Part III Gravitational Waves and Numerical Relativity

Lecture Notes

Abstract

The short lecture notes of the Part III course *Gravitational Waves and Numerical Relativity* contained in this document represent the material as displayed on the black board in the lectures. They are mainly provided as a concise summary of the course and for consumption together with the lecture course.

These notes assume that readers are already familiar with general relativity as lectured, for example, in DAMTP's Part II and Part III General Relativity lectures. Some knowledge of Part III Black Holes will be helpful, in particular a basic knowledge of the properties of the Kerr spacetime describing rotating black holes. We will also introduce some background material on the structure and properties of partial differential equations, but this should be consumable with a background in standard mathematical methods as lectured in Part IB without requiring exposure to specialized lectures dedicated to partial differential equations. There exists by now a healthy amount of text books and lecture style notes where the reader will find more in-depth discussion of some of our topics; knowledge of this literature is, however, not anticipated in our lecture.

- M. Alcubierre, *Introduction to 3+1 Numerical Relativity.* Oxford University Press (2008)
- T. W. Baumgarte, and S. L. Shapiro, *Numerical Relativity.* Cambridge University Press (2010)
- T. W. Baumgarte, and S. L. Shapiro, *Numerical Relativity – Starting from Scratch.* Cambridge University Press (2021)
- R. LeVeque, *Numerical methods for conservation laws.* Birkhäuser Verlag (1992)
- R, Courant, D. Hilbert, *Methods of Mathematical Physics II.* Wiley-Interscience, New York (1962).
- E. Gourgoulhon, *3+1 Formalism and Bases of Numerical Relativity*, Springer, New York (2012); see also `https://arxiv.org/abs/gr-qc/0703035`.
- M. Maggiore, *Gravitational Waves, Vol. 1: Theory and Experiments*, Oxford University Press (2007); *Gravitational Waves Vol. 2: Astrophysics and Cosmology*", Oxford University Press (2018).
- R. d'Inverno, "Introducing Einstein's Relativity", Clarendon Press, Oxford (1992).

Example sheets will be on Moodle and on

`http://www.damtp.cam.ac.uk/user/examples`

Lectures Webpage:

`http://www.damtp.cam.ac.uk/user/us248/Lectures/lectures.html`

Cambridge, Mar 18 2023, Ulrich Sperhake

Contents

A Introduction and conventions

A.1 Introduction and Motivation

Main goals:

- Understand how gravitational waves (GWs) arise in GR;

  This took the community almost **50 years**!!!

- How can we model their sources in full GR

  → Numerical Relativity (NR)

  → Formulation of the Einstein equations suitable for NR

  → Structure of the Einstein equations → Diagnostics for GW **observables**

A.2 Definitions and conventions

GR in 20 seconds

- $(\mathcal{M}, \boldsymbol{g}) =$ Lorentzian manifold with metric $g_{\alpha\beta}$
- Singature: $-+++ \quad = \quad +2$
- Greek indices $\alpha,\ \beta,\ \ldots = 0,\ 1,\ 2,\ 3$ "spacetime indices"
- Latin indices $i,\ j,\ \ldots = 1,\ 2,\ 3$ "spatial indices"
- Derived quantities:

  Levi-Civita connection: $\Gamma^{\mu}_{\beta\gamma} := \frac{1}{2} g^{\mu\rho} \left(\partial_\beta g_{\gamma\rho} + \partial_\gamma g_{\rho\beta} - \partial_\rho \gamma_{\beta\gamma}\right)\,,$

  Riemann tensor: $R^{\gamma}{}_{\rho\alpha\beta} := \partial_\alpha \Gamma^{\gamma}_{\rho\beta} - \partial_\beta \Gamma^{\gamma}_{\rho\alpha} + \Gamma^{\mu}_{\rho\beta}\Gamma^{\gamma}_{\mu\alpha} - \Gamma^{\mu}_{\rho\alpha}\Gamma^{\gamma}_{\mu\beta}\,,$

  Ricci tensor and scalar: $R_{\alpha\beta} := R^{\mu}{}_{\alpha\mu\beta}\,, \qquad R := R^{\mu}{}_{\mu}\,,$

  Einstein tensor: $G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R\,.$

- Properties of the curvature tensors:

  Bianchi identities: $R^{\mu}{}_{\nu[\rho\sigma;\lambda]} = \nabla_{[\lambda|} R^{\mu}{}_{\nu|\rho\sigma]} = 0\,.$

  Contracted Bianchi identities: $\nabla^{\mu} G_{\mu\alpha} = 0\,.$

- Einstein equations:

  With matter: $G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} = 8\pi T_{\alpha\beta}$ with $G = 1 = c\,,$

  Vacuum: $R_{\alpha\beta} = 0\,.$

B Linearized theory and GWs

B.1 The linearized Einstein equations

Consider small deviations from Minkowski in Cart. coords.

"Background": Manifold $\mathcal{M} = \mathbb{R}^4$, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$

"Perturbation": $h_{\mu\nu} = \mathcal{O}(\epsilon) \ll 1 \quad \Rightarrow \quad \boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}}$

regard $h_{\mu\nu}$ as a tensor field on Minkowski background

2 metrics: $\eta_{\mu\nu}$ and the "physical metric" $g_{\mu\nu}$.

inverse metric: $g^{\mu\nu} = \eta^{\mu\nu} + k^{\mu\nu}$

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho} + \mathcal{O}(\epsilon^2) \qquad \Rightarrow \qquad k^{\mu\nu} = -h^{\mu\nu} = -\eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}$$

To $\mathcal{O}(\epsilon)$: $\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(\partial_\rho h_{\sigma\nu} + \partial_\nu h_{\rho\sigma} - \partial_\sigma h_{\nu\rho})\,,$

$$R_{\mu\nu\rho\sigma} = \tfrac{1}{2}\big(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\mu\rho}\big)$$

$$R_{\mu\nu} = \partial^\rho\partial_{(\mu}h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h \qquad \Big| \quad h := h^{\mu}{}_{\mu}\,, \quad \partial^\mu := g^{\mu\rho}\partial_\rho$$

$$G_{\mu\nu} = \partial^\rho\partial_{(\mu}h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}(\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial^\rho\partial_\rho h) \stackrel{!}{=} 8\pi T_{\mu\nu}$$

$$\Rightarrow T_{\mu\nu} \ll 1$$

Def.: *Trace-reversed perturbation:* $\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} \quad \Leftrightarrow \quad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}\,,$

$\bar{h} = \bar{h}^{\mu}{}_{\mu} = -h$

$$\Rightarrow \ldots \Rightarrow G_{\mu\nu} = -\frac{1}{2}\partial^\rho\partial_\rho\bar{h}_{\mu\nu} + \partial^\rho\partial_{(\mu}\bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} = 8\pi T_{\mu\nu}$$

Gauge symmetry

New coordinates $\tilde{x}^\alpha = x^\alpha - \xi^\alpha \qquad \Leftrightarrow \qquad x^\alpha = \tilde{x}^\alpha + \xi^\alpha$

$$\Rightarrow \ldots \Rightarrow \boxed{h_{\mu\nu} \to \tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu}\,, \quad \xi_\mu = \mathcal{O}(\epsilon)$$

$$\Rightarrow \ldots \Rightarrow \quad \partial^\nu\bar{h}_{\mu\nu} \to \partial^\nu\tilde{\bar{h}}_{\mu\nu} = \partial^\nu\bar{h}_{\mu\nu} + \partial^\nu\partial_\nu\xi_\mu$$

Now choose ξ_μ such that $\partial^\nu\partial_\nu\xi_\mu = -\partial^\nu\bar{h}_{\mu\nu} \qquad \Rightarrow \qquad \tilde{G}_{\mu\nu} = -\frac{1}{2}\partial^\rho\partial_\rho\tilde{\bar{h}}_{\mu\nu}$

$\Rightarrow$ lin. Einstein eqs. (drop the tilde): $\boxed{\Box\bar{h}_{\mu\nu} = \partial^\rho\partial_\rho\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}}$ "Lorenz gauge"

B.2 Gravitational waves in the linear approximation

Linearized eqs. in vacuum: $\Box\bar{h}_{\mu\nu} = (-\partial_t^2 + \vec{\nabla}^2)\bar{h}_{\mu\nu} = 0$

Plane wave solution: $\bar{h}_{\mu\nu} = H_{\mu\nu}e^{ik_\rho x^\rho}$; $H_{\mu\nu} = \text{const}$

(1) $\Box\bar{h}_{\mu\nu} = 0 \quad\Rightarrow\quad k_\mu k^\mu = 0 \quad\to\quad$ speed of light

(2) Lorenz gauge: $\partial^\nu\bar{h}_{\mu\nu} = 0 \quad\Rightarrow\quad k^\mu H_{\mu\nu} = 0$ "transverse"

E.g. wave in z-dir.: $k^\mu = \omega\,(1,\,0,\,0,\,1) \quad\Rightarrow\quad H_{\mu 0} + H_{\mu 3} = 0$

Remaining gauge freedom: take $\xi_\mu = X_\mu e^{\mathrm{i}k_\rho x^\rho} \Rightarrow \partial^\nu\partial_\nu\xi_\mu = 0$

$\Rightarrow \ldots \Rightarrow \quad H_{\mu\nu} \to H_{\mu\nu} + \mathrm{i}(k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu}k^\rho X_\rho)$

$\Rightarrow \ldots \Rightarrow \quad \exists X_\mu : \quad H_{0\mu} = 0\,,\quad H^\mu{}_\mu = 0$ "traceless"

In this gauge: (1) $h = 0 \quad\Rightarrow\quad h_{\mu\nu} = \bar{h}_{\mu\nu}$

(2) plane wave in z-dir.: $H_{0\mu} = H_{3\mu} = H^\mu{}_\mu = 0$

$$\Rightarrow \quad H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad h_{+,\times} := H_{+,\times}e^{\mathrm{i}k_\rho x^\rho}$$

Effect on particles

Consider particle at rest in background Lorenz frame: $u_0^\alpha = (1,\,0,\,0,\,0)$

geodesic eq.: $\frac{\mathrm{d}}{\mathrm{d}\tau}u^\alpha + \Gamma^\alpha_{\mu\nu}u^\mu u^\nu = \dot{u}^\alpha + \Gamma^\alpha_{00} = 0$ (†)

$\Gamma^\alpha_{00} = \frac{1}{2}\eta^{\alpha\mu}(\partial_0 h_{\mu 0} + \partial_0 h_{0\mu} - \partial_\mu h_{00}) = 0$ since $H_{0\mu} = 0$

$\Rightarrow u^\alpha = (1,\,0,\,0,\,0)$ at all times solves (†)

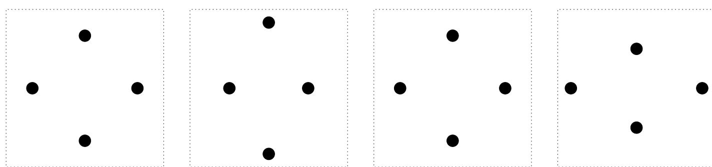
$\Rightarrow$ particle stays at $x^\mu = \text{const}$ in this gauge

Proper separation: $\mathrm{d}s^2 = -\mathrm{d}t^2 + (1+h_+)\mathrm{d}x^2 + (1-h_+)\mathrm{d}y^2 + 2h_\times \mathrm{d}x\,\mathrm{d}y + \mathrm{d}z^2$ (‡)

Case 1: $H_\times = 0\,,\ H_+ \neq 0 \quad \Rightarrow \quad h_+$ oscillates

2 particles at $(-\delta,\,0,\,0),\ (\delta,\,0,\,0) \quad \Rightarrow \quad \mathrm{d}s^2 = (1+h_+)\,4\delta^2$

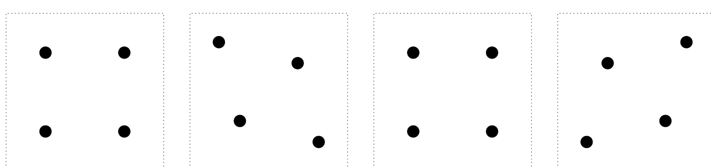
2 particles at $(0,\,-\delta,\,0),\ (0,\,\delta,\,0) \quad \Rightarrow \quad \mathrm{d}s^2 = (1-h_+)\,4\delta^2$



Case 2: $H_+ = 0\,,\ H_\times \neq 0$

2 particles at $(-\delta,\,-\delta,\,0)\,/\sqrt{2}\,, \quad (\delta,\,\delta,\,0)\,/\sqrt{2} \quad \Rightarrow \quad ds^2 = (1+h_\times)\,4\delta^2$

2 particles at $(\delta,\,-\delta,\,0)\,/\sqrt{2}\,, \quad (-\delta,\,\delta,\,0)\,/\sqrt{2} \quad \Rightarrow \quad ds^2 = (1-h_\times)\,4\delta^2$



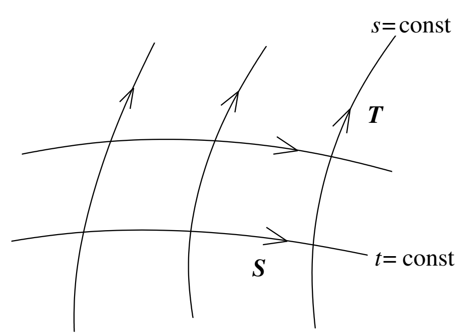
B.3 Geodesic deviation

Geodesic deviation: $\boldsymbol{T}$ along geodesic

$\boldsymbol{S}$ towards neighbouring geodesic

$$\nabla_{\boldsymbol{T}}\nabla_{\boldsymbol{T}}\boldsymbol{S} = \boldsymbol{R}(\boldsymbol{T},\boldsymbol{S})\boldsymbol{T}$$

$$\Leftrightarrow \quad T^\mu \nabla_\mu (T^\nu \nabla_\nu S^\alpha) = R^\alpha{}_{\mu\rho\sigma} T^\mu T^\rho S^\sigma$$



We need to choose a frame: use the local inertial frame where

$g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1,\ 1,\ 1,\ 1)$ and $\partial_\rho g_{\mu\nu} = 0 \quad \Rightarrow \quad \Gamma^\alpha_{\beta\gamma} = 0$

$R_{\alpha\beta\gamma\delta} = \mathcal{O}(\epsilon) \qquad \Rightarrow$ geodesic deviation small

$\Rightarrow \ \boldsymbol{T} = \boldsymbol{\partial}_t$ at background order for particles initially at rest.

$$\Rightarrow \quad \boxed{\partial_t^2 S^\alpha = R^\alpha{}_{00\sigma} S^\sigma}$$

Comment: • The components $R_{\alpha\beta\gamma\delta}$ are gauge invariant at $\mathcal{O}(\epsilon)$

$\Rightarrow$ We can compute them in TT gauge!

$$\Rightarrow \quad \boxed{R^j{}_{00k} = R_{j00k} = \tfrac{1}{2}\partial_0^2 h_{jk}} \qquad \text{and} \qquad R^0{}_{000} = R^0{}_{00k} = R^j{}_{000} = 0\,,$$

If $S^0 = \dot{S}^0 = 0$ initially, then $S^0 = 0$ always.

Also $h_{zk} = 0 \quad \Rightarrow \quad R^z{}_{00k} = 0 \quad \Rightarrow \quad S^z = \dot{S}^z = 0$ always if $S^z = 0$ initially.

$$\Rightarrow \quad \boxed{\begin{array}{rl} & \partial_t^2 S^x = \tfrac{1}{2}\left(\partial_t^2 h_+ S^x + \partial_t^2 h_\times S^y\right) \\ \wedge & \partial_t^2 S^y = \tfrac{1}{2}\left(\partial_t^2 h_\times S^x - \partial_t^2 h_+ S^y\right) \end{array}}$$

These are solved by

$$S^x = \mathrm{d}x + \tfrac{1}{2}h_+\mathrm{d}x + \tfrac{1}{2}h_\times\mathrm{d}y\,,$$

$$S^y = \mathrm{d}y + \tfrac{1}{2}h_\times\mathrm{d}x - \tfrac{1}{2}h_+\mathrm{d}y\,.$$

Recall that we are working in the local inertial frame, so with $S^0 = S^z = 0$

$$\begin{aligned} g_{\mu\nu}S^\mu S^\nu &= \eta_{\mu\nu}S^\mu S^\nu = (S^x)^2 + (S^y)^2 \qquad \Big| \ (1+\epsilon)^2 \approx 1 + 2\epsilon \\ &= \mathrm{d}x^2\left(1 + h_+ + h_\times\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \mathrm{d}y^2\left(1 + h_\times\frac{\mathrm{d}x}{\mathrm{d}y} - h_+\right) + \mathcal{O}(h_{+,\times}^2) \\ &= \mathrm{d}x^2(1+h_+) + \mathrm{d}y^2(1-h_+) + 2h_\times\mathrm{d}y\mathrm{d}x + \mathcal{O}(h_{+,\times}^2)\,, \end{aligned}$$

which at linear order is (‡) above with $\mathrm{d}z = 0 = \mathrm{d}t$.

Benefit: The gauge invariant Riemann tensor is easy to compare with the characteristic formalism below.

C Classification of Partial Differential Equations

Key difference between PDEs: propagation of information

C.1 Second-order PDEs of a single function

Order = order of highest derivatives

2nd-order PDES: Schrödinger equation, wave equation, Einstein equations, ...

We can trade order for number of variables, e.g.: $\partial_x^2 f = 0 \quad \Leftrightarrow \quad \partial_x f = g \;\wedge\; \partial_x g = 0\,.$

C.1.1 Classification of second-order PDES

Def.: Let $x_i \in \mathbb{R}^N$, $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}$. *General 2nd-order PDE*:

$$F(x_i, f, \partial_i f, \partial_i \partial_j f) = 0\,,$$

where F is a sufficiently regular function in its $(N+1)^2$ arguments.

A *linear 2nd-order PDE* is an equation of the form (sum over repeated indices)

$$A_{mn}(x_i)\partial_m \partial_n f + b_m(x_i)\partial_m f + c(x_i) f + d(x_i) = 0\,. \qquad (\star)$$

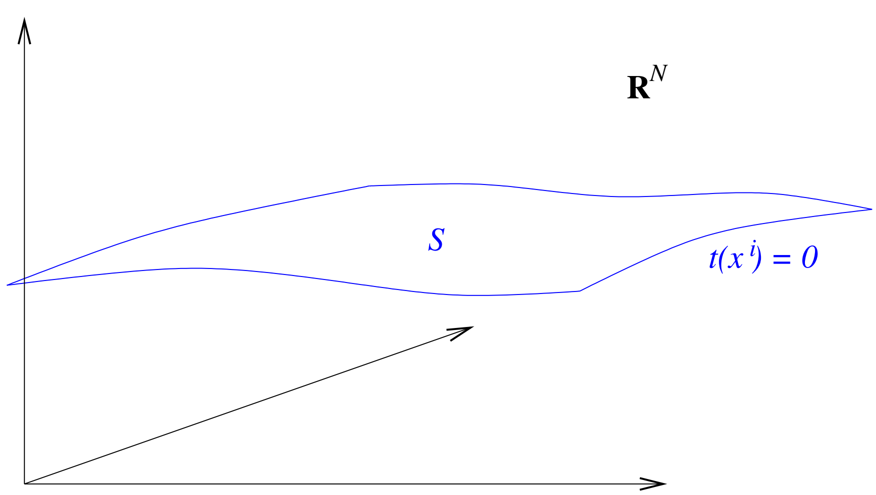
Without loss of generality: **A** symmetric: $A_{mn} = A_{nm}$.

Main or *principal part* of the PDE: the set of terms that contain the highest derivatives. For Eq. ($\star$) this is $A_{mn}\partial_m \partial_n f$.

Characteristic surfaces

Let $t(x^i)$ be a function with non-zero gradient, $\nabla t := (\partial_1 t, \ldots, \partial_N t) \neq 0$ everywhere.

Let S be the level surface $t(x_i) = 0$.



Suppose, f and $\partial_i f$ are specified on S.

Question: Are all derivatives of f determined on S?

Let's use new coordinates $\quad \xi_a = \xi_a(x_i) \quad$ for $a = 1, \ldots, N-1$,

$$\xi_N = t(x_i)\,.$$

These exist in a neighbourhood of S since $\nabla t \neq 0$. With $\partial_a := \dfrac{\partial}{\partial \xi_a}$ we get

$$\partial_i f = \frac{\partial f}{\partial x_i} = \frac{\partial \xi_a}{\partial x_i}\partial_a f\,, \qquad \partial_i\partial_j f = \frac{\partial \xi_a}{\partial x_i}\partial_a\left(\frac{\partial \xi_b}{\partial x_j}\partial_b f\right) = \frac{\partial^2 \xi_b}{\partial x_i\,\partial x_j}\partial_b f + \frac{\partial \xi^a}{\partial x^i}\frac{\partial \xi^b}{\partial x^j}\partial_a\partial_b f$$

$\Rightarrow$ Eq. ($\star$) becomes:
$$A^{mn}\frac{\partial \xi^a}{\partial x^m}\frac{\partial \xi_b}{\partial x_n}\partial_a\partial_b f + \text{lower order terms} = 0\,. \qquad (\dagger)$$

From the initial data we directly have

$$f(\xi_a) = f(\xi_1, \ldots, \xi_{N-1}, 0) = f\big(\xi_a(x_i)\big)\,,$$

$$\partial_a f(\xi_a) = \partial_a f(\xi_1, \ldots, \xi_{N-1}, 0) = \frac{\partial x_m}{\partial \xi_a}\left.\frac{\partial f}{\partial x_m}\right|_S\,.$$

For $b = 1, \ldots, N-1$ we also get the second derivatives via

$$\partial_b\partial_a f(\xi_1, \ldots, \xi_{N-1}, 0) = \lim_{h\to 0}\frac{\partial_a f(\xi_1, \ldots, \xi_b + h, \ldots, \xi_{N-1}, 0) - \partial_a f(\xi_1, \ldots, \xi_{N-1}, 0)}{h}\,.$$

For $b = N$, we substitute all known derivatives in ($\dagger$), so that

$$A_{mn}\frac{\partial \xi_N}{\partial x_m}\frac{\partial \xi_N}{\partial x_n}\frac{\partial^2 f}{\partial(\xi_N)^2} = \text{terms known on } S\,. \qquad (\text{sum over } m,\, n, \text{ but not } N)$$

We can thus calculate the missing derivative *if and only if* $\quad A_{mn}\dfrac{\partial \xi_N}{\partial x_m}\dfrac{\partial \xi_N}{\partial x_n} \neq 0\,.$

If this condition is satisfied, we can differentiate the PDE to compute all third derivatives and so on.

Def.: *Characteristic equation* associated with the PDE ($\star$):

$$\boxed{A_{mn}(x_i)\partial_m t\,\partial_n t = 0}\,, \qquad (\ddagger)$$

If $t(x_i)$ with $\nabla t \neq 0$ solves ($\ddagger$), the surface $t(x_i) = 0$ is a *characteristic surface*.

Example: If $A_{mn}(x_i)$ is positive or negative definite on Ω, then for any $t(x_i)$ with $\nabla t \neq 0$,

$$A_{mn}\partial_m t\,\partial_n t > 0 \qquad \text{or} \qquad A_{mn}\partial_m t\,\partial_n t < 0\,,$$

$$\Rightarrow\ A_{mn}\frac{\partial \xi_N}{\partial x_m}\frac{\partial \xi_N}{\partial x_n} \neq 0 \ \Rightarrow\ \text{the PDE has no characteristic surface.}$$

Def.: The PDE

$$A_{mn}(x_i)\partial_m f\,\partial_n f + b_m(x_i)\partial_m f + c(x_i)f + d(x_i) = 0\,,$$

is said to be of type (α, β, γ) at $x_i \in \Omega$ if α Eigenvalues of $A_{mn}(x_i)$ are positive, β Eigenvalues are negative and γ Eigenvalues are 0 with $\alpha + \beta + \gamma = N$.

The PDE is:

- *elliptic* if it is of type $(N,\,0,\,0)$ or $(0\,,N\,,0)$, i.e. if all Eigenvalues are non-zero and have the same sign.
- *parabolic* if it is of type $(N-1,\,0,\,1)$ or $(0,\,N-1,\,1)$, i.e. if one Eigenvalue is zero and all others are non-zero and have the same sign.
- *hyperbolic* if it is of type $(N-1,\,1,\,0)$ or $(1,\,N-1,\,0)$, i.e. all Eigenvalues are non-zero and exactly one of them has the opposite sign of all the others.

Comments:

- For parabolic and hyperbolic PDEs, we can always find non-vanishing linear combinations V_m of Eigenvectors such that $A_{mn}V_mV_n = 0$, so the PDE admits a characteristic surface.
- For elliptic PDEs, A_{mn} is positive or negative definite $\Rightarrow$ no characteristic surface.
- $\exists$ other types of PDEs; these are not relevant for us.
- The type of a PDE depends on x_i and may change.

Examples

(1) Tricomi equation: $y\partial_x^2 f + \partial_y^2 f = 0 \quad \Rightarrow \quad A^{mn} = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$.

A_{mn} has Eigenvalues y and 1

$\Rightarrow$ The Tricomi eq. is elliptic for $y > 0$ and hyperbolic for $y < 0$.

(2) Laplace eq. in 3 dims.: $\triangle f := \partial_x^2 f + \partial_y^2 f + \partial_z^2 f = 0$.

$\Rightarrow A_{mn} = \delta_{mn}$ and all 3 Eigenvalues are 1. Newsflash: the Laplace equation is elliptic.

(3) 3+1 wave equation: $\quad \Box f := -\partial_t^2 f + \partial_x^2 f + \partial_y^2 f + \partial_z^2 f = 0$.

Is hyperbolic everywhere with Eigenvalues -1, 1, 1, 1.

(4) 3+1 dmensional heat equation: $-\partial_t f + \partial_x^2 f + \partial_y^2 f + \partial_z^2 f = 0\,.$

Is parabolic, since one Eigenvalue is 0 and the others are 1.

C.1.2 Principal axes

Cf. long lecture notes.

C.1.3 Second-order PDEs in 2 dimensions

C.1.3.1 Classification of PDEs in 2 dimensions

Consider the PDE

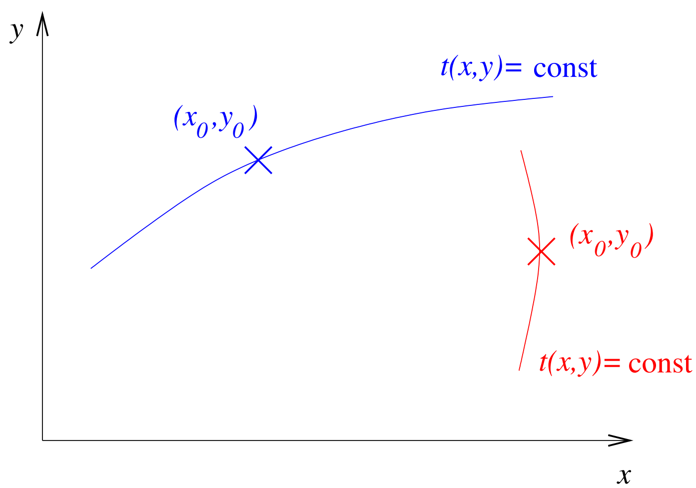
$a(x,y)\partial_x^2 f + 2b(x,y)\partial_x\partial_y f + c(x,y)\partial_y^2 f + \text{lower-order terms} = 0 \qquad \text{with } a \neq 0$

on a domain $\Omega \subset \mathbb{R}^2$.

Characteristic eq. : $\quad a(\partial_x t)^2 + 2b\partial_x t\,\partial_y t + c(\partial_y t)^2 = 0 \qquad (*)$

Let $t(x,y)$ be a solution of the char. eq.

Level sets $\big\{(x,y) \;\big|\; t(x,y) = \text{const}\big\}$ are curves.



Without loss of generality, we assume that $\partial_y t \neq 0$ at (x_0, y_0); otherwise swap x and y.

$\Rightarrow$ We can write the curve $t(x,y) = \text{const}$ as $y(x)$ in a neighbourhood of (x_0, y_0).

Parametrize $t(\lambda) = t\big(x(\lambda), y(\lambda)\big)$

$$\Rightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{\partial t}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}\lambda} + \frac{\partial t}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\lambda} = 0$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^{-1} = -\frac{\partial_x t}{\partial_y t}$$

$$\Rightarrow \quad \text{char. eq. } (*) \text{ becomes} \quad a\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2b\frac{\mathrm{d}y}{\mathrm{d}x} + c = 0$$

$$\Rightarrow \quad (y')^2 - \frac{2b}{a}y' + \frac{c}{a} = 0$$

$$\Rightarrow \quad y' = \frac{b}{a} \pm \sqrt{\frac{b^2}{a^2} - \frac{c}{a}} = \frac{1}{a}\left(b \pm \sqrt{b^2 - ac}\right)$$

$\Rightarrow$ (1) Characteristic surfaces exist for $b^2 \geq ac$

(2) No characteristic surgaces for $b^2 < ac$

Eigenvalue criterion

Let us compare this result with the classification of Sec. C.1.1

$$A_{mn} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Eigenvalues: $$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = ac - (a+c)\lambda + \lambda^2 - b^2 = 0$$

$$\Rightarrow \quad \lambda^2 - (a+c)\lambda - (b^2 - ac) = 0$$

$$\Rightarrow \quad \lambda_\pm = \frac{a+c}{2} \pm \sqrt{\frac{(a+c)^2}{4} + (b^2 - ac)} = \frac{a+c}{2}\left[1 \pm \sqrt{1 + 4\frac{b^2 - ac}{(a+c)^2}}\,\right] \overset{!}{\in} \mathbb{R}$$

$$\text{since} \quad 1 + 4\frac{b^2 - ac}{(a+c)^2} = \frac{1}{(a+c)^2}\left[(a+c)^2 + 4b^2 - 4ac\right] = \frac{(a-c)^2 + 4b^2}{(a+c)^2} \geq 0$$

$\Rightarrow$

1. elliptic if $b^2 < ac$; $\lambda_\pm$ have the same sign.
2. hyperbolic if $b^2 > ac$; $\lambda_\pm$ have opposite signs.
3. parabolic if $b^2 = ac$; in that case, $\lambda_- = 0$.

C.1.3.2 The normal form of hyperbolic PDEs in 2 dimensions

Consider a PDE $\boxed{a\partial_x^2 f + 2b\partial_x\partial_y f + c\partial_y^2 f + \text{l.o.t.} = 0}$ hyperbolic in a neighbourhood of (x_0, y_0).

Let $u(x,y)$, $v(x,y)$ be two independent solutions of the characteristic equation

$$\boxed{a(\partial_x t)^2 + 2b\partial_x t\,\partial_y t + c(\partial_y t)^2 = 0}\,.$$

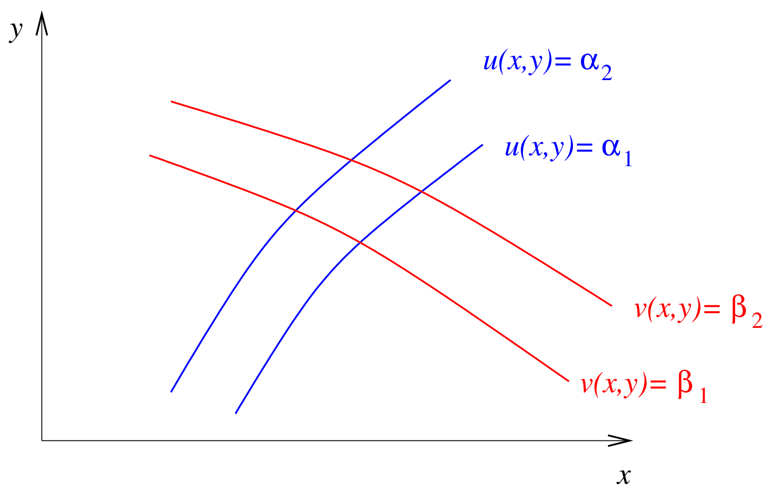
With $\nabla u \neq 0 \neq \nabla v$, we can always rotate the coordinates such that $\partial_y u \neq 0 \neq \partial_y v$.

$\Rightarrow$ We can write: Curves of constant u: $y = y_1(x)$

Curves of constant v: $y = y_2(x)$

with: $$y_1'(x) = -\frac{\partial_x u}{\partial_y u}\,, \qquad y_2'(x) = -\frac{\partial_x v}{\partial_y v} \qquad (\star)$$

$$\Rightarrow \quad \partial_x u + y_1'\partial_y u = 0 \qquad \wedge \qquad \partial_x v + y_2'\partial_y v = 0\,.$$



Lemma: (i) Curves from different families cannot touch, i.e. intersect each other *with* equal tangent direction.

(ii) The functions u and v obey the inequality

$$\partial_x u\,\partial_y v - \partial_y u\,\partial_x v \neq 0\,.$$

Proof. (i) From Page 13 the slope of characteristic curves is $y' = \frac{1}{a}\left(b \pm \sqrt{b^2 - ac}\right)$. y_1, y_2 belonging to different families have different signs, so at the point of intersection:

$y_2' - y_1' = \pm 2\frac{\sqrt{b^2-ac}}{a} \neq 0$ since $b^2 > ac$ for hyperbolic PDEs.

(ii) Plug ($\star$) into $y_2' - y_1' \neq 0$

$$-\frac{\partial_x v}{\partial_y v} + \frac{\partial_x u}{\partial_y u} \neq 0 \qquad \Rightarrow \qquad \partial_x u\, \partial_y v - \partial_x v\, \partial_y u \neq 0\,. \qquad \square$$

Def.: The solutions $u(x, y)$, $v(x, y)$ are called characteristic coordinates.

Proposition: In characteristic coordinates, the PDE $a\partial_x^2 f + 2b\partial_x\partial_y f + c\partial_y^2 f + \text{l.o.t.} = 0$ has the form

$$\boxed{\partial_u\partial_v f = \text{lower order terms}}\,.$$

Proof. (Only sketched)

Writing $f_x := \partial_x f,\ u_y := \partial_y u,\ f_u := \partial_u f$ etc. chain rule gives us

$$f_x = u_x f_u + v_x f_v\,, \quad f_y = u_y f_u + v_y f_v\,,$$

$$f_{xx} = u_x^2 f_{uu} + 2u_x v_x f_{uv} + v_x^2 f_{vv} + u_{xx} f_u + v_{xx} f_v$$

$$f_{xy} = u_{xy} f_u + v_{xy} f_v + u_y u_x f_{uu} + (u_y v_x + u_x v_y) f_{uv} + v_x v_y f_{vv}$$

$$f_{yy} = u_y^2 f_{uu} + 2u_y v_y f_{uv} + v_y^2 f_{vv} + u_{yy} f_u + v_{yy} f_v\,.$$

The principal part of the PDE then becomes $\alpha f_{uu} + 2\beta f_{uv} + \gamma f_{vv}$, where

$\alpha = \gamma = 0$ by the characteristic equation for u, v.

One also finds (`Mathematica!`): $\alpha\gamma - \beta^2 = (ac - b^2)(u_x v_y - u_y v_x)^2 \overset{!}{<} 0\,,$

since for a hyperbolic PDE $b^2 > ac$. So $\beta^2 > 0$ and $\beta \neq 0$. $\square$

Example:

$1D$ Wave equation with $x = r$, $y = t$: $\partial_t^2 f - \partial_r^2 f = 0\,.$

Characteristic eq. : $(\partial_t u)^2 - (\partial_r u)^2 = (\partial_t u + \partial_r u)(\partial_t u - \partial_r u) = 0$

$$\Rightarrow \partial_t u = -\partial_r u \qquad \vee \qquad \partial_t u = \partial_r u$$

$$\Rightarrow r_1'(t) = 1\,, \qquad r_2'(t) = -1$$

$$\Rightarrow u(t, r) = t - r\,, \qquad v(t, r) = t + r$$

$$\Rightarrow \text{Wave eq.: } \boxed{\partial_v\partial_u f = 0}\,.$$

Let f be a solution of the wave equation.

Clearly $\partial_u f$ is a function of u only, say $\partial_u f = \phi(u)$.

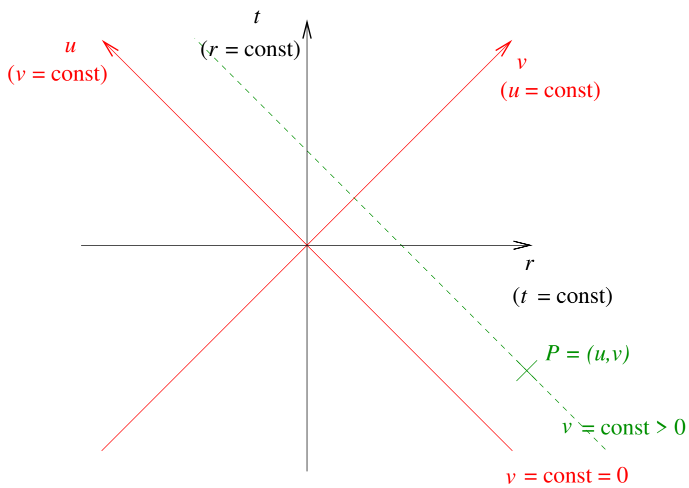
$$\Rightarrow \quad f(u,v) = \underbrace{\int \phi(u)\mathrm{d}u}_{=:F(u)} + G(v) = F(u) + G(v)\,.$$

Conversely, every function of this type satisfies the wave eq., so:

f solves $\partial_t^2 f - \partial_r^2 f = 0 \quad \Leftrightarrow \quad f(u,v) = F(t-r) + G(t+r)$ for some C^1 functions F, G.

We can also understand the deficiency of initial data on a characteristic surface.

Say, we specify initial data $f(u,0) = f_0(u)$ on the "surface" $v = 0$.



- This gives us $\partial_u f$ on the slice $v = 0$.
- The PDE predicts $\partial_u f$ for neighbouring $v \neq 0$, e.g. on the green slice.
- But we cannot reconstruct f at $v \neq 0$ since we do not know the integration constant $G(v)$.

This problem does not arise for initial data on non-characteristic surfaces like $t = 0$.

C.2 Systems of PDEs

C.f. example sheets.

D The structure of the Einstein equations

Three viewpoints for $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$

1. Given $T_{\alpha\beta}$, we look for $g_{\alpha\beta}$. This is done for the vacuum equations.
2. Specify $g_{\alpha\beta}$, compute $G_{\alpha\beta}$ which gives $T_{\alpha\beta}$. Rarely useful!
3. Regard the Einstein equations as 10 constraints on 20 functions (10 $g_{\alpha\beta}$ and 10 $T_{\alpha\beta}$).

Bianchi identies $\nabla_\mu G^{\mu\alpha} = 0 \quad \Rightarrow \quad \nabla_\mu T^{\mu\alpha} = 0$ "Energy momentum conservation"

Here we focus on the gravitational sector and set $T_{\alpha\beta} = 0 \quad \Rightarrow \quad \boxed{R_{\alpha\beta} = 0}$.

D.1 The Einstein equations in vacuum

We immediately see:

- The contracted Bianchi identities relate the $G_{\alpha\beta}$ ($= R_{\alpha\beta}$ in vacuum).

  $\Rightarrow$ We have too few equations to determine all 10 $g_{\alpha\beta}$.
- This is expected, since coordinate transformations change the $g_{\alpha\beta}$,

  $$\tilde{g}_{\tilde{\alpha}\tilde{\beta}} = \frac{\partial x^\mu}{\partial \tilde{x}^{\tilde{\alpha}}} \frac{\partial x^\nu}{\partial \tilde{x}^{\tilde{\beta}}} g_{\mu\nu}\,,$$

  without altering the spacetime.

  E.g. we can choose coordinates such that $g_{00} = -1$, $g_{0i} = 0$ and the 6 independent Einstein equations then determine the 6 components g_{ij}.

Note: The metric functions only need to be differentiable twice (e.g. shocks or surfaces).

D.2 The Cauchy problem

Cauchy problem := process of constructing a solution to a PDE given data on some boundary or initial hypersurface. In short: Start with a snapshot and evolve in "time".

In GR: Equations are tensorial; how do we get an evolution system?

<u>Def.:</u> Let $\mathcal{M}$ be a Lorentzian manifold $\mathcal{M}$ with metric $g_{\alpha\beta}$ of signature $+2$.

A *Cauchy surface* is a spacelike hypersurface Σ in $\mathcal{M}$ such that each timelike or null curve without endpoints intersects Σ exactly once.

The spacetime $(\mathcal{M}, \boldsymbol{g})$ is *globally hyperbolic* if it admits a Cauchy surface.

Proposition: Let Σ be a Cauchy surface of a globally hyperbolic spacetime $(\mathcal{M}, \boldsymbol{g})$. Then there exists a smooth function

$$t : \mathcal{M} \to \mathbb{R} \qquad \text{with} \qquad \mathbf{d}t \neq 0\,,$$

such that Σ is a level surface

$$\Sigma_{t_0} := \{p \in \mathcal{M} \quad : \quad t(p) = t_0\}\ ,$$

and two level surfaces Σ_{t_1} and Σ_{t_2} are either disjoint or equal,

$$\Sigma_{t_1} \cap \Sigma_{t_2} = \emptyset \qquad \Leftrightarrow \qquad t_1 \neq t_2\,.$$

Def.: If $(\mathcal{M}, \boldsymbol{g})$ is a globally hyperbolic spacetime and $\mathcal{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t$, then the union of the Σ_t is called a *foliation* of the spacetime.

From now on: Let $(\mathcal{M}, \boldsymbol{g})$ be a globally hyperbolic spacetime and $||\mathbf{d}t||^2 < 0$, i.e. the Σ_t are spacelike.

$\Rightarrow$ $\exists$ coordinates x^α with $x^0 = t$ "time" and x^i label points inside the Σ_t.

Question: Given $g_{\alpha\beta}$ and $\partial_\mu g_{\alpha\beta}$ on Σ_0, can we find all derivatives of the metric? Cf. Sec. C.1.1.

We answer this question for the vacuum equations $R_{\alpha\beta} = 0$.

(1) $R_{00} = R^\mu{}_{0\mu0} = \underbrace{R^0{}_{000}}_{=0} + R^m{}_{0m0} = \partial_m \Gamma^m_{00} - \partial_0 \Gamma^m_{0m} + \text{“}\Gamma \times \Gamma\text{”}$

$\Gamma^\alpha_{\beta\gamma}$ only involve first derivatives, so are known on Σ_0.

$\partial_m \Gamma^m_{00}$ involves only first time derivatives of the metric; it's also known on Σ_t. Now,

$$\partial_0 \Gamma^m_{0m} = \partial_0 \left[\frac{1}{2} g^{m\rho} \left(\partial_0 g_{m\rho} + \partial_m g_{\rho 0} - \partial_\rho g_{0m}\right)\right]$$

$$\Rightarrow \quad R_{00} = -\frac{1}{2} g^{m\rho} \partial_0^2 g_{m\rho} + \frac{1}{2} g^{m0} \partial_0^2 g_{0m} + M_{00} = -\frac{1}{2} g^{mn} \partial_0^2 g_{mn} + M_{00} \, ,$$

where M_{00} contains at most first time derivatives of $g_{\alpha\beta}$.

(2) We likewise find $R_{0i} = \frac{1}{2} g^{0m} \partial_0^2 g_{im} + M_{0i}$,

(3) and $R_{ij} = -\frac{1}{2} g^{00} \partial_0^2 g_{ij} + M_{ij}$. (†)

We note

(i) We have no terms $\partial_0^2 g_{0\alpha}$, so the Einstein equations do not determine $g_{0\alpha}$. Gauge!

(ii) We have 10 equations for 6 unknowns $\partial_0^2 g_{ij}$. Constraints!

Note that gauge and constraints are directly related!

Preliminary insight:

- If $g^{00} \neq 0$, Eq. (†) determines the missing 2nd derivatives $\partial_0^2 g_{ij}$.

  $\rightarrow$ time evolution of g_{ij}.
- If $g^{00} = 0$ everywhere, the surface is characteristic; cf. Sec. E.
- Using Eqs. (†), we get

  $$R = g^{\mu\nu} R_{\mu\nu} = \ldots = -g^{00} g^{mn} \partial_0^2 g_{mn} + g^{0m} g^{0n} \partial_0^2 g_{mn} + g^{00} M_{00} + 2 g^{0m} M_{0m} + g^{mn} M_{mn} \, ,$$

  $$\Rightarrow \quad G_0{}^0 = \ldots = \frac{1}{2} g^{00} M_{00} - \frac{1}{2} g^{mn} M_{mn}$$

  $$\wedge \quad G_i{}^0 = \ldots = g^{00} M_{0i} + g^{0m} M_{im}$$

  $\Rightarrow$ $G_\alpha{}^0$ contain no second time derivatives!

  $\rightarrow$ constraints.
- Summary: 6 evolution equations $R_{mn} = 0$ and 4 constraints $G_\alpha{}^0 = 0$.

Proposition: Let Σ be a Cauchy surface of a globally hyperbolic spacetime $(\mathcal{M}, \boldsymbol{g})$. If the constraints $G_\alpha{}^0 = 0$ are satisfied on Σ and the evolution equations $R_{mn} = 0$ are satisfied on $\mathcal{M}$, then the constraints are satisfied at all times by virtue of the Bianchi identities.

Proof. See long script. □

E The Bondi-Sachs formalism

- Initial data on characteristic surfaces does not determine solution in a neighbourhood.
- We can still do characteristic evolutions!
- Boundary conditions provide the missing information; e.g. CCM.

E.1 Characteristic coordinates

Characteristic surfaces of the Einstein eqs.:

surfaces Σ_u where $\boxed{g^{00} = \boldsymbol{g}(\mathbf{d}x^0, \mathbf{d}x^0) = ||\mathbf{d}x^0||^2 = 0}$ $\Rightarrow$ $x^0 =: u$ is a null coordinate.

Def.: $\boldsymbol{\ell} := \mathbf{d}x^0 = \mathbf{d}u$

$\Rightarrow \ell_\alpha = (\mathbf{d}x^0)_\alpha = \delta^0{}_\alpha$

$\Rightarrow$ (i) $\boldsymbol{\ell}$ is tangent to Σ_u: $\ell^\alpha(\mathbf{d}x^0)_\alpha = ||\mathbf{d}x^0||^2 = 0$

(ii) $\boldsymbol{\ell}$ is normal to Σ_u: $\mathbf{d}x^0$ is orthogonal to $x^0 = \text{const}$ by definition.

Proposition: The integral curves of ℓ^α are affinely parametrized null geodesics.

Proof. $\partial_\alpha \ell_\beta = \partial_\alpha\partial_\beta u = \partial_\beta \ell_\alpha$

$$\Rightarrow \ell^\mu\nabla_\mu\ell_\alpha = \ell^\mu\nabla_\mu\partial_\alpha u = \ell^\mu\partial_\mu\partial_\alpha u - \ell^\mu\Gamma^\rho_{\alpha\mu}\partial_\rho u = \ell^\mu\nabla_\alpha\ell_\mu = \frac{1}{2}\nabla_\alpha(\ell^\mu\ell_\mu) = 0\,,$$

since partial derivatives commute and $\Gamma^\alpha_{\beta\gamma}$ is torsion free. □

The integral curves of $\boldsymbol{\ell}$ are the curves of propagation of information, the *characteristic curves* of GR.

Def.: A spacetime $(\mathcal{M}, \boldsymbol{g})$ is *asymptotically flat*

:⇔ ∃ Cartesian coordinates such that $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$

with $\lim_{r\to\infty} h_{\alpha\beta} = \mathcal{O}(r^{-1})\,,\quad \lim_{r\to\infty} \partial_\mu h_{\alpha\beta} = \mathcal{O}(r^{-2})\,,\quad \lim_{r\to\infty} \partial_\nu\partial_\mu h_{\alpha\beta}\mathcal{O}(r^{-3})\,,$

where $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ and $r = \sqrt{x^2 + y^2 + z^2}$.

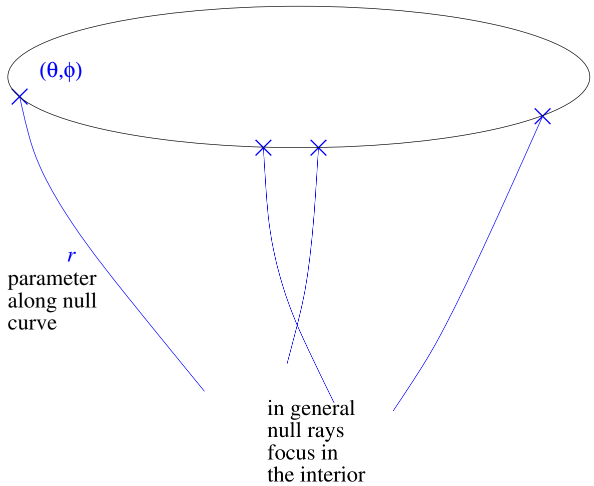
Construction of coordinates

At $r \to \infty$, we recover the light cone structure of special relativity.

(1) Consider 2-sphere $u = \text{const}$, $r \to \infty$. Label each point with standard θ, ϕ

(2) Integrate the null geodesic from this point inward, using r as a monotonic parameter.

(3) Each point has a unique $x^0 = u$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$



Comments

- The choice of $\boldsymbol{\ell}$ is not unique

  $\rightarrow$ *Bondi-Metzner-Sachs group* $\approx$ Lorentz group for *asymptotically* flat spacetimes.
- Once $\boldsymbol{\ell}$ is chosen, the congruence of null geodesics fills the characteristic surface without crossing in a neighbourhood of infinity.
- The coordinate system breaks down when the geodesics cross at sufficiently small r:

  $\rightarrow$ No unique θ, ϕ.

  But we only need a neighbourhood of infinity.

E.2 The Bondi metric

(1) Tangent vector along the null geodesics: $\dfrac{\mathrm{d}x^\alpha}{\mathrm{d}r} = \delta^\alpha{}_1$.

Null geodesics are integral curves of $\boldsymbol{\ell}$

$$\Rightarrow \quad \ell^\alpha = g^{\alpha\mu}\partial_\mu u = g^{\alpha\mu}\delta^0{}_\mu = g^{0\alpha} = \sigma\delta^\alpha{}_1 \quad \text{for} \quad \sigma \neq 0 \in \mathbb{R}$$

$$\Rightarrow \quad g^{00} = g^{02} = g^{03} = 0 \quad \Rightarrow \quad g^{\alpha\beta} = \begin{pmatrix} 0 & \sigma & 0 & 0 \\ \sigma & g^{11} & g^{12} & g^{13} \\ 0 & g^{21} & g^{22} & g^{23} \\ 0 & g^{31} & g^{32} & g^{33} \end{pmatrix}.$$

(2) Matrix inversion via co-factor matrix $C_{\mu\nu}$: $g_{\mu\nu} = \dfrac{C_{\nu\mu}}{\det g^{\alpha\beta}}$,

where $C_{\mu\nu} = (-1)^{\mu+\nu} \times$ determinant of $g^{\alpha\beta}$ with row μ and column ν struck out.

Example: $C^{12} = -\begin{vmatrix} 0 & \sigma & 0 \\ 0 & g^{21} & g^{23} \\ 0 & g^{31} & g^{33} \end{vmatrix} = 0 \quad \Rightarrow \quad g_{21} = g_{12} = 0\,.$

Likewise: $g_{11} = g_{13} = 0\,.$

(3) Fix r as *areal radius*, i.e. 2-spheres $u = \text{const}$, $r = \text{const}$ have proper area $A = 4\pi r^2$.

$$\Rightarrow \quad \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} = r^4 \sin^2\theta$$

(4) Now we simplify: axisymmetry & azimuthal reflection symmetry (no rotation!)

$\Rightarrow$ $\mathrm{d}s^2$ invariant under $\mathrm{d}\phi \to -\mathrm{d}\phi$ $\Rightarrow$ $g_{03} = g_{13} = g_{23} = 0\,.$

$\wedge$ $\exists$ coordinate ϕ such that $\partial_\phi g_{\alpha\beta} = 0\,.$

We thus get $$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & 0 \\ g_{01} & 0 & 0 & 0 \\ g_{02} & 0 & g_{22} & 0 \\ 0 & 0 & 0 & r^4\sin^2\theta/g_{22} \end{pmatrix}.$$

Bondi uses four variables β, γ, V, U: $$g_{\alpha\beta} = \begin{pmatrix} -\frac{V}{r}e^{2\beta} + r^2U^2e^{2\gamma} & -e^{2\beta} & -r^2Ue^{2\gamma} & 0 \\ -e^{2\beta} & 0 & 0 & 0 \\ -r^2Ue^{2\gamma} & 0 & r^2e^{2\gamma} & 0 \\ 0 & 0 & 0 & r^2e^{-2\gamma}\sin^2\theta \end{pmatrix} \tag{†}$$

$$\Rightarrow \quad g^{\alpha\beta} = \begin{pmatrix} 0 & -e^{-2\beta} & 0 & 0 \\ -e^{-2\beta} & \frac{V}{r}e^{-2\beta} & -Ue^{-2\beta} & 0 \\ 0 & -Ue^{-2\beta} & r^{-2}e^{-2\gamma} & 0 \\ 0 & 0 & 0 & r^{-2}e^{2\gamma}\sin^{-2}\theta \end{pmatrix}$$

E.3 The characteristic field equations

(1) We consider vacuum $\Rightarrow$ Field equations $R_{\alpha\beta} = 0$.

Plugging **(†)** in yields: $\boxed{R_{03} = R_{13} = R_{23} = 0}$.

(2) Next, assume we have somehow solved the *main equations* $R_{11} = R_{12} = R_{22} = R_{33} = 0$.

Write the Bianchi identies as:

<u>Lemma:</u> $$\nabla^\mu G_{\alpha\mu} = g^{\mu\rho}\left(\partial_\rho R_{\alpha\mu} - \Gamma^\sigma_{\mu\rho}R_{\alpha\sigma} - \frac{1}{2}\partial_\alpha R_{\mu\rho}\right) \tag{$\star$}$$

Proof. Exercise □

Using
$$\begin{aligned} g^{00} = g^{02} = g^{03} = 0\,, &\qquad R_{03} = 0\,, \\ g^{13} = 0\,, &\qquad R_{11} = R_{12} = R_{13} = 0\,, \\ g^{20} = g^{23} = 0\,, &\qquad R_{21} = R_{22} = R_{23} = 0\,, \\ g^{30} = g^{31} = g^{32} = 0\,, &\qquad R_{30} = R_{31} = R_{32} = R_{33} = 0\,, \end{aligned}$$

Eq. (⋆) gives us 0 for $\alpha = 3$ and

$$\text{for } \alpha = 1: \quad -g^{\mu\rho}\Gamma^0_{\mu\rho}R_{10} = 0\,,$$

$$\text{for } \alpha = 2: \quad g^{01}\partial_1 R_{20} - g^{\mu\rho}\Gamma^0_{\mu\rho}R_{20} - g^{01}\partial_2 R_{01} = 0\,,$$

$$\text{for } \alpha = 0: \quad g^{0\rho}\partial_\rho R_{00} + g^{1\rho}\partial_\rho R_{01} + g^{2\rho}\partial_\rho R_{02} - g^{\mu\rho}\Gamma^0_{\mu\rho}R_{00} - g^{\mu\rho}\Gamma^1_{\mu\rho}R_{01} - g^{\mu\rho}\Gamma^2_{\mu\rho}R_{02}$$
$$-\frac{1}{2}g^{01}\partial_0 R_{01} - \frac{1}{2}g^{10}\partial_0 R_{10} = 0\,.$$

Some crunching gives us: $g^{\mu\rho}\Gamma^0_{\mu\rho} = \dfrac{2}{re^{2\beta}} > 0$

We conclude: for $\alpha = 1$: $R_{10} = R_{01} = 0$

for $\alpha = 2$: $g^{01}\partial_1 R_{20} - g^{\mu\rho}\Gamma^0_{\mu\rho}R_{20} \;=\; -\,e^{-2\beta}\partial_r R_{20} - 2r^{-1}e^{-2\beta}R_{20} \;=\; 0$

$$\Rightarrow \quad -\,e^{-2\beta}\left(\partial_r R_{02} + \frac{2}{r}R_{02}\right) = 0$$

$$\Rightarrow \quad -\,e^{-2\beta}r^{-2}\partial_r(r^2 R_{02}) = 0$$

$$\Rightarrow \quad \boxed{R_{02} = f(u,\theta)r^{-2}}\,.$$

So if $f(u,\theta) = 0$ at some r, then $R_{02} = 0$ everywhere. Then ...

for $\alpha = 0$: $g^{01}\partial_1 R_{00} - g^{\mu\rho}\Gamma^0_{\mu\rho}R_{00} = 0$

$$\Rightarrow \quad \boxed{R_{00} = g(u,\theta)r^{-2}}$$

Again, if $g(u,\theta) = 0$ at some r, then $R_{00} = 0$ everywhere.

We call $R_{00} = 0$ and $R_{02} = 0$ the *supplementary equations.*

(3) This leaves us with the *main equations...*

$$R_{11} = -2(\partial_r\gamma)^2 + \frac{4}{r}\partial_r\beta = \frac{4}{r}\left[\partial_r\beta - \frac{1}{2}r(\partial_r\gamma)^2\right] = 0\,, \tag{E.1}$$

$$2r^2R_{12} = \partial_r\left[r^4e^{2(\gamma-\beta)}\partial_r U\right] - 2r^2\left[\partial_r\partial_\theta\beta - \partial_r\partial_\theta\gamma + 2\partial_r\gamma\,\partial_\theta\gamma - 2\cot\theta\,\partial_r\gamma - 2\frac{\partial_\theta\beta}{r}\right] = 0\,, \tag{E.2}$$

$$\begin{aligned} &e^{2(\beta-\gamma)}R_{22} + r^2e^{2\beta}{R^3}_3 \\ &\quad= -2\partial_r V - \frac{r^4}{2}e^{2(\gamma-\beta)}(\partial_r U)^2 + r^2\partial_r\partial_\theta U + r^2\cot\theta\,\partial_r U + 4r(\partial_\theta U + \cot\theta\, U) \\ &\qquad + 2e^{2(\beta-\gamma)}\left[1 + \cot\theta\,(3\partial_\theta\gamma - \partial_\theta\beta) + \partial_\theta^2\gamma - \partial_\theta^2\beta - (\partial_\theta\beta)^2 - 2(\partial_\theta\gamma)^2 + 2\partial_\theta\beta\,\partial_\theta\gamma\right] \\ &\quad= 0\,, \end{aligned} \tag{E.3}$$

$$\begin{aligned} -r^2e^{2\beta}{R^3}_3 &= e^{2(\beta-\gamma)}\left[-1 - \cot\theta\,(3\partial_\theta\gamma - 2\partial_\theta\beta) - \partial_\theta^2\gamma + 2\partial_\theta\gamma(\partial_\theta\gamma - \partial_\theta\beta)\right] + 2r\partial_r\partial_u(r\gamma) \\ &\quad + (1 - r\partial_r\gamma)\partial_r V - (r\partial_r^2\gamma + \partial_r\gamma)V - r(1 - r\partial_r\gamma)\partial_\theta U - r^2(\cot\theta - \partial_\theta\gamma)\partial_r U \\ &\quad + rU(2r\partial_\theta\partial_r\gamma + 2\partial_\theta\gamma + r\cot\theta\,\partial_r\gamma - 3\cot\theta) = 0\,. \end{aligned} \tag{E.4}$$

Note that there is only one time derivative in this mess!

(4) Ignoring constants of integration, we have the evolution scheme:

a) γ is given on a hypersurface Σ_u.
b) then Eq. (E.1) determines β on Σ_u
c) then Eq. (E.2) determines U on Σ_u
d) then Eq. (E.3) determines V on Σ_u
e) from Eq. (E.4) we can compute $\partial_u\gamma$ on Σ_u
f) with $\partial_u\gamma$ we can update γ to $\Sigma_{u+\mathrm{d}u}$.

Characteristic evolutions, if they work, are nice!

(5) Constants of integration:

- Eq. (E.1) needs one function of integration $H(u,\theta)$ to determine β.
- Eq. (E.2) needs one for the integration of $\partial_r[r^4e^{2(\gamma-\beta)}\partial_r U]$; we call it $-6N(u,\theta)$. We need a second, $L(u,\theta)$, for integrating U.
- Eq. (E.3) needs one for integrating V; we call it $-2M(u,\theta)$.
- Eq. (E.4) determines $\partial_u\gamma$ except for a function of integration we call $\partial_u c(u,\theta)$.

Interpretation

- Say, we know the system on a light cone $u = u_0$.
- If the system does anything new, this must be encoded in the functions of integration!
- We'll soon see there's only 1 independent function of integration: the *Bondi news function c*.
- Without axisymmetry, c is complex.

From now on assume:

(i) Asymptotic flatness $\Rightarrow \quad \gamma \propto r^{-1}$ as $r \to \infty$

(ii) No incoming radiation as $r \to \infty \quad \Rightarrow \quad \gamma = \dfrac{f(u,\theta)}{r} + \mathcal{O}(r^{-2}) \qquad$ "Sommerfeld condition"

Plug in for γ in Eq. (E.4): $\quad \partial_u(r\gamma) = \partial_u f(u,\theta) + \mathcal{O}(r^{-1})$

This is the function of integration we called $\partial_u c(u,\theta)$, so $\boxed{\gamma = \dfrac{c(u,\theta)}{r} + \mathcal{O}(r^{-2})}$

Plug this into Eq. (E.1)

$$\Rightarrow \quad \partial_r\beta - \frac{1}{2}r(\partial_r\gamma)^2 = 0$$

$$\Rightarrow \quad \partial_r\beta = \frac{1}{2}r\left[-\frac{c(u,\theta)}{r^2} + \mathcal{O}(r^{-3})\right]^2 = \frac{c(u,\theta)^2}{2r^3} + \mathcal{O}(r^{-4})$$

$$\Rightarrow \quad \beta = H(u,\theta) - \frac{c(u,\theta)^2}{4r^2} + \mathcal{O}(r^{-3})$$

Likewise, Eq. (E.3) and then (E.4) give
$$U = L + 2e^{2H}\partial_\theta H\, r^{-1} + \mathcal{O}(r^{-2})\,,$$
$$V = [L\cot\theta + \partial_\theta L]r^2 + \mathcal{O}(r)\,.$$

Proposition: The function of integration L vanishes, $L(u,\theta) = 0$.

Proof. By construction, $\boldsymbol{\partial}_u$ is a timelike vector. But

$\boldsymbol{g}(\boldsymbol{\partial}_u, \boldsymbol{\partial}_u) = g_{00} = -\frac{V}{r}e^{2\beta} + U^2r^2e^{2\gamma} = L^2r^2 + \mathcal{O}(r^1) > 0 \quad$ at large r unless $L = 0$. □

Proposition: We can choose the coordinates $(u,\ r,\ \theta,\ \phi)$ such that the form of the Bondi metric is preserved *and* $H(u,\theta) = 0$.

Proof. Long script. □

Note that we have used here the remaining freedom in choosing the vector $\boldsymbol{\ell}$ at infinity.

With $L = H = 0$, we can do the series expansion to higher order.

One quickly finds that the $\mathcal{O}(r^{-2})$ term in γ leads to $\ln r$ terms in U. We exclude that, so:

$$\begin{aligned}
\gamma(u,r,\theta) &= cr^{-1} + \left[e - \frac{1}{6}c^3\right] r^{-3} + \mathcal{O}(r^{-4}), \\
\beta(u,r,\theta) &= -\frac{1}{4}c^2 r^{-2} + \left[\frac{c^4}{8} - \frac{3}{4}ce\right] r^{-4} + \mathcal{O}(r^{-6}), \\
U(u,r,\theta) &= [-\partial_\theta c - 2c\cot\theta]\, r^{-2} + \left[2N + \frac{4}{3}c\partial_\theta c + \frac{8}{3}c^2\cot\theta\right] r^{-3} + \mathcal{O}(r^{-4}) \\
V(u,r,\theta) &= r - 2M + \left[\frac{37}{6}c\partial_\theta c\cot\theta - N\cot\theta + 4c^2\cot^2\theta + \frac{5}{6}c\partial_\theta^2 c - \partial_\theta N - \frac{c^2}{6} + \frac{11}{6}(\partial_\theta c)^2\right] r^{-1} \\
&\quad + \mathcal{O}(r^{-2}).
\end{aligned}$$

Supplementary equations

Recall $R_{00} = R_{02} = 0$. Their series expansion only had $\propto r^{-2}$ terms!

They are very lengthy (`Mathematica` or `Maple`!!!). At order r^{-2} they give:

$$\partial_u M = -\partial_u c + \frac{1}{2}\partial_\theta^2\partial_u c + \frac{3}{2}\cot\theta\partial_\theta\partial_u c - (\partial_u c)^2,$$

$$3\partial_u N = -\partial_\theta M - \frac{1}{2}c\partial_\theta\partial_u c + \frac{3}{2}\partial_\theta\partial_u c$$

$\Rightarrow$ c is the only independent function of integration!

Note: The 3rd derivative comes from comes from the 2nd derivative of U which already has a $\partial_\theta c$.

E.4 Interpretation of the functions of integration

Schwarzschild in outgoing Eddington Finkelstein coordinates:

$$ds^2 = -\left(1 - \frac{2M_S}{r}\right) du^2 - 2du\,dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

That's Bondi with $\gamma = \beta = U = 0$ and $V = r - 2M_S$.

Def.: M is called the *mass aspect* and we define the *Bondi mass* as

$$m(u) := \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} M\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{1}{2}\int_0^{\pi} M(u,\theta)\,\sin\theta\,\mathrm{d}\theta\,.$$

Lemma: For axisymmetric spacetimes with no conical singularity, $\lim_{\theta\to 0} c = \lim_{\to\pi} c = 0\,.$

Proof. Example sheets. □

Proposition: The time evolution of the Bondi mass is given by

$$\partial_u m = -\frac{1}{2}\int_0^{\pi}(\partial_u c)^2\sin\theta\mathrm{d}\theta \overset{!}{\leq} 0\,.$$

Proof. Example sheets. □

Note: the Bondi mass remains constant or decreases; $c \neq 0 \quad \rightarrow \quad$ GW emission.

Interlude: $2D$ wave equation

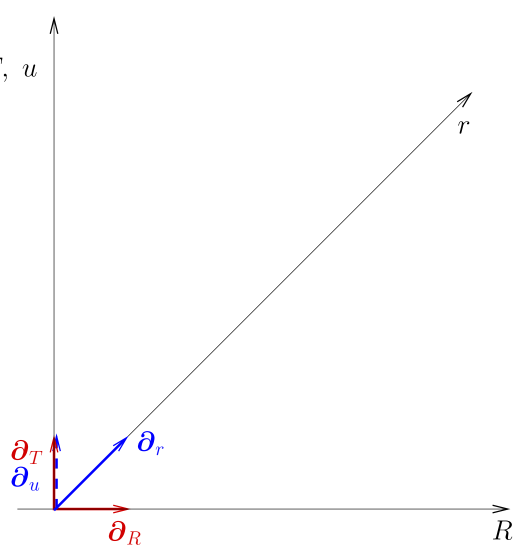
$\mathrm{d}s^2 = g_{\tilde{\alpha}\tilde{\beta}}\mathrm{d}x^{\tilde{\alpha}}x^{\tilde{\beta}} = -\mathrm{d}T^2 + \mathrm{d}R^2$

Clearly $\boldsymbol{g}(\boldsymbol{\partial}_T,\boldsymbol{\partial}_T) = -1\,,\quad \boldsymbol{g}(\boldsymbol{\partial}_R,\boldsymbol{\partial}_R) = 1.$

characteristic coordinates:

$$\begin{matrix} u = T - R \\ r = R \end{matrix} \quad\Leftrightarrow\quad \begin{matrix} T = u + r \\ R = r \end{matrix}\,.$$

$$\Rightarrow\quad \begin{matrix} \boldsymbol{\partial}_u = \boldsymbol{\partial}_T \\ \boldsymbol{\partial}_r = \boldsymbol{\partial}_T + \boldsymbol{\partial}_R \end{matrix} \quad\Leftrightarrow\quad \begin{matrix} \boldsymbol{\partial}_T = \boldsymbol{\partial}_u \\ \boldsymbol{\partial}_R = \boldsymbol{\partial}_r - \boldsymbol{\partial}_u \end{matrix}$$



Bondi news c and strain $h_+,\ h_\times$

(1) Take $r \to \infty$, so the GW becomes planar.

(2) Coordinates $(u, r, \theta, \phi) \quad \leftrightarrow \quad (T, x, y, z)$:

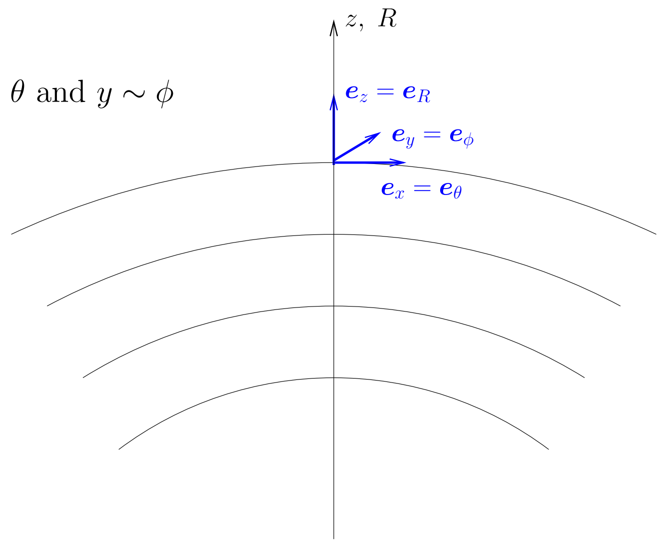
Rotate Cartesian axes such that z is radial, $x \sim \theta$ and $y \sim \phi$

$$\Rightarrow \quad \mathbf{e}_z = \mathbf{e}_R = -\boldsymbol{\partial}_u + \boldsymbol{\partial}_r\,,$$

$$\mathbf{e}_x = \mathbf{e}_\theta = \frac{1}{r}\boldsymbol{\partial}_\theta\,,$$

$$\mathbf{e}_y = \mathbf{e}_\phi = \frac{1}{r\sin\theta}\boldsymbol{\partial}_\phi\,,$$

$$\mathbf{e}_T = \mathbf{e}_u = \boldsymbol{\partial}_u\,,$$



(3) Riemann tensor of the Bondi metric in Cartesian coordinates:

12 non-zero components: $\quad R_{xTxT} = \boldsymbol{R}(\mathbf{e}_x, \mathbf{e}_T, \mathbf{e}_x, \mathbf{e}_T) = \frac{1}{r^2}R_{\theta u\theta u} \quad$ etc.

$$\Rightarrow \quad R_{xTxT} = -R_{yTyT} = -R_{xzxT} = R_{yzyT} = R_{xzxz} = -R_{yzyz} = -\partial_u^2 c r^{-1} + \mathcal{O}(r^{-2})\,,$$

$$R_{zTxT} = -R_{zTxz} = -R_{yxyT} = R_{yxyz} = -\left(\partial_\theta\partial_u c + 2\cot\theta\,\partial_u c\right) r^{-2} + \mathcal{O}(r^{-3})\,,$$

$$R_{z0z0} = -R_{xyxy} = -(2M + 2c\partial_u c)r^{-3} + \mathcal{O}(r^{-4})\,.$$

(4) Riemann tensor of the linearized formalism: $\quad R_{j00k} = \frac{1}{2}\partial_0^2 h_{jk}; \quad$ cf. Sec. B.3.

One also finds: $R_{jz0k} = -\partial_z\partial_0 h_{jk}, \quad R_{jzzk} = \partial_z^2 h_{jk}$.

For a planar wave in z dir.: $\quad \partial_z h_{\alpha\beta} = -\partial_0 h_{\alpha\beta}$ and $\partial_x h_{\alpha\beta} = \partial_y h_{\alpha\beta} = 0$.

$$\Rightarrow \quad R_{x0x0} = -R_{y0y0} = -R_{xzx0} = R_{yzy0} = R_{xzxz} = -R_{yzyz} = -\frac{1}{2}\partial_0^2 h_+\,,$$

$$R_{z0x0} = -R_{zTxz} = -R_{yxy0} = R_{yxyz} = 0\,,$$

$$R_{z0z0} = -R_{xyxy} = 0\,.$$

(5) They agree if $\boxed{h_+ = \dfrac{2c}{r}}$

No $h_\times$ in axial and reflection symmetry.

E.5 The characteristic formalism for general spacetimes

Sachs 1962.

Line element

$$\mathrm{d}s^2 = \frac{\tilde{V}e^{2\beta}}{r}\mathrm{d}u^2 - 2e^{2\beta}\mathrm{d}u\mathrm{d}r + r^2 h_{AB}\left(\mathrm{d}x^A\mathrm{d}x^B - \mathrm{d}x^A U^B \mathrm{d}u - \mathrm{d}x^B U^A \mathrm{d}u + U^A U^B \mathrm{d}u^2\right)\,,$$

where $A,\ B = 2,\ 3$ and $h_{AB}\mathrm{d}x^A\mathrm{d}x^B = \frac{e^{2\gamma}+e^{2\delta}}{2}\mathrm{d}\theta^2 + 2\sin\theta\sinh(\gamma-\delta)\mathrm{d}\theta\mathrm{d}\phi + \sin^2\theta\frac{e^{-2\gamma}+e^{-2\delta}}{2}\mathrm{d}\phi^2\,.$

We recover Bondi's axisymmetry for $\tilde{V} = -V,\ U^\theta = U,\ U^\phi = 0,\ \gamma = \delta.$

Null tetrad

$\mathbf{k},\ \boldsymbol{\ell},\ \mathbf{m}$ with $\boldsymbol{g}(\mathbf{k},\boldsymbol{\ell}) = 1,\quad \boldsymbol{g}(\mathbf{m},\overline{\mathbf{m}}) = 1;$ all other products vanish

$\Rightarrow\quad \boldsymbol{g} = \mathbf{k}\otimes\boldsymbol{\ell} + \boldsymbol{\ell}\otimes\mathbf{k} + \mathbf{m}\otimes\overline{\mathbf{m}} + \overline{\mathbf{m}}\otimes\mathbf{m}$

At $r\to\infty$:

$$\mathsf{k}^\alpha \simeq \left[-1,\ \tfrac{1}{2},\ 0,\ 0\right] \quad\Rightarrow\quad \mathbf{k} \simeq -\boldsymbol{\partial}_u + \frac{1}{2}\boldsymbol{\partial}_r = -\frac{1}{2}(\mathbf{e}_T - \mathbf{e}_R)$$

$$\ell^\alpha \simeq [0,\ 1,\ 0,\ 0] \quad\Rightarrow\quad \boldsymbol{\ell} \simeq \boldsymbol{\partial}_r = \mathbf{e}_T + \mathbf{e}_R$$

$$\mathsf{m}^\alpha \simeq \left[0,\ 0,\ \tfrac{i+\mathrm{i}}{2r},\ \tfrac{1-\mathrm{i}}{2r\sin\theta}\right] \quad\Rightarrow\quad \mathbf{m} \simeq \frac{\mathbf{e}_\theta + \mathbf{e}_\phi}{2} + \mathrm{i}\,\frac{\mathbf{e}_\theta - \mathbf{e}_\phi}{2}$$

Einstein equations

(i) 6 *main equations*

(a) 4 *hypersurface equations* $R_{\alpha\beta}\ell^\alpha\ell^\beta = R_{\alpha\beta}\ell^\alpha\mathsf{m}^\beta = R_{\alpha\beta}\mathsf{m}^\alpha\overline{\mathsf{m}}^\beta = 0$

(b) 2 *standard equations* $R_{\alpha\beta}\mathsf{m}^\alpha\mathsf{m}^\beta = 0$

(ii) 1 *trivial equation* $R_{\alpha\beta}\ell^\alpha\mathsf{k}^\beta = 0$

(iii) 3 *supplementary equations* $R_{\alpha\beta}\mathsf{k}^\alpha\mathsf{m}^\beta = R_{\alpha\beta}\mathsf{k}^\alpha\mathsf{k}^\beta$

If the main equations hold:

- The trivial equation holds.
- The supplementary eqs. hold *if* they hold at some r

The 2 standard eqs. contain time derivatives: $\partial_u\gamma$ and $\partial_u\delta$.

Evolution of the equations

(1) We have a complex Bondi news function c: $\boxed{\frac{1}{2}[(\delta+\mathrm{i}\gamma)(1-\mathrm{i})] = cr^{-1} + \mathcal{O}(r^{-2})}$

(2) Initial data at $u = u_0$ for γ, δ: 2 functions of (r, θ, ϕ)

Initial data for Integration constants N (complex) and M: 3 functions of (θ, ϕ)

Boundary data at $r = r_0$ for $\partial_u c$: 2 functions of (u, θ, ϕ)

(3) Integrate hypersurface equations along r to get β, U^A, $\tilde{V}$.

We need constants of integration N, M.

(4) Evolve γ and δ in time with the standard equations.

We need $\partial_u c$ as functions of integration.

(5) Evolve N, M at r_0 in time using the supplementary equations.

The source is given by the news c.

Series expansions

$$\beta = -\frac{c\bar{c}}{4} r^{-2} + \mathcal{O}(r^{-3}) \,,$$

$$U^\theta + \mathrm{i}U^\phi = -\left(\partial_\theta c + 2\cot\theta\, c - \frac{\mathrm{i}}{\sin\theta}\partial_\phi c\right) r^{-2} + \mathcal{O}(r^{-3}) \,,$$

$$\tilde{V} = -r + 2M + \mathcal{O}(r^{-1}) \,.$$

GW strain

The leading-order ($\propto r^{-1}$) components of the Riemann tensor are

$$\boxed{R_{\mu\nu\rho\sigma}\mathsf{k}^\mu \mathsf{m}^\nu \mathsf{k}^\rho \mathsf{m}^\sigma = -\frac{\mathrm{i}}{r}\partial_u^2 \bar{c} + \mathcal{O}(r^{-2})}$$

Using $\mathbf{e}_R = \mathbf{e}_z$, $\mathbf{e}_\theta = \mathbf{e}_x$, $\mathbf{e}_\phi = \mathbf{e}_y$ as before,

$$\Rightarrow \quad \lim_{r\to\infty} R_{\mu\nu\rho\sigma}\mathsf{k}^\mu \mathsf{m}^\nu \mathsf{k}^\rho \mathsf{m}^\sigma = R_{TxTy} + \mathrm{i}\,\frac{R_{TxTx} - R_{TyTy}}{2} \overset{!}{=} -\mathrm{i}\frac{\partial_u^2 \bar{c}}{r}$$

In the linearized regime we have:

$$R_{TxTx} = -\frac{1}{2}\partial_T^2 h_{xx} = -\frac{1}{2}\partial_T^2 h_+ \,, \qquad R_{TxTy} = -\frac{1}{2}\partial_T^2 h_{xy} = -\frac{1}{2}\partial_T^2 h_\times \,, \qquad R_{TyTy} = -\frac{1}{2}\partial_T^2 h_{yy} = +\frac{1}{2}\partial_T^2 h_+$$

They agree if $\boxed{h_+ = \frac{2}{r}\mathsf{Re}(c)\,, \qquad h_\times = \frac{2}{r}\mathsf{Im}(c)\,.}$

Bondi mass

$$m(u) \;:=\; \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} M(u,\theta,\phi)\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta$$

$$\Rightarrow \quad \partial_u m \;=\; -\frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} |\partial_u c|^2 \sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta \;=\; -\lim_{r\to\infty}\frac{r^2}{16\pi}\int_0^\pi \int_0^{2\pi} [(\partial_T h_+)^2 + (\partial_T h_\times)^2]\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,.$$

Say, a system evolves from one stationary state to another: $E_{\mathrm{GW}} = m_{\mathrm{ini}} - m_{\mathrm{fin}}$

F The ADM 3+1 formulation

F.1 Spacetime foliations, induced metric and extrinsic curvature

<u>Def.:</u> Let $\mathcal{M}$ be a manifold with metric $\boldsymbol{g}$ and Σ a hypersurface $t(x^\alpha) = \text{const}$.

$$\textit{Lapse function:}\quad \frac{1}{\alpha := \sqrt{\mp||\mathbf{d}t||^2}} = \begin{cases} \sqrt{-||\mathbf{d}t||^2}^{\,-1} & \text{if } \mathbf{d}t \text{ is timelike} \\ \sqrt{||\mathbf{d}t||^2}^{\,-1} & \text{if } \mathbf{d}t \text{ is spacelike} \end{cases}.$$

unit normal on Σ: $\boldsymbol{n} := \mp\alpha\mathbf{d}t \quad \Rightarrow \quad ||\boldsymbol{n}|| = \mp 1$

Projector: $\perp^\alpha{}_\beta := \delta^\alpha{}_\beta \pm n^\alpha n_\beta$

Acceleration: $a_\beta := n^\mu \nabla_\mu n_\beta$.

<u>Def.:</u> A vector $\boldsymbol{X}$ is *tangent to* Σ $:\Leftrightarrow \quad \langle \mathbf{d}t, \boldsymbol{X}\rangle = \langle \boldsymbol{n}, \boldsymbol{X}\rangle = 0$.

Projection of a tensor $\boldsymbol{T}$: $\perp T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} := \perp^\alpha{}_\mu \perp^\beta{}_\nu \ldots \perp^\rho{}_\gamma \perp^\sigma{}_\delta \ldots T^{\mu\nu\ldots}{}_{\rho\sigma\ldots}$

<u>Corollary</u>:

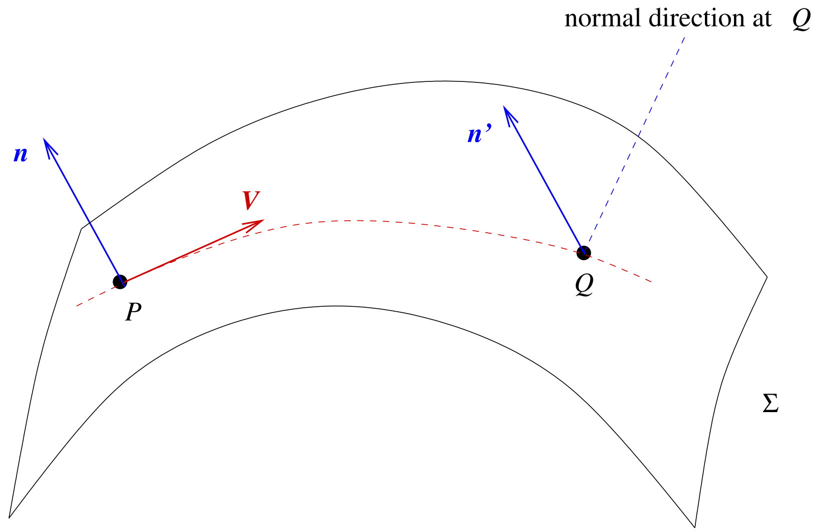
1. $\perp^\alpha{}_\mu n^\mu = n^\alpha \pm n^\alpha (n_\mu n^\mu) = 0$
2. $n^\mu a_\mu = n^\mu n^\rho \nabla_\rho n_\mu = \frac{1}{2} n^\rho \nabla_\rho (n^\mu n_\mu) = 0$
3. $\perp^\alpha{}_\mu \perp^\mu{}_\beta = \perp^\alpha{}_\mu (\delta^\mu{}_\beta \pm n^\mu n_\beta) = \perp^\alpha{}_\beta$
4. For any vector $\boldsymbol{V}$, $\perp V$ is tangent to Σ: $\perp^\alpha{}_\mu V^\mu n_\alpha = 0$

   If $\boldsymbol{V}$ is already tangent to Σ, then $\perp^\alpha{}_\mu V^\mu = (\delta^\alpha{}_\mu \pm n^\alpha n_\mu) V^\mu = V^\alpha$.
5. For any vectors $\boldsymbol{V}$, $\boldsymbol{W}$ tangent to Σ, $\quad g_{\alpha\beta} V^\alpha W^\beta = \perp_{\alpha\beta} V^\alpha W^\beta$.

<u>Def.</u>: *induced metric on* Σ: $\quad \gamma_{\alpha\beta} := \perp_{\alpha\beta} = g_{\alpha\beta} \pm n_\alpha n_\beta$

Let $\boldsymbol{V}$, $\boldsymbol{Y}$ be vector fields everywhere tangent to Σ.

Parallel transport $\boldsymbol{n}$ from P to Q along $\boldsymbol{V}$.



In general, $\boldsymbol{n}$ will not remain normal to Σ, since

$$V^\mu \nabla_\mu (Y^\alpha n_\alpha) = Y^\alpha \underbrace{V^\mu \nabla_\mu n_\alpha}_{=0} + n_\alpha V^\mu \nabla_\mu Y^\alpha$$

Def.: Let $\boldsymbol{n}$ be extended in a neghbourhood of Σ such that $n_\mu n^\mu = \mp 1$. Let $\boldsymbol{V}$, $\boldsymbol{W}$ be vector fields

Extrinsic curavture: $\boldsymbol{K} : (\boldsymbol{V}, \boldsymbol{W}) \mapsto \boldsymbol{n}\big(\nabla_{\perp \boldsymbol{V}}(\perp \boldsymbol{W})\big)$

$$\Leftrightarrow \quad K_{\mu\nu} V^\mu W^\nu := n_\nu \perp V^\mu \nabla_\mu (\perp W^\nu)\,.$$

Proposition: Independent of the expansion,

$$K_{\alpha\beta} = -\perp^\mu{}_\alpha \perp \nu_\beta \nabla_\mu n_\nu = -\perp^\mu{}_\alpha \nabla_\mu n_\beta = -\nabla_\alpha n_\beta \mp n_\alpha a_\beta$$

Proof. $K_{\mu\nu} V^\mu W^\nu = n_\nu \perp V^\mu \nabla_\mu (\perp W^\nu) = -\perp V^\mu \perp W^\nu \nabla_\mu n_\nu = -\perp^\mu{}_\alpha V^\alpha \perp^\nu{}_\beta W^\beta \nabla_\mu n_\nu\,.$

This holds for all $\boldsymbol{V}$, $\boldsymbol{W}$, so $K_{\alpha\beta} = -\perp^\mu{}_\alpha \perp^\nu{}_\beta \nabla_\mu n_\nu$

Also: $\perp^\nu{}_\beta \nabla_\mu n_\nu = \delta^\nu{}_\beta \nabla_\mu n_\nu \pm n_\beta \underbrace{n^\nu \nabla_\mu n_\nu}_{=0}$

For the independence on the extension, see long script. □

Proposition: $K_{\alpha\beta}$ is symmetric and tangent to Σ: $K_{\alpha\beta} = K_{\beta\alpha}\,, \quad K_{\alpha\beta} n^\alpha = 0 = K_{\alpha\beta} n^\beta$.

Its trace is $K := g^{\mu\nu} K_{\mu\nu} = \gamma^{\mu\nu} K_{\mu\nu}$.

Proof. $\nabla_\mu n_\nu = \mp \nabla_\mu (\alpha \mathbf{d} t_\nu) = \mp \alpha \nabla_\mu \nabla_\nu t + (\nabla_\mu \alpha) \frac{n_\nu}{\alpha}$

$$\Rightarrow \quad K_{\alpha\beta} = -\perp^\mu{}_\alpha \perp^\nu{}_\beta \nabla_\mu n_\nu = \pm \alpha \perp^\mu{}_\alpha \perp^\nu{}_\beta \nabla_\mu \nabla_\nu t + 0\,,$$

∇ is torsion free, so $\nabla_\mu \nabla_\nu t = \nabla_\nu \nabla_\mu t$ □

From now on: $\mathbf{d}t$ is timelike, so only upper sign!

F.2 Intrisic curavture

Intrisic curvature of Σ independent of embedding: E.g. geodesic deviation

$\rightarrow$ $3D$ Riemann tensor $\mathcal{R}^\alpha{}_{\beta\gamma\delta}$

Def.: Let $T^{\alpha\ldots}{}_{\beta\ldots}$ a rank $\binom{r}{s}$ tensor tangent to Σ in all components.

3D or *spatial* covariant derivative: $D_\mu T^{\alpha\ldots}{}_{\beta\ldots} := \perp^\rho{}_\mu \perp^\alpha{}_\sigma \perp^\tau{}_\beta \ldots \nabla_\rho T^{\sigma\ldots}{}_{\tau\ldots}$

For a vector $\boldsymbol{X}$ tangent to Σ: $\perp \boldsymbol{X} = \boldsymbol{X} \quad \Rightarrow \quad D_{\boldsymbol{X}} \boldsymbol{T} = \perp(\nabla_{\boldsymbol{X}} \boldsymbol{T})$

Proposition: The derivative D_μ is a covariant derivative for tensors tangent to Σ, it is torsion free and $D_\mu \gamma_{\alpha\beta} = 0$.

Proof. Sketched; cf. example sheets.

Let $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{V}$ be vectors tangent to Σ and f, g scalar functions. One shows that

(1) $X^\mu D_\mu f = X^\mu \partial_\mu f$,

(2) $D_{f\boldsymbol{X}+g\boldsymbol{Y}} \boldsymbol{V} = f D_{\boldsymbol{X}} \boldsymbol{V} + g D_{\boldsymbol{Y}} \boldsymbol{V}$,

(3) $D_{\boldsymbol{X}}(\boldsymbol{V} + \boldsymbol{W}) = D_{\boldsymbol{X}} \boldsymbol{V} + D_{\boldsymbol{X}} \boldsymbol{W}$

(4) $D_{\boldsymbol{X}}(f\boldsymbol{V}) = f D_{\boldsymbol{X}} \boldsymbol{V} + \boldsymbol{V} D_{\boldsymbol{X}} f$.

So D is a covariant derivative. Metric compatibility and torsion free nature are inherited from the $4D$ covariant derivative ∇_μ. □

Def.: For vectors $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{V}$ tangent to Σ, the $3D$ *Riemann tensor* is defined by

$$\boldsymbol{\mathcal{R}}(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{V} = D_{\boldsymbol{X}} D_{\boldsymbol{Y}} \boldsymbol{V} - D_{\boldsymbol{Y}} D_{\boldsymbol{X}} \boldsymbol{V} - D_{[\boldsymbol{X},\boldsymbol{Y}]} \boldsymbol{V}$$

$$\left(\boldsymbol{\mathcal{R}}(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{V}\right)^\alpha = \mathcal{R}^\alpha{}_{\beta\gamma\delta} V^\beta X^\gamma Y^\delta .$$

Ricci tensor, scalar: $\mathcal{R}_{\alpha\beta} := \mathcal{R}^\mu{}_{\alpha\mu\beta}\,, \qquad \mathcal{R} := \gamma^{\mu\nu} \mathcal{R}_{\mu\nu}$

Proposition: *Ricci identity* for a vector tangent to Σ: $(D_\gamma D_\delta - D_\delta D_\gamma) V^\alpha = \mathcal{R}^\alpha{}_{\mu\gamma\delta} V^\mu$

F.3 The Gauss, Codazzi and Ricci equations

Goal: Projections of the Riemann tensor onto $\boldsymbol{n}$, $\perp$.

Proposition: *Gauss equation:* $\boxed{\perp R^{\alpha}{}_{\beta\gamma\delta} = \mathcal{R}^{\alpha}{}_{\beta\gamma\delta} + K^{\alpha}{}_{\gamma}K_{\delta\beta} - K^{\alpha}{}_{\delta}K_{\gamma\beta}}$

Contracted Gauss: $\perp^{\mu}{}_{\alpha}\perp^{\nu}{}_{\beta}R_{\mu\nu}+\perp^{\mu}{}_{\alpha}\perp^{\rho}{}_{\beta}n^{\nu}n^{\sigma}R_{\mu\nu\rho\sigma} = \mathcal{R}_{\alpha\beta}+KK_{\alpha\beta}-K_{\alpha\mu}K^{\mu}{}_{\beta}$

Scalar Gauss: $R + 2n^{\mu}n^{\nu}R_{\mu\nu} = \mathcal{R} + K^2 - K^{\mu\nu}K_{\mu\nu}$

Proof. (Sketched)

Compute $\nabla_{\mu}\perp^{\sigma}{}_{\nu} = n_{\nu}\nabla_{\mu}n^{\sigma} + n^{\sigma}\nabla_{\mu}n_{\nu}$

For a vector field $\boldsymbol{V}$ tangent to Σ, show

$D_{\alpha}D_{\beta}V^{\gamma} = -K_{\alpha\beta}\perp^{\gamma}{}_{\lambda}n^{\sigma}\nabla_{\sigma}V^{\lambda} - K_{\alpha}{}^{\gamma}K_{\beta\lambda}V^{\lambda} + \perp^{\mu}{}_{\alpha}\perp^{\sigma}{}_{\beta}\perp^{\gamma}{}_{\lambda}\nabla_{\mu}\nabla_{\sigma}V^{\lambda}$

Use the 3D Ricci identity $D_{\alpha}D_{\beta}V^{\gamma} - D_{\beta}D_{\alpha}V^{\gamma} = \mathcal{R}^{\gamma}{}_{\rho\alpha\beta}V^{\rho}$

Use the $4D$ Ricci identity to replace $\nabla_{\alpha}\nabla_{\beta}V^{\gamma} - \nabla_{\beta}\nabla_{\alpha}V^{\gamma} = R^{\gamma}{}_{\rho\alpha\beta}V^{\rho}$

This holds for all spatial $\boldsymbol{V}$, so gives us $\perp R^{\alpha}{}_{\beta\gamma\delta}$.

Contract over α, γ, then with $\gamma^{\beta\delta}$. □

Proposition: *Codazzi Eq.:* $\boxed{\perp^{\rho}{}_{\alpha}\perp^{\sigma}{}_{\beta}\perp^{\gamma}{}_{\mu}n^{\nu}R^{\mu}{}_{\nu\rho\sigma} = D_{\beta}K_{\alpha}{}^{\gamma} - D_{\alpha}K_{\beta}{}^{\gamma}}$

Contracted Codazzi: $\perp^{\mu}{}_{\alpha}n^{\nu}R_{\mu\nu} = D_{\alpha}K - D_{\mu}K_{\alpha}{}^{\mu}$

Proof. (Sketched)

$4D$ Ricci identity applied to n^{μ} : $\perp(\nabla_{\alpha}\nabla\beta - \nabla_{\beta}\nabla_{\alpha})n^{\gamma} = \perp R^{\gamma}{}_{\mu\alpha\beta}n^{\mu}$

Recall $K_{\alpha\beta} = -\nabla_{\alpha}n_{\beta} - n_{\alpha}a_{\beta}$ to compute

$\perp^{\rho}{}_{\alpha}\perp^{\sigma}{}_{\beta}\perp^{\gamma}{}_{\tau}\nabla_{\rho}\nabla_{\sigma}n^{\tau} = -D_{\alpha}K_{\beta}{}^{\gamma} + a^{\gamma}K_{\alpha\beta}$

Antisymmetrize on α, β. Contract on γ, α. □

For the final projection we need Lie derivatives along $\boldsymbol{n}$:

Lemma: $\mathcal{L}_{\boldsymbol{n}}\perp^{\alpha\beta} = n^\alpha a^\beta + n^\beta a^\alpha + 2K^{\alpha\beta}\,,$

$\mathcal{L}_{\boldsymbol{n}}\perp^{\alpha}{}_{\beta} = n^\alpha a_\beta\,,$

$\mathcal{L}_{\boldsymbol{n}}\perp_{\alpha\beta} = -2K_{\alpha\beta}$

For any spatial tensor $T_{\alpha\beta} = \perp T_{\alpha\beta}$: $\quad \mathcal{L}_{\boldsymbol{n}} T_{\alpha\beta} = \perp(\mathcal{L}_{\boldsymbol{n}} T_{\alpha\beta})$

$a_\mu = D_\mu \ln\alpha$

Proof. Example sheets and:

$$\begin{aligned}
a_\beta &= n^\mu\nabla_\mu n_\beta = -n^\mu\nabla_\mu(\alpha\nabla_\beta t) = -\alpha n^\mu \underbrace{\nabla_\mu\nabla_\beta t}_{\nabla_\beta\nabla_\mu t} - n^\mu \underbrace{(\nabla_\beta t)}_{=-\frac{1}{\alpha}n_\beta}\nabla_\mu\alpha \\
&= \alpha n^\mu\nabla_\beta\frac{n_\mu}{\alpha} + \frac{n^\mu n_\beta}{\alpha}\nabla_\mu\alpha = \underbrace{n^\mu\nabla_\beta n_\mu}_{=0} - \alpha n^\mu n_\mu\frac{\nabla_\beta\alpha}{\alpha^2} + \frac{n^\mu n_\beta}{\alpha}\nabla_\mu\alpha \\
&= \frac{1}{\alpha}\big(\delta^\mu{}_\beta\nabla_\mu\alpha + n^\mu n_\beta\nabla_\mu\alpha\big) = \perp^\mu{}_\beta\frac{\nabla_\mu\alpha}{\alpha} = \frac{D_\beta\alpha}{\alpha} = D_\beta\ln\alpha\,.
\end{aligned}$$

□

Proposition: *Ricci Eq.:*

$$\boxed{\perp^\mu{}_\alpha n^\nu \perp^\rho{}_\gamma n^\sigma R_{\mu\nu\rho\sigma} = \mathcal{L}_{\boldsymbol{n}} K_{\alpha\gamma} + \frac{1}{\alpha}D_\alpha D_\gamma\alpha + K_{\rho\gamma}K_\alpha{}^\rho}$$

Proof. (Sketched)

4D Ricci identity applied to $\boldsymbol{n}$: $\quad \nabla_\rho\nabla_\sigma n^\mu - \nabla_\sigma\nabla_\rho n^\mu = R^\mu{}_{\nu\rho\sigma}n^\nu$

Project to get: $\quad \perp_{\alpha\mu} n^\nu \perp^\rho{}_\gamma n^\sigma R^\mu{}_{\nu\rho\sigma} = -K_{\alpha\sigma}K^\sigma{}_\gamma + D_\gamma a_\alpha + a_\alpha a_\gamma + \perp^\mu{}_\alpha\perp^\rho{}_\gamma n^\sigma\nabla_\sigma K_{\rho\mu}$

Show: $\quad \perp^\mu{}_\alpha\perp^\nu{}_\beta n^\sigma\nabla_\sigma K_{\mu\nu} = \mathcal{L}_{\boldsymbol{n}}K_{\alpha\beta} + K_{\rho\beta}K_\alpha{}^\rho + K_{\alpha\rho}K_\beta{}^\rho$

Show $\; D_\beta a_\alpha + a_\alpha a_\beta = \frac{1}{\alpha}D_\alpha D_\beta\alpha$ □

Proposition: $\perp^\mu{}_\alpha\perp^\nu{}_\beta R_{\mu\nu} = -\mathcal{L}_{\boldsymbol{n}}K_{\alpha\beta} - \frac{1}{\alpha}D_\alpha D_\beta\alpha - 2K_{\alpha\rho}K^\rho{}_\beta + \mathcal{R}_{\alpha\beta} + KK_{\alpha\beta}\,,$

$$R = -2\mathcal{L}_{\boldsymbol{n}}K - \frac{2}{\alpha}D^\mu D_\mu\alpha + \mathcal{R} + K^2 + K_{\mu\nu}K^{\mu\nu}$$

Proof. (Sketched)

Combine *contracted Gauss* and *Ricci* Eqs. to get the first result.

Contract with $\perp^{\alpha\beta}$. Use the first Lemma to show: $\perp^{\alpha\beta}\mathcal{L}_{\boldsymbol{n}}K_{\alpha\beta} = \mathcal{L}_{\boldsymbol{n}}K - 2K_{\alpha\beta}K^{\alpha\beta}$.

Combine with the *scalar Gauss* Eq. □

F.4 The 3+1 version of the Einstein equations

Einstein Eqs. : $$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad (\dagger) \qquad \Leftrightarrow \qquad R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) + \Lambda g_{\alpha\beta} \quad (\ddagger)$$

Def.: *energy density*, *momentum density* and *stress tensor*:

$$\rho := n^\mu n^\nu T_{\mu\nu} , \qquad j_\alpha := -\perp^\mu{}_\alpha n^\nu T_{\mu\nu} , \qquad S_{\alpha\beta} := \perp^\mu{}_\alpha \perp^\nu{}_\beta T_{\mu\nu}$$

$$\Rightarrow \quad T_{\alpha\beta} = \rho n_\alpha n_\beta + j_\alpha n_\beta + n_\alpha j_\beta + S_{\alpha\beta} \qquad \text{and} \qquad T = S - \rho$$

Proposition: The projections of the Einstein equations give:

$$\mathcal{H} := \mathcal{R} + K^2 - K_{\mu\nu} K^{\mu\nu} - 2\Lambda - 16\pi\rho = 0 ,$$

$$\mathcal{M}_\alpha := D_\alpha K - D_\mu K_\alpha{}^\mu + 8\pi j_\alpha = 0 ,$$

$$\mathcal{L}_{\boldsymbol{n}} \gamma_{\alpha\beta} = -2K_{\alpha\beta} ,$$

$$\mathcal{L}_{\boldsymbol{n}} K_{\alpha\beta} = -\frac{1}{\alpha} D_\alpha D_\beta \alpha - 2K_{\alpha\mu} K^\mu{}_\beta + \mathcal{R}_{\alpha\beta} + K K_{\alpha\beta} - \Lambda\gamma_{\alpha\beta} - 8\pi \left[S_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}(S - \rho) \right] .$$

Proof. (Sketched)

(1) Project **(†)** onto $n^\alpha n^\beta$ and use *scalar Gauss*.

(2) Project **(†)** onto $\perp^\alpha{}_\mu n^\beta$ and use *Codazzi*.

(3) The 3rd Eq. is our 3rd Lemma above.

(4) Project **(‡)** onto $\perp^\alpha{}_\mu \perp^\beta{}_\nu$ and use the last proposition of Sec. F.3.

□

F.5 Adapted coordinates

Def.: Let $(\mathcal{M}, \boldsymbol{g})$ be globally hyperbolic spacetime with foliation Σ_t given by $t : \mathcal{M} \to \mathbb{R}$ with $\mathbf{d}t \neq 0$.

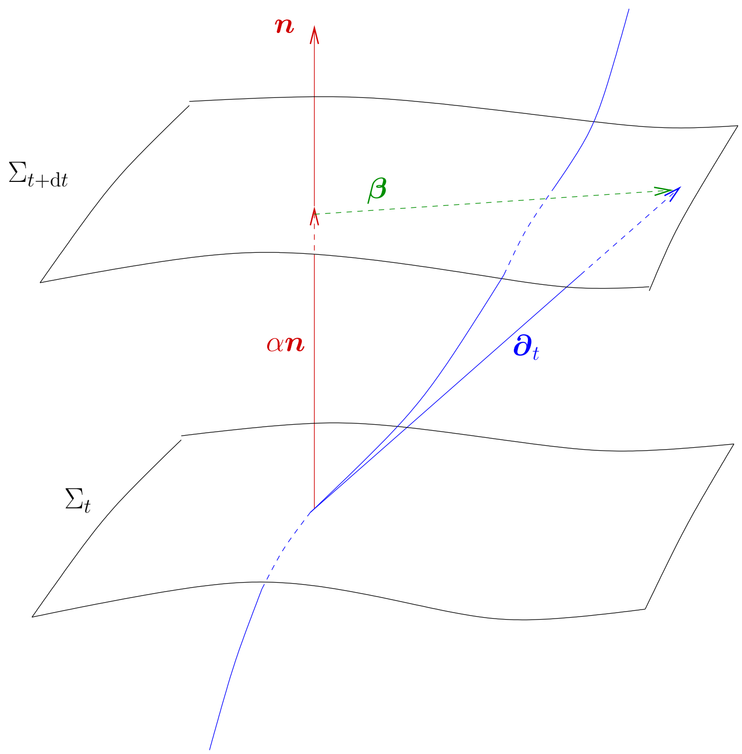
Adapted coordinates: $x^\alpha = (t,\ x^i)$. x^i label points inside Σ_t.

Coordinate basis: $(\boldsymbol{\partial}_t,\ \boldsymbol{\partial}_i)$ and $(\mathbf{d}t,\ \mathbf{d}x^i)$.

Def.: *Shift vector*: $\boldsymbol{\beta} = \boldsymbol{\partial}_t - \alpha \boldsymbol{n}$

$\boldsymbol{\beta}$ is spatial: $\langle \mathbf{d}t, \boldsymbol{\beta} \rangle = \langle \mathbf{d}t, \boldsymbol{\partial}_t \rangle - \alpha \langle \mathbf{d}t, \boldsymbol{n} \rangle = 1 + \langle \boldsymbol{n}, \boldsymbol{n} \rangle = 0$

$\boldsymbol{\beta}$ measures deviations of $\boldsymbol{\partial}_t$ from the normal direction and fixes x^i on new slices



Metric components

$\beta^0 = \langle \mathbf{d}t, \boldsymbol{\beta}\rangle = 0$, so $\beta_m = g_{mn}\beta^n$ and:

$$g_{00} \;=\; \boldsymbol{g}(\boldsymbol{\partial}_t, \boldsymbol{\partial}_t) \;=\; \boldsymbol{g}(\alpha\boldsymbol{n} + \boldsymbol{\beta}, \alpha\boldsymbol{n} + \boldsymbol{\beta}) \;=\; -\alpha^2 + \beta^m\beta_m\,,$$

$$g_{0i} \;=\; \boldsymbol{g}(\boldsymbol{\partial}_t, \boldsymbol{\partial}_i) \;=\; \boldsymbol{g}(\alpha\boldsymbol{n} + \boldsymbol{\beta}, \boldsymbol{\partial}_i) \;=\; -\langle \mathbf{d}t, \boldsymbol{\partial}_i\rangle + \langle \beta_m \mathbf{d}x^m, \boldsymbol{\partial}_i\rangle \;=\; \beta_i\,,$$

$$g_{ij} \;=\; \boldsymbol{g}(\boldsymbol{\partial}_i, \boldsymbol{\partial}_j) \;=\; \boldsymbol{\gamma}(\boldsymbol{\partial}_i, \boldsymbol{\partial}_j) \;=\; \gamma_{ij}\,,$$

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^2 + \beta^m\beta_m & \beta_j \\ \hline \beta_i & \gamma_{ij} \end{array}\right) \qquad \Leftrightarrow \qquad g^{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \hline \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{array}\right)$$

$$\Rightarrow \quad \mathrm{d}s^2 = (-\alpha^2 + \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i \mathrm{d}t\,\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\,\mathrm{d}x^j\,.$$

$$n_\alpha = (-\alpha,\, 0)\,, \qquad n^\alpha = \left(\frac{1}{\alpha},\, -\frac{\beta^i}{\alpha}\right)\,.$$

Proposition: An observer moving with 4-velocity $u^\alpha = n^\alpha$ from Σ_t to $\Sigma_{t+\mathrm{d}t}$ measures proper time $\mathrm{d}\tau = \alpha \mathrm{d}t$.

Proof. Tangent vector to the observer's worldline parametrized with t:

$$m^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = \left(1,\ \frac{\mathrm{d}x^i}{\mathrm{d}t}\right) \propto n^\alpha = \left(\frac{1}{\alpha},\ -\frac{\beta^i}{\alpha}\right).$$

$$\Rightarrow\quad m^\alpha = (1,\ -\beta^i)$$

$$\Rightarrow\quad \mathrm{d}\tau = \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}\,\mathrm{d}t = \sqrt{-g_{00}m^0m^0 - 2g_{0i}m^0m^i - g_{ij}m^im^j}\,\mathrm{d}t = \alpha\mathrm{d}t\,. \qquad \square$$

So α fixes the slicing. Lapse and shift are the gauge variables.

Changing to spatial indices

- For any spatial vector $\boldsymbol{V}$: $V^0 = \langle \mathbf{d}t, \boldsymbol{V}\rangle = 0$.
- Likewise for any spatial tensor $T^{0\alpha\ldots}{}_{\beta\ldots} = 0$.
- $V_0 \neq 0$ in general, but still only 3 independent components!
- Contractions of spatial tensors $T_{\mu\alpha}V^\mu = T_{m\alpha}V^m$.

$\Rightarrow$ in adapted coordinates we can replace in spatial equations Greek with Latin indices!

3D Christoffel symbols: $\Gamma^i_{jk} = \frac{1}{2}\gamma^{im}\left(\partial_j\gamma_{km} + \partial_k\gamma_{mj} - \partial_m\gamma_{jk}\right)$,

3D Riemann tensor: $\mathcal{R}^j{}_{kmn} = \partial_m\Gamma^j_{kn} - \partial_n\Gamma^j_{km} + \Gamma^l_{kn}\Gamma^j_{lm} - \Gamma^l_{km}\Gamma^j_{ln}$.

Proposition: In adapted coordinates:

$$\mathcal{L}_{\boldsymbol{n}}\gamma_{\mu\nu} = \frac{1}{\alpha}\partial_t\gamma_{\mu\nu} - \frac{1}{\alpha}\mathcal{L}_{\boldsymbol{\beta}}\gamma_{\mu\nu}\,,$$

$$\mathcal{L}_{\boldsymbol{n}}K_{\mu\nu} = \frac{1}{\alpha}\partial_t K_{\mu\nu} - \frac{1}{\alpha}\mathcal{L}_{\boldsymbol{\beta}}K_{\mu\nu}$$

Proof. (Sketched)

Show for a scalar f and spatial tensors: $\mathcal{L}_{f\boldsymbol{n}}T_{\alpha\beta} = f\mathcal{L}_{\boldsymbol{n}}T_{\alpha\beta}$.

Use this for $\boldsymbol{n} = \frac{1}{\alpha}(\boldsymbol{\partial}_t - \boldsymbol{\beta})$, bearing in mind $\mathcal{L}_{\boldsymbol{\partial}_t} = \partial_t$. $\square$

Using $\mathcal{L}_{\boldsymbol{\beta}}\gamma_{ij} = \beta^m\partial_m\gamma_{ij} + 2\gamma_{m(i}\partial_{j)}\beta^m$, we get York's version of the ADM equations:

$$\mathcal{H} := \mathcal{R} + K^2 - K_{mn}K^{mn} - 2\Lambda - 16\pi\rho = 0$$

$$\mathcal{M}_i := D_iK - D_mK_i{}^m + 8\pi j_i = 0$$

$$\partial_t\gamma_{ij} = \beta^m\partial_m\gamma_{ij} + 2\gamma_{m(i}\partial_{j)}\beta^m - 2\alpha K_{ij}$$

$$\partial_t K_{ij} = \beta^m\partial_m K_{ij} + 2K_{m(i}\partial_{j)}\beta^m - D_iD_j\alpha + \alpha\left[\mathcal{R}_{ij} + KK_{ij} - 2K_{im}K^m{}_j\right] - \alpha\Lambda\gamma_{ij} - 8\pi\alpha\left[S_{ij} - \gamma_{ij}\frac{S-\rho}{2}\right]$$

Note: 10 variables, 10 equations, 4 constraints, 4 gauge variables, 2 degrees of freedom

3+1 matter equations:

$$\partial_t \rho = \beta^m \partial_m \rho - 2 j^m D_m \alpha + \alpha \left(\rho K + S^{mn} K_{mn} - D_m j^m \right) ,$$

$$\partial_t j_i = \beta^m \partial_m j_i + j_m \partial_i \beta^m - \rho D_i \alpha - S^m{}_i D_m \alpha + \alpha \left(j_i K - D_m S^m{}_i \right) .$$

Note: No equation for $\partial_t S_{ij}$; need EOS!

G Well-posedness, strong hyperbolicity and BSSNOK

G.1 The concept of well-posedness

Def.: An initial value (aka Cauchy) problem is *well-posed* if a solution exists, is unique and depends continuously on the initial data in the sense of a norm $||f(t,.)||$ of a function $f(t,x^i)$. Otherwise, it is *ill-posed*.

We take for $||f(t,.)||$ the L_2 norm of $f(t,x^i)$ at fixed time.

Example

$2D$ Laplace equation: $\triangle\phi(t,x) = \partial_t^2\phi + \partial_x^2\phi = 0$

$$\text{with} \quad \phi(0,x) = f_n(x) := 0\,, \qquad \partial_t\phi(0,x) = g_n(x) := e^{-\sqrt{n}}\sin(nx)$$

Solution: $\phi_n(t,x) = \dfrac{e^{-\sqrt{n}}}{n}\sinh(nt)\,\sin(nx)$

But: $\lim_{n\to\infty} f_n(x) =: f_\infty = 0\,, \qquad \lim_{n\to\infty} g_n(x) =: g_\infty = 0\,,$

$$\lim_{n\to\infty}\phi_n(t,x)\to\infty \quad \text{for any} \quad t>0$$

This problem does not arise for the wave equation $-\partial_t^2\phi + \partial_x^2\phi = 0$.

G.2 Well-posedness of first-order systems

Consider PDEs $\quad \mathbf{A}\partial_t\boldsymbol{u} + \mathbf{P}^i\partial_i\boldsymbol{u} + \mathbf{C}\boldsymbol{u} = 0 \quad$ for $\quad \boldsymbol{u}:\Omega\subset\mathbb{R}^{d+1}\to\mathbb{R}^N$, (†)

where $\mathbf{A}$, $\mathbf{P}^i$, $\mathbf{C}$ are real $N\times N$ matrices and $\mathbf{A}$ is invertible.

<u>**Constant coefficient systems**</u>: $\mathbf{A}$, $\mathbf{P}^i$, $\mathbf{C}$ constant.

Def.: *Fourier transformation:*

$$\tilde{f}(k_i) = \mathcal{F}[f](k_i) = \frac{1}{\sqrt{2\pi}^d}\int f(x_i)e^{-\mathrm{i}k_m x_m}\mathrm{d}^d x\,,$$

$$f(x_i) = \mathcal{F}^{-1}[\tilde{f}](x_i) = \frac{1}{\sqrt{2\pi}^d}\int \tilde{f}(k_i)e^{\mathrm{i}k_m x_m}\mathrm{d}^d x$$

$$\Rightarrow \quad \mathcal{F}[\partial_i f](k_i) = \frac{1}{\sqrt{2\pi}^d}\int \partial_i f\, e^{-\mathrm{i}k_m x_m}\mathrm{d}^d x = \mathrm{i}k_i\tilde{f}(k_i)\,.$$

Transformed PDE: $\quad \mathbf{A}\partial_t\tilde{\boldsymbol{u}} + \mathbf{P}^m\mathrm{i}k_m\tilde{\boldsymbol{u}} + \mathbf{C}\tilde{\boldsymbol{u}} = 0$

$$\Rightarrow \quad \partial_t\tilde{\boldsymbol{u}} + \mathbf{A}^{-1}\left(\mathrm{i}\mathbf{P}^m k_m + \mathbf{C}\right)\tilde{\boldsymbol{u}} = \partial_t\tilde{\boldsymbol{u}} - \mathrm{i}\mathbf{M}(k_m)\tilde{\boldsymbol{u}} \quad \text{with} \quad \mathbf{M} = \mathbf{A}^{-1}(-\mathbf{P}^m k_k + \mathrm{i}\mathbf{C})$$

Solution: $\tilde{\boldsymbol{u}}(t,k_i) = e^{\mathrm{i}\mathbf{M}t}\tilde{\boldsymbol{u}}(0,k_i) \quad \Rightarrow \quad \boldsymbol{u}(t,x_i) = \dfrac{1}{\sqrt{2\pi}^d}\displaystyle\int e^{\mathrm{i}\mathbf{M}t}\tilde{\boldsymbol{u}}(0,k_i)e^{\mathrm{i}k_m x_m}\mathrm{d}^d k$ **(∗)**

The integral converges for $t = 0$, but how about $t > 0$?

Proposition: If there exists a regular function f with $\left|\left|e^{\mathrm{i}\mathbf{M}t}\right|\right| \leq f(t)$ **(‡)**

the integral in (∗) converges and the PDE (†) is well-posed.

Proof. By Parseval's theorem

$$||\boldsymbol{u}||(t) \;=\; ||\tilde{\boldsymbol{u}}||(t) \;=\; \left|\left|e^{\mathrm{i}\mathbf{M}t}\right|\right| \times ||\tilde{\boldsymbol{u}}||(0) \;\leq\; f(t)||\tilde{\boldsymbol{u}}||(0) \;=\; f(t)||\boldsymbol{u}||(0)$$

Two solutions $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ for initial data $\boldsymbol{u}_1(0,x_i)$ and $\boldsymbol{u}_2(0,x_i)$ satisy

$$||\boldsymbol{u}_1 - \boldsymbol{u}_2||(t) \leq f(t)||\boldsymbol{u}_1 - \boldsymbol{u}_2||(0)$$

→ unique solutions and continuous dependence on initial data. □

Def.: The PDE $\mathbf{A}\partial_t\boldsymbol{u} + \mathbf{P}^i\partial_i\boldsymbol{u} + \mathbf{C}\boldsymbol{u} = 0$ is *weakly hyperbolic*

$:\Leftrightarrow \;\; \forall \;\; \hat{k}^i$ with $|\hat{k}^i| = 1$, all Eigenvalues of $\mathbf{Q}(\hat{k}_i) := -\mathbf{A}^{-1}\mathbf{P}^m\hat{k}_m$ are real.

Proposition: Weak hyperbolicity is necessary for $\left|\left|e^{\mathrm{i}\mathbf{M}t}\right|\right| \leq f(t)$.

Proof. With $\hat{t} := |k|\,t\,, \quad \hat{k}_i := \dfrac{k_i}{|k|}$, Eq. **(‡)** becomes

$$\left|\left|e^{\mathrm{i}\mathbf{M}\hat{t}/|k|}\right|\right| = \left|\left|e^{\mathrm{i}\left[\mathbf{A}^{-1}(-\mathbf{P}^m k_m + \mathrm{i}\mathbf{C})\right]\hat{t}/|k|}\right|\right| = \left|\left|e^{\mathrm{i}\left[\mathbf{A}^{-1}(-\mathbf{P}^m \hat{k}_m + \mathrm{i}\mathbf{C}/|k|)\right]\hat{t}}\right|\right| \leq f\left(\tfrac{\hat{t}}{|k|}\right)$$

Take $|k^i| \to \infty \quad \Rightarrow \quad \left|\left|e^{\mathrm{i}\mathbf{Q}(\hat{k}_i)\hat{t}}\right|\right| \leq f(0) \quad$ with $\quad \mathbf{Q}(\hat{k}_i) = -\mathbf{A}^{-1}\mathbf{P}^m\hat{k}_m$

Note: Short-wavelength modes need large $\hat{t}$ and dominate this!

Let $\lambda = \lambda_1 + \mathrm{i}\lambda_2$ be an Eigenvalue of $\mathbf{Q}$. We need $\lambda_2 \geq 0$.

But $\mathbf{Q}$ is real $\Rightarrow \;\; \lambda_1 - \mathrm{i}\lambda_2$ is also an Eigenvalue. So we need $\lambda_2 = 0$. □

The ADM equations are weakly hyperbolic. But that's not sufficient.

Lemma: If $\mathbf{J}_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \lambda \in \mathbb{C},$ then $e^{\mathrm{i}\mathbf{J}_2\hat{t}} = e^{\mathrm{i}\lambda\hat{t}}\begin{pmatrix} 1 & \mathrm{i}\hat{t} \\ 0 & 1 \end{pmatrix}.$

Proof. Example sheets. □

We cannot bound $\left|\left|\exp\left[\mathrm{i}\mathbf{Q}(\hat{k}_i)\hat{t}\right]\right|\right|$ with such a Jordan block.

Larger Jordan blocks give similar $\hat{t}^p,\ p \geq 1$ growth.

Def.: The PDE $\mathbf{A}\partial_t \boldsymbol{u} + \mathbf{P}^m\partial_m \boldsymbol{u} + \mathbf{C}\boldsymbol{u} = 0$ is *strongly hyperbolic* if for all $\hat{k}_i$ with $|\hat{k}| = 1$, $\mathbf{Q} = -\mathbf{A}^{-1}\mathbf{P}^m\hat{k}_m$ has only real Eigenvalues and is diagonalizable. If the symmetrizer $\mathbf{S}$ in $\mathbf{Q} = \mathbf{S}\Lambda\mathbf{S}^{-1}$ does not depend on $\hat{k}_i$, the PDE is *symmetric hyperbolic*.

Linear PDEs: Use the constant-coefficient criterion for all (t, x_i).

Non-linear PDEs: Linearize around all backgrounds. This is challenging!

Note: Stability tests often need empirical tests.

G.3 The BSSNOK formulation

Baumgarte-Shapiro-Shibata-Nakamura-Oohara-Kojima. There are other well-posed formulations.

Proposition: $\dfrac{\partial g}{\partial g_{\alpha\beta}} = g\, g^{\alpha\beta}\,, \qquad \dfrac{\partial g}{\partial g^{\alpha\beta}} = -g\, g_{\alpha\beta}$

in n dimension and for any signature.

$$\Rightarrow \quad \partial_\alpha g = g\, g^{\mu\nu}\partial_\alpha g_{\mu\nu} = -g\, g_{\mu\nu}\partial_\alpha g^{\mu\nu} = 2g\Gamma^\mu_{\mu\alpha}$$

Proof. Use the cofactor matrix and $\nabla_\alpha g_{\mu\nu} = 0$. □

Def.: BSSNOK variables:

$$\chi = \gamma^{-1/3}\,, \qquad K = \gamma^{mn}K_{mn},$$

$$\tilde{\gamma}_{ij} = \chi\gamma_{ij} \quad \Leftrightarrow \quad \tilde{\gamma}^{ij} = \frac{1}{\chi}\gamma^{ij},$$

$$\tilde{A}_{ij} = \chi\left(K_{ij} - \tfrac{1}{3}\gamma_{ij}K\right) \quad \Leftrightarrow \quad K_{ij} = \frac{1}{\chi}\left(\tilde{A}_{ij} + \frac{1}{3}\tilde{\gamma}_{ij}K\right),$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn}\,,$$

Extra constraints: $\tilde{\gamma} = 1\,, \quad \tilde{\gamma}^{mn}\tilde{A}_{mn} = 0\,, \quad \mathcal{G}^i := \Gamma^i - \tilde{\gamma}^{mn}\Gamma^i_{mn} = 0\,.$

To translate the ADM equations into BSSNOK, we need some auxiliary relations.

Lemma: $\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} - \dfrac{1}{2\chi}\left(\delta^i{}_k\partial_j\chi + \delta^i{}_j\partial_k\chi - \tilde{\gamma}_{jk}\tilde{\gamma}^{im}\partial_m\chi\right)$

Proof. Product rule for $\gamma_{ij} = \frac{1}{\chi}\tilde{\gamma}_{ij}$, and $\tilde{\gamma}^{km}\tilde{\gamma}_{ml} = \delta^k{}_l$. □

Lemma: For any metric:
$$\partial_i \gamma^{jk} = -\gamma^{jm}\gamma^{kn}\partial_i\gamma_{mn}\,, \qquad \partial_i\gamma_{jk} = -\gamma_{jm}\gamma_{kn}\partial_i\gamma^{mn}$$
$$\Rightarrow \quad \partial_k\gamma^{jk} = -\gamma^{jm}\gamma^{kn}\partial_k\gamma_{mn}\,.$$

For the conformal metric $\tilde{\gamma} = 1$ implies:
$$\tilde{\Gamma}^i = \tilde{\gamma}^{mn}\tilde{\gamma}^{il}\partial_m\tilde{\gamma}_{nl} = -\partial_m\tilde{\gamma}^{mi}\,,$$
$$\tilde{\Gamma}^m_{im} = \frac{1}{2}\tilde{\gamma}^{mn}\partial_i\tilde{\gamma}_{mn} = 0\,.$$

Proof. $0 = \partial_i\delta^j{}_k = \partial_i(\gamma^{jm}\gamma_{mk}) = \gamma^{jm}\partial_i\gamma_{mk} + \gamma_{mk}\partial_i\gamma^{jm} \quad \Big| \times\gamma^{kl} \;\text{ or }\; \times\gamma_{jl}$

$$\Rightarrow \quad \partial_i\gamma^{jl} = -\gamma^{jm}\gamma^{kl}\partial_i\gamma_{mk} \qquad \wedge \qquad \partial_i\gamma_{lk} = -\gamma_{mk}\gamma_{jl}\partial_i\gamma^{jm}$$

With $\tilde{\gamma} = 1$: $\quad \partial_i\tilde{\gamma} = \tilde{\gamma}\,\tilde{\gamma}^{mn}\partial_i\tilde{\gamma}_{mn} = -\tilde{\gamma}\,\tilde{\gamma}_{mn}\partial_i\tilde{\gamma}^{mn} = 0$

$$\Rightarrow \quad \tilde{\Gamma}^i = \frac{1}{2}\tilde{\gamma}^{mn}\tilde{\gamma}^{il}\left(\partial_m\tilde{\gamma}_{nl} + \partial_n\tilde{\gamma}_{lm} - \partial_l\tilde{\gamma}_{mn}\right) = \tilde{\gamma}^{mn}\tilde{\gamma}^{il}\partial_m\tilde{\gamma}_{nl}$$

$$\wedge \quad \tilde{\Gamma}^m_{im} = \frac{1}{2}\tilde{\gamma}^{mn}(\partial_i\tilde{\gamma}_{mn} + \partial_m\tilde{\gamma}_{ni} - \partial_n\tilde{\gamma}_{im}) = \frac{1}{2}\tilde{\gamma}^{mn}\partial_i\tilde{\gamma}_{mn} = 0$$

□

Proposition:

$$\begin{aligned}
\mathcal{H} &= \mathcal{R} - \tilde{A}^{mn}\tilde{A}_{mn} + \frac{2}{3}K^2 - 2\Lambda - 16\pi\rho = 0\,,\\
\mathcal{M}_i &= \frac{2}{3}\partial_i K - \tilde{\gamma}^{mn}\tilde{D}_m\tilde{A}_{in} + \frac{3}{2}\tilde{A}_i{}^m\frac{\partial_m\chi}{\chi} + 8\pi j_i = 0\,,\\
\partial_t\chi &= \beta^m\partial_m\chi - \frac{2}{3}\chi\partial_m\beta^m + \frac{2}{3}\alpha\chi K\,,\\
\partial_t\tilde{\gamma}_{ij} &= \beta^m\partial_m\tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{\gamma}_{ij}\partial_m\beta^m - 2\alpha\tilde{A}_{ij}\,,\\
\partial_t K &= \beta^m\partial_m K - \chi\tilde{\gamma}^{mn}D_mD_n\alpha + \alpha\left[\tilde{A}^{mn}\tilde{A}_{mn} + \frac{1}{3}K^2 - \Lambda + 4\pi(S+\rho)\right]\,,\\
\partial_t\tilde{A}_{ij} &= \beta^m\partial_m\tilde{A}_{ij} + 2\tilde{A}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{A}_{ij}\partial_m\beta^m + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{im}\tilde{A}^m{}_j\\
&\quad +\chi\left(\alpha\mathcal{R}_{ij} - D_iD_j\alpha - 8\pi S_{ij}\right)^{\text{TF}}\,,\\
\partial_t\tilde{\Gamma}^i &= \beta^m\partial_m\tilde{\Gamma}^i - \tilde{\Gamma}^m\partial_m\beta^i + \frac{2}{3}\tilde{\Gamma}^i\partial_m\beta^m + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + \frac{1}{3}\tilde{\gamma}^{im}\partial_m\partial_n\beta^n + 2\alpha\tilde{\Gamma}^i_{mn}\tilde{A}^{mn}\\
&\quad -2\tilde{A}^{im}\partial_m\alpha - \frac{4}{3}\alpha\tilde{\gamma}^{im}\partial_m K - 3\alpha\tilde{A}^{im}\frac{\partial_m\chi}{\chi} - 16\pi\alpha\tilde{\gamma}^{im}j_m - \sigma\mathcal{G}^i\,,
\end{aligned}$$

Comments

- "TF" means "tracefree part".
- Constraints only used as diagnostic $\rightarrow$ *free* evolutions.
- $\sigma > 0$ for damping $\mathcal{G}^i$.
  Alternatively: Replace undifferentiated $\tilde{\Gamma}^i$ in terms of $\tilde{\gamma}_{ij}$.
- One needs to enforce $\tilde{A}^m{}_m = 0$, but not $\tilde{\gamma} = 1$.

Proof. $\partial_t K$ as an example. Use $\partial\gamma_{ij} = -\gamma_{im}\gamma_{jn}\partial\gamma^{mn}\,,\quad \partial\gamma^{ij} = -\gamma^{im}\gamma^{jn}\partial\gamma_{mn}\,,$

$\Rightarrow \partial_t K = \partial_t(\gamma^{ij}K_{ij}) = \gamma^{ij}\partial_t K_{ij} + K_{ij}\partial_t\gamma^{ij} = \gamma^{ij}\partial_t K_{ij} - K_{ij}\gamma^{im}\gamma^{jn}\partial_t\gamma_{mn} = \gamma^{ij}\partial_t K_{ij} - K^{ij}\partial_t\gamma_{ij}\,.$

Use the ADM equations for $\partial_t\gamma_{ij}$, $\partial_t K_{ij}$

$$\begin{aligned}
\Rightarrow\quad \partial_t K &= \gamma^{ij}\beta^m\partial_m K_{ij} + 2\gamma^{ij}K_{m(i}\partial_{j)}\beta^m - \gamma^{ij}D_iD_j\alpha + \alpha(\mathcal{R} + K^2 \underset{\sim}{-2K^{mn}K_{mn}})\\
&\quad - 3\alpha\Lambda - 8\pi\alpha\left[S - \frac{3}{2}(S-\rho)\right] - K^{ij}\left[\beta^m\partial_m\gamma_{ij} + 2\gamma_{m(i}\partial_{j)}\beta^m \underset{\sim}{-2\alpha K_{ij}}\right]\\
&= \beta^m\gamma^{ij}\partial_m K_{ij} - \beta^m K^{ij}\left(-\gamma_{ik}\gamma_{jl}\partial_m\gamma^{kl}\right) + \alpha(\mathcal{R} + K^2) - 3\alpha\Lambda + 4\pi\alpha(S - 3\rho)
\end{aligned}$$

$$- \gamma^{ij} D_i D_j \alpha + \underset{\sim\sim\sim}{\gamma^{ij} K_{mi} \partial_j \beta^m} + \underset{\sim\sim\sim}{\gamma^{ij} K_{mj} \partial_i \beta^m} - \underset{\sim\sim\sim}{K^{ij} \gamma_{mi} \partial_j \beta^m} - \underset{\sim\sim\sim}{K^{ij} \gamma_{mj} \partial_i \beta^m}$$

$$= \beta^m \gamma^{ij} \partial_m K_{ij} + \beta^m K_{ij} \partial_m \gamma^{ij} + \alpha(\mathcal{R} + K^2) - 3\alpha\Lambda + 4\pi\alpha(S - 3\rho) - \gamma^{mn} D_m D_n \alpha$$

$$= \beta^m \partial_m K + \alpha(\mathcal{R} + K^2) - 3\alpha\Lambda + 4\pi\alpha(S - 3\rho) - \chi\tilde{\gamma}^{mn} D_m D_n \alpha \,.$$

Finally subtract $\alpha\mathcal{H} = \alpha \left[\mathcal{R} + \frac{2}{3} K^2 - \tilde{A}^{mn} \tilde{A}_{mn} - 2\Lambda - 16\pi\rho \right]$

Note: we subtract zero, but change the principal part of the PDE! □

We need the auxiliary expressions:

Proposition:

$$\begin{aligned}
\mathcal{R}_{ij} &= \tilde{R}_{ij} + \tilde{R}^{\chi}_{ij} \,, \\
\tilde{\mathcal{R}}_{ij} &= -\frac{1}{2} \tilde{\gamma}^{mn} \partial_m \partial_n \tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i} \partial_{j)} \tilde{\Gamma}^m + \tilde{\Gamma}^m \tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn} \tilde{\Gamma}^k_{im} \tilde{\Gamma}_{kjn} + 2\tilde{\gamma}^{mn} \tilde{\Gamma}^k_{m(i} \tilde{\Gamma}_{j)kn} \,, \\
\mathcal{R}^{\chi}_{ij} &= \frac{1}{2\chi} \left(\tilde{D}_i \tilde{D}_j \chi + \tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \tilde{D}_m \tilde{D}_n \chi \right) - \frac{1}{4\chi^2} \left(\partial_i \chi \partial_j \chi + 3\tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \partial_m \chi \partial_n \chi \right) \,, \\
D_i D_j \alpha &= \tilde{D}_i \tilde{D}_j \alpha + \frac{1}{\chi} \partial_{(i} \alpha \partial_{j)} \chi - \frac{1}{2\chi} \tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \partial_m \chi \partial_n \alpha \,.
\end{aligned}$$

Proof. Long script. □

Comments

- BSSNOK can be shown to be strongly hyperbolic for suitable gauge.
- There are other strongly hyperbolic formulations, e.g. CCZ4, GHG.
- We still need initial data and gauge conditions...

H Gauge and initial data

H.1 Initial data

Two goals: 1) Solve constraints

2) Get physically realistic snapshot

Degrees of freedom: 6 γ_{ij}, 6 K_{ij}. One constraint $\mathcal{H}$ for γ_{ij}, three $\mathcal{M}^i$ for K_{ij}

4 coordinate choices, leaving 2 γ_{ij} and 2 K_{ij}.

H.1.1 Conformal transformations

Def.: *Conformal transformation*: $\bar{\gamma}_{ij} = e^{2\varphi}\gamma_{ij} \quad \Leftrightarrow \quad \gamma_{ij} = e^{-2\varphi}\bar{\gamma}_{ij}$

Proposition:

$$\begin{aligned}
\Gamma^i_{jk} &= \bar{\Gamma}^i_{jk} - \left(\delta^i{}_j\partial_k\varphi + \delta^i{}_k\partial_j\varphi - \bar{\gamma}_{jk}\bar{\gamma}^{im}\right), \\
e^{2\varphi}\mathcal{R}_{ijkl} &= \bar{\mathcal{R}}_{ijkl} + \bar{\gamma}_{ik}X_{jl} - \bar{\gamma}_{il}X_{jk} + \bar{\gamma}_{jl}X_{ik} - \bar{\gamma}_{jk}X_{il} \\
&\quad \text{with} \quad X_{jl} = X_{lj} = \bar{D}_j\bar{D}_l\varphi + \partial_j\varphi\partial_l\varphi - \frac{1}{2}\bar{\gamma}_{jl}(\bar{\gamma}^{mn}\partial_m\varphi\partial_n\varphi), \\
\mathcal{R}_{ij} &= \bar{\mathcal{R}}_{ij} + (n-2)\big(\bar{D}_i\bar{D}_j\varphi + \partial_i\varphi\partial_j\varphi\big) + \bar{\gamma}_{ij}\bar{\gamma}^{mn}\big[\bar{D}_m\bar{D}_n\varphi - (n-2)\partial_m\varphi\partial_n\varphi\big], \\
\mathcal{R} &= e^{2\varphi}\left\{\bar{\mathcal{R}} + (n-1)\bar{\gamma}^{mn}\left[2\bar{D}_m\bar{D}_n\varphi - (n-2)\partial_m\varphi\partial_n\varphi\right]\right\}
\end{aligned}$$

Inversion: (i) $\varphi \to -\varphi$ and (ii) swap bar and non-bar.

Proof. Long script. □

H.1.2 The York-Lichnerowicz split

Goal: Rearrange degrees of freedom into specifiable and derived parts.

Def.: Conformal traceless split:

$$\gamma_{ij} = \psi^4\bar{\gamma}_{ij} = e^{-2\varphi}\bar{\gamma}_{ij} \quad \Leftrightarrow \quad \gamma^{ij} = \psi^{-4}\bar{\gamma}^{ij} = e^{2\varphi}\bar{\gamma}^{ij},$$

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K,$$

$$A_{ij} = \psi^{-2}\bar{A}_{ij} \quad \Leftrightarrow \quad A^{ij} = \psi^{-10}\bar{A}^{ij}.$$

Note: $\psi \stackrel{!}{=} e^{-\varphi/2}$ is free and $\det\bar{\gamma}_{ij}$ arbitrary!

Lemma: $$\Gamma^i_{jk} = \bar{\Gamma}^i_{jk} + \frac{2}{\psi}\left(\delta^i{}_j \partial_k \psi + \delta^i{}_k \partial_j \psi - \bar{\gamma}_{jk}\bar{\gamma}^{im}\partial_m \psi\right),$$

$$\mathcal{R} = \psi^{-4}\bar{\mathcal{R}} - \frac{8}{\psi^5}\bar{\gamma}^{mn}\bar{D}_m\bar{D}_n\psi\,.$$

Proof. Long script. □

Proposition: The constraints are

$$\boxed{\bar{\mathcal{H}} := 8\bar{\gamma}^{mn}\bar{D}_m\bar{D}_n\psi - \psi\bar{\mathcal{R}} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{mn}\bar{A}^{mn} + 2\psi^5\Lambda + 16\pi\psi^5\rho = 0\,,}$$

$$\bar{\mathcal{M}}^i := \bar{D}_m\bar{A}^{mi} - \frac{2}{3}\psi^6\bar{\gamma}^{mi}\partial_m K - 8\pi\psi^{10}j^i = 0\,.$$

Proof. (Sketched)

Use the ADM constraints. Show $K^2 - K_{mn}K^{mn} = \frac{2}{3}K^2 - A_{mn}A^{mn}$.

For $\bar{\mathcal{M}}^i$, show $D_m(\gamma^{mi}K - K^{mi}) = \frac{2}{3}\gamma^{mi}D_m K - D_m A^{mi}$.

Use the Lemma for Γ^i_{jk} to compute $D_m A^{im} = \psi^{-10}\bar{D}_m\bar{A}^{mi}$.

Our choice $A_{ij} = \psi^{-2}\bar{A}_{ij}$ leeds to nice cancelations here! □

Proposition: Let $\bar{A}^{ij}$ be symmetric and traceless.

$\Rightarrow$ $\exists$ symmetric Q^{ij} with $\bar{D}_m Q^{mi} = 0$, $Q^m{}_m = 0$ and a vector field X^i:

$$\bar{A}^{ij} = Q^{ij} + (\mathbb{L}X)^{ij} := Q^{ij} + \bar{D}^i X^j + \bar{D}^j X^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_m X^m\,.$$

Note: $(\mathbb{L}X)_m{}^m = 0$ by construction. Q^{ij} is "transverse" and "traceless"

Proposition: $$\boxed{\bar{\mathcal{M}}^i := \bar{D}^m\bar{D}_m X^i + \frac{1}{3}\bar{D}^i\bar{D}_m X^m + \bar{\mathcal{R}}^i{}_m X^m - \frac{2}{3}\psi^6\bar{\gamma}^{mi}\partial_m K - 8\pi\psi^{10}j^i = 0\,.}$$

Proof. Example sheets. □

Henceforth: vacuum, asymptotic flatness, i.e. $\Lambda = \rho = j^i = 0$

Summary:
- Specify: 5 $\bar{\gamma}_{ij}$, 1 K, 2 Q_{ij}
- Solve $\bar{H}$ for ψ and $\bar{M}^i$ for X^i

Examples

(1) For $K = 0$ the constraints decouple: Solve $\bar{M}^i$ for X^i and then $\bar{H}$ for ψ.

(2) Time symmetry and conformal flatness, $K_{ij} = 0$, $\bar{\gamma}_{ij} = \delta_{ij}$

$$\Rightarrow \quad \tfrac{1}{8}\bar{\mathcal{H}} = \delta^{mn}\partial_m\partial_n\psi = \triangle\psi = 0 \quad \wedge \quad \bar{M}^i = 0 \text{ manifestly}$$

$$\Rightarrow \quad \psi = \frac{A}{r} + B \quad \text{on} \quad \mathbb{R}^3 \setminus \{0\}.$$ $\boldsymbol{r} = 0$ is called a *puncture*.

For $A = \frac{M}{2}$, $B = 1$ this is isotropic Schwarzschild: $\mathrm{d}s^2 = -\left(\frac{2r - M}{2r + M}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{M}{2r}\right)^4 (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$

$\bar{H} = 0$ is linear in ψ $\Rightarrow$ We can superpose:

$$\psi = 1 + \sum_{i=1}^{n} \frac{M_{(i)}}{2|\boldsymbol{r} - \boldsymbol{r}_{(i)}|}$$ *Brill-Lindquist data* for n BHs at rest

H.1.3 Bowen-York and puncture data

Goal: spinning and boosted BHs.

Impose $\bar{\gamma}_{ij} = \delta_{ij}$, $K = 0$, $Q_{ij} = 0$, but allow for $X^i \neq 0$

$$\Rightarrow \quad \boxed{\bar{\mathcal{M}}^i = \partial^m\partial_m X^i + \frac{1}{3}\partial^i\partial_m X^m = 0}$$

Linear in X^i $\Rightarrow$ We can superpose solutions!

<u>**Proposition:**</u> $\boxed{X^i = \epsilon^{ijk}\frac{x_j}{r^3}J_k}$, ϵ^{ijk} = Levi-Civita tensor, $J_k = \text{const}$ (†)

solves $\bar{\mathcal{M}}^i = 0$ and endows the spacetime with angular momentum J_k.

Proof. (Sketched)

Using $\partial_i r = x_i/r$, one shows $\partial_m X^m = 0$ and $\partial^m\partial_m X^i = 0$.

The total angular momentum of an asymptotically flat spacetime is Eq. (8.83) in `gr-qc/0703035`

$$J_m^\infty = \frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_r}(K_{ij} - K\gamma_{ij})(\boldsymbol{\phi_m})^i\frac{x^j}{r}r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,, \quad \text{where} \quad (\boldsymbol{\phi_m})^i = \epsilon_{mj}{}^i x^j.$$

Using $\epsilon_{ijk}\epsilon^{imn} = \delta_j{}^m\delta_k{}^n - \delta_j{}^n\delta_k{}^m$, and $\bar{A}_{ij} = \partial_i X_j + \partial_j X_i$, we find

$$\bar{A}_{ij}(\boldsymbol{\phi_m})^i x^j = \frac{3}{r}\left(J_m - \frac{x_m x_k}{r^2}J^k\right).$$

$\lim_{r\to\infty}\psi = 1$, so $K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K = \psi^{-2}\bar{A}_{ij} \to \bar{A}_{ij}$

Rotate the coordinates such that $J_k = (0, 0, J)$. Integrands $\sim z \sim \cos\theta$ lead to zero, so

$\Rightarrow \quad J_z^\infty = J, \quad J_x^\infty = J_y^\infty = 0$ □

Proposition: $$\boxed{X^i = -\frac{1}{4r}\left(7P^i + \frac{x^i x_k}{r^2}P^k\right)}, \quad P^i = \text{const} \qquad (\ddagger)$$

solves $\bar{\mathcal{M}}^i = 0$ and endows the spacetime with linear momentum P^i.

Proof. (Sketched)

With $\partial_i r = x_i/r$, one computes

$$\partial_m\partial_m X^i = \frac{1}{2}\left(3\frac{x^i x_k P^k}{r^5} - \frac{P^i}{r^3}\right) \quad \text{and} \quad \partial^i\partial_m X^m = \frac{3}{2}\left(\frac{P^i}{r^3} - 3\frac{x^i x_m P^m}{r^5}\right).$$

The linear momentum is Eq. (8.78) in `gr-qc/0703035`

$$P_i^{\text{ADM}} = \frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_r}(K_{ik} - K\gamma_{ik})\frac{x^k}{r}r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

As before, $K_{ik} - K\gamma_{ik} \to \bar{A}_{ik}$, and one finds: $\bar{A}_{ik}x^k = \frac{3}{2r^3}P^k(r^2\delta_{ik} + x_i x_k)$

Rotate coordinates such that $P^k = (0, 0, P)$.

$\Rightarrow \quad P_z^{\text{ADM}} = P, \quad P_x^{\text{ADM}} = P_y^{\text{ADM}} = 0.$ □

Comments

- These are *Bowen-York* data.
- One similarly shows that (†) has $P_i^{\text{ADM}} = 0$ and (‡) has J_i^∞=0.
- We can superpose in two ways:

  (i) joint X^i that carry linear *and* angular momentum.

  (ii) Sources at multiple points, x_{A}^i, x_{B}^i etc.

Hamiltonian constraint

ToDo: $\bar{H} = 8\partial^m\partial_m\psi + \psi^{-7}\bar{A}_{mn}\bar{A}^{mn} = 0$

Puncture data

(1) Recall Brill-Lindquist data for $\bar{A}_{ij} = 0$: $\psi_{\text{BL}} = 1 + \sum_{i=1}^{n} \frac{M_{(i)}}{2|\boldsymbol{r} - \boldsymbol{r}_{(i)}|}$

(2) Ansatz: $\psi = \psi_{\text{BL}} + u$ on $\mathbb{R} \setminus \{\boldsymbol{r}_{(i)}\}$

$\Rightarrow \quad 8\partial^m\partial_m + (\psi_{\text{BL}} + u)^{-7}\bar{A}_{mn}\bar{A}^{mn} = 0$

Needs numerical solving.

(3) Brandt & Brügmann (PRL 1996) have shown that $\exists$ unique solutions u regular on all $\mathbb{R}^3$.

(4) Regularity implies that near $\boldsymbol{r}_{(i)}$ the Brill-Lindquist ψ_{BL} dominates.

$\Rightarrow$ The solutions are still BHs, but with spin and velocity.

(5) Kerr has no conformally flat slice $\Rightarrow$ *junk radiation*

H.2 Gauge conditions

$g_{0\alpha}$ or $(\alpha,\, \beta^i)$ freely specifiable. Four options:

1. $\alpha, \beta^i =$ functions of (t, x^i).
2. Functions of other variables, e.g. γ or $\tilde{\Gamma}^i$.
3. Elliptic PDEs. E.g. *maximal slicing* $K = 0$

   $\Rightarrow \ldots \Rightarrow \triangle\alpha = \alpha K_{mn}K^{mn}$ in vacuum.
4. Hyperbolic or parabolic PDEs for $\alpha,\, \beta^i$.

H.2.1 What can go wrong?

<u>**Kruskal-Szekeres BH**</u>

Schwarzschild: $\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$

Transform:

$$\bar{t} = t + 2M\ln|r - 2M|\,, \qquad \tilde{t} = t - 2M\ln|r - 2M|\,,$$

$$v = \bar{t} + r\,, \qquad u = \tilde{t} - r\,,$$

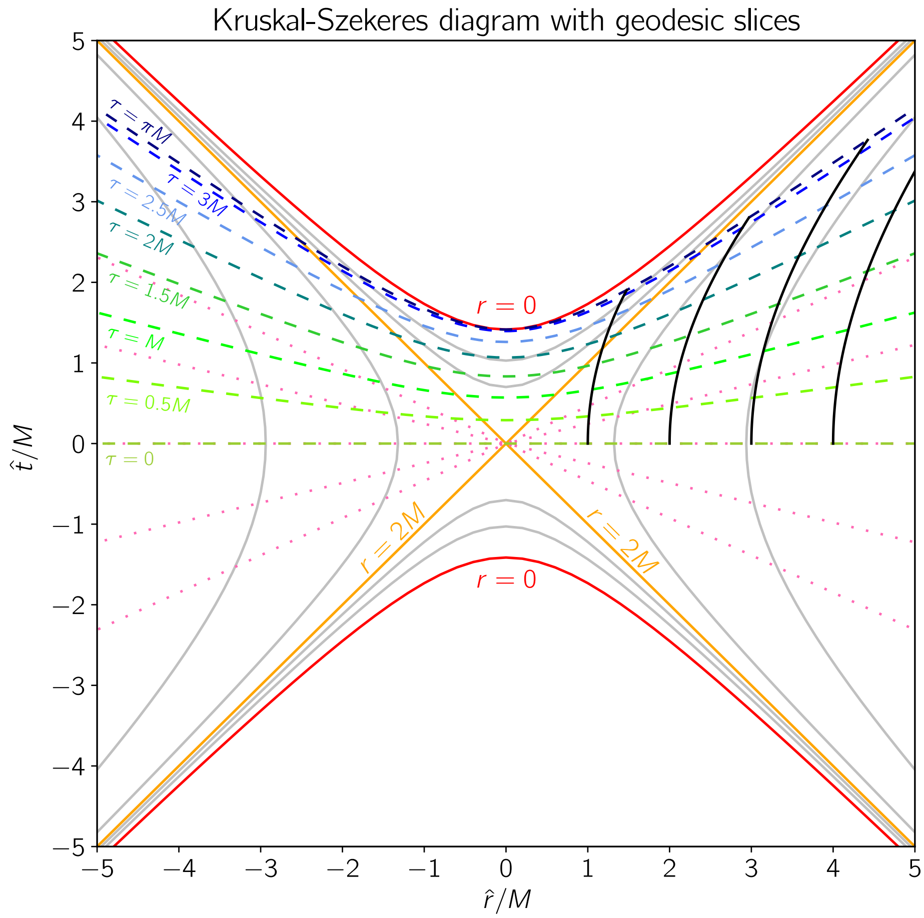
$$\hat{v} = e^{\frac{v}{4M}}\,, \qquad \hat{u} = -e^{-\frac{u}{4M}}\,,$$

$$\hat{t} = \frac{1}{2}(\hat{v} + \hat{u})\,, \qquad \hat{r} = \tfrac{1}{2}(\hat{v} - \hat{u})\,.$$

$$\Rightarrow \quad \mathrm{d}s^2 = \frac{16M^2}{r}e^{-\frac{r}{2M}}(-\mathrm{d}\hat{t}^2 + \mathrm{d}\hat{r}^2) + r^2\mathrm{d}\omega^2$$

Slices $t = \text{const}$: $\boxed{\dfrac{\hat{t}}{\hat{r}} = \tanh\dfrac{t}{4M}}$ for $r > 2M$ and $\boxed{\dfrac{\hat{r}}{\hat{t}} = \tanh\dfrac{t}{4M}}$ for $r < 2M$,

Slices $r = \text{const}$: $\boxed{\hat{t}^2 - \hat{r}^2 = -e^{\frac{r}{2M}}(r-2M) =: C(r)}$.



Proposition: An evolution starting at $\hat{t} = 0$ with *geodesic slicing*, $\alpha = 1$, $\beta^i = 0$ reaches $r = 0$, after πM time units.

Proof. Recall $\mathrm{d}\tau = \alpha \mathrm{d}t_{\text{num}}$. Now $\alpha = 1$

$\Rightarrow$ coordinate time $t_{\text{num}} = \tau =$ proper time of observers moving with 4-velocity $u^\mu = n^\mu$.

$a_\mu = n^\rho \nabla_\rho n_\mu = D_\mu \alpha = 0 \quad \Rightarrow \quad$ normal observers follow geodesics.

$\beta^i = 0 \quad \Rightarrow$ Observers start with $\mathrm{d}r/\mathrm{d}\tau = 0$.

$\hat{r} \geq 0 \quad \Rightarrow \quad r \geq 2M$, $\hat{r} < 0$ by symmetry.

So $t_{\text{num}} = \tau$ of freely infalling observers from $r = r_0 \geq 2M$.

Timelike geodesics in Schwarzschild (e.g. Part II GR):

(1) $\left(1-\frac{2M}{r}\right)\dot{t} = E\,, \qquad -E^2+\dot{r}^2 = -1+\frac{2M}{r}$

(2) Observers start with $\dot{r}$ at $r(0)=r_0$

$$\Rightarrow \quad E = \sqrt{1-\frac{2M}{r_0}} \;\in\; [0,1) \text{ for } r_0 \in [2M,\infty)$$

$$\Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{2M}{r}-\frac{2M}{r_0}} \qquad \Rightarrow \qquad r_0\frac{\mathrm{d}x}{\mathrm{d}\tau} = -\sqrt{\frac{2M}{r_0}}\sqrt{\frac{1}{x}-1} \;\text{ with }\; x=\frac{r}{r_0}.$$

$$\Rightarrow \quad -\sqrt{\frac{2M}{r_0^3}}\int \mathrm{d}\tau = \int\sqrt{\frac{x}{1-x}}\mathrm{d}x = \ldots = -\sqrt{x(1-x)}+\arcsin\sqrt{x}$$

$$\Rightarrow \quad \tau = \underbrace{\tau_0}_{=0} + r_0\sqrt{\frac{r_0}{2M}}\left(\frac{\pi}{2}-\arcsin\sqrt{\frac{r}{r_0}}\right) + r_0\sqrt{\frac{r}{2M}}\sqrt{1-\frac{r}{r_0}}$$

(3) $r=0$ reached at $\tau = \frac{\pi}{2}r_0\sqrt{\frac{r_0}{2M}}\,.$

□

Observers starting at $r=2M$ remain at $\hat{r}=0$ by symmetry.

For other observers, we invert $\tau(r)\to r(\tau)$ and calculate $v(\tau)$ numerically.

$\Rightarrow$ $\hat{t}(\tau)$ and $\hat{r}(\tau)$ trajectories for all observers.

Two problems:

- The code crashes at the singularity before much physics is calculated.
- Observers diverge in $\hat{r}$ $\rightarrow$ *slice stretching.*

H.2.2 Singularity avoiding slicing

Goal: Reduce α near singularity.

Tool: Often the volume element $\sqrt{\gamma}$ vanishes at physical or coordinate singularities!

Def.: Bona-Massó slicing: $\frac{\mathrm{d}\alpha}{\mathrm{d}t} := (\partial_t - \mathcal{L}_{\boldsymbol{\beta}})\alpha = (\partial_t - \beta^m\partial_m)\alpha = -\alpha^2 f(\alpha)K\,,$

where $f(\alpha)>0$ but otherwise free.

Proposition: Bona-Massó slicing implies $\left[g^{\mu\nu}+\left(1-\frac{1}{f}\right)n^\mu n^\nu\right]\nabla_\mu\nabla_\nu t = 0 \qquad (\star)$

Proof. (Sketched)

In adapted coordinates: $\nabla_\mu \nabla_\nu t = \partial_\mu \partial_\nu t - \Gamma^\rho_{\nu\mu}\partial_\rho t = \partial_\mu \delta^0{}_\rho - \Gamma^\rho_{\nu\mu}\delta^0{}_\rho = -\Gamma^0_{\mu\nu}$.

Show that the spacetime Christoffel symbols in adapted coordinates are

$$\Gamma^0_{00} = \frac{1}{\alpha}(\partial_0\alpha + \beta^m\partial_m\alpha) - \frac{1}{\alpha}\beta^m\beta^l K_{ml}\,, \quad \Gamma^0_{0i} = \frac{\partial_i\alpha}{\alpha} - \frac{1}{\alpha}\beta^m K_{im}\,, \quad \Gamma^0_{ij} = -\frac{1}{\alpha}K_{ij}\,.$$

Use $\gamma^{\alpha\beta} = g^{\alpha\beta} + n^\alpha n^\beta$ to compute

$$-\left[g^{\mu\nu} + \left(1 - \frac{1}{f}\right) n^\mu n^\nu\right] \nabla_\mu \nabla_\nu t = -\frac{1}{f\alpha^3}\left[(\partial_0 - \beta^m\partial_m)\alpha + \alpha^2 f K\right] = 0$$

□

Def.: *Focussing singularity* :≔ a point where $\sqrt{\gamma}$ vanishes at a bounded rate as a function of proper time τ of normal observers.

Proposition: $\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\gamma^{1/2} = (\partial_t - \mathcal{L}_{\boldsymbol{\beta}})\gamma^{1/2} = -\alpha\gamma^{1/2}K\,.}$

Proof. (Sketched)

BSSNOK equation: $(\partial_t - \beta^m\partial_m)\chi = -\frac{2}{3}\chi\partial_m\beta^m + \frac{2}{3}\alpha\chi K\,.$

Use $\chi = \gamma^{-1/3}$ and use the Lie derivative for $\sqrt{\gamma}$ which is a tensor density of weight $w = 1$. □

Examples

(1) $f(\alpha) = 1 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\alpha = -\alpha^2 K = \frac{\alpha}{\sqrt{\gamma}}\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\gamma}$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\ln\alpha = \frac{\mathrm{d}}{\mathrm{d}t}\ln\sqrt{\gamma} \quad \Rightarrow \quad \alpha = h(x^i)\sqrt{\gamma}\,.$$

In Eq. ($\star$), $f = 1$ is: $\quad \Box t = \nabla^\mu\nabla_\mu t = 0 \quad$ *harmonic slicing*

Likewise for $f(\alpha) = N$: $\quad \alpha = h(x^i)\sqrt{\gamma}^N$.

(2) $f = \frac{N}{\alpha} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\alpha = -\alpha N K = \frac{N}{\sqrt{\gamma}}\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\gamma} \quad \Rightarrow \quad \alpha = h(x^i) + N\ln\sqrt{\gamma}\,.$

For $N = 2$, $h(x^i) = 1$: $\quad \frac{\mathrm{d}}{\mathrm{d}t}\alpha = -2\alpha K \quad \Rightarrow \quad \alpha = 1 + \ln\gamma \quad$ "1+log" slicing

(3) In general: $\quad \frac{\mathrm{d}}{\mathrm{d}t}\alpha = \frac{\alpha f}{\sqrt{\gamma}}\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\gamma} \quad \Rightarrow \quad \frac{1}{\alpha f}\frac{\mathrm{d}}{\mathrm{d}t}\alpha = \frac{1}{\sqrt{\gamma}}\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\gamma}$

$$\Rightarrow \quad \boxed{\ln\sqrt{\gamma} + \tilde{h}(x^i) = \int \frac{1}{\alpha f}\frac{\mathrm{d}}{\mathrm{d}t}\alpha \mathrm{d}t = \int \frac{\mathrm{d}\alpha}{\alpha f}} \quad \Rightarrow \quad \boxed{\sqrt{\gamma} = h(x^i)\exp\left\{\int \frac{\mathrm{d}\alpha}{\alpha f}\right\}} \qquad (\dagger)$$

Comment: α finite $\Rightarrow$ $\int \frac{\mathrm{d}\alpha}{\alpha f}$ finite. So if $\gamma \to 0$, the lapse must collapse, $\alpha \to 0$.

Assume now a focussing singularity is encountered at finite $\tau = \tau_s$.

With $\mathrm{d}\tau = \alpha \mathrm{d}t$ this is coordinate time $\Delta t = \displaystyle\int_0^{\tau_s} \frac{\mathrm{d}\tau}{\alpha}$. Three possibilities:

(1) $\sqrt{\gamma} \to 0$ and α finite. Cannot happen for Bona-Massó slicing.

(2) $\gamma \to 0$, $\alpha \to$ simultaneously. The singularity can then be reached at finite or infinite t.

If it happens as $t \to \infty$, we have *Marginal singularity avoidance*.

(3) $\alpha \to 0$ before $\sqrt{\gamma} \to 0$. This is *strong singularity avoidance*.

Now let: $\sqrt{\gamma} \sim (\tau_s - \tau)^m$, $m > 1$ as $\gamma \to 0$,

$f(\alpha) = A\alpha^n$, $A > 0$ as $\alpha \to 0$.

Proposition:

1. For $n < 0$: strong singularity avoidance.
2. For $n = 0$, $mA \geq 1$: marginal singularity avoidance.
3. For $n > 0$ or ($n = 0$ and $mA < 1$): no singularity avoidance.

Proof. (Sketched)

$$\int \frac{\mathrm{d}\alpha}{\alpha f(\alpha)} = \frac{1}{A}\int \frac{\mathrm{d}\alpha}{\alpha^{n+1}} = \begin{cases} \ln\left(\alpha^{1/A}\right) & \text{for } n = 0 \\ -\frac{1}{nA}\alpha^{-n} & \text{for } n \neq 0 \end{cases}$$

For $n < 0$ in ($\dagger$): $\sqrt{\gamma} = h(x^i)\exp\left\{\dfrac{-1}{nA}\alpha^{-n}\right\}$, so γ is finite as $\alpha \to 0$

$\Rightarrow$ strong singularity avoidance.

For $n \geq 0$, we evaluate $\Delta t = \displaystyle\int_0^{\tau_s} \frac{\mathrm{d}\tau}{\alpha} = \int_{\alpha_0}^{0} \frac{\mathrm{d}\tau/\mathrm{d}\alpha}{\alpha}\mathrm{d}\alpha$

(i) If $\mathrm{d}\tau/\mathrm{d}\alpha$ vanishes faster than α^p for some $p > 0$, the integral and Δt are finite.

$\to$ no singularity avoidance.

(ii) If $\mathrm{d}\tau/\mathrm{d}\alpha$ is finite or larger, the integral and Δt diverge.

$\to$ marginal singularity avoidance.

Differentiating (†) gives $\dfrac{\mathrm{d}\ln\sqrt{\gamma}}{\mathrm{d}\tau} = \dfrac{1}{\alpha f}\dfrac{\mathrm{d}\alpha}{\mathrm{d}\tau}$.

Insert $f(\alpha)$ to compute $\tau(\alpha)$ and, thus, $\mathrm{d}\tau/\mathrm{d}\alpha$, which confirms the proposition. □

H.2.3 Shift conditions

Goal: Ensure neighbouring observers see "similar" evolution of $\tilde{\gamma}_{ij}$. Recall $\tilde{\gamma} = 1$!

Def.: *Distortion tensor*: $\Sigma_{ij} := \dfrac{1}{2}\gamma^{1/3}\partial_t\tilde{\gamma}_{ij} = \dfrac{1}{2\chi}\partial_t\tilde{\gamma}_{ij}$.

Minimizing the integral $\Sigma_{mn}\Sigma^{mn}$ over the hypersurface

→ ellitic PDE for β^i, *minimal distortion shift*

Lemma: $$\Sigma^{ij} = \frac{\chi}{2}\left[-\beta^m\partial_m\tilde{\gamma}^{ij} + \tilde{\gamma}^{jl}\partial_l\beta^i + \tilde{\gamma}^{il}\partial_l\beta^j - \frac{2}{3}\tilde{\gamma}^{ij}\partial_m\beta^m - 2\alpha\tilde{A}^{ij}\right],$$

and $\gamma_{mn}\Sigma^{mn} = 0$.

Proof. Example sheets. □

Proposition: $$2\partial_j(\chi^{-1}\Sigma^{ij}) = \frac{2}{\chi}\left(D_j\Sigma^{ij} - \tilde{\Gamma}^i_{jk}\Sigma^{jk} + \frac{3}{2}\frac{\partial_j\chi}{\chi}\Sigma^{ij}\right) = \partial_t\tilde{\Gamma}^i\,.$$

Proof. Example sheets. □

This motivates the *Gamma freezing shift condition*: $\tilde{\Gamma}^i = 0$.

Comments:
- Minimal distortion and Gamma freezing only differ by terms ∝ 1st metric derivatives × Σ_{mn}.
- Setting $\partial_t\tilde{\Gamma}^i = 0$ in the BSSNOK eq. for $\tilde{\Gamma}^i$ gives an elliptic PDE for β^i.
- However, we only need to solve it once! Just don't evolve $\tilde{\Gamma}^i$.

Better yet, *Gamma driver* shift: $\partial_t\beta^i = F\partial_t\tilde{\Gamma}^i$ or $\partial_t^2\beta^i = F\partial_t\tilde{\Gamma}^i - \tilde{\eta}\partial_t\beta^i$,

where $F > 0$ and $\tilde{\eta}\beta^i$ is added to avoid oscillations in β^i.

In practice:
- $\partial_t\beta^i = \dfrac{3}{4}B^i$, $\partial_t B^i = \partial_t\tilde{\Gamma}^i - \eta B^i$ with $M\tilde{\eta} = \mathcal{O}(1)$ work well.
- It also works well to add advection derivatives: $\partial_t \to \partial_t - \beta^m\partial_m$.

Together with 1+log slicing this is called *moving puncture gauge* ($\kappa = 1$ or 0):

$$
\begin{aligned}
\partial_t \alpha &= \kappa \beta^m \partial_m \alpha - 2\alpha K \,, \\
\partial_t \beta^i &= \kappa \beta^m \partial_m \beta^i + \frac{3}{4} B^i \\
\partial_t B^i &= \kappa \beta^m \partial_m B^i + (\partial_t - \kappa \beta^m \partial_m) \tilde{\Gamma}^i - \eta B^i \,.
\end{aligned}
$$

I Gravitational-wave diagnostics

I.1 GW strain and the Newman-Penrose scalar

Recall from Bondi-Sachs: $R_{\mu\nu\rho\sigma}\mathsf{k}^\mu\mathsf{m}^\nu\mathsf{k}^\rho\mathsf{m}^\sigma = -\frac{\mathrm{i}}{r}\partial_u^2\bar{c} = -\frac{1}{2}(\mathrm{i}\partial_u^2 h_+ + \partial_u^2 h_\times)\,,$ (†)

where $\mathbf{k} \simeq -\frac{1}{2}(\mathbf{e}_T - \mathbf{e}_R)\,,\qquad \boldsymbol{\ell} \simeq \mathbf{e}_T + \mathbf{e}_R\,,\qquad \mathbf{m} \simeq \frac{\mathbf{e}_\theta + \mathbf{e}_\phi}{2} + \mathrm{i}\frac{\mathbf{e}_\theta - \mathbf{e}_\phi}{2}$

Rescale: $\widetilde{\mathbf{k}} := -\sqrt{2}\mathbf{k} \simeq \frac{\mathbf{e}_T - \mathbf{e}_R}{\sqrt{2}}\,,\qquad \widetilde{\boldsymbol{\ell}} := \sqrt{\ell} \simeq \frac{\mathbf{e}_T + \mathbf{e}_R}{\sqrt{2}}\,,\qquad \widetilde{\mathbf{m}} := \frac{1+\mathrm{i}}{\sqrt{2}}\mathbf{m} \simeq \frac{\mathbf{e}_\theta + \mathrm{i}\mathbf{e}_\phi}{\sqrt{2}}$

$$\Rightarrow R_{\mu\nu\rho\sigma}\mathsf{k}^\mu\mathsf{m}^\nu\mathsf{k}^\rho\mathsf{m}^\sigma = \frac{\mathrm{i}}{2}R_{\mu\nu\rho\sigma}\tilde{\mathsf{k}}^\mu\,\overline{\widetilde{\mathsf{m}}}^\nu\,\tilde{\mathsf{k}}^\rho\,\overline{\widetilde{\mathsf{m}}}^\sigma$$

<u>Def.:</u> *Newman Penrose scalar* $\Psi_4 = C_{\mu\nu\rho\sigma}\tilde{\mathsf{k}}^\mu\,\overline{\widetilde{\mathsf{m}}}^\nu\,\tilde{\mathsf{k}}^\rho\,\overline{\widetilde{\mathsf{m}}}^\sigma\,,$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor. In vacuum $C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$.

<u>Proposition:</u> In vacuum $\Psi_4 = -\ddot{h}_+ + \mathrm{i}\ddot{h}_\times = \partial_T^2 H$ with $H := -h_+ + \mathrm{i}h_\times\,.$

Proof. $\frac{2}{\mathrm{i}}R_{\mu\nu\rho\sigma}\mathsf{k}^\mu\mathsf{m}^\nu\mathsf{k}^\rho\mathsf{m}^\sigma \overset{(\dagger)}{=} -\partial_u^2 h_+ + \mathrm{i}\partial_u^2 h_\times$ and $\partial_T = \partial_u$. □

I.2 GW energy and momentum

In 2nd-order perturbation theory, quadratic 1st-order perturbations source the 2nd-order perturbations.

$\rightarrow$ *Isaacson stress-energy tensor*: $t_{\mu\nu} = \frac{1}{32\pi}\left\langle \partial_\mu\bar{h}_{\rho\sigma}\,\partial_\nu\bar{h}^{\rho\sigma} - \frac{1}{2}\partial_\mu\bar{h}\,\partial_\nu\bar{h} - 2\partial_\sigma\bar{h}^{\rho\sigma}\,\partial_{(\mu}\bar{h}_{\nu)\rho}\right\rangle\,,$

where $\langle\,.\,\rangle$ averages over large volumes.

One can show: $t_{\mu\nu}$ is gauge invariant, so use TT gauge

$$\Rightarrow t_{\mu\nu} = \frac{1}{32\pi}\langle\partial_\mu h_{ij}^{\mathrm{TT}}\,\partial_\nu h_{ij}^{\mathrm{TT}}\rangle \qquad \text{(sum over } i,\ j)$$

Energy flux across surface $x^k = \text{const}$: $t^{0k} = \frac{1}{32\pi}\langle\partial^0 h_{ij}\,\partial^k h_{ij}\rangle = \frac{1}{32\pi}\langle-\partial_0 h_{ij}\,\partial_k h_{ij}\rangle\,.$

$\Rightarrow$ flux across surface element $\mathrm{d}A$ with normal n^k: $\frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A} = t^{0k}n_k$

On sphere $R = \text{const}$: $n^k = \frac{x^k}{R},\ \ \frac{x^k}{R}\partial_k = \partial_R$

$$\Rightarrow \quad \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A} = \frac{1}{32\pi}\left\langle -\partial_0 h_{ij}\,\partial_k h_{ij}\,\frac{x_k}{R}\right\rangle = \frac{1}{32\pi}\langle-\partial_T h_{ij}\,\partial_R h_{ij}\rangle\,,$$

Outgoing radiation: $h_{ij} = h_{ij}(T-R) \quad \Rightarrow \quad \partial_R h_{ij} = -\partial_T h_{ij} = -\partial_0 h_{ij}$

$$\Rightarrow \quad \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A} = \frac{1}{32\pi}\langle \partial_0 h_{ij}\,\partial_0 h_{ij}\rangle$$

At each point on the sphere, rotate coordinates such that z becomes the radial direction

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \partial_0 h_{ij}\,\partial_0 h_{ij} = 2(\partial_0 h_+)^2 + 2(\partial_0 h_\times)^2$$

$$\Rightarrow \quad \frac{\mathrm{d}E}{\mathrm{d}t\,\mathrm{d}A} = \frac{1}{16\pi}\langle(\partial_0 h_+)^2 + (\partial_0 h_\times)^2\rangle\,. \quad \text{Same as in Bondi-Sachs!}$$

With $\Psi_4 = \partial_T^2 H\,,\ H = -h_+ + \mathrm{i}h_\times \quad \Rightarrow \quad |\partial_T H|^2 = (\partial_T h_+)^2 + (\partial_T h_\times)^2 \stackrel{!}{=} \left|\int_{-\infty}^{T} \Psi_4 \mathrm{d}\tilde{t}\right|^2$,

$$\Rightarrow \quad \boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \lim_{R\to\infty}\frac{R^2}{16\pi}\oint |\partial_T H|^2 \mathrm{d}\Omega = \lim_{R\to\infty}\frac{R^2}{16\pi}\oint \left|\int_{-\infty}^{T}\Psi_4 \mathrm{d}\tilde{t}\right|^2 \mathrm{d}\Omega\,.}$$

Radiated linear momentum:

As $R\to\infty$: $\quad \dfrac{\partial}{\partial x^l}h_{ij} \to \dfrac{\partial R}{\partial x^l}\partial_R h_{ij}$

$$\Rightarrow \quad \frac{\mathrm{d}P^l}{\mathrm{d}t\,\mathrm{d}A} = t^{lk}n_k = \frac{1}{32\pi}\left\langle \partial_l h_{ij}\,\partial_k h_{ij}\frac{x^k}{r}\right\rangle = \frac{1}{32\pi}\left\langle \frac{x_l}{R}\partial_R h_{ij}\,\partial_R h_{ij}\right\rangle = \frac{1}{32\pi}n_l\langle \partial_T h_{ij}\,\partial_T h_{ij}\rangle$$

with $\quad n_l = \dfrac{x_l}{R} = (\sin\theta\cos\phi,\ \ \sin\theta\sin\phi,\ \ \cos\theta)$

$$\Rightarrow \quad \boxed{\frac{\mathrm{d}P_l}{\mathrm{d}t} = \lim_{R\to\infty}\frac{R^2}{16\pi}\oint n_l|\partial_t H|^2 \mathrm{d}\Omega = \lim_{R\to\infty}\frac{R^2}{16\pi}\oint n_l\left|\int_{-\infty}^{T}\Psi_4 \mathrm{d}\tilde{t}\right|^2 \mathrm{d}\Omega\,.}$$

Without proof, the angular momentum in GWs is

$$\boxed{\frac{\mathrm{d}J_i}{\mathrm{d}t} = -\lim_{R\to\infty}\frac{R^2}{16\pi}\mathsf{Re}\left\{\oint \hat{J}_i H\,\partial_T \bar{H}\,\mathrm{d}\Omega\right\} = -\lim_{R\to\infty}\frac{R^2}{16\pi}\mathsf{Re}\left\{\oint \hat{J}_i\left(\int_{-\infty}^{T}\int_{-\infty}^{\hat{t}}\Psi_4 \mathrm{d}\tilde{t}\,\mathrm{d}\hat{t}\right)\times\left(\int_{-\infty}^{T}\bar{\Psi}_4 \mathrm{d}\tilde{t}\right)\right\}}\,,$$

with $\ \hat{J}_x = -\sin\phi\,\partial_\theta - \cos\phi\left(\cot\theta\,\partial_\phi - \dfrac{2\mathrm{i}}{\sin\theta}\right)$,

$\qquad \hat{J}_y = \cos\phi\,\partial_\theta - \sin\phi\left(\cot\theta\,\partial_\phi - \dfrac{2\mathrm{i}}{\sin\theta}\right) \qquad$ and $\qquad \hat{J}_z = \partial_\phi\,.$

I.3 The mutipolar decomposition

Project Ψ_4 (or H) onto spherical harmonics of spin weight $s = -2$:

$$\psi_{lm} := \langle Y^{-2}_{lm}, \Psi_4 \rangle := \int_0^{2\pi} \int_0^{\pi} \Psi_4 \bar{Y}^{-2}_{lm} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, .$$

$$\Rightarrow \quad \Psi_4(t,\theta,\phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \psi_{lm}(t) Y^{-2}_{lm}(\theta,\phi)$$

Y^{-2}_{lm} form a complete orthonormal basis, so $\quad \langle Y^{-2}_{lm}, \bar{Y}^{-2}_{l'm'} \rangle = \delta_{ll'}\delta_{mm'}$

E.g. $Y^{-2}_{22}(\theta,\phi) = \sqrt{\frac{5}{64\pi}}\,(1+\cos\theta)^2 e^{2\mathrm{i}\phi}\,, \ldots$

Energy in individual multipoles: $\quad \dfrac{\mathrm{d}E}{\mathrm{d}t} = \sum_{l,m} \dot{E}_{lm} \quad$ with $\quad \dot{E}_{lm} = \lim_{R\to\infty} \dfrac{R^2}{16\pi} \left| \int_{-\infty}^{T} \psi_{lm} \mathrm{d}\tilde{t} \right|^2 .$

Angular and linear momentum arise from overlap of different multipoles.

I.4 An example of a GW signal

11 orbit inspiral of a non-spinning 1 : 4 mass-ratio BH binary.