ \frac{d}{d\phi}$$

$$\Rightarrow y'' + y = \frac{M}{L^2} + 3My^2$$

Expand solution in $\alpha := 3\frac{M^2}{L^2} \sim \mathcal{O}(10^{-7})$ for Mercury

$$\Rightarrow \quad y = y_0 + \alpha y_1 + \mathcal{O}(\alpha^2)$$

$$\Rightarrow \quad y_0'' + y_0 - \frac{M}{L^2} + \alpha\left(y_1'' + y_1 - \frac{L^2}{M}y_0^2\right) + \mathcal{O}(\alpha^2) = 0$$

Order α^0: $y_0'' + y_0 - \frac{M}{L^2} = 0 \quad \Rightarrow \quad y_0 = \frac{M}{L^2}(1 + \epsilon\cos\phi)$ Newtonian case!

Order α^1: Plug in y_0 and use $\cos^2\phi = (1 + \cos 2\phi)/2$

$$\Rightarrow \quad y_1'' + y_1 = \frac{M}{L^2}\left(1 + \frac{\epsilon^2}{2}\right) + \frac{2M}{L^2}\epsilon\cos\phi + \frac{M}{2L^2}\epsilon^2\cos 2\phi$$

Ansatz: $\quad y_1 = A + B\phi\sin\phi + C\cos 2\phi$

$$\Rightarrow \quad \ldots \quad \Rightarrow \quad A = \frac{M}{L^2}\left(1 + \frac{\epsilon^2}{2}\right), \qquad B = \frac{M\epsilon}{L^2}, \qquad C = -\frac{M\epsilon^2}{6L^2}.$$

Solution: $y = y_0 + \alpha y_1 = \frac{M}{L^2}(1 + \epsilon\cos\phi) + \alpha\frac{M}{L^2}\left[1 + \epsilon\phi\sin\phi + \epsilon^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\phi\right)\right]$

Ignore last term $\propto \epsilon^2$

$\Rightarrow y \approx \frac{M}{L^2}(1 + \alpha + \epsilon\cos\phi + \alpha\epsilon\phi\sin\phi)$

$\alpha \ll 1 \quad \Rightarrow \quad \cos(\phi - \alpha\phi) = \cos\phi\,\cos\alpha\phi + \sin\phi\,\sin\alpha\phi \approx \cos\phi + \alpha\phi\sin\phi$

$\Rightarrow \boxed{y \approx \frac{M}{L^2}\left\{1 + \alpha + \epsilon\cos[\phi(1-\alpha)]\right\}}$

Key result: y returns to the same value as $(1-\alpha)\phi$ increases by 2π

$\Rightarrow$ Perihelion period: $\phi_{n+1} - \phi_n = \frac{2\pi}{1-\alpha} \approx 2\pi(1+\alpha) = 2\pi\left(1 + \frac{3M^2}{L^2}\right)$

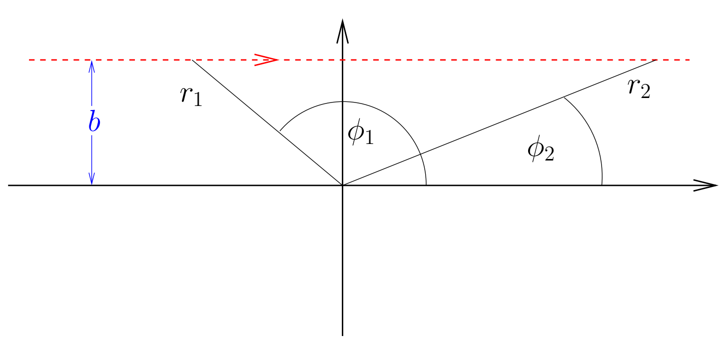
Circular timelike geodesic $\Rightarrow r = r_+ \Rightarrow \ldots \Rightarrow L^2 = \frac{Mr}{1 - 3M/r} \approx Mr$

$\Rightarrow \quad \Delta\phi \approx 6\pi\frac{M}{r} = \frac{43''}{\text{century}}$

Light bending

1) Newtonian

Without gravitational field: $y'' + y = 0 \quad \Rightarrow \quad y = \frac{1}{b}\sin\phi\,,$ straight line



light from left ($\phi = \pi$) to right ($\phi = 0$); $\quad b =$ impact parameter

With field: Recall $y = \dfrac{M}{L^2}(1 + \epsilon \sin\phi)$, we shifted Phase: $\cos \to \sin$

Small deflection $\Rightarrow$ $y = 0$ at $\phi = -\Delta\phi$, $\pi + \Delta\phi$ with $\Delta\phi \ll 1$

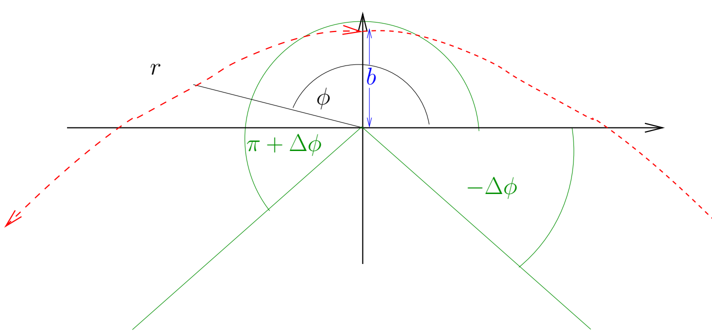
$$\Rightarrow \sin(-\Delta\phi) \approx -\Delta\phi = -\frac{1}{\epsilon}\,, \qquad \sin(\pi + \Delta\phi) \approx -\Delta\phi = -\frac{1}{\epsilon}$$

It follows: $\dfrac{1}{\epsilon} \ll 1 \quad \Rightarrow \quad \epsilon \gg 1$

Impact parameter $\dfrac{1}{b} = y(\pi/2) = \dfrac{M}{L^2}(1+\epsilon) \approx \dfrac{M}{L^2}\epsilon$

Angular momentum: $mL = |\vec{r} \times \vec{p}| = bmc = bm \quad \Rightarrow \quad L = b$

Total deflection $2\Delta\phi = \dfrac{2}{\epsilon} = \dfrac{2M}{b}$



2) GR

Geodesic equation for $Q = 0$: $L^2(y')^2 + L^2 y^2 - 2ML^2 y^3 = E^2 \quad \Big| \; \dfrac{d}{d\phi}$

$$\Rightarrow \quad y'' + y = 3My^2$$

Without field: $M = 0 \quad \Rightarrow \quad y_0'' + y_0 = 0 \quad \Rightarrow \quad y_0 = \dfrac{1}{b}\sin\phi$ (like Newtonian)

Small deflection: perturb around straightline $y_0 = (\sin\phi)/b$

$$\Rightarrow \quad y = y_0 + \frac{M}{b}\Delta y + \mathcal{O}(M^2/b^2)$$

$$\Rightarrow \ldots \Rightarrow \quad \Delta y'' + \Delta y = \frac{3}{b}\frac{1 - \cos 2\phi}{2}$$

Homogeneous DE: $\Delta y'' + \Delta y = 0$ solved by $\Delta \tilde{y} = A\cos\phi + B\sin\phi$

Particular solution: $\Delta \hat{y} = \frac{1}{2b}(3 + \cos 2\phi)$

With $A = \frac{2}{b}$, $B = 0$, we get $y = 0$ for $\phi \to \pi$

$$\Rightarrow \quad y = y_0 + \frac{M}{b}\Delta y = \frac{1}{b}\sin\phi + \frac{M}{2b^2}(3 + \cos 2\phi) + \frac{2M}{b^2}\cos\phi$$

Deflection $\delta\phi$ from $y = 0$ at $\phi = 0 + \delta\phi$

$$\Rightarrow 0 \approx \frac{\delta\phi}{b} + \frac{M}{2b^2}(3+1) + \frac{2M}{b^2}$$

$$\Rightarrow \quad \delta\phi \approx -\frac{4M}{b}$$

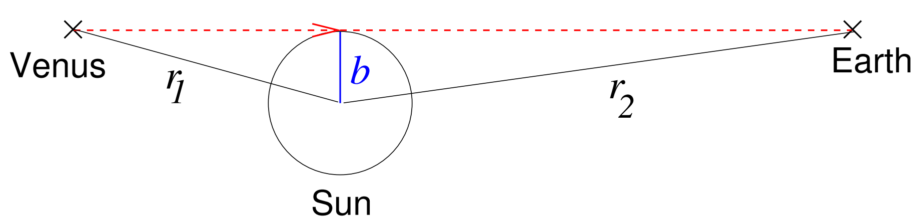
For sun: $M_\odot = 1.5$ km, $b \approx R_\odot = 7 \times 10^5$ km: $|\delta\phi| \approx 1.77''$

Measured by Eddington expedition in 1919.

Shapiro time delay

Radio signal past sun to Venus and back. Measure time delay.

1) Without field: Pythagoras $\Rightarrow$ $T = 2\left(\sqrt{r_1^2 - b^2} + \sqrt{r_2^2 - b^2}\right)$



2) With field

Null geodesic (Q=0): $\dot{r}^2 + \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} = E^2\,, \qquad \dot{t} = \left(1 - \frac{2M}{r}\right)^{-1} E$

$$\Rightarrow \ldots \Rightarrow \frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right)\sqrt{1 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2E^2}}$$

At point of closest approach: $r = b\,, \quad dr/dt = 0$

$$\Rightarrow \quad \left(1 - \frac{2M}{b}\right) \frac{L^2}{b^2 E^2} = 1 \qquad \Rightarrow \qquad \frac{L^2}{E^2} = \frac{b^2}{1 - \frac{2M}{b}} \quad \text{eliminates } L,\ E$$

$$\Rightarrow T = 2\int_b^{r_1} \frac{dr}{f(r)} + 2\int_b^{r_2} \frac{dr}{f(r)}\,, \qquad f(r) = \left(1 - \frac{2M}{r}\right)\sqrt{1 - \frac{b^2}{r^2}\frac{1-2M/r}{1-2M/b}}$$

Solve integral by Taylor expanding $f(r)$ in M/r using $M/b \ll 1$

$$\Rightarrow \ldots \Rightarrow T \;\approx\; \underbrace{2\left(\sqrt{r_1^2 - b^2} + \sqrt{r_2^2 - b^2}\right)}_{=:T_{\text{Mink}}} + 4M\left(\ln\frac{r_1 + \sqrt{r_1^2 - b^2}}{b} + \ln\frac{r_2 + \sqrt{r_2^2 - b^2}}{b}\right)$$

$$+ 2M\left(\sqrt{\frac{r_1 - b}{r_1 + b}} + \sqrt{\frac{r_2 - b}{r_2 + b}}\right).$$

For Venus and Earth: $\Delta T \approx 77\text{ km} = 257\ \mu\text{s}$

D.4 The causal structure of the Schwarzschild spacetime

Recall Schwarzschild: $ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

singular at $r = 0$, $r = 2M$; what's happening?

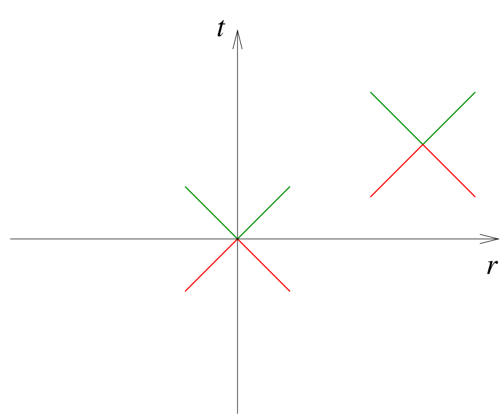
Light cones

Timelike (null) curves travel inside (on) light cones

$\Rightarrow$ Lightcones display causal structure of spacetime!

E.g. Minkowski

in spherical coords.:



Radial null geodesics: $d\theta = d\phi = 0$; Let λ be an affine parameter

t component: We already know $\left(1 - \frac{2M}{r}\right)\dot{t} = E = \text{const}$

r component: EL eqs.: $\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r}$

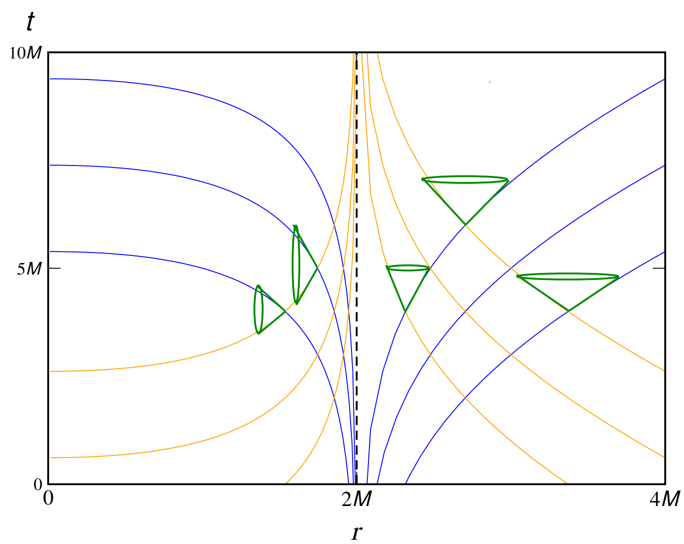
$$\Rightarrow \ldots \Rightarrow \quad \left(1 - \frac{2M}{r}\right)\frac{r^2}{M}\ddot{r} = \dot{r}^2 - E^2$$

Solved by $\dot{r} = \pm E$

$\Rightarrow \quad r = \pm E\lambda + r_0$ is also an affine parameter. Let's use r.

$$\Rightarrow \quad \frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = \pm\frac{r}{r - 2M}$$

$$\Rightarrow \ldots \Rightarrow \boxed{t(r) = \pm(r + 2M\ \ln|r - 2M|) + k}, \qquad k = \text{const}$$



$r > 2M$: In/Outgoing null geodesics for the $+/-$ sign.

Note: $r = \text{const}$ timelike (inside light cone)

$$r < 2M: \quad ds^2 = -\left(\frac{2M}{r} - 1\right)^{-1} dr^2 + \left(\frac{2M}{r} - 1\right) dt^2$$

$\Rightarrow \quad g_{rr} < 0\,, \quad g_{tt} > 0 \quad \Rightarrow \quad r$ is the timelike coordinate, t spacelike

Light cones tilted horizontally, but In/Outgoing direction unclear.

Infalling observers

Consider timelike geodesic starting at large r:

$$\left(1-\frac{2M}{r}\right)\dot{t}=E\,, \qquad -\left(1-\frac{2M}{r}\right)\dot{t}^2+\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2=Q=-1$$

$$\Rightarrow -E^2+\dot{r}^2=-1+\frac{2M}{r}$$

Observer starting at rest at $r\to\infty$ $\quad\Rightarrow\quad$ $E=1$ (cf. Sec. D.2)

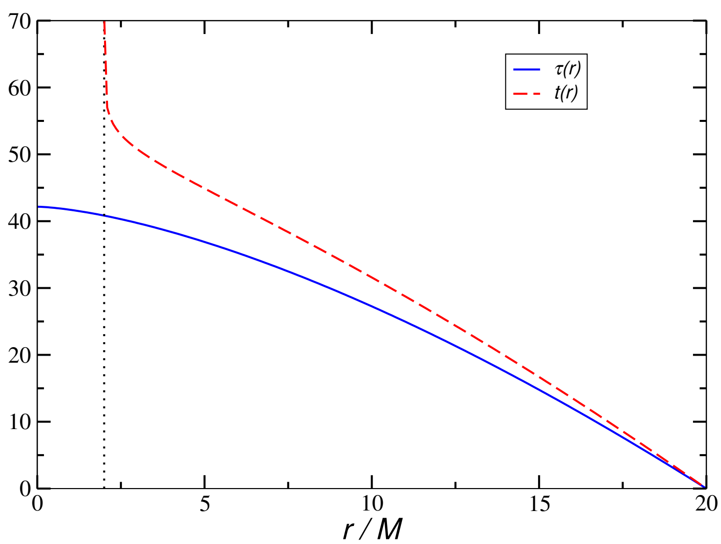
Use proper time: $\dot{r}^2=\dfrac{2M}{r} \quad\Rightarrow\quad \dfrac{d\tau}{dr}=-\sqrt{\dfrac{r}{2M}}<0$ infalling!

$$\Rightarrow\ldots\Rightarrow\quad \tau-\tau_0=\frac{2}{3\sqrt{2M}}\left(r_0^{3/2}-r^{3/2}\right)$$

How about coordinate time t?

$$\frac{dt}{dr}=\frac{\dot{t}}{\dot{r}}=-\sqrt{\frac{r}{2M}}\left(1-\frac{2M}{r}\right)^{-1}$$

$$\Rightarrow\ldots\Rightarrow\quad t-t_0=-\frac{2}{3\sqrt{2M}}\left[r^{3/2}-r_0^{3/2}+6M(\sqrt{r}-\sqrt{r_0})\right]+2M\,\ln\frac{\left(\sqrt{r}+\sqrt{2M}\right)\left(\sqrt{r_0}-\sqrt{2M}\right)}{\left(\sqrt{r_0}+\sqrt{2M}\right)\left(\sqrt{r}-\sqrt{2M}\right)}$$



Interpretation:

- $t=$ proper time of observer at infinity
- geodesic crosses $r=2M$ at finite τ but $t\to\infty$
- one falls in finite time, but process infinitely redshifted for outside observer

Ingoing Eddington Finkelstein (IEF) coordinates

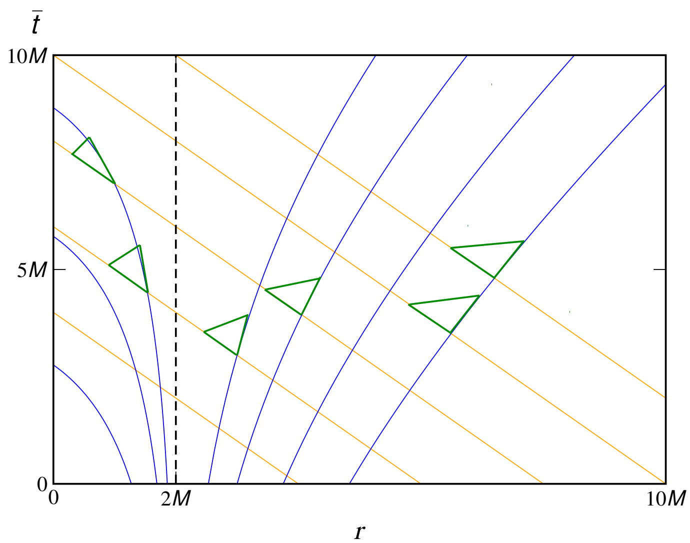
Ingoing null geodesics: $t + 2M \ln|r - 2M| = -r + \text{const}$

New time coordinate: $\bar{t} = t + 2M \ln|r - 2M| \quad \Rightarrow \quad d\bar{t} = dt + \frac{2M}{r - 2M} dr$

$$\Rightarrow \ldots \Rightarrow \quad \boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) d\bar{t}^2 + \frac{4M}{r} d\bar{t} dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi)}$$

Ingoing geodesics: $\bar{t} = -r + \text{const}$

Outgoing geodesics: $\bar{t} = r + 4M \ln|r - 2M| + \text{const}$



Light cones tilt over inwards at $r = 2M$.

Def.: Event horizon: The outermost boundary of a region of spacetime from which no null geodesics or timelike curves can escape to infinity.

Israel's theorem: If a spacetime is static, asymptotically flat and contains a regular horizon then it is a Schwarzschild spacetime.

We can use a null coordinate $v = \bar{t} + r \quad \Rightarrow \quad d\bar{t} = dv - dr$

$$\Rightarrow \ldots \Rightarrow \quad ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dr\, dv + r^2 d\Omega^2$$

Note: ∂_r is tangent to curves $v = \text{const}$. Clearly $\boldsymbol{g}(\partial_r, \partial_r) = 0$.

Outgoing Eddington Finkelstein (OEF) coordinates

Outgoing null geodesics: $t - 2M \ln|r - 2M| = r + \text{const}$

New time coordinate: $\tilde{t} = t - 2M \ln|r - 2M| \quad \Rightarrow \quad d\tilde{t} = dt - \frac{2M}{r - 2M} dr$

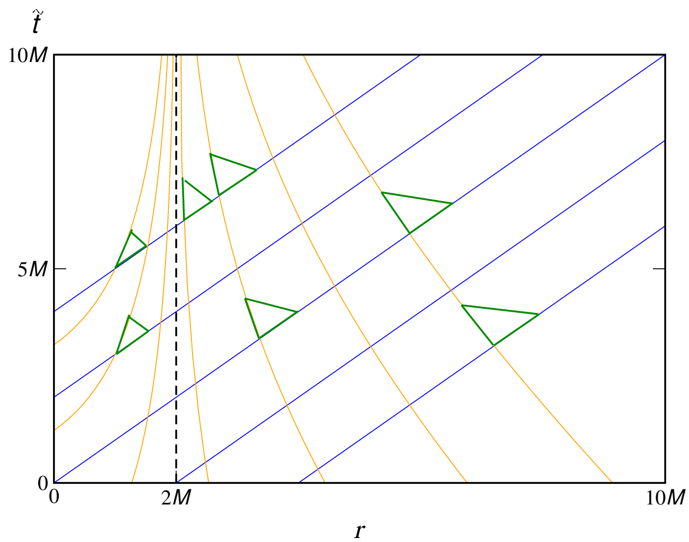
$$\Rightarrow \ldots \Rightarrow \quad \boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 - \frac{4M}{r} d\tilde{t} dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi)} \qquad (*)$$

We can use a null coordinate $u = \tilde{t} - r \quad \Rightarrow \quad d\tilde{t} = du + dr$

$$\Rightarrow \ldots \Rightarrow \quad ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dr\, du + r^2 d\Omega^2$$

Ingoing geodesics: $\tilde{t} = -r - 4M \ln|r - 2M| + \text{const}$

Outgoing geodesics: $\tilde{t} = r + \text{const}$



Now all light cones point outwards.

But above we showed that light cones point inwards at $r < 2M$. WTF?!?

On the other hand: Schwarzschild should be time symmetric. What's going on?

Kruskal-Szekeres coordinates

Step 1: Combine IEF and OEF: $v = t + r + 2M \ln\left(\frac{r-2M}{r_*}\right)$; $\quad r_*$: integration constant

$$u = t - r - 2M \ln\left(\frac{r-2M}{r_*}\right)$$

$$\Rightarrow \quad ds^2 = -\left(1 - \frac{2M}{r}\right) du\, dv + r^2 d\Omega^2$$

Step 2: Use an exponential version of u, v:

$$\tilde{v} = e^{\frac{v}{4M}}\,, \qquad \tilde{u} = -e^{-\frac{u}{4M}}$$

$$\Rightarrow \tilde{u}\tilde{v} = -e^{\frac{v-u}{4M}} = -\frac{r-2M}{r_*}\, e^{\frac{r}{2M}}$$

$$\Rightarrow \ldots \Rightarrow \quad ds^2 = -\frac{16M^2}{r/r_*} e^{-\frac{r}{2M}} d\tilde{u}\, d\tilde{v} + r^2 d\Omega^2$$

Step 3: Go back to time and radius:

$$\hat{t} = \frac{1}{2}(\tilde{v} + \tilde{u})\,, \quad \hat{r} = \frac{1}{2}(\tilde{v} - \tilde{u})$$

$$\Rightarrow \ldots \Rightarrow \quad \boxed{ds^2 = \frac{16M^2}{r/r_*} e^{-\frac{r}{2M}} (-d\hat{t}^2 + d\hat{r}^2) + r^2 d\Omega^2}\,, \qquad \text{from now on set } r_* = 1$$

r implicitly given through $\quad \hat{t}^2 - \hat{r}^2 = -e^{\frac{r}{2M}}(r - 2M)$

Comments:

- Metric regular at $r = 2M$
- Radial null geodesics are: $\hat{t} = \pm\hat{r} + \text{const}$
- Coordinate range:

  a) $r = 2M \quad \Rightarrow \quad \hat{t}^2 - \hat{r}^2 = 0 \quad \Rightarrow \quad \hat{t} = \pm\hat{r}$

  b) $r = 0 \quad \Rightarrow \quad \hat{t}^2 - \hat{r}^2 = 2M \quad \Rightarrow \quad \hat{t} = \pm\sqrt{\hat{r}^2 + 2M}$

  c) $\hat{t}^2 - \hat{r}^2 = -e^{r/(2M)}(r - 2M)$ monotonically decreasing in r

  $\Rightarrow \quad \hat{t}^2 - \hat{r}^2 < 2M \quad$ for all $\quad r > 0$

  d) No further restrictions on $\hat{t}$, $\hat{r}$: $\quad \boxed{\hat{r} \in (-\infty, \infty)\,, \quad \hat{t}^2 \leq \hat{r}^2 + 2M}$

Kruskal diagram

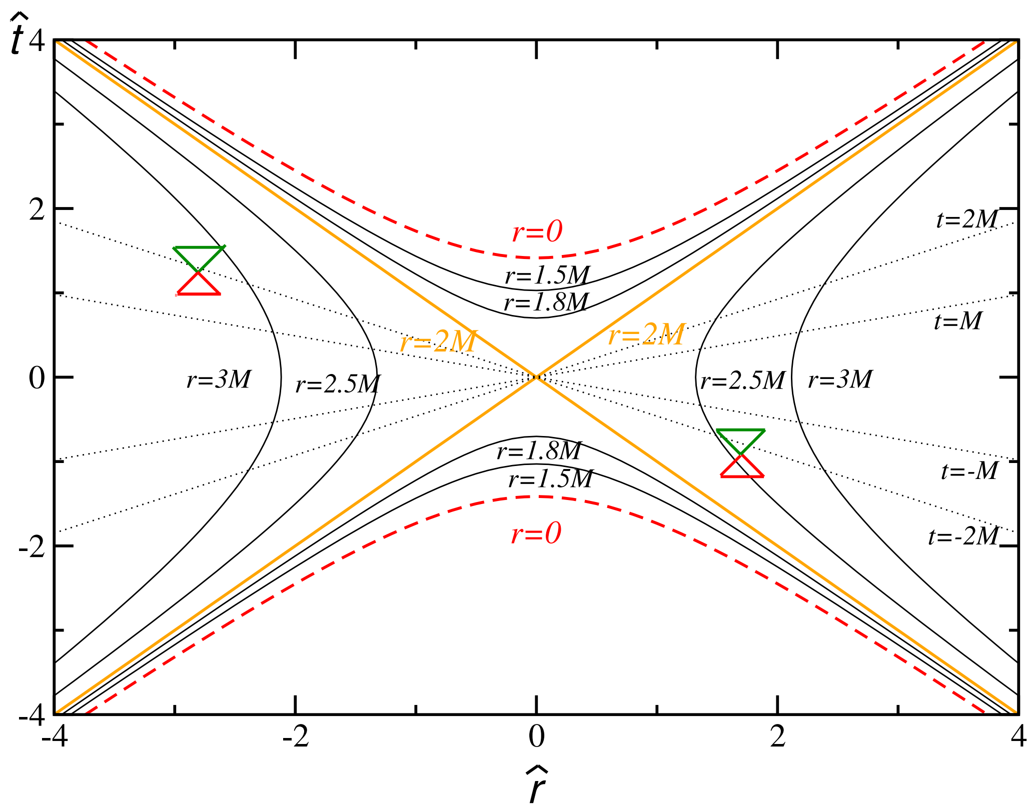
(1) Curves $r = r_0$: $\hat{t}^2 - \hat{r}^2 = -e^{\frac{r_0}{2M}}(r_0 - 2M) =: C$

$$\Rightarrow \boxed{\hat{t} = \pm\sqrt{\hat{r}^2 + C}}$$

(2) Curves $t = t_0$: $\hat{t} = \frac{1}{2}(\tilde{v} + \tilde{u}) = \sqrt{r - 2M}e^{\frac{r}{4M}} \sinh \frac{t}{4M}$

$$\hat{r} = \frac{1}{2}(\tilde{v} - \tilde{u}) = \sqrt{r - 2M}e^{\frac{r}{4M}} \cosh \frac{t}{4M}$$

$$\Rightarrow \boxed{\tanh \frac{t}{4M} = \frac{\hat{t}}{\hat{r}}}$$



Comments:
- Spacetime extended: white hole, black hole, 2 asymptotically flat regions
- Resolves puzzle of light cones in IEF/OEF coordinates.

D.5 Hawking radiation

GR is a classical theory and we do not yet have a quantum theory of gravity.

Semi-classical calculations: QM on curved background spacetime.

Hawking effect: Pair creation of virtual particles

$\rightarrow$ one falls into black hole; the other escapes

Hawking radiation depends on temperature $T = \dfrac{\hbar c^3}{8\pi G M k_B}$

$$-\frac{1}{A}\frac{dM}{dt} = \sigma T^4\,, \qquad \sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$$

$$\Rightarrow \frac{dM}{dt} = -\frac{\hbar c^6}{15360\pi\, G^2\, M^2} \qquad \Rightarrow \qquad t = 5120\frac{\pi G^2}{\hbar c^6} M^3\,; \quad \text{for } \; M_\odot :\; t = \mathcal{O}(10^{60}) \text{ yr}$$

Note: Energy loss $\rightarrow$ higher T $\rightarrow$ more radiation!

E Cosmology

Goal: Simplified model of the entire universe.

E.1 Homogeneity and isotropy

Observations:

- Electromagnetic observations of universe out to $\sim 10^{11}$ pc.
- Galaxies have $R \sim 10^5$ km $\quad\rightarrow\quad$ point like on scale 10^{11} pc.
- On scales $\sim 10^9$ pc, the universe looks the same on average and the same in every direction.
- Hubble redshift $\quad\rightarrow\quad$ universe seems to expand.

$\Rightarrow$ Model universe as an isotropic, spatially homogeneous spacetime with fluid matter.

Cosmological principle: At a given time, the universe is spatially homogeneous and isotropic when viewed on large scales.

Weyl's postulate: The world lines of the universe's fluid elements are orthogonal to hypersurfaces of constant time, Σ_t, to which the cosmological principle applies.

Comments:

- Isotropy is observer dependent: a boosted observer will not see isotropy.
- Which observer sees isotropy? The one comoving with the cosmological fluid.
- We do *not* require homogeneity in time!

Adapted coordinates

Let x^i be spatial coordinates comoving with the cosmological fluid.

Let t be proper time along the world lines of observers comoving with the fluid.

Isotropy $\Rightarrow$ no spatial metric component has a prefered time dependency

$$\Rightarrow \quad ds^2 = -dt^2 + g_{0i} dt\, dx^i + a(t)^2 h_{ij}(x^k) dx^i\, dx^j \,.$$

Consider basis $\mathbf{e}_0 = \partial_t$, $\mathbf{e}_i = \partial_i$

$\Rightarrow \quad \mathbf{e}_0 =$ four-velocity of comoving observers and $\mathbf{e}_i$ are tangent to Σ_t

By Weyl's postulate, $\mathbf{e}_0$ is orthogonal to Σ_t, i.e. to the $\mathbf{e}_i$

$\Rightarrow \quad g_{0i} = \boldsymbol{g}(\mathbf{e}_0, \mathbf{e}_i) = 0$

$$\Rightarrow \quad ds^2 = -dt^2 + a(t)^2 h_{ij}(x^k) dx^i \, dx^j \qquad (\dagger)$$

Note: Observers moving relative to the fluid do not move orthogonally to Σ_t.

$\Rightarrow \quad \tilde{g}_{0i} \neq 0 \quad \Rightarrow \quad$ this observer does not see isotropy.

Recall: Spatial part of spherically symmetric metric: $\quad d\ell^2 = C(t,r)dr^2 + D(t,r)d\Omega^2 \qquad (\ddagger)$

Spherical symmetry = isotropy around one point

Isotropy around *every* point = "Maximal symmetry" = Spherical symmetry + more

$\Rightarrow$ we can combine ($\dagger$) and ($\ddagger$)

$\Rightarrow$ factorize $C(t,r) = a(t)^2 e^{2\beta(r)}$, $D(t,r) = a^2(t) r^2 \quad \Rightarrow \quad d\ell^2 = a^2(t) \left[e^{2\beta(r)} dr^2 + r^2 d\Omega^2 \right]$

Next use our differential geometry on the hypersurface $t = \text{const}$

3-dimensional Ricci scalar $\mathcal{R} = \mathcal{R}^i{}_i\,, \quad i = 1,\, 2,\, 3$

Spatial homogeneity $\Rightarrow$ $\mathcal{R} = \ldots = \dfrac{2}{r^2}\left[1 - \partial_r(re^{-2\beta})\right] = \text{const} =: \tilde{k}$

$$\Rightarrow \quad e^{2\beta} = \frac{1}{1 - \frac{1}{6}\tilde{k}r^2 - \frac{A}{r}}$$

No conical singularity $\Rightarrow \quad \lim\limits_{r\to 0} d\ell^2 \propto (dr^2 + r^2 d\Omega^2) \quad \Rightarrow \quad A = 0$

$\Rightarrow$ **Robertson-Walker** metric: $\boxed{ds^2 = -dt^2 + a(t)^2 \left[\dfrac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \right]}$

Note: We can always rescale r, a so that $k = +1,\ 0$ or -1

1) $k = 0$ **:** $d\ell^2 = dr^2 + r^2 d\Omega^2 = dx^2 + dy^2 + dz^2$

Flat metric on $\mathbb{R}^3$; $\quad k = 0$ models are called *flat*.

2) $k = +1$**:** $r = \sin\chi \quad \Rightarrow \quad d\ell^2 = d\chi^2 + \sin^2\chi d\Omega^2$

Metric on a 3-sphere ($w^2 + x^2 + y^2 + z^2 = r^2$ in $\mathbb{R}^4$); *closed* models

3) $k = -1$**:** $r = \sinh\psi \quad \Rightarrow \quad d\ell^2 = d\psi^2 + \sinh^2\psi d\Omega^2$

Metric of a "saddle"; *open* models

E.2 The Friedmann equations

<u>The matter fields</u>

Recall: Perfect fluids are isotropic by definition!

We therefore set $T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$

Comoving frame: $u^\mu = (1,\, 0,\, 0,\, 0) \quad \Rightarrow \quad T^\mu{}_\nu = \mathrm{diag}(-\rho,\, P,\, P,\, P)$

$$\Rightarrow \quad T = T^\mu{}_\mu = -\rho + 3P \quad \text{coordinate invariant!}$$

Conservation: $\nabla_\mu T^\mu{}_\nu = 0 \quad \Rightarrow \quad \ldots \Rightarrow \quad \dot\rho = -3\dfrac{\dot a}{a}(\rho + P)$

Equation of state: Cosmological matter typically has $P = w\rho\,, \quad w = \mathrm{const}$

$$\Rightarrow \quad \frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a} \quad \Rightarrow \quad \rho \propto a^{-3(1+w)}$$

(1) ***Dust***

$w = 0 \quad \Rightarrow \quad P = 0 \quad \Rightarrow \quad \rho \propto a^{-3}$

Pressure between galaxies $\approx$ 0.

(2) ***Radiation***

Statistical Physics: photons are a gas with $P = \rho/3$

$\Rightarrow \quad w = \dfrac{1}{3} \quad \Rightarrow \quad \rho \propto a^{-4}$

Fourth power of a comes from redshift (cf. below).

(3) ***Dark energy***

Recall Lovelock's theorem: We can add $\Lambda g_{\alpha\beta}$ to the Einstein equations.

Perfect fluid with $w = -1$: $8\pi T_{\mu\nu} = 8\pi P g_{\mu\nu} \stackrel{!}{=} -\Lambda g_{\mu\nu} \quad \Rightarrow \quad -P = \rho = \dfrac{\Lambda}{8\pi}$

$w = -1 \quad \Rightarrow \quad \rho \propto a^0$

Interpretation: Non-zero of the vacuum. Density = const, independent of volume

Do not confuse with *dark matter*!

<u>Def.:</u> $H := \frac{\dot a}{a}$ is the Hubble parameter.

$q := -\frac{a\ddot a}{\dot a^2}$ is the deceleration parameter.

$\rho_{\text{crit}} := \frac{3H^2}{8\pi}$ is the critical density; its significance will be revealed below.

$\Omega = \frac{\rho}{\rho_{\text{crit}}} = \frac{8\pi}{3H^2}\rho$ is the density parameter.

Note: These are time dependent variables. The "parameters" are their present day values.

<u>The Einstein equations</u>

Plug Robertson-Walker metric into $G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$

$$\Rightarrow \ldots \Rightarrow \quad \boxed{3\frac{\dot a^2 + k}{a^2} - \Lambda = 8\pi\rho} \quad \text{(I)}, \qquad \boxed{\frac{2a\ddot a + \dot a^2 + k}{a^2} - \Lambda = -8\pi P} \quad \text{(II)}$$

$$\boxed{\frac{\ddot a}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3}} \quad \text{(III)}$$

Note: Eq. (III) follows from (I), (II) but can be useful.

Differentiate (I) and combined with (I), (II)

$$\Rightarrow \ldots \Rightarrow \quad \dot\rho + 3\frac{\dot a}{a}(\rho + P) = 0 \qquad \Big| \cdot a^3$$

$$\Rightarrow \quad \frac{d}{dt}(a^3\rho) + P\frac{d}{dt}a^3 = 0$$

Volume $V \propto a^3$, energy $E = V\rho$, so this can be written as $dE + PdV = 0$

E.3 Cosmological redshift

Radial null curves: $ds^2 = -dt^2 + \frac{a^2}{1-kr^2}dr^2 = 0 \quad \Rightarrow \quad \frac{dt}{a(t)} = \pm\frac{dr}{\sqrt{1-kr^2}}$

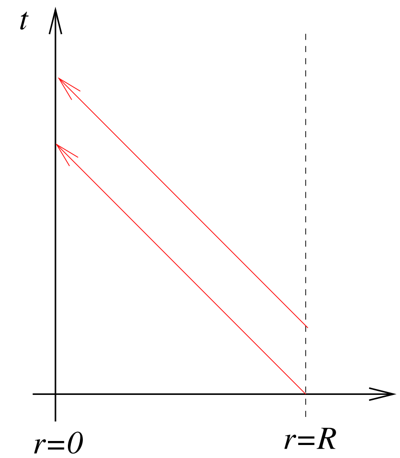
A galaxy at $r = R$ emits at times t_e and $t_e + \Delta t_e$.

These reach an observer at $r = 0$ at times t_o, $t_o + \Delta t_o$.

$$\Rightarrow \quad \int_{t_e}^{t_o} \frac{dt}{a} = -\int_R^0 \frac{dr}{\sqrt{1-kr^2}} = \int_{t_e+\Delta t_e}^{t_o+\Delta t_o} \frac{dt}{a}$$

('$-$' since ingoing geodesic)

$$\Rightarrow \quad \int_{t_o}^{t_o+\Delta t_o} \frac{dt}{a} = \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a}$$



E.g. crests of light wave: $\Delta t_e,\ \Delta t_o \ll t_o - t_e \quad \Rightarrow \quad a \approx \text{const}$

$$\Rightarrow \quad \frac{\Delta t_o}{a(t_o)} = \frac{\Delta t_e}{a(t_e)} \qquad \Rightarrow \qquad \boxed{\frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)}}$$

Nearby galaxies: Taylor expand $a(t_e) \approx a(t_o) - (t_o - t_e)\dot{a}(t_o)$

$$\Rightarrow \quad \frac{a(t_o)}{a(t_e)} \approx 1 + (t_o - t_e)\frac{\dot{a}(t_o)}{a(t_o)} = 1 + (t_o - t_e)H(t_o)$$

Hubble's law with distance $= c(t_o - t_e)$, $c = 1$.

Luminosity distance: Surface of constant radius, time: $ds^2 = a(t)^2 r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$

Area of sphere of constant r: $4\pi a^2 r^2$

Intensity of light collected at $r = 0$ from source at $r = R$:

$$I := \frac{\text{energy}}{\text{area}} = \frac{E}{4\pi a^2 R^2 (1+z)^2}$$

Factors $(1+z)$ from (i) redshift, (ii) reduced rate of photon hits.

Luminosity distance $\quad D_L^2 := \dfrac{E}{4\pi I (1+z)^2}$

E.4 Cosmological models

General considerations

(1) Eq. (I) $\Rightarrow$ $\dfrac{k}{\dot a^2} = \Omega - 1 + \dfrac{\Lambda}{3H^2}$

For $\Lambda = 0$: $\rho > \rho_{\text{crit}} \quad \Leftrightarrow \quad \Omega > 1 \quad \Leftrightarrow \quad k = +1$ "closed"

$\rho = \rho_{\text{crit}} \quad \Leftrightarrow \quad \Omega = 1 \quad \Leftrightarrow \quad k = 0$ "flat'

$\rho < \rho_{\text{crit}} \quad \Leftrightarrow \quad \Omega < 1 \quad \Leftrightarrow \quad k = -1$ "open"

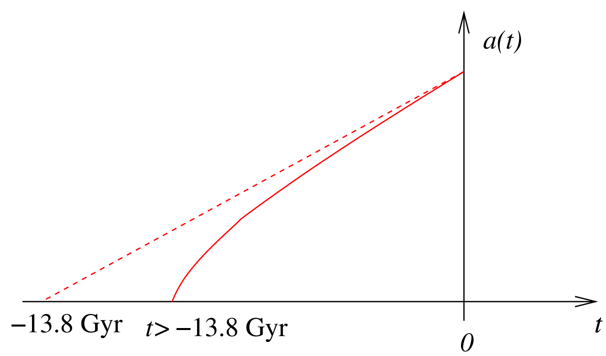
(2) Consider $\Lambda = 0$, $\rho > 0$, $P \geq 0$. Then: Eq. (III) $\Rightarrow$ $\ddot a < 0$.

Observations: $H_0 \approx 71$ km/(s Mpc) $\Rightarrow$ $\dot a > 0$

If $\ddot a = 0$, then $a = 0$ at $t = -13.8$ Gyr

"Big Bang"!

$\ddot a < 0 \quad \Rightarrow$ Universe is less old



(3) Again $\Lambda = 0$, $\rho > 0$, $P \geq 0$. Then Eq. (I) $\Rightarrow$ $\dot a^2 = \dfrac{8\pi}{3} a^2 \rho - k$.

For $k = 0, \ -1$: $\dot a^2 > 0$ always $\Rightarrow$ $\dot a > 0$ always (since $\dot a > 0$ today)

Recall: $\dfrac{d}{dt}\left(a^3 \rho\right) = -P \dfrac{d}{dt} a^3 = -3a^2 P \dot a \leq 0$

But: $\rho a^3 \geq 0$, so $\lim\limits_{t\to\infty} a^2 \rho = 0$

Then: Eq. (I) $\Rightarrow$ $\dot a^2 = a^2 H^2 = \dfrac{8\pi}{3} a^2 \rho - k \to k \quad \Rightarrow \quad \lim\limits_{t\to\infty} \dot a = |k|$

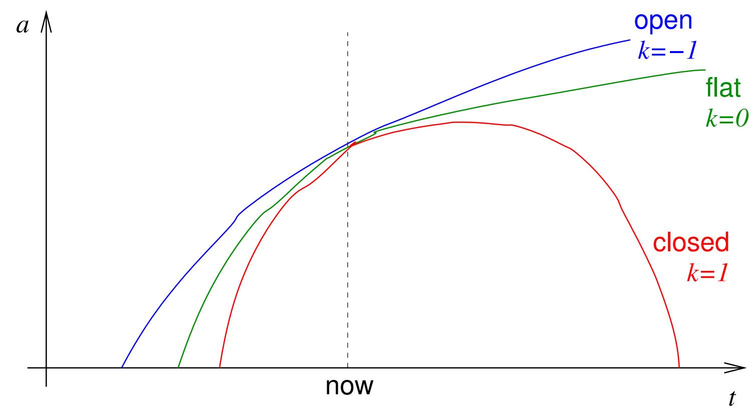
Expansion never stops.

For $k = +1$: Eq. (I) $\Rightarrow$ $\dot a^2 = \dfrac{8\pi}{3} a^2 \rho - 1$. As before $\lim\limits_{a\to\infty} a^2 \rho = 0$

$\Rightarrow$ $\dot a = 0$ at $a = a_{\max} = \sqrt{3/(8\pi\rho)}$.

Eq. (III) $\Rightarrow$ $\lim\limits_{a\to a_{\max}} \ddot a = -\dfrac{4\pi}{3}(\rho + P) a_{\max} < 0$

$\Rightarrow$ contraction back to $a = 0$: "Big Crunch"



Let's calculate solutions of the Friedmann Eqs.

1) Flat, matter dominated: $k = 0$, $P = 0$

Recall: $a^3\rho = \text{const}$ for $P = 0$.

Eq. (I) $\Rightarrow$ $8\pi a^3\rho = 3\left(a\dot{a}^2 + ka - \frac{1}{3}\Lambda a^3\right) \stackrel{!}{=} \text{const} =: 3C$ $\Rightarrow$ $\boxed{C = \frac{8\pi}{3}a^3\rho}$.

$$\Rightarrow \quad \dot{a}^2 = \frac{C}{a} + \frac{1}{3}\Lambda a^2 - k \qquad (\dagger)$$

$\Lambda > 0$: Set $k = 0$ in ($\dagger$) and use variable

$$u = \frac{2\Lambda}{3C}a^3 \qquad \Rightarrow \qquad \dot{u} = \frac{2\Lambda}{C}a^2\dot{a}$$

$$\Rightarrow \dot{u}^2 = \frac{4\Lambda^2}{C^2}a^4\left(\frac{C}{a} + \frac{\Lambda}{3}a^2\right) = \frac{4\Lambda^2}{C}a^3 + \frac{4\Lambda^3}{3C^2}a^6 = 6\Lambda u + 3\Lambda u^2$$

$$\Rightarrow \dot{u}^2 = 3\Lambda(2u + u^2)$$

$$\Rightarrow \dot{u} = \sqrt{3\Lambda}(2u + u^2)^{1/2}$$

Initial conditions: $a = u = 0$ at $t = 0$ $\quad\Rightarrow\quad$ $\displaystyle\int_0^u \frac{1}{\sqrt{2\tilde{u} + \tilde{u}^2}}d\tilde{u} = \sqrt{3\Lambda}\,t$

Use $u = -1 + \cosh w$

$$\Rightarrow \int_0^u \frac{d\tilde{u}}{\sqrt{\tilde{u}^2 + 2\tilde{u}}} = \int_0^u \frac{d\tilde{u}}{\sqrt{(\tilde{u}+1)^2 - 1}} = \int_0^w \frac{\sinh\tilde{w}\, d\tilde{w}}{\sqrt{\cosh^2\tilde{w} - 1}} = \int_0^w d\tilde{w} = w$$

$$\Rightarrow u + 1 = \cosh w = \cosh(\sqrt{3\Lambda}t)$$

$$\Rightarrow \frac{2\Lambda}{3C}a^3 = \cosh(\sqrt{3\Lambda}\,t) - 1 \qquad \Rightarrow \qquad \boxed{a^3 = \frac{3C}{2\Lambda}\left[\cosh(\sqrt{3\Lambda}\,t) - 1\right]}$$

<u>$\Lambda < 0$</u>: Use $u = -\dfrac{2\Lambda}{3C}a^3$

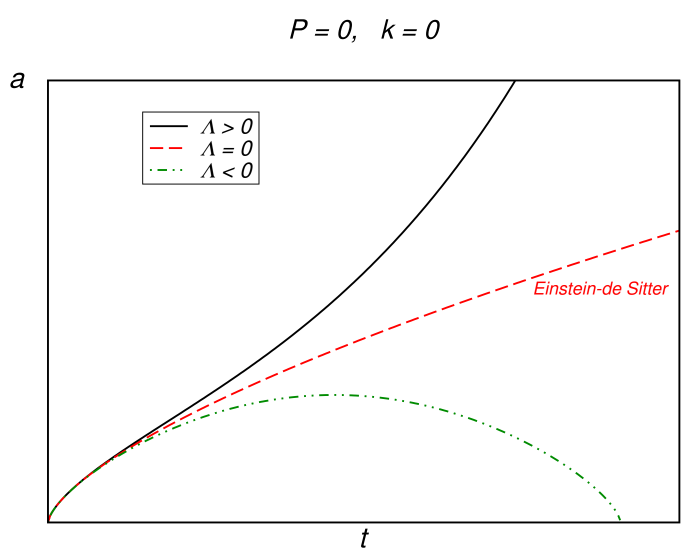
$$\Rightarrow \ldots \Rightarrow \quad \boxed{a^3 = \frac{3C}{2(-\Lambda)}\left[1 - \cos(\sqrt{-3\Lambda}\,t)\right]}$$

<u>$\Lambda = 0$</u>: Set $k = 0$ in (†) $\Rightarrow \quad \dot{a}^2 = \dfrac{C}{a} \qquad \Rightarrow \qquad \displaystyle\int \sqrt{a}da = \int \sqrt{C}dt$

$$\Rightarrow \frac{2}{3}a^{3/2} = \sqrt{C}t \qquad \Rightarrow \qquad \boxed{a^3 = \frac{9C}{4}t^2}$$

$$\Rightarrow H = \frac{\dot{a}}{a} = \frac{2}{3t}, \qquad q = -\frac{a\ddot{a}}{\dot{a}^2} = -\left(\frac{\dot{a}}{a}\right)^{-1}\left(\frac{\ddot{a}}{\dot{a}}\right) = \frac{1}{2}$$

"Einstein-de Sitter model"



2) Matter dominated, no cosmological constant: $\Lambda = 0$, $P = 0$

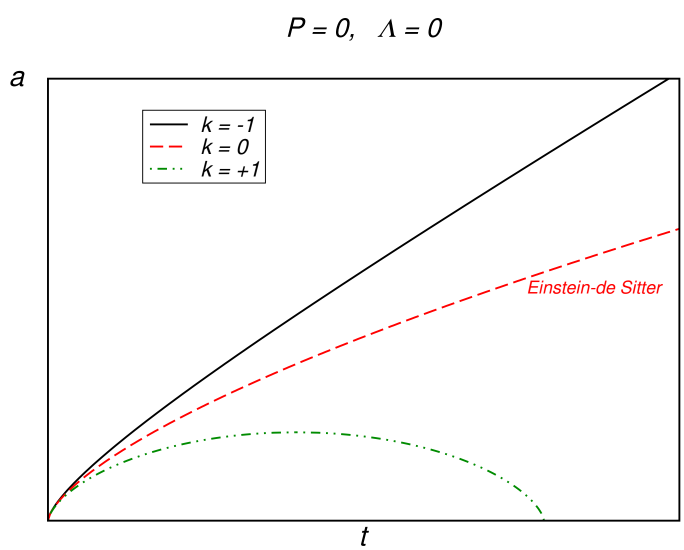
Eq. (†) on page 65 $\Rightarrow \quad \dot{a}^2 = \frac{C}{a} - k$

$\Rightarrow \ldots \Rightarrow$ For $k = +1$: $\boxed{C\left[\arcsin\sqrt{\frac{a}{C}} - \sqrt{\frac{a}{C}}\sqrt{1-\frac{a}{C}}\right] = \pm t + b_\pm}$, $\quad b_\pm = \text{const}$

For $k = -1$: $\boxed{C\left[\sqrt{\frac{a}{C}}\sqrt{1+\frac{a}{C}} - \operatorname{arsinh}\sqrt{\frac{a}{C}}\right] = \pm t + b_\pm}$

For $k = 0$: Einstein-de Sitter model from above.

WLoG $b_\pm = 0$. Also $t \geq 0$ for future oriented models.



3) The static Einstein Universe: $\dot{a} = \ddot{a} = 0$, $P = 0$

Eqs. (I), (II) $\Rightarrow \quad \frac{3k}{a^2} = \Lambda + 8\pi\rho\,, \qquad \frac{k}{a^2} = \Lambda \quad \Rightarrow \quad k = 4\pi a^2 \rho$

$\rho > 0 \quad \Rightarrow \quad k = +1$ necessarily for “sensible” matter.

Eq. (†) on page 65 $\Rightarrow \quad 3a = 3C + \Lambda a^3 = 3C + a$

$\Rightarrow \quad \boxed{a = \frac{3C}{2} \qquad \wedge \qquad \Lambda = \frac{4}{9C^2}}$

Note: Conceptually nice model: homogeneous in time as well!

Problem: unstable to perturbations! Say, $a = a_0 + \epsilon\,, \quad \epsilon \ll a_0$

$$\text{Eq. (III)} \;\Rightarrow a^2\ddot{a} = -\frac{C}{2} + \frac{4}{27C^2}a^3$$

$$\Rightarrow a_0^2\ddot{\epsilon} \approx \underbrace{-\frac{C}{2} + \frac{4}{27C^2}(a_0^3}_{=0} + 3a_0^2\epsilon + \ldots) = \frac{4}{9C^2}\,\frac{9}{4}C^2\epsilon = \epsilon$$

Has solutions $\exp(\pm\alpha t)$, i.e. also exponentially growing modes.

4) de Sitter Universe: $\rho = P = 0\,,\ \Lambda > 0$

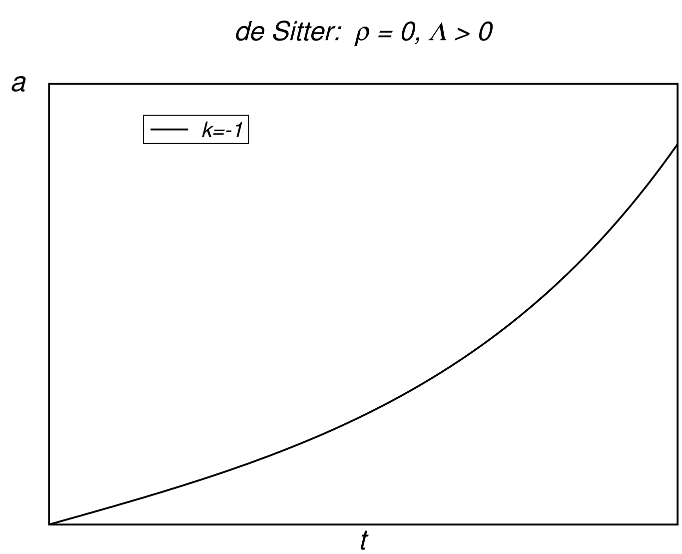
$$\text{Eq. (I)} \;\Rightarrow\; 3\frac{\dot{a}^2 + k}{a^2} = \Lambda$$

$$k = -1 \quad \Rightarrow \quad a(t) = \sqrt{\frac{3}{\Lambda}}\sinh\left(\sqrt{\frac{\Lambda}{3}}\,t\right)$$

$$k = 0 \quad \Rightarrow \quad a(t) \propto e^{\pm\sqrt{\Lambda/3}\,t}$$

$$k = +1 \quad \Rightarrow \quad a(t) = \sqrt{\frac{3}{\Lambda}}\cosh\left(\sqrt{\frac{\Lambda}{3}}\,t\right)$$

All three turn out to be the same spacetime, just in different coordinates.



5) $\Lambda = 0$, radiation dominated: $P = \rho/3$

From Sec. E.2: in general $\frac{d}{dt}(a^3\rho) + P\frac{d}{dt}a^3 = 0$

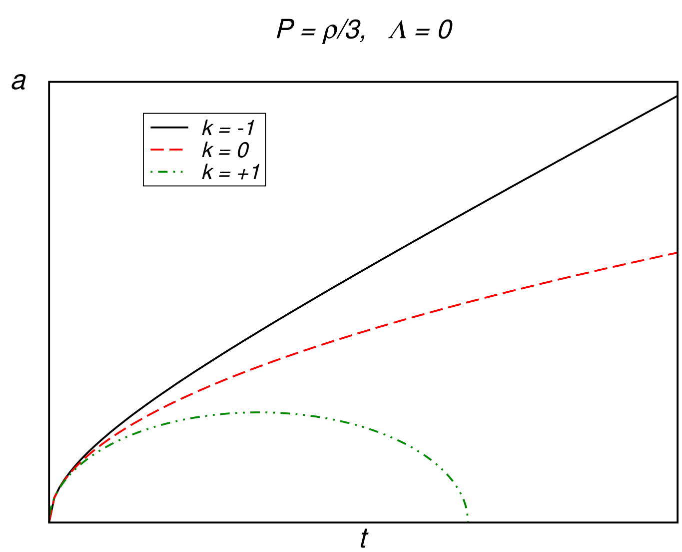
$$P = \rho/3 \quad \Rightarrow \quad 0 = \frac{d}{dt}(a^3\rho) + \rho a^2 \frac{da}{dt} = \ldots = \frac{1}{a}\frac{d}{dt}(a^4\rho)$$

$$\Rightarrow \; B := \frac{8\pi}{3}a^4\rho = \text{const}$$

We find $k = 0 \quad \Rightarrow \quad a = \sqrt{2}B^{1/4}\sqrt{t}$

$$k = +1 \quad \Rightarrow \quad a = \sqrt{B}\sqrt{1 - \left(1 - \frac{t}{\sqrt{B}}\right)^2}$$

$$k = -1 \quad \Rightarrow \quad a = \sqrt{B}\sqrt{\left(1 + \frac{t}{\sqrt{B}}\right)^2 - 1}$$



Summary:

- Radiation dominated: $\rho a^4 = \text{const}$

  Matter dominated: $\rho a^3 = \text{const}$

  Dark energy: $\rho \propto \Lambda = \text{const}$

- Radiation dominant in early universe, dark energy in late expanding universe.

- Presently: ~ 75 % dark energy, 4 % visible matter, ~ 21 % dark matter, radiation negligible

F Singularities and geodesic incompleteness

Metric components are sometimes 0 or ∞. What happens there?

F.1 Coordinate vs. physical singularities

Schwarzschild: $ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$

$r = 2M$: cured with Kruskal coordinates

Questions: 1) Can we predict when better coordinates exist?

2) Is there a "super-Kruskal" that cures $r = 0$?

Note: Scalars are invariant under coordinate transformations!

Def. : "Kretschmann scalar" $\kappa := R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$

In Schwarzschild: $\kappa = \ldots = \frac{48\ M^2}{r^6} \quad \Rightarrow \quad r = 0$ is a genuine, physical singularity.

Einstein-de Sitter Universe: $\kappa = \ldots = \frac{80}{27 t^4} \quad \Rightarrow \quad t = 0$ is a physical singularity.

F.2 Geodesic incompleteness

Consider geodesics in "Kasner V" spacetime: $ds^2 = -\frac{1}{z} dt^2 + z^2 (dx^2 + dy^2) + z dz^2\,, \quad z > 0$

Noether $c_0 = \frac{\dot{t}}{z}\,, \quad c_1 = z^2 \dot{x}\,, \quad c_2 = z^2 \dot{y}\,, \quad \dot{} = \frac{d}{d\lambda}\,, \quad \lambda =$ affine

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -\frac{1}{z} \dot{t}^2 + z^2 (\dot{x}^2 + \dot{y}^2) + z \dot{z}^2 = \epsilon$$

$c_i =$ const, $\epsilon = +1, -1, 0$ for space, timelike, null geodesic ($\lambda = \tau$ or s)

Geodesic Eq. $\Rightarrow \quad \ldots \quad \Rightarrow \quad \dot{z}^2 + \frac{c_1^2 + c_2^2}{z^3} - \frac{\epsilon}{z} = c_0^2$

Consider null geodesics with $\dot{x} = \dot{y} = 0\,, \quad \dot{z} < 0\,, \quad z = z_0\,, \quad t = 0$ at $\lambda = 0$

$\Rightarrow \quad x = y = \text{const}\,, \quad z = -c_0 \lambda + z_0$

Reaches $z = 0$ at finite λ.

$\kappa = \frac{12}{z^6} \quad \Rightarrow \quad$ singularity is physical.

Def.: A geodesic is incomplete if it cannot be extended to arbitrarily large values of its parameter, either to the future or the past. The termination point is a singularity.

Can this happen too at coordinate singularities? Yes! Example:

Rindler spacetime: $ds^2 = -z^2 dt^2 + dx^2 + dy^2 + dz^2\,, \quad t,\, x,\, y,\, z,\, \in \mathbb{R}\,,\; z > 0$

Noether: $c_0 = z^2 \dot{t}\,, \quad c_1 = \dot{x}\,, \quad c_2 = \dot{y}$ all const.

$$-z^2\dot{t}^2 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \epsilon = +1,\, -1 \text{ or } 0 \quad \text{as in Kasner V}$$

Consider timelike geodesics with $\dot{x} = \dot{y} = 0\,, \quad \dot{z} < 0\,, \quad z = z_0\,, \quad t = 0$ at $\tau = 0$

$$\Rightarrow \quad \ldots \quad \Rightarrow \quad z(\tau) = \sqrt{z_0^2 - \tau^2}\,, \qquad t(\tau) = \operatorname{artanh}\frac{\tau}{z_0}$$

geodesic hits singular $z = 0$ at finite τ

Now transform coordinates: $\quad x = X\,, \quad y = Y\,, \quad z = \sqrt{Z^2 - T^2}\,, \quad t = \operatorname{artanh}\dfrac{T}{Z}$

$$\Rightarrow \quad \ldots \quad \Rightarrow \quad ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2$$

Rindler spacetime is just a wedge of Minkowski spacetime!

Here the singular point $z = 0$ can be cured with better coordinates.

Is there a systematic way to find such better coordinates?

In 4 dimensions no, but in 2 dimensions (e.g. spherical symm.) there is: Sec. 6.4 in Wald.

G Linearized theory and gravitational waves

G.1 Plane waves and *pp* metrics

Plane waves are common in physics. E.g. electromagnetic waves $\vec{E},\ \vec{B} \propto e^{i(\vec{k}\vec{x}-\omega t)}$

Rotate coordinates such that $\vec{k} = (0,\, 0,\, k) \quad \Rightarrow \quad \vec{E},\ \vec{B} \propto e^{i(kz-\omega t)} = e^{ik(z-vt)}$

phase velocity: $v = \dfrac{\omega}{k}$

The components satisfy the wave equation $\Box f = -\partial_t^2 f + \vec{\nabla}^2 f = 0$

With $f \propto e^{i(\vec{k}\vec{x}-\omega t)} \quad \Rightarrow \quad \omega^2 - \vec{k}^2 = 0 \quad \Rightarrow \quad k = |\vec{k}| = \pm\omega \quad \Rightarrow \quad v = \dfrac{\omega}{k} = \pm 1 = \pm c$

SR notation: $f \propto e^{ik_\alpha x^\alpha}$ with $k_\alpha = (-\omega,\, \vec{k})$

$$\Box f = \eta^{\alpha\beta}\partial_\alpha\partial_\beta f = 0 \quad \Rightarrow \quad k_\alpha k^\alpha = 0$$

plane wave in $+z$ direction: $k_\alpha = (-\omega,\, 0,\, 0,\, k)$

Plane waves also exist in full GR!

<u>**Def.:**</u> Spacetimes admitting a covariantly constant vector field $\boldsymbol{V}$ are called *pp* wave spacetimes. They admit plane wave solutions.

Example: *Brinkmann metrics*: $ds^2 = H(u,x,y)du^2 + 2du\,dv + dx^2 + dy^2$

$\boldsymbol{V} := \partial_v$ satisfies $V^\beta = \partial_\alpha V^\beta + \Gamma^\beta_{\mu\alpha}V^\mu = 0 + \Gamma^\beta_{\mu\alpha}\delta^\mu{}_v = \Gamma^\beta_{v\alpha} = \ldots = 0$

Furthermore: $R_{uu} = 0 \quad \Rightarrow \quad \ldots \quad \Rightarrow \quad \partial_x^2 H + \partial_y^2 H = 0$

Clearly solved by $H = H_0\, e^{ik_\alpha x^\alpha}\,, \qquad H_0 = \text{const}\,, \qquad k_\alpha = (-\omega,\, 0,\, 0,\, \omega)$

pp metrics are important in the construction of analytic solutions in GR.

E.g. Aichelburg-Sexl metric: Schwarzschild BH boosted to the speed of light.

G.2 Linearized theory

Consider a system close to a known solution: Metric = *background* + *perturbation*

For us *background* is Minkowski, but idea works for general backgrounds: *perturbation theory*

$\Rightarrow \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1,\, 1,\, 1,\, 1)\,, \qquad h_{\mu\nu} = \mathcal{O}(\epsilon) \ll 1$

We regard $h_{\mu\nu}$ as a tensor field on Minkowski background inverse metric: $g^{\mu\nu} = \eta^{\mu\nu} + k^{\mu\nu}$

$$\Rightarrow g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho + k^{\mu\nu}\eta_{\nu\rho} + \eta^{\mu\nu} h_{\nu\rho} + \underbrace{k^{\mu\nu} h_{\nu\rho}}_{=\mathcal{O}(\epsilon^2)\to 0} \stackrel{!}{=} \delta^\mu{}_\rho$$

$$\Rightarrow k^{\mu\nu} = -\eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma} =: -h^{\mu\nu} = \mathcal{O}(\epsilon)$$

To $\mathcal{O}(\epsilon)$: $\Gamma^\mu_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(\partial_\rho h_{\sigma\nu} + \partial_\nu h_{\rho\sigma} - \partial_\sigma h_{\nu\rho})\,,$

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\tau}\big(\partial_\rho \Gamma^\tau_{\nu\sigma} - \partial_\sigma \Gamma^\tau_{\nu\rho}\big) \qquad \Big|\ \Gamma\cdot\Gamma = \mathcal{O}(\epsilon^2)$$

$$= \tfrac{1}{2}\big(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\mu\rho}\big)$$

$$R_{\mu\nu} = \partial^\rho\partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h \qquad \Big|\ h := h^\mu{}_\mu\,,\quad \partial^\mu := g^{\mu\rho}\partial_\rho$$

$$G_{\mu\nu} = \partial^\rho\partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}(\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial^\rho\partial_\rho h) \stackrel{!}{=} 8\pi T_{\mu\nu}$$

$$\Rightarrow T_{\mu\nu} \ll 1$$

<u>Def.:</u> "trace-reversed perturbation" $\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} h\eta_{\mu\nu} \quad \Leftrightarrow \quad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}\,,$

$$\bar{h} = \bar{h}^\mu{}_\mu = -h$$

$$\Rightarrow \ldots \Rightarrow G_{\mu\nu} = -\frac{1}{2}\partial^\rho\partial_\rho \bar{h}_{\mu\nu} + \partial^\rho\partial_{(\mu}\bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} = 8\pi T_{\mu\nu}$$

<u>Coordinate freedom</u>

$$\tilde{x}^\alpha = x^\alpha - \xi^\alpha\,,\quad \xi^\alpha = \mathcal{O}(\epsilon) \quad \Rightarrow \quad \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} = \delta^\alpha{}_\mu - \partial_\mu \xi^\alpha\,,\quad \frac{\partial x^\nu}{\partial \tilde{x}^\beta} = \delta^\nu{}_\beta + \tilde{\partial}_\beta \xi^\nu$$

$$\Rightarrow \quad \tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}\,,\quad \boxed{\tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \mathcal{O}(\epsilon^2)}$$

Note:
- Background unchanged
- 4 free functions: We choose ξ_μ such that $\partial^\nu\partial_\nu\xi_\mu = -\partial^\nu\bar{h}_{\mu\nu}$

$$\Rightarrow \quad \ldots \quad \Rightarrow \quad \partial^\nu \bar{\tilde{h}}_{\mu\nu} = 0 \quad \text{“Lorentz gauge”}$$

$$\Rightarrow \quad -2\tilde{G}_{\mu\nu} = \boxed{\Box\bar{\tilde{h}}_{\mu\mu} = \partial^\rho\partial_\rho\bar{\tilde{h}}_{\mu\nu} = -16\pi T_{\mu\nu}}$$

Wave equation! drop ˜ from now on.

G.3 The Newtonian limit

Newtonian gravity: $\vec{\nabla}^2\Phi = 4\pi\rho\,;\quad \Phi \sim v^2 \ll 1$ (Sec. A.1) $,\quad \epsilon := \dfrac{v^2}{c^2} = v^2$

$\Rightarrow$ matter sources weak: $\rho \sim \mathcal{O}(\epsilon)$

Newtonian matter:
$$T_{00} = \rho + \mathcal{O}(\epsilon^2)$$
$$T_{0i} \sim T_{00}\, v_i \sim \mathcal{O}(\epsilon^{3/2})$$
$$T_{ij} \sim T_{00}\, v_i v_j \sim \mathcal{O}(\epsilon^2)$$

E.g. perfect fluid: $T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}\,,\quad P \sim \rho v^2 \approx 10^{-5}\,\rho$ in sun

In Newt. gravity temporal changes in Φ are caused by motion of sources

$$\Rightarrow \frac{\partial}{\partial t} \sim v\frac{\partial}{\partial x^i} = \mathcal{O}(\epsilon^{1/2})\frac{\partial}{\partial x^i}$$

$$\Rightarrow \Box \bar{h}_{\mu\nu} = \partial^\rho\partial_\rho \bar{h}_{\mu\nu} = \partial^i\partial_i \bar{h}_{\mu\nu} = \vec{\nabla}^2 \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

$$\Rightarrow \vec{\nabla}^2 \bar{h}_{00} = -16\pi T_{00} = -16\pi\rho + \mathcal{O}(\epsilon^2)\,,\quad \bar{h}_{0i} = \mathcal{O}(\epsilon^{3/2})\,,\quad \bar{h}_{ij} = \mathcal{O}(\epsilon^2)$$

Newton's law with $\bar{h}_{00} = -4\Phi$

$$\Rightarrow \bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = 4\Phi + \mathcal{O}(\epsilon^2) = -h$$

$$\Rightarrow h_{00} = \bar{h}_{00} - \frac{1}{2}\eta_{00}\bar{h} = -2\Phi\,,\quad h_{ij} = \bar{h}_{ij} - \frac{1}{2}\eta_{ij}\bar{h} = -2\Phi\delta_{ij}$$

or $\boxed{ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2)}$ cf. Sec. A.2

<u>Geodesics</u>

$$\mathcal{L} = (1+2\Phi)\dot{t}^2 - \delta_{ij}(1-2\Phi)\dot{x}^i\dot{x}^j \overset{!}{=} 1 \quad\Rightarrow\quad \dot{t}^2 = (1+2\Phi)^{-1}\big[1+\delta_{ij}\dot{x}^i\dot{x}^j + \mathcal{O}(\epsilon^2)\big]$$

$$\Rightarrow \dot{t} = 1 - \Phi + \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + \mathcal{O}(\epsilon^2)$$

EL for x^k: $$\frac{d}{d\tau}\frac{\mathcal{L}}{\partial\dot{x}^k} = \frac{d}{d\tau}\big[-2\delta_{jk}(1-2\Phi)\dot{x}^j\big] = \frac{\partial\mathcal{L}}{\partial x^k} = 2\partial_k\Phi\,\underbrace{(\dot{t}^2 + \delta_{ij}\dot{x}^i\dot{x}^j)}_{=1+\mathcal{O}(\epsilon)}$$

$$\Rightarrow \ddot{x}_k = \frac{d^2x_k}{d\tau^2} = \frac{d^2x_k}{dt^2} = -\partial_k\Phi \quad \text{to } \mathcal{O}(\epsilon)\,;\text{ test body in Newt. gravity}$$

G.4 Gravitational waves

weak field but now: vacuum; no longer "$\partial_t \ll \partial_x$"

$\Rightarrow \Box \bar{h}_{\mu\nu} = (\partial_t^2 - \vec{\nabla}^2)\bar{h}_{\mu\nu} = 0$

Plane wave solution: $\bar{h}_{\mu\nu} = H_{\mu\nu} e^{ik_\rho x^\rho}$; $H_{\mu\nu} = \text{const}$

(i) $\Box \bar{h}_{\mu\nu} = 0 \quad \Rightarrow \quad k_\mu k^\mu = 0 \quad \rightarrow$ speed of light

(ii) Lorentz gauge: $\partial^\nu \bar{h}_{\mu\nu} = 0 \quad \Rightarrow \quad k^\mu H_{\mu\nu} = 0$ "transverse"

E.g. wave in z-dir.: $k^\mu = \omega\,(1,\,0,\,0,\,1) \quad \Rightarrow \quad H_{\mu 0} + H_{\mu 3} = 0$

Remaining gauge freedom: take $\xi_\mu = X_\mu e^{ik_\rho x^\rho} \Rightarrow \partial^\nu \partial_\nu \xi_\mu = 0$

$\Rightarrow \ldots \Rightarrow \quad H_{\mu\nu} \rightarrow H_{\mu\nu} + i(k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu} k^\rho X_\rho)$

$\Rightarrow \ldots \Rightarrow \quad \exists X_\mu : \quad H_{0\mu} = 0\,, \quad H^\mu{}_\mu = 0$ "traceless"

In this gauge: 1) $h = 0 \Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu}$

2) plane wave in z-dir.: $H_{0\mu} = H_{3\mu} = H^\mu{}_\mu = 0$

$$\Rightarrow \quad H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Effect on particles

Consider particle at rest in background Lorentz frame: $u_0^\alpha = (1,\,0,\,0,\,0)$

geodesic eq.: $\displaystyle \frac{d}{d\tau} u^\alpha + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = \dot{u}^\alpha + \Gamma^\alpha_{00} = 0$

$$\Gamma^\alpha_{00} = \frac{1}{2}\eta^{\alpha\beta}(\partial_0 h_{\beta 0} + \partial_0 h_{0\beta} - \partial_\beta h_{00}) = 0 \quad \text{since } H_{0\mu} = 0$$

$\Rightarrow u^\alpha = (1,\,0,\,0,\,0)$ always

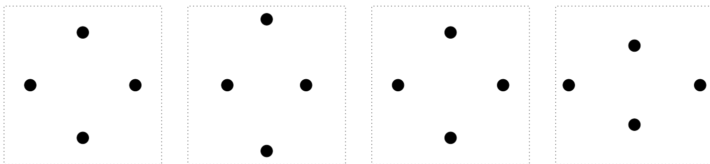
$\Rightarrow$ particle stays at $x^i = \text{const}$ in this gauge

Proper separation: $ds^2 = -dt^2 + (1+h_+)dx^2 + (1-h_+)dy^2 + 2h_\times dx\, dy + dz^2$

Case 1: $H_\times = 0\,,\ H_+ \neq 0 \quad \Rightarrow \quad h_+$ oscillates

2 particles at $(-\delta,\, 0,\, 0)$, $(\delta,\, 0,\, 0) \quad \Rightarrow \quad ds^2 = (1+h_+)\, 4\delta^2$

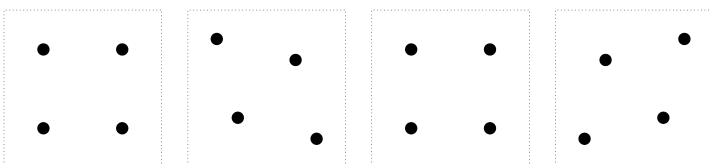
2 particles at $(0,\, -\delta,\, 0)$, $(0,\, \delta,\, 0) \quad \Rightarrow \quad ds^2 = (1-h_+)\, 4\delta^2$



Case 2: $H_+ = 0\,,\ H_\times \neq 0$

2 particles at $(-\delta,\, -\delta,\, 0)\,/\sqrt{2}\,,\quad (\delta,\, \delta,\, 0)\,/\sqrt{2} \quad \Rightarrow \quad ds^2 = (1+h_\times)\, 4\delta^2$

2 particles at $(\delta,\, -\delta,\, 0)\,/\sqrt{2}\,,\quad (-\delta,\, \delta,\, 0)\,/\sqrt{2} \quad \Rightarrow \quad ds^2 = (1-h_\times)\, 4\delta^2$



G.5 The quadrupole formula

Consider energy density $\rho(t, \vec{y})$ with compact support

Def.: Quadrupole tensor $I_{ij} := \int \rho(t,\vec{y}) y^i\, y^j\, d^3y$

Reduced quadrupole tensor $Q_{ij} := I_{ij} - \frac{1}{3} I_{kk}\, \delta_{ij}\,,$ sum over repeated indices!

Averaged energy flux in GWs: $\boxed{\langle p \rangle_t = \frac{G}{5c^5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle_{t-r}}$ sum over i, j

Emitted at $t-r$, observed at t, $\langle . \rangle$ = time average.

Example: Two equal point masses on Newtonian circular orbit

$$\rho(\vec{x}) = m\delta(\vec{x} - \vec{x}_1) + m\delta(\vec{x} - \vec{x}_2)\,,$$

$$x_1^i = r\,(\cos\phi,\, \sin\phi,\, 0)\,, \qquad x_2^i = -r\,(\cos\phi,\, \sin\phi,\, 0)$$

Newtonian orbit: $\omega = \dot{\phi} = \sqrt{G \frac{m}{4r^3}}$

$$\Rightarrow \; I_{xx} = 2mr^2 \cos^2 \omega t = mr^2(1 + \cos 2\omega t)$$

$$I_{yy} = 2mr^2 \sin^2 \omega t = 2mr^2(1 - \cos^2 \omega t) = mr^2(1 - \cos 2\omega t)$$

$$I_{xy} = I_{yx} = 2mr^2 \cos \omega t \, \sin \omega t = mr^2 \sin 2\omega t$$

$$Q_{ij} = I_{ij} - \frac{2}{3} mr^2 = I_{ij} - \text{const}$$

$$\Rightarrow \; \dddot{Q}_{xx} = 8\omega^3 mr^2 \sin 2\omega t$$

$$\dddot{Q}_{yy} = -8\omega^3 mr^2 \sin 2\omega t$$

$$\dddot{Q}_{xy} = \dddot{Q}_{yx} = -8\omega^3 mr^2 \cos 2\omega t$$

Adding all up: $\langle p \rangle_t = \frac{2}{5} \frac{G^4}{c^5} \frac{m^5}{r^5}$

Gravitational waves directly detected on 14 Sep 2015: New window to the universe!

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