Introduction to Numerical Relativity

Ulrich Sperhake

DAMTP, University of Cambridge

Summerschool

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Gravitational Waves: Ripples in spacetime

- Unusual news headlines on 11/12 February 2016
- First direct detection of gravitational waves: GW150914


GRAVITY AI

Scots scientists played key part in 'monumenta leap in physics'

theguardia

THE TIMES

SCOTLAND

Country life

Black hole proves Einstein's theory (and it's about time)

The INDEPENDENT

Give to GOSH: Our thanks for the £3.56m raised

The theory of relativity proved

Huge explosion in a galaxy far, far away confirms Einstein's space-time theory

THE SCOTSMAN

University team sheds fresh light on mysteries of universe

EL PAÍS

EL PERIÓDICO GLOBAL

La sombra de una nueva crisis bancaria hunde los mercados

El Ibex cae un 4,88% y la prima de riesgo llega a 169, máximo desde 2010

El PP madrileño, investigado también por finanzas ilegales

La Guardia Civil busca en la sede de Génova y en la casa de López Madrid pruebas de la conexión de la red Púnica con pagos al partido

Una empresa con un contrato público pagó la campaña de la líder popular valenciana

Un motín salvaje en una cárcel de México causa 52 muertos

La detección de ondas gravitacionales confirma el "sonido del cosmos" y abre vías inéditas en la observación del universo

Einstein tenía razón

Un nuevo sentido para percibir la realidad

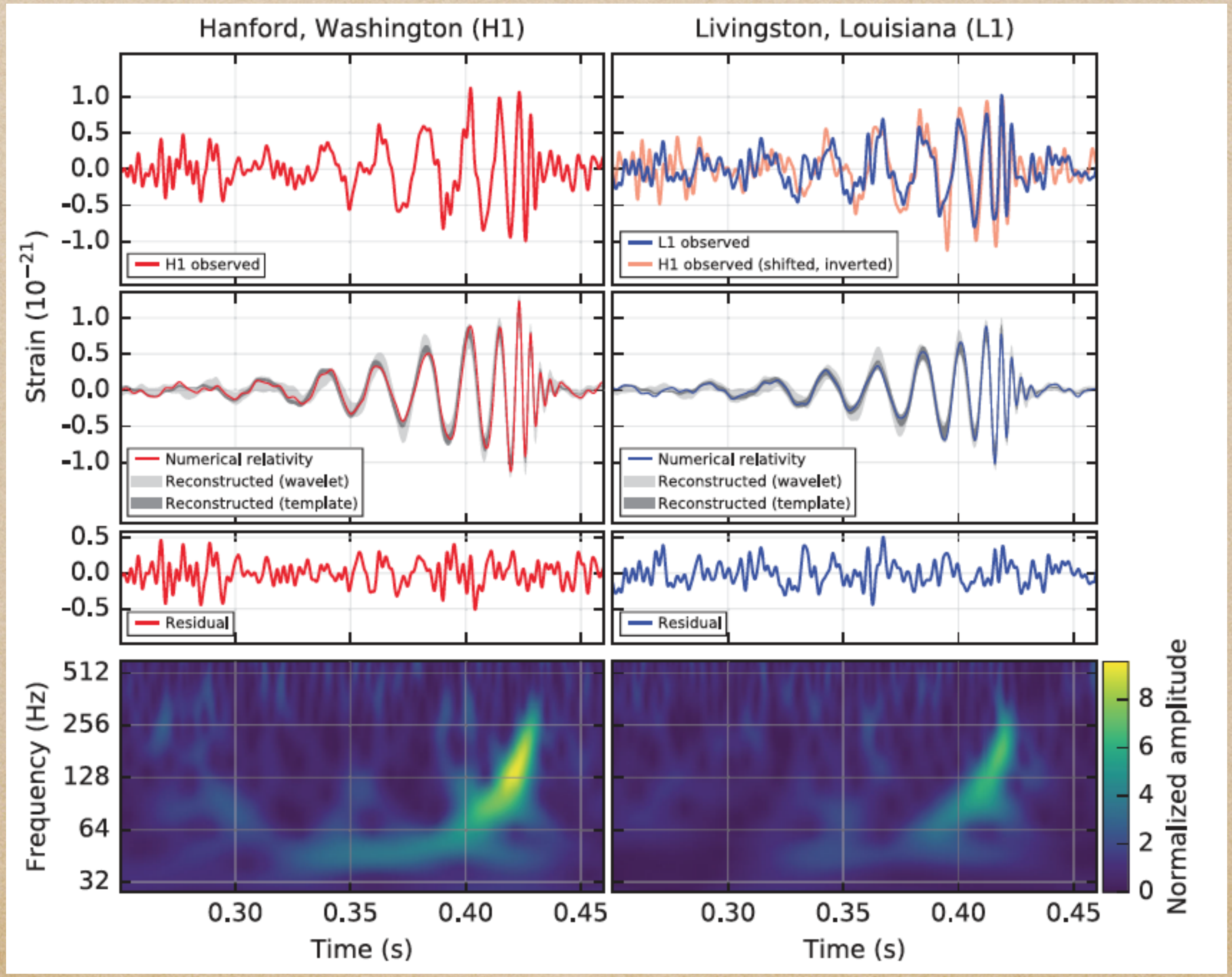
idealista

So, what happened?

- Sep 14, 2015 at 09:50:45 UTC: SNR ~ 24

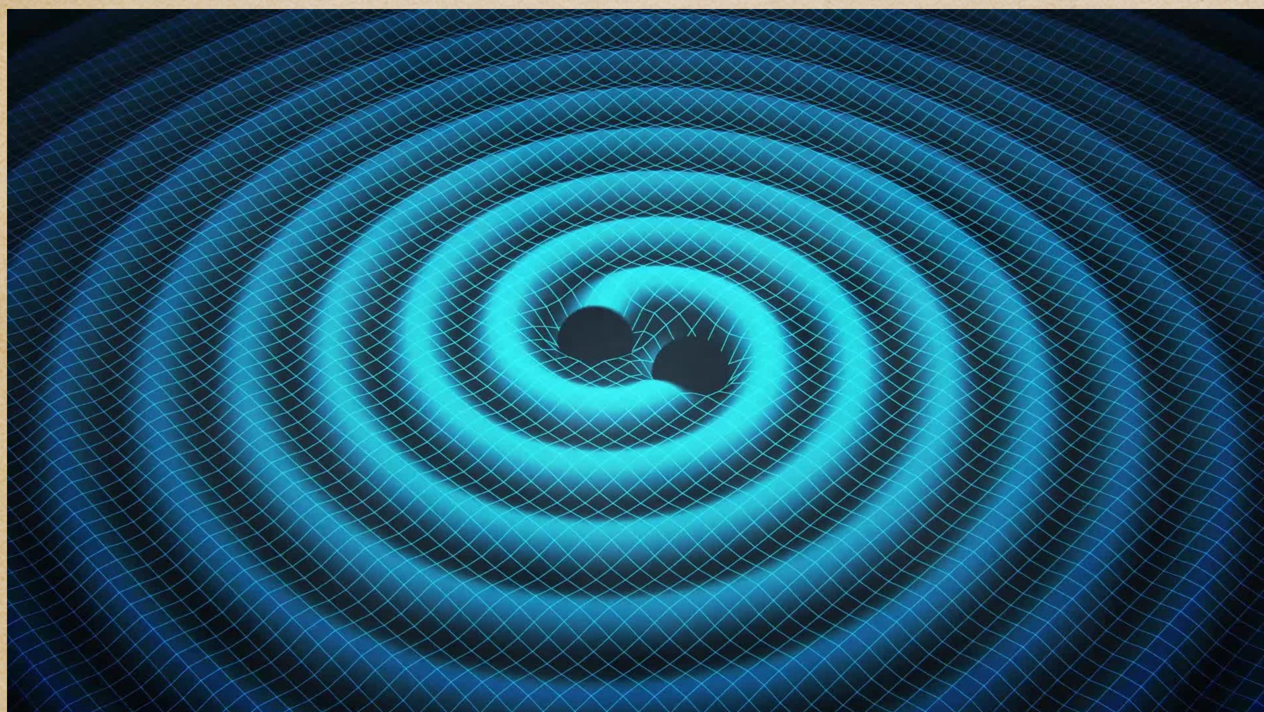
  Abbott et al. PRL 1602.03837, Abbott et al. 1606.01210

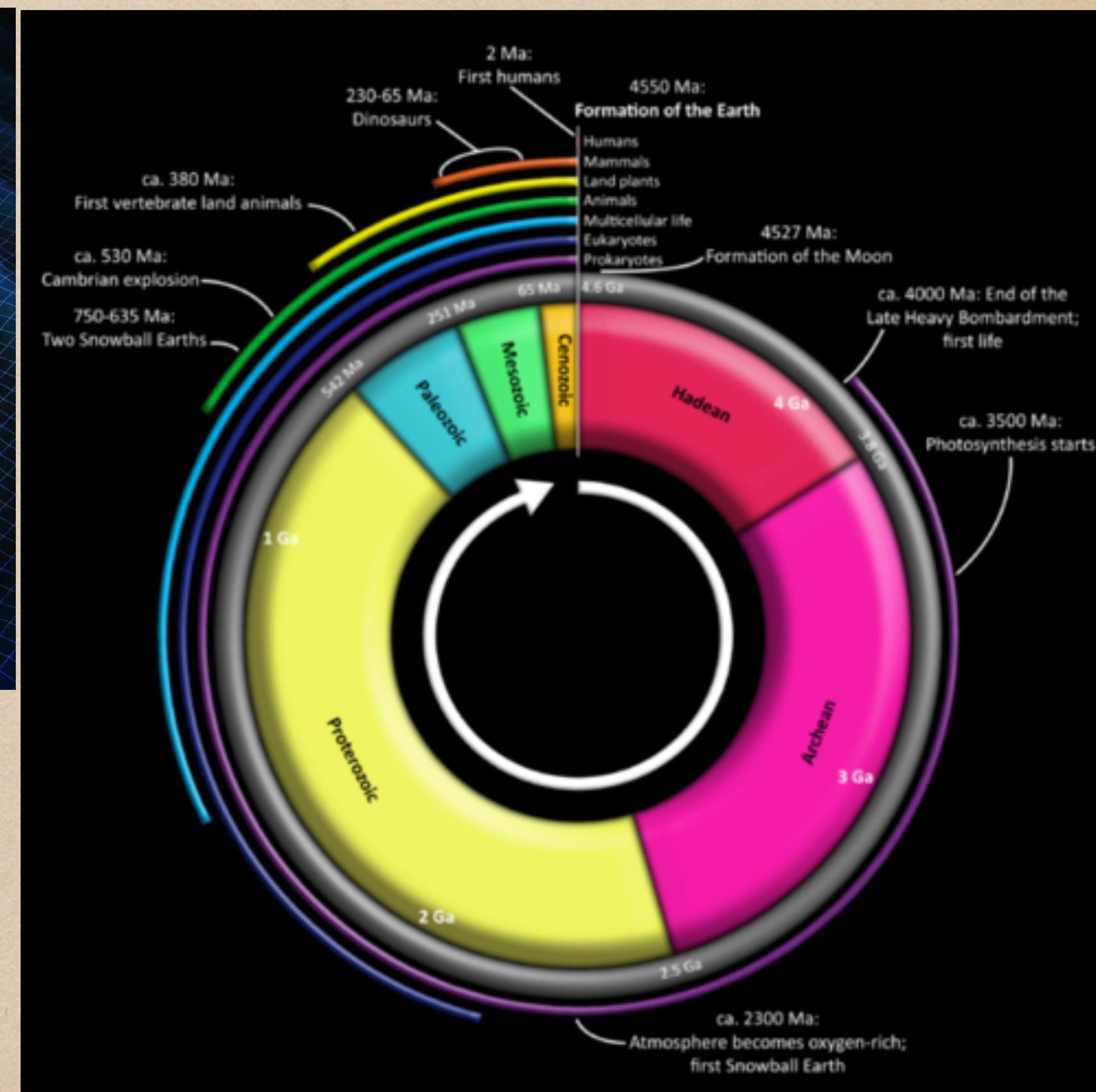
- BBH inspiral, merger and ringdown: $m_1 = 35^{+5}_{-3}\, m_\odot\,, \quad m_2 = 30^{+3}_{-4}\, M_\odot$



What really happened…

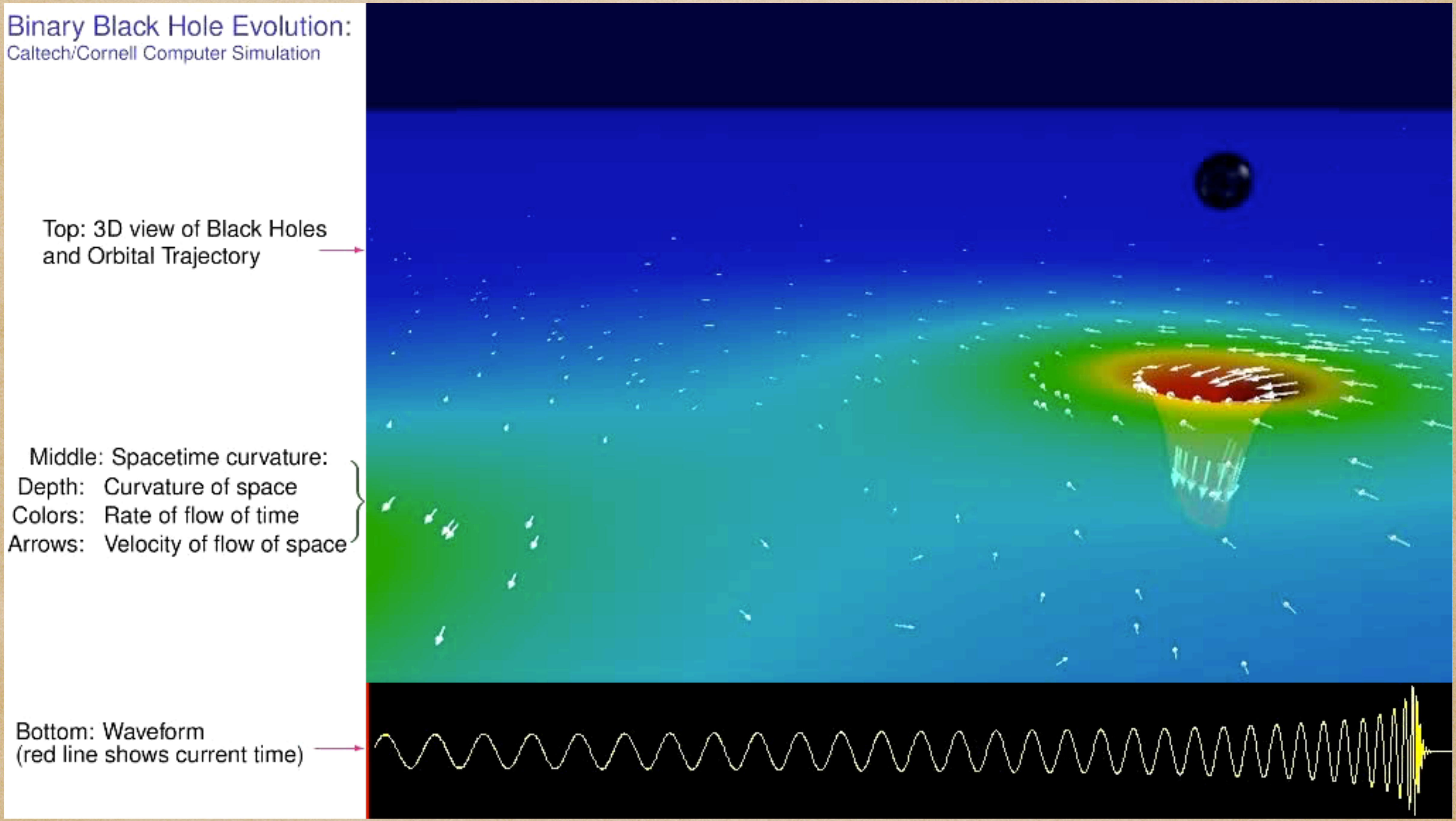
- Once upon a time: $1.34^{+0.52}_{-0.59}\ \mathrm{Gyr}$ ago, somewhere in the universe





- Deep Precambrian

We can model this with NR



Thanks to Caltech-Cornell groups

Overview

- Introduction, Motivation
- Foundations of numerical relativity
  - Formulations of Einstein's Eqs.: 3+1, BSSN, GHG, characteristic
  - Initial data, gauge
  - Technical ingredients: Discretization, AMR, boundaries…
  - Diagnostics: Horizons, momenta, GWs,…
- Applications and selected results
  - Astrophysics
  - Gravitational wave physics
  - Fundamental properties of gravity

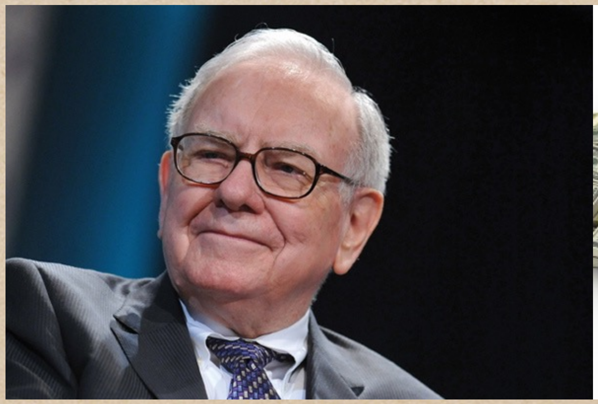
1. Introduction, Motivation

Strong gravity = non linearity

- What is non-linearity?   Think of the stock market

⇒ linear

⇒ NON-LINEAR!

Strongest possible gravity: Black holes

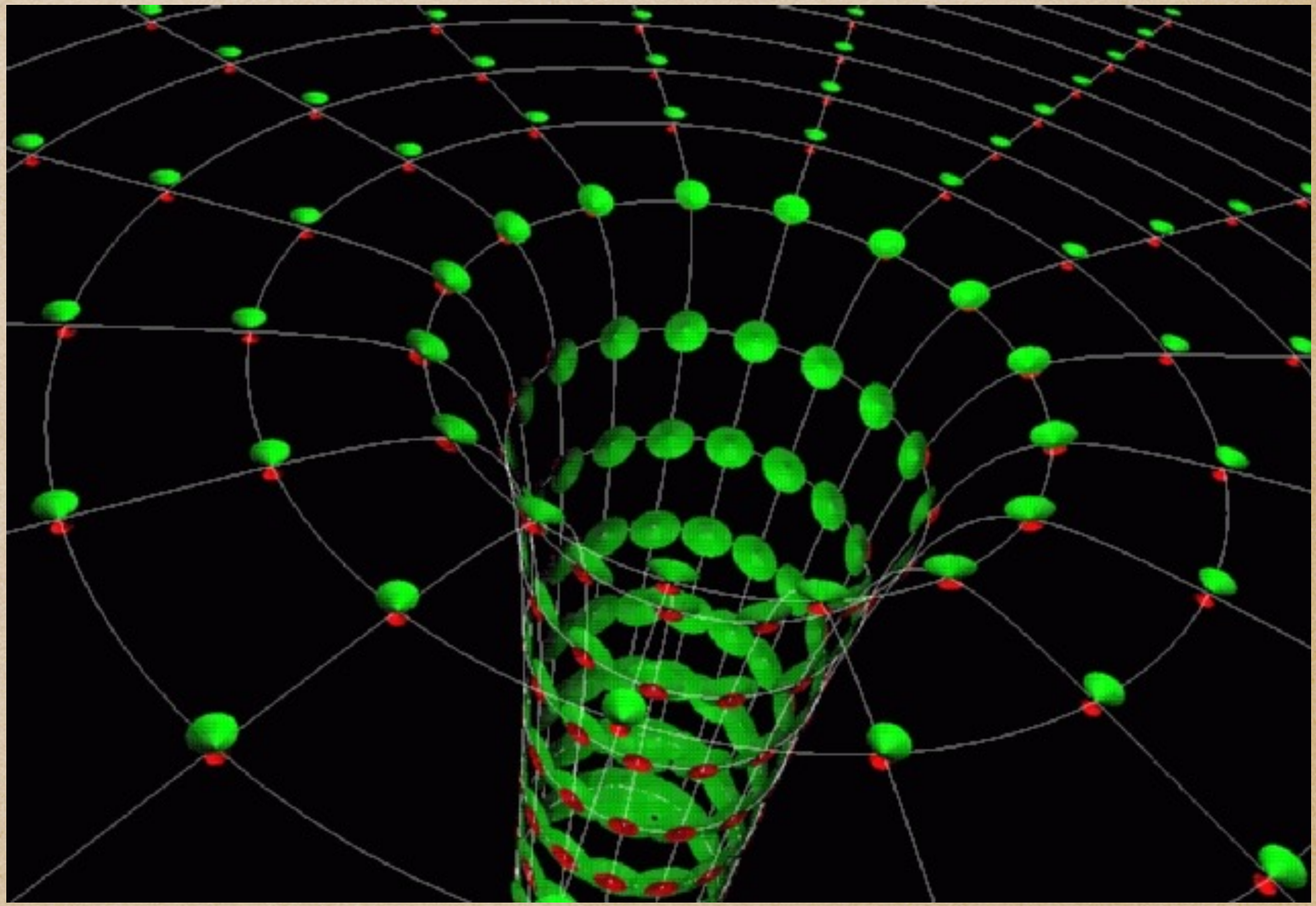
- Einstein 1915: General Relativity; geometric theory of gravity
- Schwarzschild 1916: Solution to Einstein's equations

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 d\phi^2$$

- Singularities

  $r = 0$: physical

  $r = 2M$: singularity



- Horizon at $r = 2M$

  Light cones tilt over

- Newtonian escape velocity

$$v = \sqrt{\frac{2M}{r}}$$

Black-hole analogy

Evidence for astrophysical BHs

- LIGO observation of GWs (above)
- X-ray binaries

  e.g. Cygnus X-1 (1964)

  MS star + compact star

  $\Rightarrow$ Stellar Mass BH

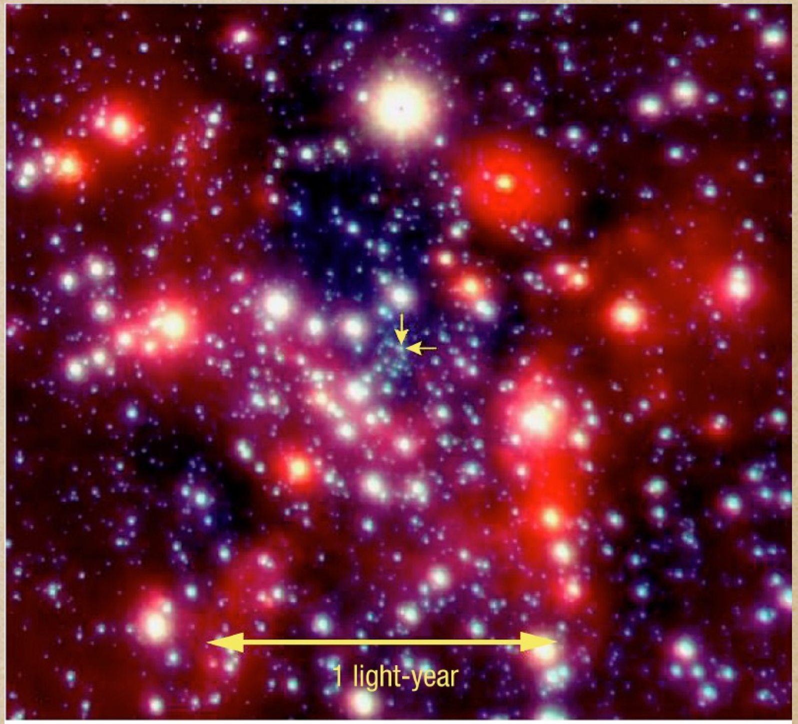
  $5 \ldots 50\ M_\odot$
- Stellar dynamics near galactic center

  Iron emission line profiles

  $\Rightarrow$ Supermassive BHs

  $10^6 \ldots 10^{10}\ M_\odot$

  AGN engines



The Centre of the Milky Way
(VLT YEPUN + NACO)
ESO PR Photo 23a/02 (9 October 2002)

Conjectured BHs

- Intermediate mass BHs

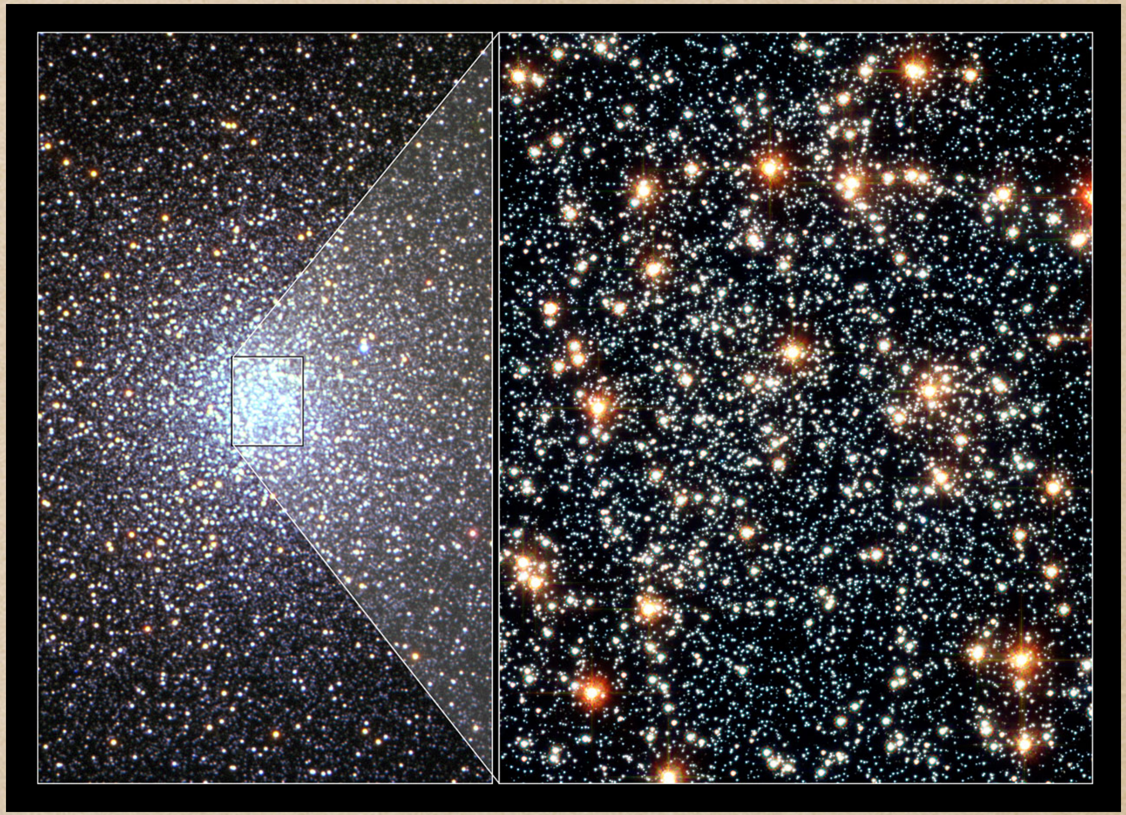
  $10^2 \ldots 10^5 M_\odot$

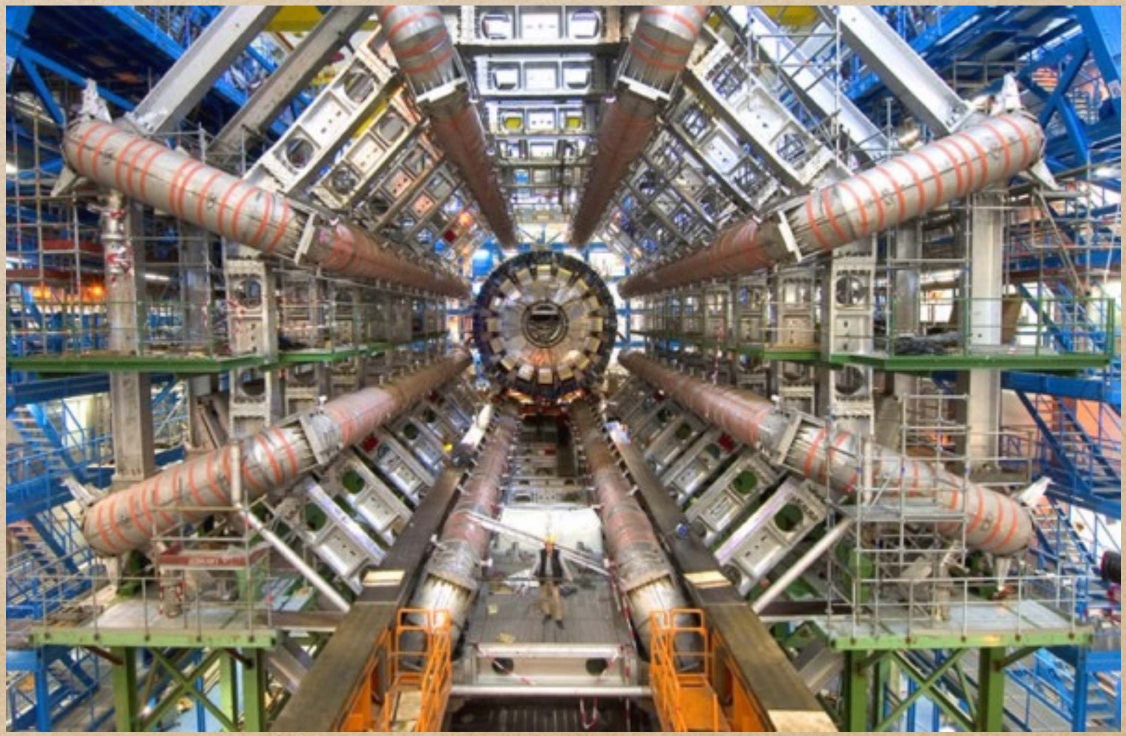
- Primordial BHs

  $\leq M_{\rm Earth}$

- Microscopic BHs, LHC

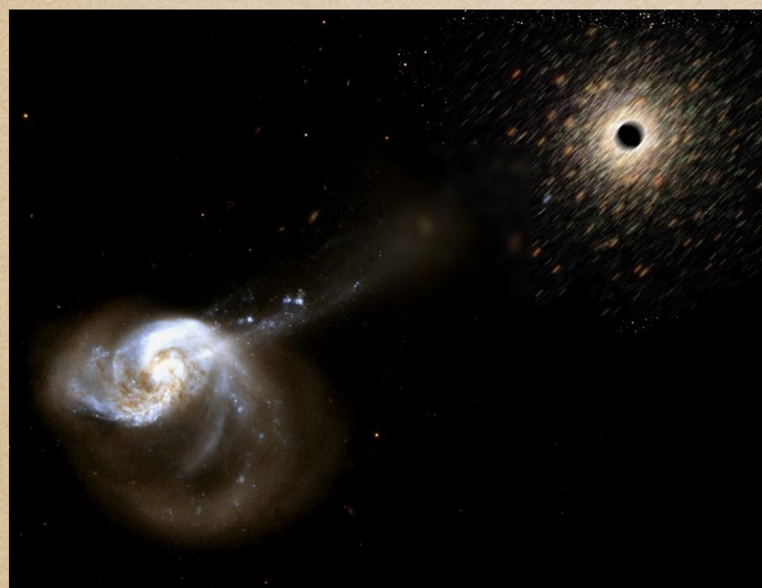
  $\sim \mathrm{TeV}$



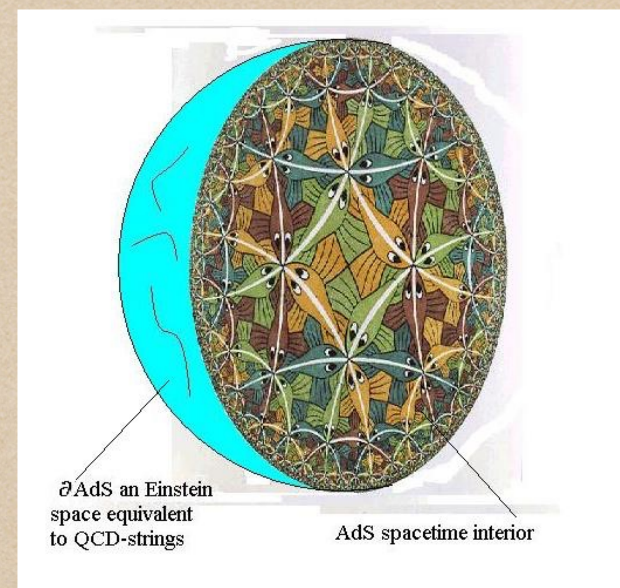


Research areas: BHs have come a long way

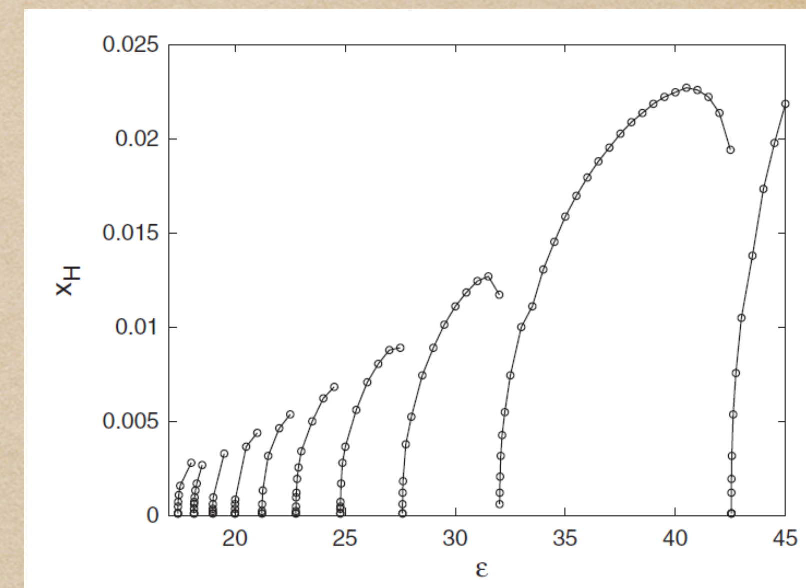
Astrophysics



Gauge gravity duality

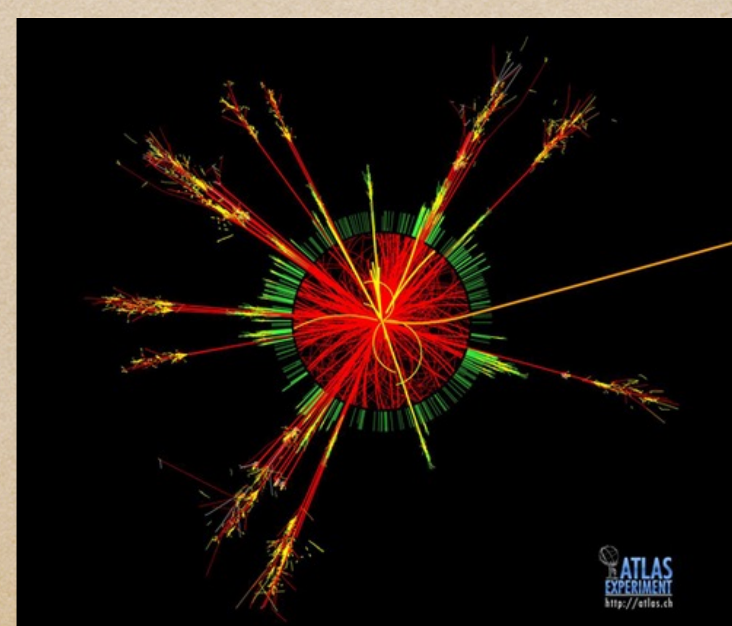


Fundamental studies

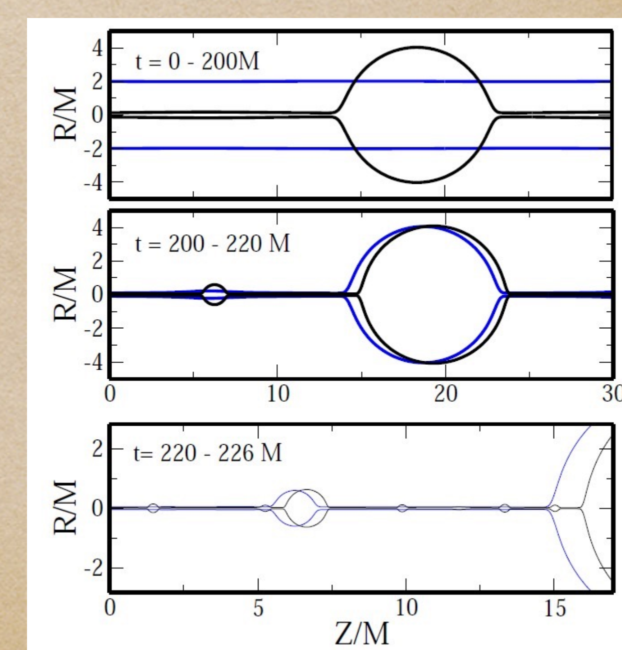


GW physics

High-energy physics



Fluid analogs



Vitor's talk in 30 seconds

- Spacetime as a curved manifold
- Key quantity: spacetime metric $g_{\alpha\beta}$
- Curvature, geodesics etc. all follow
- Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

10 non-linear PDEs for $g_{\alpha\beta}$

$T_{\alpha\beta}$ = Matter fields

- Conceptually simple,

  hard in practice
- E.g. Schwarzschild

$$g_{\mu\nu} = \begin{pmatrix} \left(1 - \frac{2GM}{rc^2}\right) & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2GM}{rc^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

$$ds^2 = c^2 dt^2 \left(1 - \frac{2GM}{rc^2}\right) - \frac{dr^2}{1 - 2GM/rc^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

How do we get the metric?

Solving this equation is our job

How do we get the metric?

- The metric must obey $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$
- Ricci tensor, Einstein tensor, matter tensor

$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$

$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{\mu}{}_{\mu}$ "Trace reverse Ricci"

$T_{\alpha\beta}$ "Matter"; see Talk by Luciano Rezzolla

Λ "Cosmological constant"

- Solutions: Easy!

Take metric $g_{\alpha\beta}$

$\Rightarrow$ Calculate $G_{\alpha\beta}$

$\Rightarrow$ Use that for $T_{\alpha\beta}$

- Physically meaningful solutions: That's the hard part!

Solving Einstein's Eqs.: The toolbox

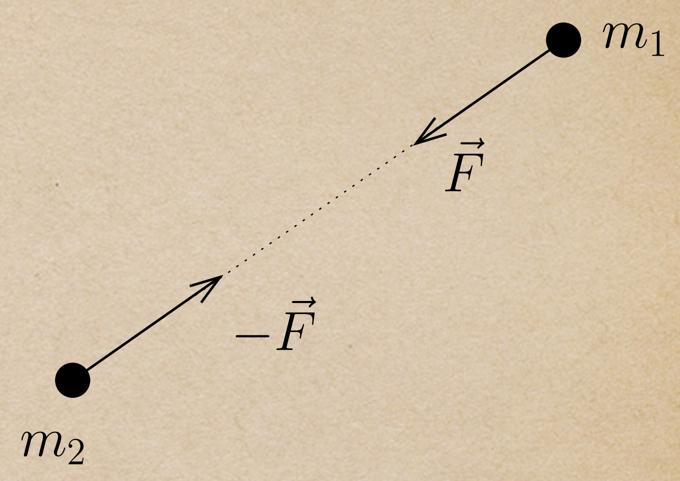
- **Analytic solutions**
  - Symmetry assumptions
  - Schwarzschild, Kerr, FLRW, Vaidya, Tangherlini, Myers-Perry, ...
- **Perturbation theory**
  - Assume solution is close to a known "background" $g^{(0)}_{\alpha\beta}$
  - Expand $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots \quad \Rightarrow$ linear system
  - Regge-Wheeler-Zerilli-Moncrief, Teukolsky, QNMs, EOB, ...
- **Post-Newtonian theory**
  - Assume small velocities $\Rightarrow$ Expansion in $\frac{v}{c}$
  - N^{th} order expressions for GWs, momenta, orbits, ...
  - Blanchet, Buonanno, Damour, Kidder, Schäfer, Will, ...
- **Numerical Relativity**

2. Foundations of Numerical Relativity

The Newtonian 2-body problem

- Eqs. of motion

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = \vec{F} = -G\frac{m_1 m_2}{r^2}\hat{\vec{r}} = -m_2\frac{d^2\vec{r}_2}{dt^2}$$



- Solution: Keppler ellipses, parabolic, hyperolic

$$r = \frac{r_0}{1+\epsilon\cos\theta}$$

e.g. Sperhake CQG 1411.3997

- What's different in GR?
  - No point particles in GR!
  - GR is non-linear
  - No "background" time and space
  - Systems typically are dissipative $\Rightarrow$ Gravitational waves
  - No obvious formulation as time evolution problem

A list of tasks in NR

- **Target:** Predict time evolution of a physical system in GR
- **Einstein eqs.:** 1) Cast as evolution system

  2) Choose a “good” formulation

  3) Discretize for a computer
- **Gauge:** Choose “good” coordinates
- **Technical aspects:** 1) Mesh refinement / spectral domains

  2) Singularity handling (excision)

  3) Parallelization
- **Initial data:** 1) Solve constraints

  2) Get “realistic” initial data
- **Diagnostics:** 1) GW extraction, kicks, …

  2) Horizon data, ADM mass,…

Notation

- Spacetime indices: Greek $\alpha,\,\beta,\,\ldots = 0,\,\ldots,\,D-1$
- Spatial indices: middle Latin $i,\,j,\,\ldots = 1,\,\ldots,\,D-1$
- Signature: $-+\ldots+$
- Christoffel symbols: $\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}(-\partial_{\mu}g_{\beta\gamma} + \partial_{\beta}g_{\gamma\mu} + \partial\gamma g_{\mu\beta})$
- Riemann tensor $R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\rho} + \Gamma^{\tau}_{\nu\sigma}\Gamma^{\mu}_{\tau\rho} - \Gamma^{\tau}_{\nu\rho}\Gamma^{\mu}_{\tau\sigma}$
- Units: $c = 1 = G$
- Spatial metric γ_{ij}
- Spatial Riemann, Ricci tensor: $\mathcal{R}^{i}{}_{jkl}\,,\ \ \mathcal{R}_{ij}$
- We use Γ for the spatial and spacetime Christoffel symbols.

  Unlike for Riemann, it will always be clear from the context.

2.1 Formulations of Einstein's equations

The Einstein equations

- Recall: $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$

- In this form, the mathematical character is unclear!

  hyperbolic, elliptic, parabolic?

- Coordinates x^α are on equal footing.

  Time singled out only through signature of the metric!

- Well-posedness of the equations? Suitable for numerics?

- There are various ways to address these questions

  $\rightarrow$ Formulations of the equations

2.1.1 ADM type 3+1 formulations

The 3+1 decomposition

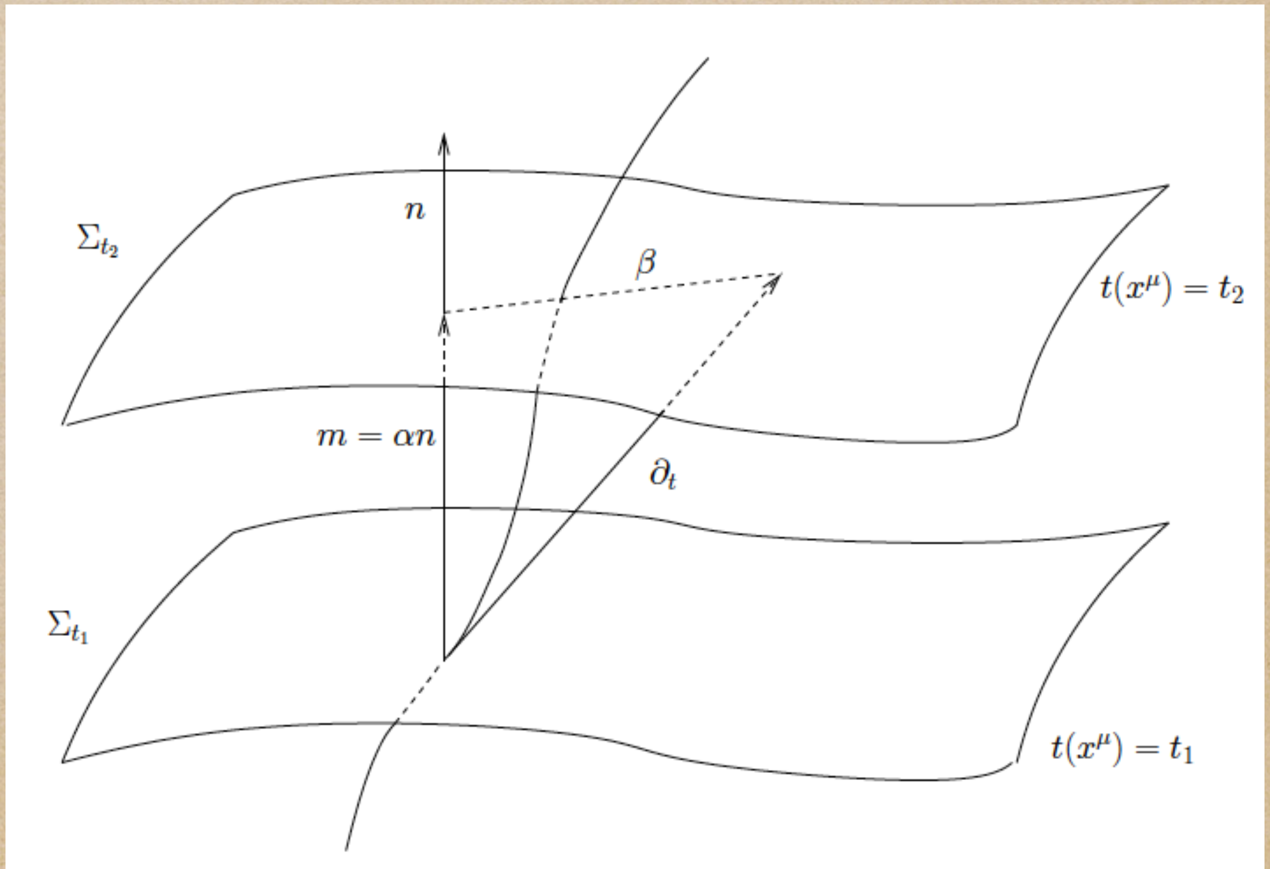
- ADM 3+1 split: Arnowitt, Deser & Misner 1962

  York 1979, Choquet-Bruhat & York 1980

Def.: Spacetime := $(\mathcal{M}, g)$

= Manifold with metric of signature $-+++$

Def.: Cauchy surface := A spacelike hypersurface Σ in $\mathcal{M}$ such that each timelike or null curve without endpoint intersects Σ exactly once.



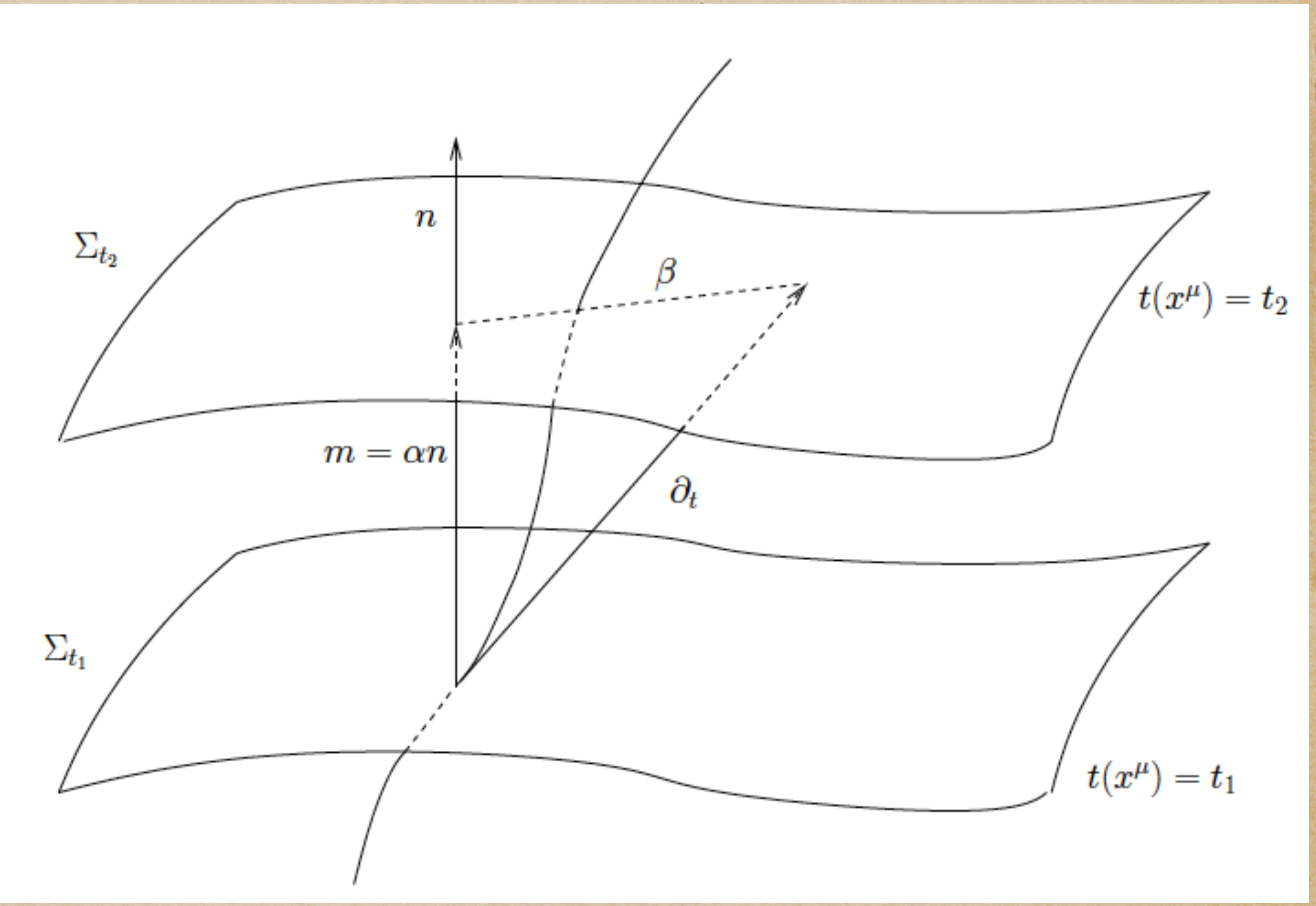
The 3+1 decomposition

Def.: A spacetime is globally hyperbolic

:$\Leftrightarrow$ it admits a Cauchy surface

- From now on:

  Let $(\mathcal{M}, g)$ be glob.hyp.



Then one can show:

$\exists$ smooth $t : \mathcal{M} \mapsto \mathbb{R}$

such that

1) The gradient $\mathbf{d}t \neq 0$

   everywhere

2) level surfaces $t = \text{const}$ are hypersurfaces:

$$\forall_{t_1 \in \mathbb{R}} \ \ \Sigma_{t_1} = \{p \in \mathcal{M} \ \ : \ \ t(p) = t_1\} \ , \ \ \Sigma_{t_1} \cap \Sigma_{t_2} = \emptyset \ \Leftrightarrow \ t_1 \neq t_2$$

The 3+1 decomposition

- 1-Form: $\mathbf{d}t$; vector: $\frac{\partial}{\partial t} =: \partial_t \quad \Rightarrow \quad \langle \mathbf{d}t, \partial_t \rangle = 1$

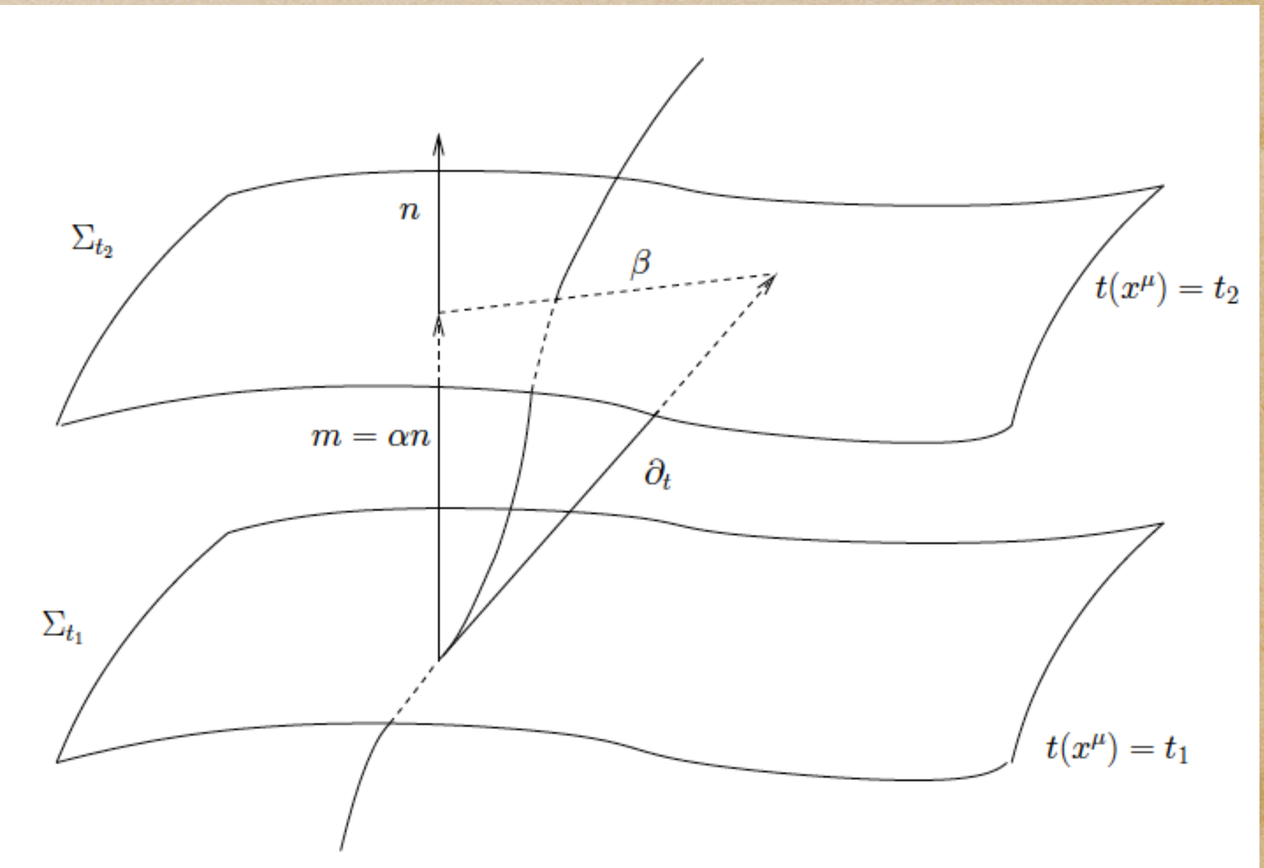
Def.: Time like unit field: $n_\mu := -\alpha (\mathbf{d}t)_\mu$

Lapse function: $\alpha := \frac{1}{||\mathbf{d}t||}$ Shift vector: $\beta^\mu := (\partial_t)^\mu - \alpha n^\mu$

Adapted coordinates: $(t, x^i)\,,\quad x^i$ label points in Σ_t

Adapted coordinate basis:

$$\partial_t = \alpha n + \beta\,, \quad \partial_i := \frac{\partial}{\partial x^i}$$



The 3+1 decomposition

Def.: A vector $\boldsymbol{v}^\alpha$ is tangent to Σ_t $:\Leftrightarrow$ $\langle \mathbf{d}t, \boldsymbol{v}\rangle = (\mathbf{d}t)_\mu v^\mu = 0$

Def.: Projector $\perp^\alpha{}_\mu := \delta^\alpha{}_\mu + n^\alpha n_\mu$ $\Rightarrow$ $\perp^\alpha{}_\mu n^\mu = 0$

- For a vector tangent to Σ_t one easily shows: $n_\mu v^\mu = 0$

$$\perp^\alpha{}_\mu v^\mu = v^\alpha$$

- Projection of the metric

$$\gamma_{\alpha\beta} := \perp^\mu{}_\alpha \perp^\nu{}_\beta g_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad \Rightarrow \quad \gamma_{\alpha\beta} = \perp_{\alpha\beta}$$

For $\boldsymbol{v}^\alpha$ tangent to Σ_t : $g_{\mu\nu} v^\mu v^\nu = \gamma_{\mu\nu} v^\mu v^\nu$

- In adapted coordinates $(t,\, x^i)$:

1) we can ignore t components for tensors tangential to Σ_t

2) γ_{ij} , $i = 1, \ldots, D-1$ is the metric on Σ_t "First fundamental form"

(D-1)+1 decomposition of the metric

- In adapted coordinates, we write the spacetime metric

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^2 + \beta_m \beta^m & \beta_j \\ \hline \beta_i & \gamma_{ij} \end{array}\right)$$

$$\Leftrightarrow \quad g^{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \hline \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{array}\right)$$

$$\Leftrightarrow \quad ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- Gauge variables: Lapse α , shift β^i

- For tensors tangent in all components to Σ_t we lower indices with γ_{ij} : $S^i{}_{jk} = \gamma_{jm} S^{im}{}_k$, etc.

Projections and spatial covariant derivative

Def.: Projections of an arbitrary tensor $\boldsymbol{S}$ of type $\binom{p}{q}$:

$$(\perp S)^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q} = \perp^{\alpha_1}{}_{\mu_1} \dots \perp^{\alpha_p}{}_{\mu_p} \perp^{\nu_1}{}_{\beta_1} \dots \perp^{\nu_q}{}_{\beta_q} S^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}$$

"Project every free index"

Def.: Spatial covariant derivative of a tensor $\boldsymbol{S}$ tangential to Σ_t :

$$D\boldsymbol{S} := \perp(\nabla \boldsymbol{S})$$

$$D_\rho S^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q} := \perp^{\alpha_1}{}_{\mu_1} \dots \perp^{\alpha_p}{}_{\mu_p} \perp^{\nu_1}{}_{\beta_1} \dots \perp^{\nu_q}{}_{\beta_q} \perp^{\sigma}{}_{\rho} \nabla_\sigma S^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}$$

Def.: One can show that

1) $D = \perp \nabla$ is torsion free on Σ_t if ∇ is on $\mathcal{M}$
2) $\nabla g_{\alpha\beta} = 0 \quad \Rightarrow \quad (D\gamma)_{ij} = 0$ "Metric compatible"
3) $D = \perp \nabla$ is unique in satisfying these properties

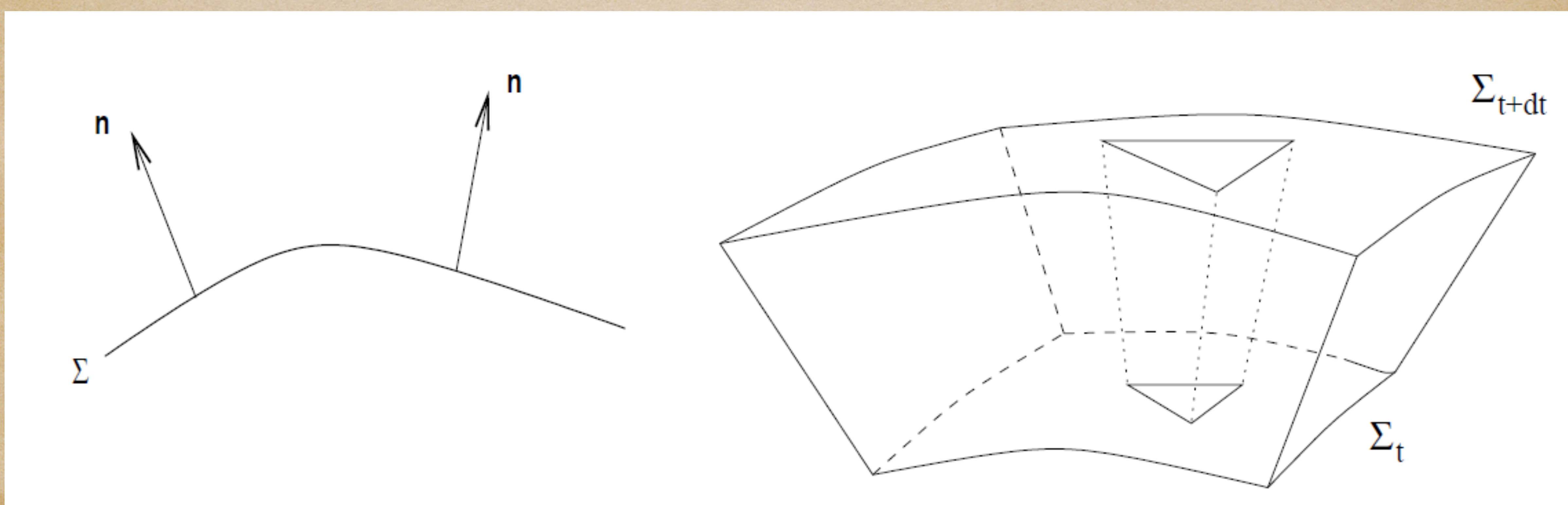
Extrinsic curvature

Def.: Extrinsic curvature: $K_{\alpha\beta} := -\perp\nabla_\beta n_\alpha$

- $\nabla_\beta n_\alpha$ is not symmetric, but $\perp\nabla_\beta n_\alpha$ is!
- The minus sign is a non-universal convention
- One can show that

$$\mathcal{L}_n \gamma_{\alpha\beta} = n^\mu \nabla_\mu \gamma_{\alpha\beta} + \gamma_{\mu\beta}\nabla_\alpha n^\mu + \gamma_{\alpha\mu}\nabla_\beta n^\mu = -2K_{\alpha\beta}$$

- Interpretation of $K_{\alpha\beta}$ $\rightarrow$ Embedding of Σ_t in $\mathcal{M}$



The projections of the Riemann tensor

- Projections of Riemann: $\perp R^{\alpha}{}_{\beta\gamma\delta}\,, \quad \perp R^{\alpha}{}_{\beta\gamma\mu} n^{\mu}\,, \quad \perp R^{\alpha}{}_{\mu\gamma\nu} n^{\mu} n^{\nu}$
- Starting point: Ricci identity $(\nabla_\gamma \nabla_\delta - \nabla_\delta \nabla_\gamma) Z^\alpha = R^{\alpha}{}_{\beta\gamma\delta} Z^\beta$

  Then a lengthy calculation yields   Gourgoulhon gr-qc/0703035

$$\perp R^{\alpha}{}_{\beta\gamma\delta} = \mathcal{R}^{\alpha}{}_{\beta\gamma\delta} + 2K^{\alpha}{}_{[\gamma} K_{\delta]\beta} \qquad \text{Gauss}$$

$$\perp R_{\alpha\beta} + \perp(R_{\alpha\delta\beta\nu}\, n^\delta n^\nu) = \mathcal{R}_{\alpha\beta} + K K_{\alpha\beta} - K_{\alpha\gamma} K^{\gamma}{}_{\beta} \qquad \text{contracted Gauss}$$

$$R + 2R_{\gamma\delta}\, n^\gamma n^\delta = \mathcal{R} + K^2 - K_{\gamma\delta} K^{\gamma\delta} \qquad \text{scalar Gauss}$$

$$\perp(R_{\alpha\beta\gamma\lambda}\, n^\lambda) = -D_\alpha K_{\beta\gamma} + D_\beta K_{\alpha\gamma} \qquad \text{Codazzi}$$

$$\perp(R_{\beta\delta}\, n^\delta) = -D_\alpha K^{\alpha}{}_{\beta} + D_\beta K \qquad \text{contracted Codazzi}$$

$$\perp(R_{\alpha\nu\beta\mu}\, n^\mu n^\nu) = \frac{1}{\alpha}\mathcal{L}_{\boldsymbol{m}} K_{\alpha\beta} + K_{\alpha\gamma} K^{\gamma}{}_{\beta} + \frac{1}{\alpha} D_\alpha D_\beta \alpha$$

$$\perp R_{\alpha\beta} = -\frac{1}{\alpha}\mathcal{L}_{\boldsymbol{m}} K_{\alpha\beta} - \frac{1}{\alpha} D_\alpha D_\beta \alpha + \mathcal{R}_\alpha \beta + K\, K_{\alpha\beta} - 2K_{\alpha\gamma} K^{\gamma}{}_{\beta}$$

$$R = \frac{2}{\alpha}\mathcal{L}_{\boldsymbol{m}} K - \frac{2}{\alpha} D_\gamma D^\gamma \alpha + \mathcal{R} + K^2 + K_{\gamma\delta} K^{\gamma\delta}$$

- Here: $\mathcal{L}$ is the Lie derivative and $m^\mu = \alpha n^\mu$
- Summation over spatial tensors: Can ignore time components

Decomposition of the Einstein eqs.

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}$$

$$\Leftrightarrow R_{\alpha\beta} = 8\pi\left(T_{\alpha\beta} - \frac{1}{D-2}g_{\alpha\beta}T\right) + \frac{2}{D-2}\Lambda g_{\alpha\beta}$$

- Energy momentum tensor

$$\rho := T_{\mu\nu}n^{\mu}n^{\nu}\,,$$

$$j_{\alpha} := -\perp^{\mu}{}_{\alpha}T_{\mu\nu}n^{\nu}\,,$$

$$S_{\alpha\beta} := \perp T_{\alpha\beta}\,, \quad S = \gamma^{\mu\nu}S_{\mu\nu}\,,$$

$$T_{\alpha\beta} = S_{\alpha\beta} + n_{\alpha}j_{\beta} + n_{\beta}j_{\alpha} + \rho n_{\alpha}n_{\beta}\,, \quad T = S - \rho\,.$$

- Lie derivative

$$\mathcal{L}_{\boldsymbol{m}}K_{ij} = \partial_t K_{ij} - \beta^m\partial_m K_{ij} - K_{mj}\partial_i\beta^m - K_{im}\partial_j\beta^m$$

$$\mathcal{L}_{\boldsymbol{m}}\gamma_{ij} = \partial_t\gamma_{ij} - \beta^m\partial_m\gamma_{ij} - \gamma_{mj}\partial_i\beta^m - \gamma_{im}\partial_j\beta^m$$

The ADM version of the Einstein eqs.

- Introduction of the extrinsic curvature:

$$\mathcal{L}_{\boldsymbol{m}} \gamma_{ij} = -2\alpha K_{ij}$$

- $\perp^{\mu}{}_{\alpha} \perp^{\nu}_{\beta}$ projection

$$\mathcal{L}_{\boldsymbol{m}} K_{ij} = -D_i D_j \alpha + \alpha(\mathcal{R}_{ij} + K K_{ij} - 2K_{im} K^{m}{}_{j}) + 8\pi\alpha \left(\frac{S-\rho}{D-2}\gamma_{ij} - S_{ij} \right) - \frac{2}{D-2}\Lambda\gamma_{ij}$$

"Evolution equations"

- $n^{\mu} n^{\nu}$ projection

$$\mathcal{R} + K^2 - K^{mn} K_{mn} = 2\Lambda + 16\pi\rho$$

"Hamiltonian constraint"

- $\perp^{\mu}{}_{\alpha} n^{\nu}$ projection

$$D_i K - D_m K^m_i = -8\pi j_i$$

"Momentum constraints"

Well-posedness in 30 seconds

- Consider a field ϕ evolved with a first-order system of PDEs
- The system has a well-posed initial-value formulation

  $:\Leftrightarrow$ there exists a norm and a smooth function

  $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\forall_t \;\; ||\phi(t)|| \leq F(||\phi(0)||, t) \times ||\phi(0)||$
- Well-posed systems have unique solutions for given initial data
- There can still be rapid divergence, e.g. exponential
- A necessary condition for well-posedness is strong hyperbolicity
- The general ADM equations are only weakly hyperbolic
- Key part of PDEs: Principle part = highest derivative terms
- Details depend on gauge, constraints, discretization

Sarbach & Tiglio 1203.6443; Gundlach & Martín-García gr-qc/0604035;

Reula gr-qc/0403007

The BSSN system

- Goal: Modify ADM eqs. to get a strongly hyperbolic system

Shibata & Nakamura PRD 52 (1995), Baumgarte & Shapiro PRD gr-qc/9810065

- Use (i) conformal desomposition, (ii) trace split, (iii) aux. variables

$$\gamma := \det \gamma_{ij}\,, \qquad \chi := \gamma^{-1/3}\,, \qquad K = \gamma^{mn} K_{mn}\,,$$

$$\tilde{\gamma}_{ij} := \chi \gamma_{ij} \qquad \Leftrightarrow \qquad \tilde{\gamma}^{ij} = \frac{1}{\chi} \gamma^{ij}\,,$$

$$\tilde{A}_{ij} := \chi \left(K_{ij} - \frac{1}{3} \gamma_{ij} K \right) \qquad \Leftrightarrow \qquad K_{ij} = \frac{1}{\chi} \left(\tilde{A}_{ij} + \frac{1}{3} \tilde{\gamma}_{ij} K \right)\,,$$

$$\tilde{\Gamma}^i := \tilde{\gamma}^{mn} \tilde{\Gamma}^i_{mn}\,.$$

- Auxiliary constraints

$$\tilde{\gamma} = 1, \qquad \tilde{\gamma}^{mn} \tilde{A}_{mn} = 0, \qquad \mathcal{G}^i \equiv \tilde{\Gamma}^i - \tilde{\gamma}^{mn} \tilde{\Gamma}^i_{mn} = 0\,.$$

The BSSN equations

$$\mathcal{H} := \mathcal{R} + \frac{2}{3}K^2 - \tilde{A}^{mn}\tilde{A}_{mn} - 16\pi\rho - 2\Lambda = 0\,,$$

$$\mathcal{M}_i := \tilde{\gamma}^{mn}\tilde{D}_m\tilde{A}_{ni} - \frac{2}{3}\partial_i K - \frac{3}{2}\tilde{A}^m{}_i\frac{\partial_m\chi}{\chi} - 8\pi j_i = 0\,,$$

$$\partial_t\chi = \beta^m\partial_m\chi + \frac{2}{3}\chi(\alpha K - \partial_m\beta^m)\,,$$

$$\partial_t\tilde{\gamma}_{ij} = \beta^m\partial_m\tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{\gamma}_{ij}\partial_m\beta^m - 2\alpha\tilde{A}_{ij}\,,$$

$$\partial_t K = \beta^m\partial_m K - \chi\tilde{\gamma}^{mn}D_m D_n\alpha + \alpha\tilde{A}^{mn}\tilde{A}_{mn} + \frac{1}{3}\alpha K^2 + 4\pi\alpha(S+\rho) - \alpha\Lambda\,,$$

$$\partial_t\tilde{A}_{ij} = \beta^m\partial_m\tilde{A}_{ij} + 2\tilde{A}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{A}_{ij}\partial_m\beta^m + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{im}\tilde{A}^m{}_j + \chi(\alpha\mathcal{R}_{ij} - D_iD_j\alpha - 8\pi\alpha S_{ij})^{\mathrm{TF}}\,,$$

$$\partial_t\tilde{\Gamma}^i = \beta^m\partial_m\tilde{\Gamma}^i + \frac{2}{3}\tilde{\Gamma}^i\partial_m\beta^m - \tilde{\Gamma}^m\partial_m\beta^i + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + \frac{1}{3}\tilde{\gamma}^{im}\partial_m\partial_n\beta^n - \tilde{A}^{im}\left(3\alpha\frac{\partial_m\chi}{\chi} + 2\partial_m\alpha\right) + 2\alpha\tilde{\Gamma}^i_{mn}\tilde{A}^{mn} - \frac{4}{3}\alpha\tilde{\gamma}^{im}\partial_m K - 16\pi\frac{\alpha}{\chi}j^i - \sigma\mathcal{G}^i\partial_m\beta^m\,.$$

- Note: there exist slight variations of the exact equations

The BSSN equations

- Auxiliary expressions we have used:

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} - \frac{1}{2\chi}(\delta^i{}_k\,\partial_j\chi + \delta^i{}_j\,\partial_k\chi - \tilde{\gamma}_{jk}\tilde{\gamma}^{im}\partial_m\chi)$$

$$\mathcal{R}_{ij} = \tilde{R}_{ij} + \mathcal{R}^\chi_{ij}\,,$$

$$\mathcal{R}^\chi_{ij} = \frac{\tilde{\gamma}_{ij}}{2\chi}\left(\tilde{\gamma}^{mn}\tilde{D}_m\tilde{D}_n\chi - \frac{3}{2\chi}\tilde{\gamma}^{mn}\partial_m\chi\,\partial_n\chi\right) + \frac{1}{2\chi}\left(\tilde{D}_i\tilde{D}_j\chi - \frac{1}{2}\partial_i\chi\,\partial_j\chi\right)\,,$$

$$\tilde{R}_{ij} = -\frac{1}{2}\tilde{\gamma}^{mn}\partial_m\partial_n\tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i}\partial_{j)}\tilde{\Gamma}^m + \tilde{\gamma}^m\tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn}\left[2\tilde{\Gamma}^k_{m(i}\tilde{\Gamma}_{j)kn} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{kjn}\right]\,,$$

$$D_iD_j\alpha = \tilde{D}_i\tilde{D}_j\alpha + \frac{1}{\chi}\partial_{(i}\chi\,\partial_{j)}\alpha - \frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_m\partial_n\alpha\,.$$

Beyond BSSN

- BSSN has a zero speed mode in the constraint-subsystem;

  May result in large constraint violations

- BSSN does not have systematic constraint damping

- This can be implemented by considering Generalized Einstein Eqs.

  Bona et al. PRD gr-qc/0302083 "Z4" system

- Conformal version of Z4: Very like BSSN but has constraint damping

  Alic et al. PRD 1106.2254, Hilditch et al. PRD 1212.2901

- Also allows for constraint preserving boundary conditions

  Bona et al. CQG gr-qc/0411110, Ruiz et al. PRD 1010.0523

2.1.2 Generalized harmonic formulation

The generalized harmonic gauge (GHG)

- Harmonic gauge: Choose coordinates such that

$$\Box x^\alpha = \nabla^\mu \nabla_\mu x^\alpha = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0$$

- 4 dimensional Einstein eqs. in harmonic gauge:

$$R_{\alpha\beta} = -\frac{1}{2} g^{\mu\nu}\partial_\mu \partial_\nu g_{\alpha\beta} + \ldots$$

principle part of wave equation $\Rightarrow$ Manifestly hyperbolic!

- Problem: Start with a hyper surface $t = \mathrm{const}$

Does t remain timelike?

- Goal: Generalize the harmonic gauge

Garfinkle PRD gr-qc/0110013; Pretorius CQG gr-qc/0407110;
Lindblom et al CQG gr-qc/0512093

$\rightarrow$ Source function $H^\alpha = \nabla^\mu \nabla_\mu x^\alpha = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu}$

The generalized harmonic equations

- Any spacetime in any coordinates can formulated in GH form!

  Problem: find the corresponding H^α

- Promote the H^α to evolution variables

- Einstein equations in GH form:

$$\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = -\partial_\nu g_{\mu(\alpha}\partial_{\beta)}g^{\mu\nu} - \partial_{(\alpha}H_{\beta)} + H_\mu\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\nu\alpha}\Gamma^\nu_{\mu\beta}$$
$$- \frac{2}{3}\Lambda g_{\alpha\beta} - 8\pi\left(T_{\mu\nu} - \frac{1}{2}T\, g_{\alpha\beta}\right).$$

with constraints $\mathcal{C}^\alpha = H^\alpha - \Box x^\alpha = 0$

- Still has principle part of the wave equation!!! Manifestly hyperbolic

Friedrich Comm.Math.Phys. 1985; Garfinkle PRD gr-qc/0110013;
Pretorius CQG gr-qc/0407110

Constraint damping in GHG system

- One can show: GHG constraints related to ADM constraints

$$\mathcal{C}^\alpha = 0, \quad \partial_t \mathcal{C}^\alpha = 0 \ \text{ at } \ t = 0 \quad \Rightarrow \quad \mathcal{H} = 0\,, \quad \mathcal{M}_i = 0$$

- Bianchi identies imply evolution of the $\mathcal{C}^\alpha$:

$$\Box \mathcal{C}_\alpha = -\mathcal{C}^\mu \nabla_{(\mu} \mathcal{C}_{\alpha)} - \mathcal{C}^\mu \left[8\pi \left(T_{\mu\alpha} - \frac{1}{2} T\, g_{\mu\alpha} \right) + \Lambda g_{\mu\alpha} \right] .$$

- In practice: Numerical violations of $\mathcal{C}^\mu = 0 \quad \Rightarrow$ unstable modes!

- Solution: Add constraint damping terms

$$\frac{1}{2}\partial_\mu \partial_\nu g_{\alpha\beta} = -\,\partial_\nu g_{\mu(\alpha} \partial_{\beta)} g^{\mu\nu} - \partial_{(\alpha} H_{\beta)} + H_\mu \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\nu\alpha} \Gamma^\nu_{\mu\beta}$$
$$-\,\Lambda g_{\alpha\beta} - 8\pi \left(T_{\mu\nu} - \frac{1}{2} T\, g_{\alpha\beta} \right) - \kappa \left[2 n_{(\alpha} \mathcal{C}_{\beta)} - \lambda g_{\alpha\beta} n^\mu \mathcal{C}_\mu \right]$$

Gundlach et al CQG (2005)

- E.g. Pretorius PRL gr-qc/0507014 uses $\kappa = 1.25/m\,, \quad \lambda = 1$

Summary of the GHG formulation

- Specify initial data $g_{\alpha\beta}$, $\partial_t g_{\alpha\beta}$ at $t=0$

  that satisfy the constraints $\mathcal{C}_\alpha = \partial_t \mathcal{C}_\alpha = 0$

- Constraints preserved due to Bianchi identities

- Alternative first-order version of GH formulation

  Lindblom et al CQG gr-qc/0512093

  - Auxiliary variables $\rightarrow$ First-order system
  - Symmetric hyperbolic system

    $\rightarrow$ constraint preserving boundary conditions
  - Used in spectral code SXS

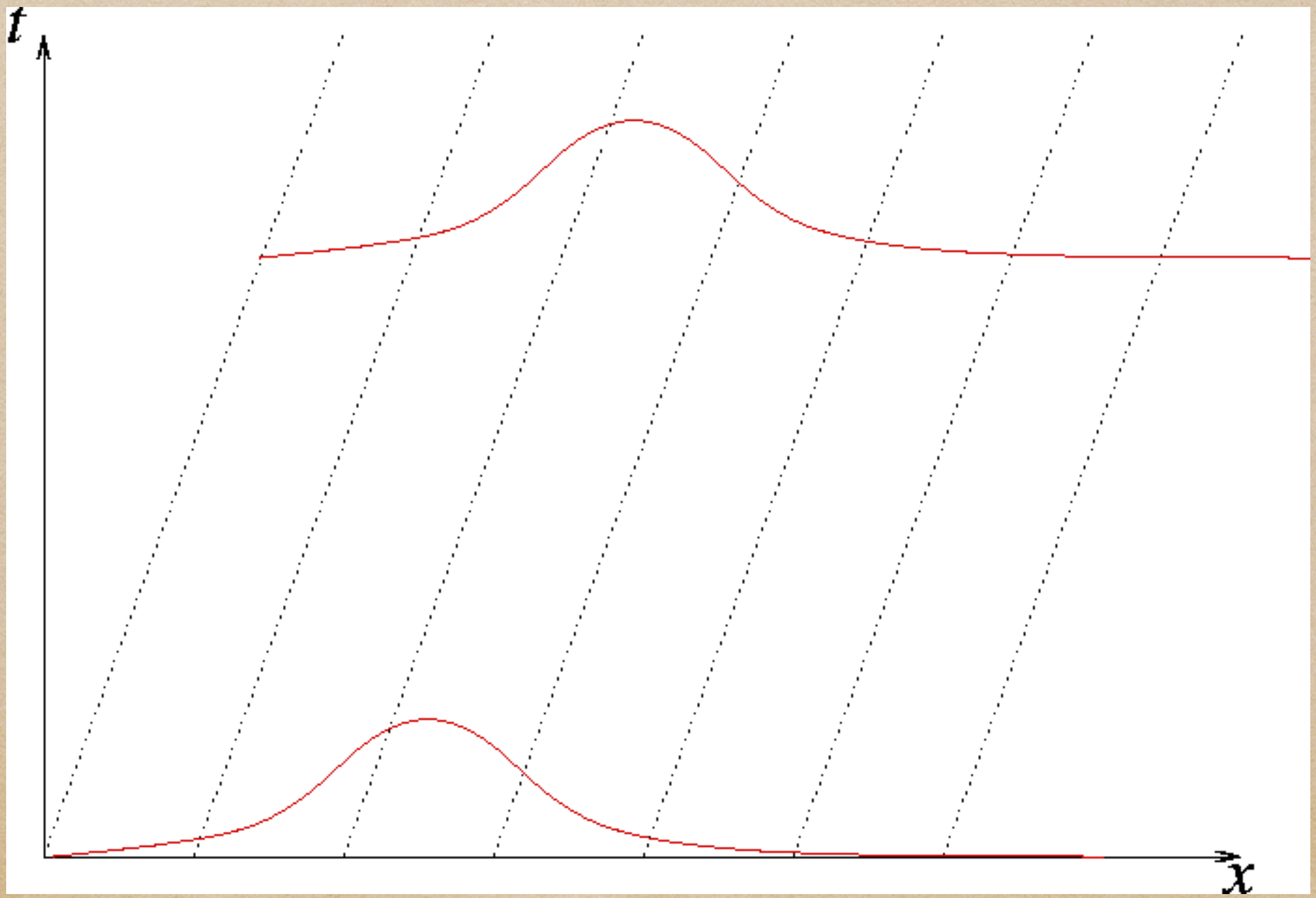
    Caltech, Cornell, CITA

2.1.3 Characteristic formulation

Characteristic coordinates

- Consider advection equation $\partial_t f + a\partial_x f = 0$
- Characteristics: Curves $\mathcal{C} : x \mapsto at + x_0 \quad \Leftrightarrow \quad \frac{dx}{dt} = a$

$$\Rightarrow \quad \left.\frac{df}{dt}\right|_{\mathcal{C}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\left.\frac{dx}{dt}\right|_{\mathcal{C}} = \frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad f = \text{const} \quad \text{along} \quad \mathcal{C}$$



Characteristic "Bondi-Sachs" formulation

Here: $D = 4\,, \quad \Lambda = 0\,, \quad T_{\alpha\beta} = 0$

- Write metric as

$$ds^2 = V\frac{e^{2\mathcal{B}}}{r}du^2 - 2e^{2\mathcal{B}}du\,dr + r^2 h_{\mu\nu}(dx^\mu - U^\mu du)(dx^\nu - U^\nu du)$$

$$2h_{\mu\nu}dx^\mu dx^\nu = (e^{2\mathcal{C}} + e^{2\mathcal{D}})d\theta^2 + 2\sin\theta\,\sinh(\mathcal{C} - \mathcal{D})d\theta\,d\phi + \sin^2\theta\,(e^{-2\mathcal{C}} + e^{-2\mathcal{D}})d\phi^2$$

- Introduce tetrad $\mathbf{k},\ \mathbf{l},\ \mathbf{m},\ \bar{\mathbf{m}}$ such that

$\boldsymbol{g}(\mathbf{k},\mathbf{l}) = 1\,, \quad \boldsymbol{g}(\mathbf{m},\bar{\mathbf{m}}) = 1$ and all other products vanish

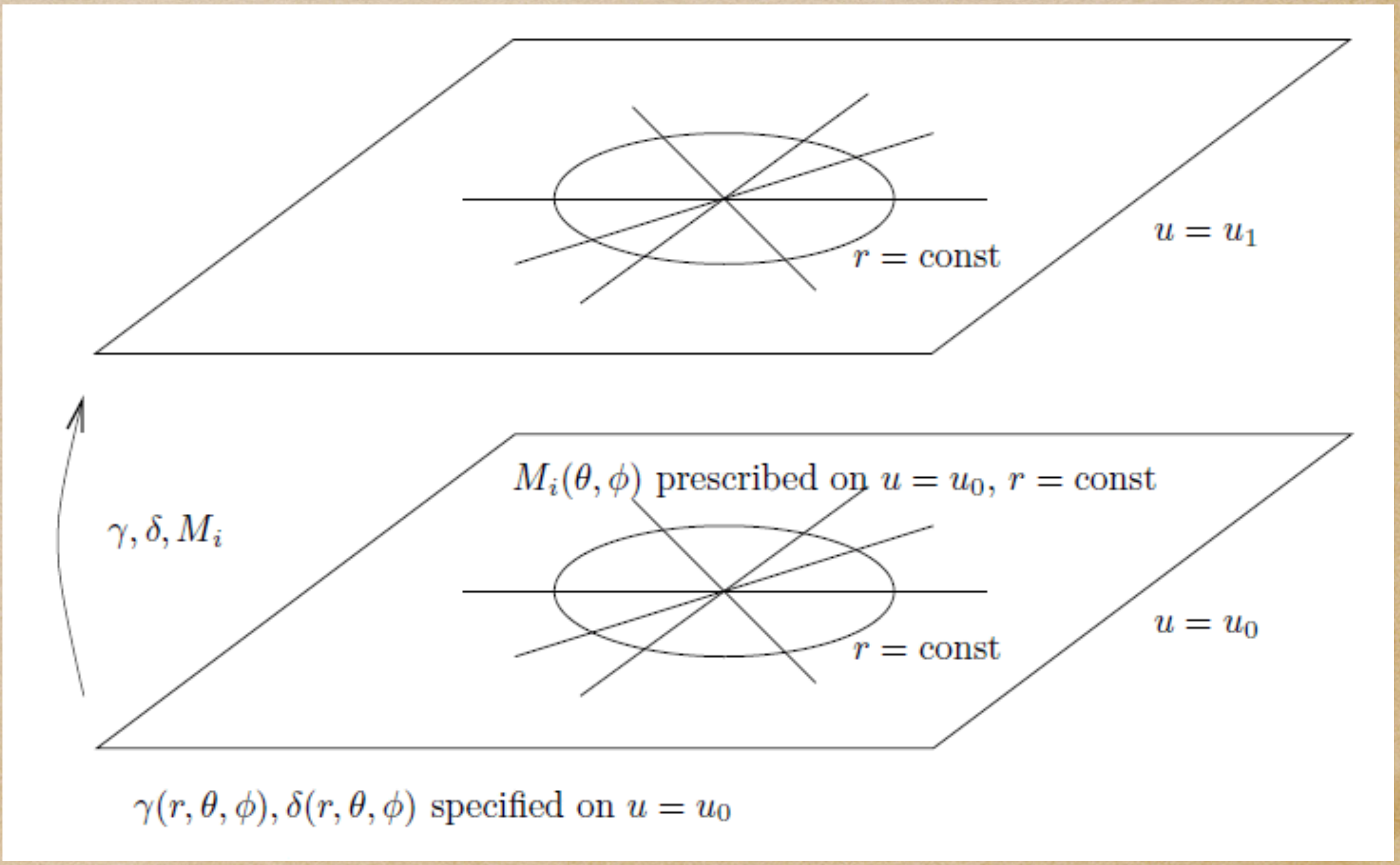
- Then the Einstein equations become
  - 4 Hypersurface equations $R_{\mu\nu}\mathbf{k}^\mu\mathbf{k}^\nu = R_{\mu\nu}\mathbf{k}^\mu\mathbf{m}^\nu = R_{\mu\nu}\mathbf{m}^\mu\bar{\mathbf{m}}^\nu = 0\,,$
  - 2 evolution equations $R_{\mu\nu}\mathbf{m}^\mu\mathbf{m}^\nu = 0\,,$
  - 1 trivial equation $R_{\mu\nu}\mathbf{k}^\mu\mathbf{l}^\mu = 0\,,$
  - 3 supplementary equations $R_{\mu\nu}\mathbf{l}^\mu\mathbf{m}^\nu = R_{\mu\nu}\mathbf{l}^\mu\mathbf{l}^\nu = 0\,.$

Integration of the characteristic eqs.

- Provide initial data for $\mathcal{C}, \mathcal{D}$ on hyper surface $u = \text{const}$
- Integrate hypersurface eqs. along r $\rightarrow$ $\mathcal{B}, V, U^\alpha$ on $u = \text{const}$

  $\rightarrow$ 3 "constants" of integration $M_i(\theta,\phi)$
- Evolve $\mathcal{C}, \mathcal{D}$ using the evolution equations

  $\rightarrow$ 2 "constants" of integration $\rightarrow$ complex news $\partial_u c(u,\theta,\phi)$
- Evolve the M_i through the supplementary eqs.



Features of the characteristic formulation

- Naturally adapted to the causal structure of GR
- Clear hierarchy of equations $\rightarrow$ isolated degrees of freedom
- Problem: Caustics $\rightarrow$ breakdown of coordinates
- Well suited for symmetric spacetimes, planar BHs
- Solution for binary problem?

  Work in progress; see e.g. Babiuc, Kreiss & Winicour 1305.7179
- Application to characteristic GW extraction

  Babiuc et al PRD 1011.4223; Reisswig et al CQG 0912.1285

Direct methods

- Use symmetry to write line element; e.g.

$$ds^2 = -a^2(\mu,t)dt^2 + b^2(\mu,t)d\mu^2 + R^2(\mu,t)(d\theta^2 + \sin^2\theta d\phi^2)$$

May & White PR (1966)

- Energy momentum tensor

$${T^0}_0(1+\epsilon)\,, \quad {T^1}_1 = {T^2}_2 = {T^3}_3 = 0$$

Lagrangian coordinates

- GRTensor (Maple), Mathematica,...

$\Rightarrow$ Field equations

$$a' = \ldots$$
$$b' = \ldots$$
$$\ddot{R} = \ldots$$

Further reading

- 3+1 formalism

  Gourgoulhon gr-qc/0703035, Cardoso et al LRR-2015-1 1310.7590

- Characteristic formalism

  Winicour LRR-2012-2 gr-qc/0102085

- Numerical relativity in general

  Alcubierre: "Introduction to 3+1 Numerical Relativity" Oxford Univ. Press

  Baumgarte & Shapiro: "Numerical Relativity" Cambridge Univ. Press

  Bona, Palenzuela, Bona-Casas: "Elements of Numerical Relativity and Relativistic Hydrodynamics" Springer

- Well-posedness, Einstein eqs. as an initial-value problem

  Sarbach & Tiglio LRR-2012-15 1203.6443

2.3 Initial data, gauge

2.3.1 Initial data

Analytic initial data

- Schwarzschild, Kerr, Tangherlini, Myers-Perry,...

  e.g. Schwarzschild in isotropic coordinates

$$ds^2 = -\left(\frac{2r-M}{2r+M}\right)^2 dt^2 + \left(1+\frac{M}{2r}\right)^4 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

- Time symmetric initial data with n BHs:

  Brill & Lindquist PR 131 (1963) 471, Misner PR 118 (1960) 1110

- Problem: Find initial data for dynamic systems
- Goals: 1) Solve constraints

  2) Realistic snapshot of physical system
- This is mostly done using the ADM 3+1 split

The York-Lichnerowicz split

- We work in $D=4$ dimensions; generalization to $D>4$ possible
- Conformal metric $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$

Lichnerowicz J.Math.Pures Appl. 23 (1944) 37

York PRL 26 (1971) 1656, PRL 28 (1972) 1082

- Note: In contrast to BSSN, we do not require $\det \bar{\gamma}_{ij} = 1$
- Conformal traceless split of the extrinsic curvature

$$K_{ij} = A_{ij} + \frac{1}{3} K \gamma_{ij} ,$$

$$A^{ij} = \psi^{-10} \bar{A}^{ij} \quad \Leftrightarrow \quad A_{ij} = \psi^{-2} \bar{A}_{ij}$$

Bowen-York data

- By further splitting $\bar{A}_{ij}$ into a longitudinal and a transverse traceless part, the momentum constraints simplify substantially

  Cook LRR gr-qc/0007085

- Further assume: Vacuum, $K = 0\,, \quad \bar{\gamma}_{ij} = f_{ij}\,, \quad \lim_{r\to\infty} \psi = 0$,

  where f_{ij} is the flat metric in arbitrary coords.

  In words: Traceless E.Curv., conformal flatness, assymptotic flatness

- Then there exists an analytic solution to the momentum constraints

$$\bar{A}_{ij} = \frac{3}{2r^2}\left[P_i n_j + P_j n_i - (f_{ij} - n_i n_j) P^k n_k\right] + \frac{3}{r^3}\left(\epsilon_{kil} S^l n^k n_j + \epsilon_{kjl} S^l n^k n_i\right) ,$$

  where r is a coordinate radius and $n^i = \frac{x^i}{r}$

  Bowen & York PRD (1980)

Properties of the Bowen-York solution

- The momentum in an asymptotically flat hyper surface associated with asymptotic translational and rotational Killing vectors $\xi^i_{(a)}$ is

$$\sum_i \Pi^i \xi^i_{(a)} = \frac{1}{8\pi} \oint_\infty (K^j{}_i - \delta^j{}_i K) \xi^i_{(a)} \, d^2 A_j$$

$\Rightarrow \ldots \Rightarrow$ P^i and S^i are the physical linear and angular momentum of the spacetime

- The momentum constraint is linear

$\Rightarrow$ we can superpose Bowen-York data. The momenta simply add up.

- Bowen-York data generalizes (analytically!) to higher D

Yoshino, Shiromizu & Shibata PRD gr-qc/0610110

Puncture data

Brandt & Brügmann PRL gr-qc/9703066

- The Hamiltonian constraint is then given by

$$\bar{\nabla}^2\psi + \frac{1}{8}\psi^{-7}\bar{A}_{mn}\bar{A}^{mn} = 0$$

- Ansatz for conformal factor $\psi = \psi_{\rm BL} + u$

  where $\psi_{\rm BL} = \sum_{a=1}^{N} \frac{m_a}{|\vec{r} - \vec{r}_a|}$ is the Brill-Lindquist conformal factor,

  i.e. the solution for $\bar{A}_{ij} = 0$.

- There then exist unique $\mathcal{C}^2$ solutions u to the Hamiltonian constr.

- The Hamiltonian constraint in this form is particularly suitable

  for numerical solution.

  E.g. Ansorg, Brügmann & Tichy PRD gr-qc/0404056

Properties of the puncture solutions

- m_a and $\vec{r}_a$ are the bare mass and position of the a^{th} BH
- In the limit of vanishing Bowen-York parameters $P^i = S^i = 0$,

  the puncture solution reduces to Brill-Lindquist data

$$\gamma_{ij} dx^i \, dx^j = \left(1 + \sum_a \frac{m_a}{2|\vec{r} - \vec{r}_a|} \right)^4 (dx^2 + dy^2 + dz^2)$$

- The numerical solution of the Hamiltonian constraint generalizes

  rather straightforwardly to higher D

  Yoshino, Shirumizu & Shibata PRD gr-qc/0610110

  Zilhão et al PRD 1109.2149

- Punctures also generalize to asymptotically de Sitter BHs

  Zilhão et al PRD 1204.2019

  using McVittie coordinates McVittie MNRAS (1933)

Beyond conformally flat initial data

- Problem: Conformally flat data limits spins to $S/M^2 \lesssim 0.928$

  Dain et al. PRD gr-qc/0201062

- Similar problems arise for large linear momenta

- Solution: Non-conformally flat initial data
  - Superpose Kerr-Schild data Lovelace et al. PRD 0805.4192

    Solve constraints with Conformal Thin Sandwich approach

    York PRL 82 (1999) 1350
  - Superpose boosted conformal Kerr BHs; attenuation functions

    Zlochower et al. PRD 1706.01980, Ruchlin et al. PRD 1410.8607

    Evolve with CCZ4 (constraint damping variant of BSSN)

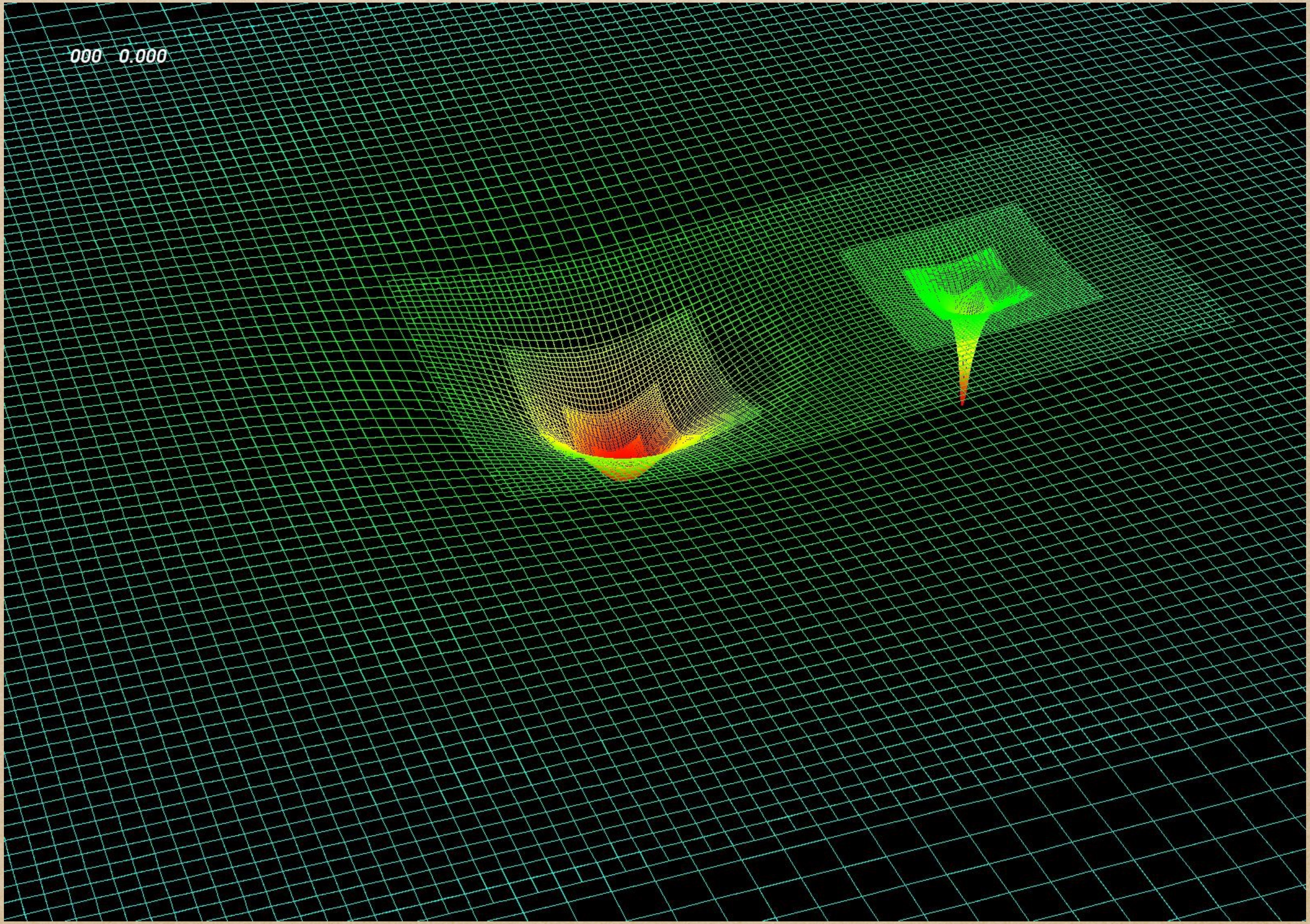
    Alic et al. PRD 1106.2254, Hilditch et al. PRD 1212.2901

2.3.2 Gauge

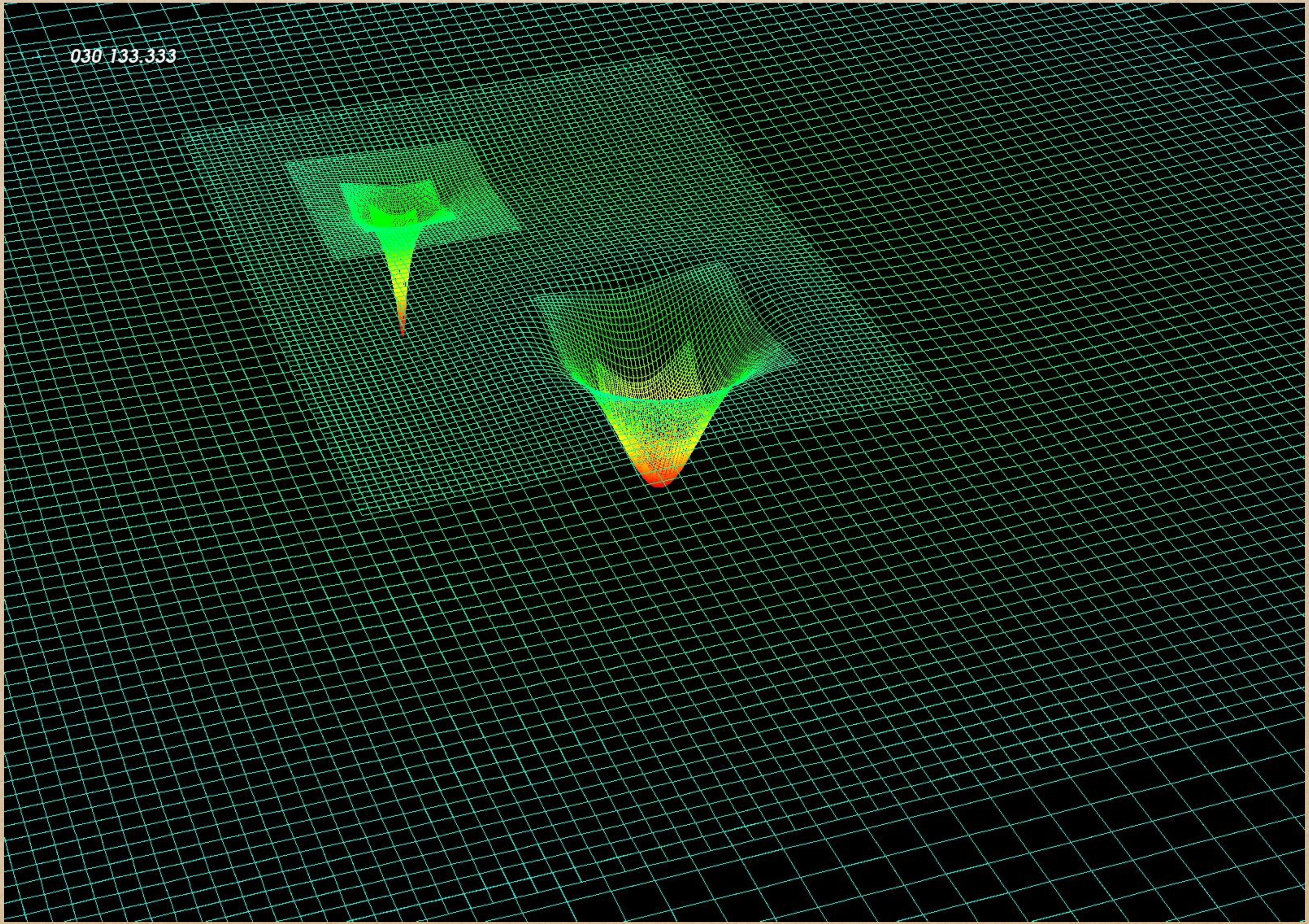
The gauge freedom

- Recall: Einstein's equations say nothing about $\alpha\,,\quad \beta^i$
- Any choice of lapse and shift gives a solution to Einstein's eqs.
- This is the coordinate or gauge freedom of GR
- If the physics do not depend on $\alpha\,,\quad \beta^i$, then why bother?
- Answer: The performance of the numerics DO depend very sensitively on the gauge!

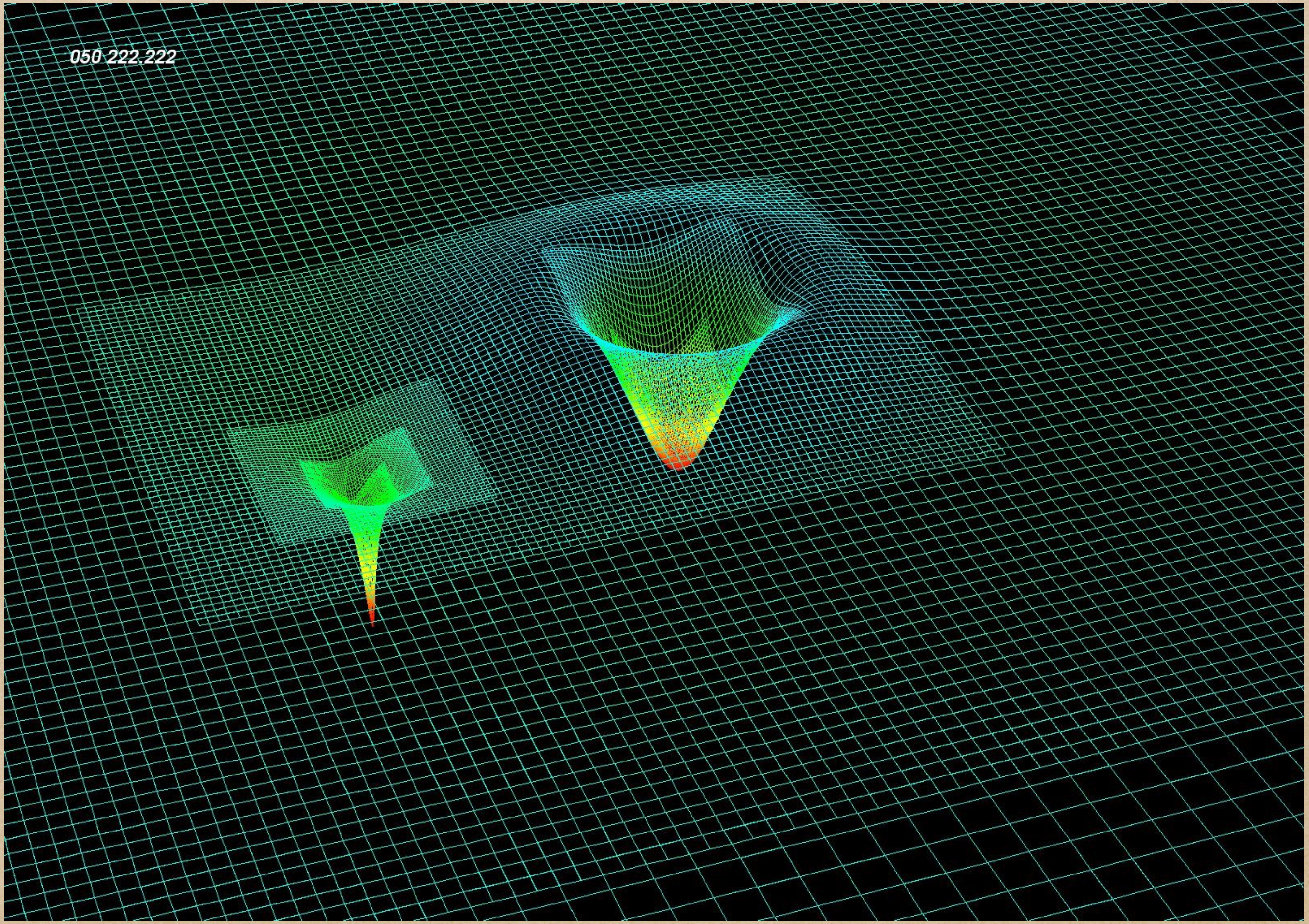
What goes wrong with bad gauge?



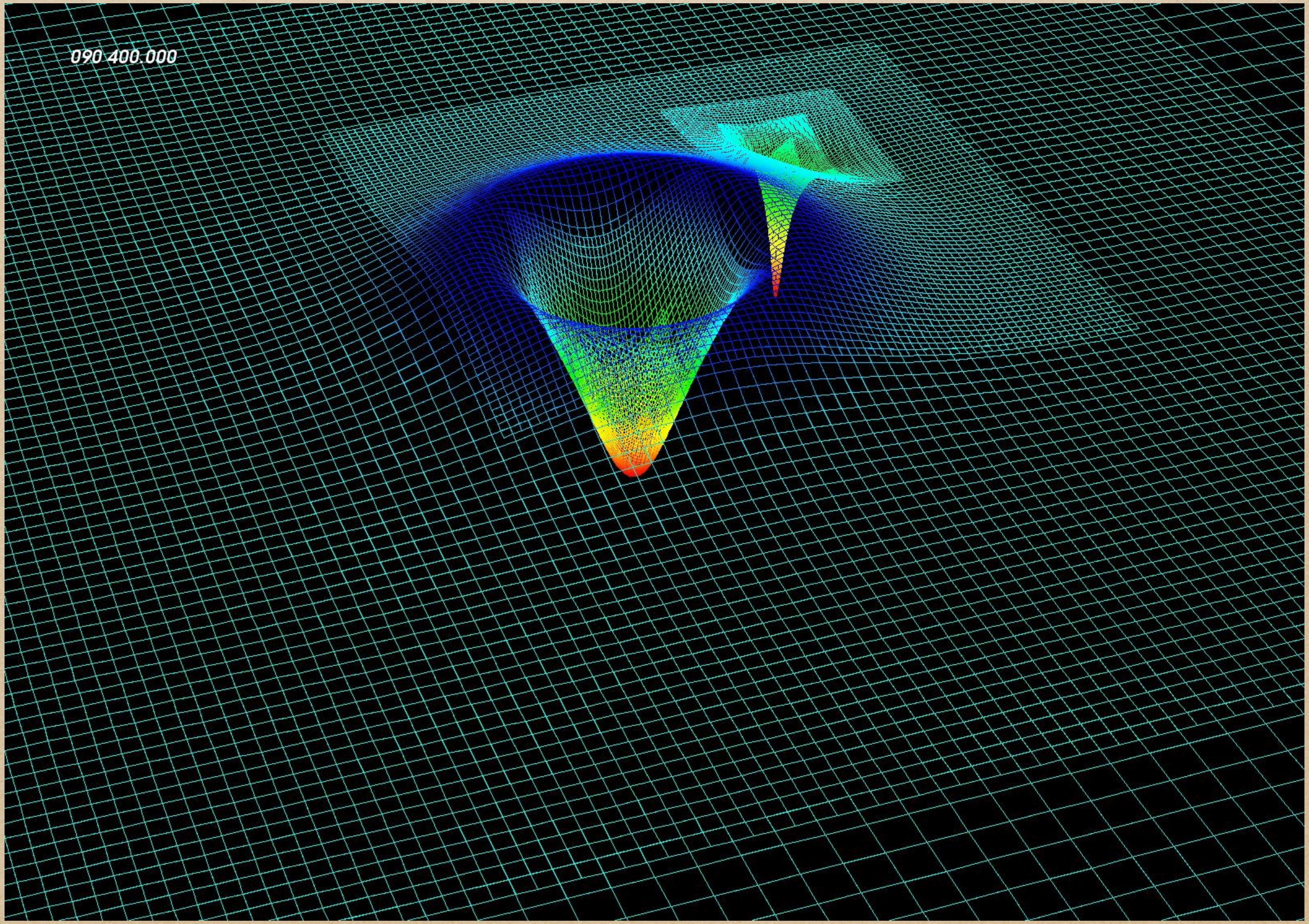
What goes wrong with bad gauge?



What goes wrong with bad gauge?



What goes wrong with bad gauge?



Ingredients for good gauge

- Singularity avoidance
- Avoid slice stretching
- Aim for stationarity in a co-moving frame
- Well-posedness of the system of PDEs
- Generalize “good” gauge, e.g. harmonic
- Lots of good luck!

Bona et al PRL (1995)

Alcubierre et al PRD gr-qc/0206072

Alcubierre CQG gr-qc/0210050

Garfinkle PRD gr-qc/0110013

Moving puncture gauge

- Moving punctures is one of the NR breakthrough methods

  Baker et al PRL gr-qc/0511103; Campanelli et al PRL gr-qc/0511048

- Gauge played a key role

- Variant of $1 + \log$ slicing and $\Gamma-$driver shift

  Alcubierre et al PRD gr-qc/0206072

- Now in use as $\partial_t \alpha = \beta^m \partial_m \alpha - 2\alpha K$

  and $\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} B^i$

  $$\partial_t B^i = \beta^m \partial_m B^i + \partial_t \tilde{\Gamma}^i - \beta^m \partial_m \tilde{\Gamma}^i - \eta B^i$$

  or $\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} \tilde{\Gamma}^i - \eta \beta^i$

  e.g. van Meter et al PRD gr-qc/0605030

Moving puncture gauge

Comments:

- Some people drop the advection terms $\beta^m \partial_m \ldots$
- η is a damping parameter or position-dependent function

  Alic et al CQG 1008.2212; Schnetter CQG 1003.0859;

  Müller et al PRD 1003.4681
- Modifications in higher D :
  - Change numerical values of the parameters: Trial & Error?

    Yoshino & Shibata PTPS 189 269
  - Dim. reduction by isometry: add scalar terms to Eqs.

    Zilhão et al PRD 1001.2302

Gauge conditions in the GH formulation

- How to choose the H_μ ? $\rightarrow$ Also requires some trial & error

- Pretorius' breakthrough simulations used

$$\Box H_t = -\xi_1 \frac{\alpha - 1}{\alpha^\eta} + \xi_2 n^\mu \partial_\mu H_t \quad \text{with}$$

$\xi_1 = 19/m\,, \quad \xi_2 = 2.5/m\,, \quad \eta = 5$, where $m =$ mass of 1 BH

- Caltech-Cornell-CITA spectral code:

Initialize H_α to minimize time derivatives of the metric,

adjust H_α to harmonic and damped harmonic gauge condition.

Lindblom & Szilágyi PRD 0904.4873; with Scheel PRD 80 (2009)

- The H_α are related to lapse and shift:

$$n^\mu H_\mu = -K - n^\mu \partial_\mu \ln \alpha\,,$$

$$\perp^{\mu i} H_\mu = -\gamma^{mn}\Gamma^i_{mn} + \gamma^{im}\partial_m \ln \alpha + \frac{1}{\alpha} n^\mu \partial_\mu \beta^i\,.$$

Further reading

- Initial data construction

  Cook LRR gr-qc/0007085

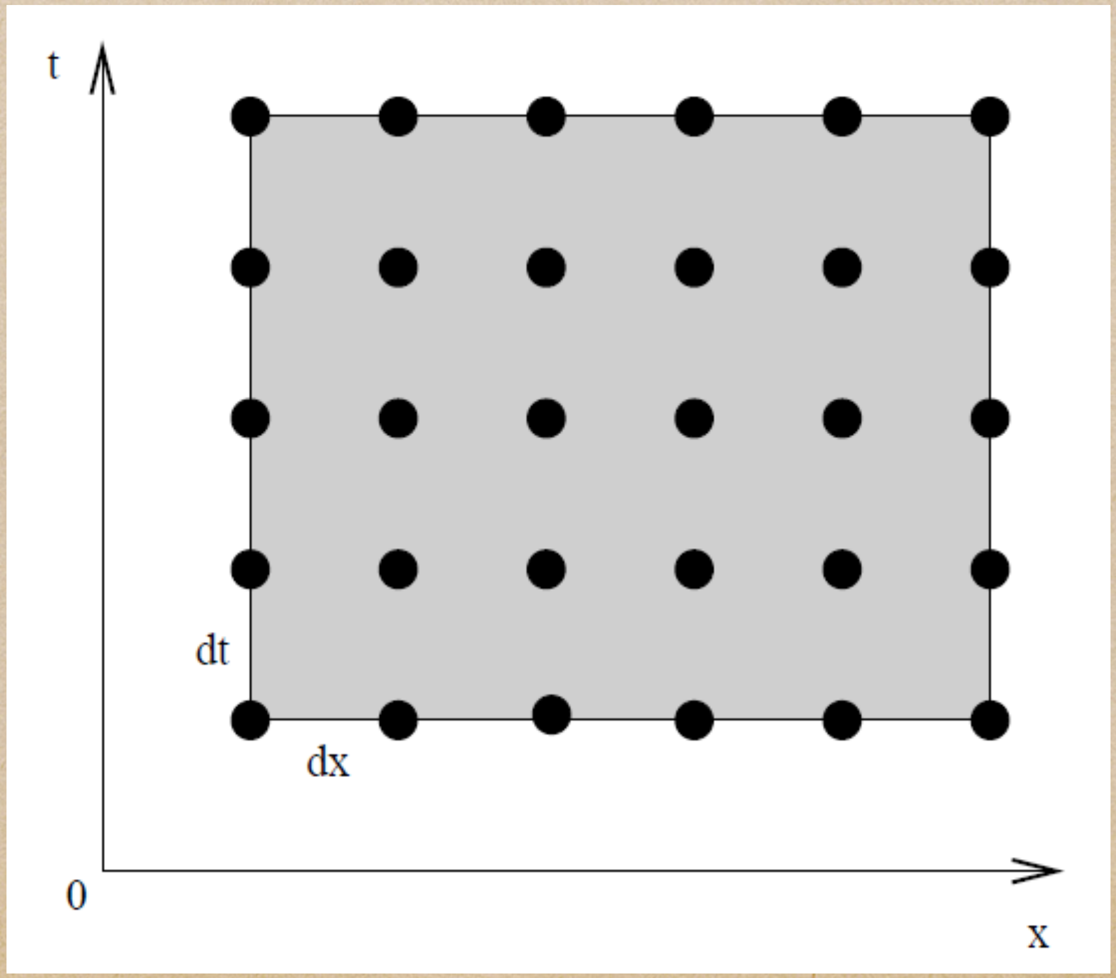
  Pfeiffer Thesis gr-qc/0510016

2.4 Discretization of the equations

Finite differencing

- Consider one spatial and one time dimension: $t, \quad x$
- Replace computational domain by discrete points

  $x_i = x_0 + i\,dx\,, \quad t_n = t_0 + n\,dt$
- Approximate function: $f(t_n, x_i) \approx f_{n,i}$



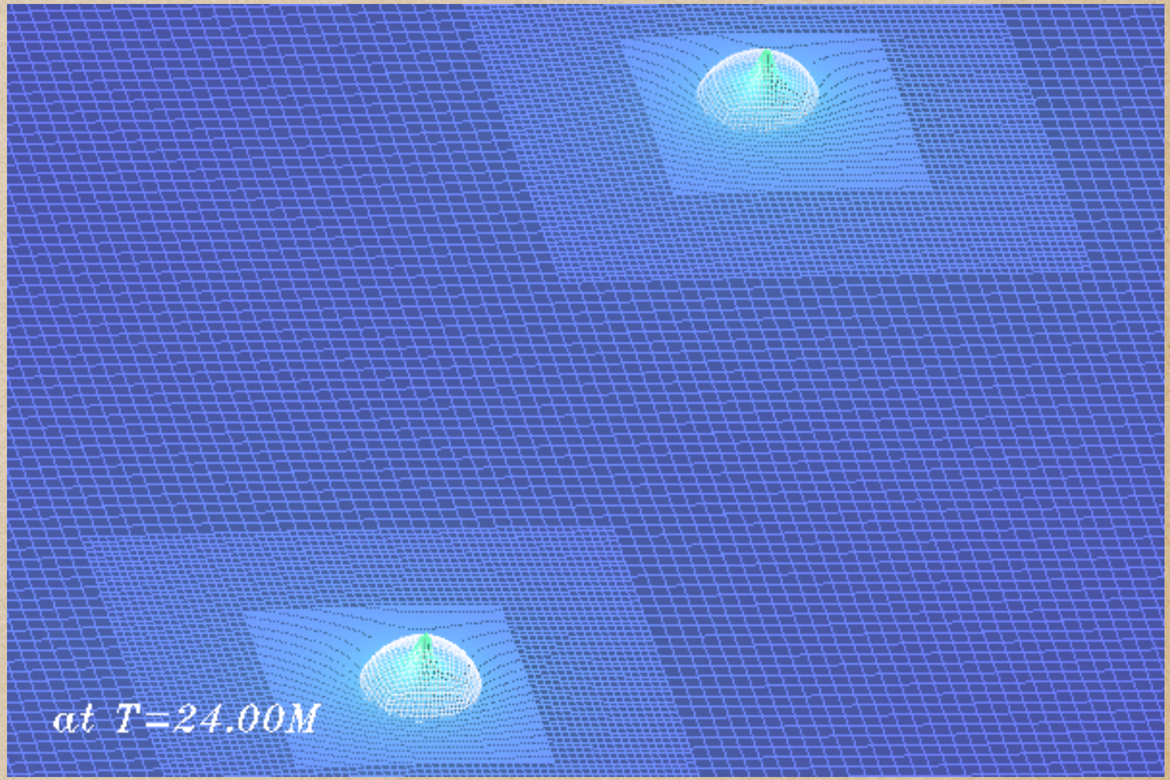
Derivatives and finite differences

- Goal: Represent $\frac{\partial^m f}{\partial x^m}$ in terms of $f_{n,i}$
- Fix index n; Taylor expand
$$f_{i-1} = f_i - f_i' dx + \frac{1}{2} f_i'' \, dx^2 + \mathcal{O}(dx^3)$$
$$f_i = f_i$$
$$f_i + 1 = f_i + f_i' \, dx + \frac{1}{2} f_i'' \, dx^2 + \mathcal{O}(dx^3)$$
- Write f_i' as linear combination: $f_i' = A f_{i-1} + B f_i + C f_{i+1}$
- Insert Taylor expressions and compare coefficients on both sides
$$\Rightarrow \; 0 = A + B + C \,, \;\; 1 = (-A + B) dx \,, \;\; 0 = \frac{1}{2} A \, dx^2 + \frac{1}{2} C \, dx^2$$
$$\Rightarrow \; A = -\frac{1}{2dx} \,, \;\; B = 0 \,, \;\; C = \frac{1}{2dx}$$
$$\Rightarrow \; f_i' = \frac{f_{i+1} - f_{i-1}}{2dx} + \mathcal{O}(dx^2)$$
- Same method in time direction; higher accuracy $\rightarrow$ more points

Mesh refinement

- 3 length scales: BH $\sim 1\ M$
  - Wavelength $\sim 10 \ldots 100\ M$
  - Wave zone $\sim 100 \ldots 1000\ M$
- First mesh refinement in GR: Critical phenomena

  Choptuik PRL **70** 9-12



- First use for BBHs

  Brügmann PRD gr-qc/9608050

- Available packages

  SAMRAI

  Paramesh MacNeice et al Comp.Phys.Comm. 136 (2000) 330

  Carpet Schnetter et al gr-qc/0310042

  Chombo Clough et al 1503.03436

Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature, …
- Here for $1 + 1$ dimensions

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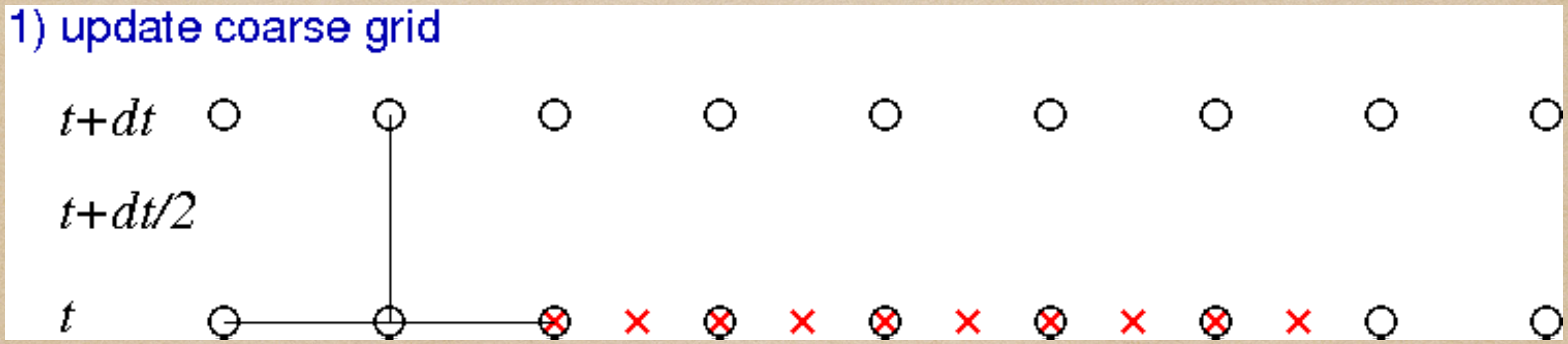

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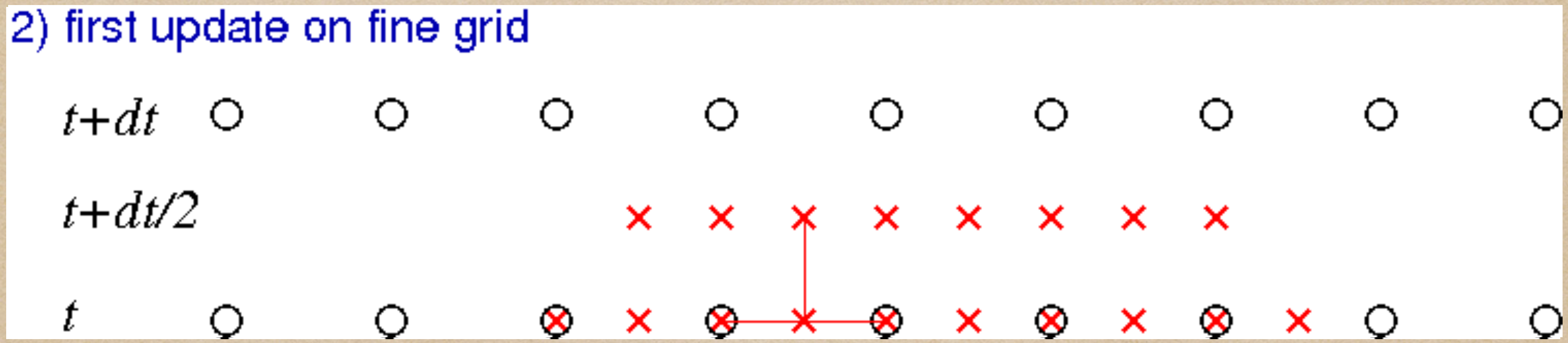

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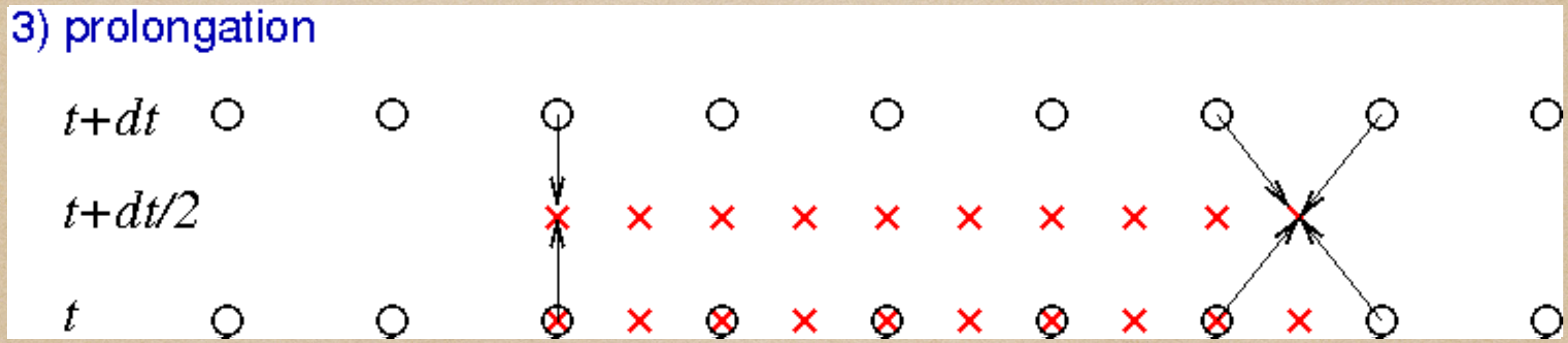

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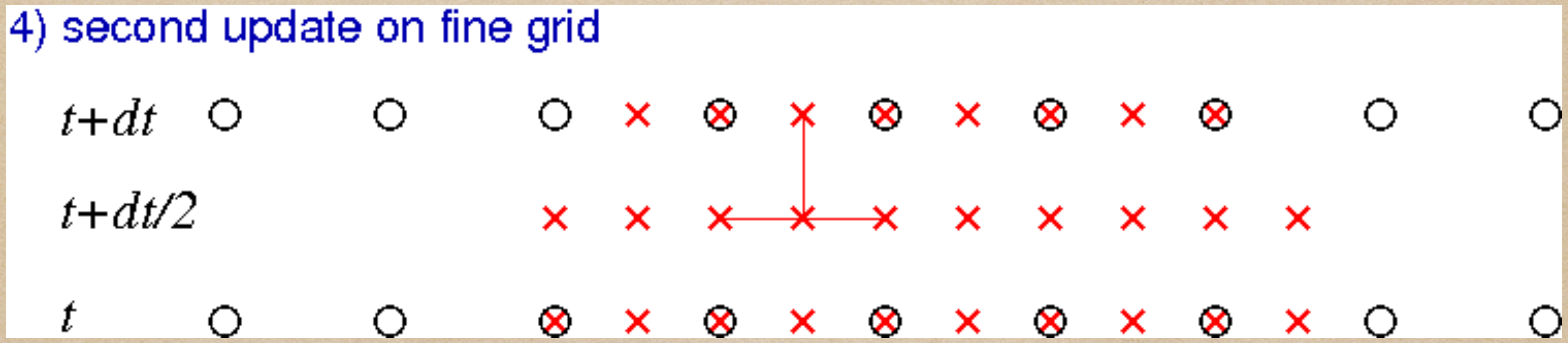

Berger-Oliger mesh refinement

- Goal: Update from t to $t+dt$
- Refinement criteria: numerical error, curvature, …
- Here for $1+1$ dimensions

Berger-Oliger mesh refinement

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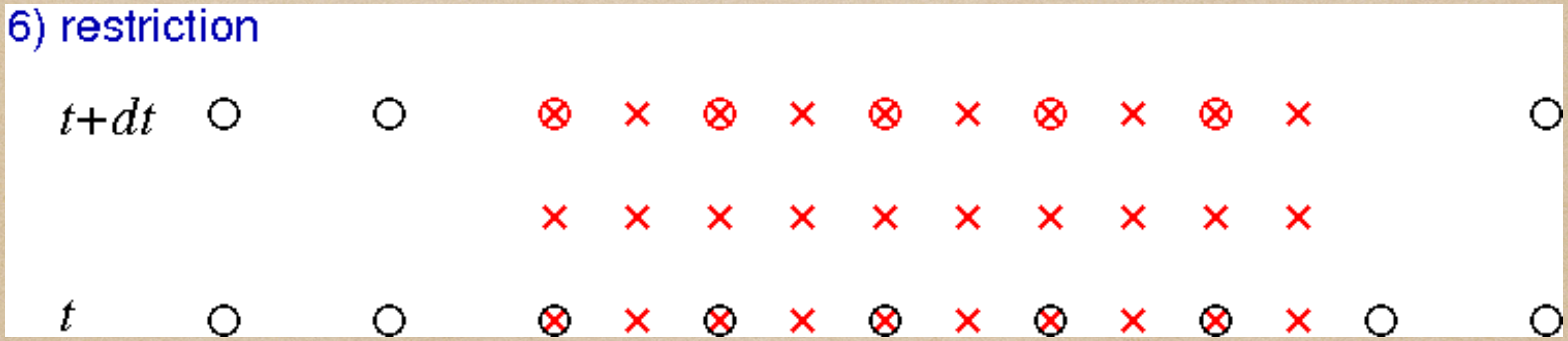
- Here for $1 + 1$ dimensions



Alternative discretization schemes

- Spectral methods: high accuracy, efficiency, complexity

  e.g. Caltech-Cornell-CITA code; http://www.black-holes.org/SpEC.html

  Application to moving punctures hard

  e.g. Tichy PRD 0911.0973

  Also used in symmetric asymptotically AdS spacetimes

  e.g. Chesler & Yaffe PRL 1011.3562; Santos & Sopuerta PRL 1511.04344

- Finite volume methods

- Finite element methods

  e.g. Arnold, Mukherjee & Pouly gr-qc/9709038

  Sopuerta et al CQG gr-qc/0507112

  Sopuerta & Laguna PRD gr-qc/0512028

Further reading

- Numerical Methods

  Press et al "Numerical Recipes" Cambridge University Press

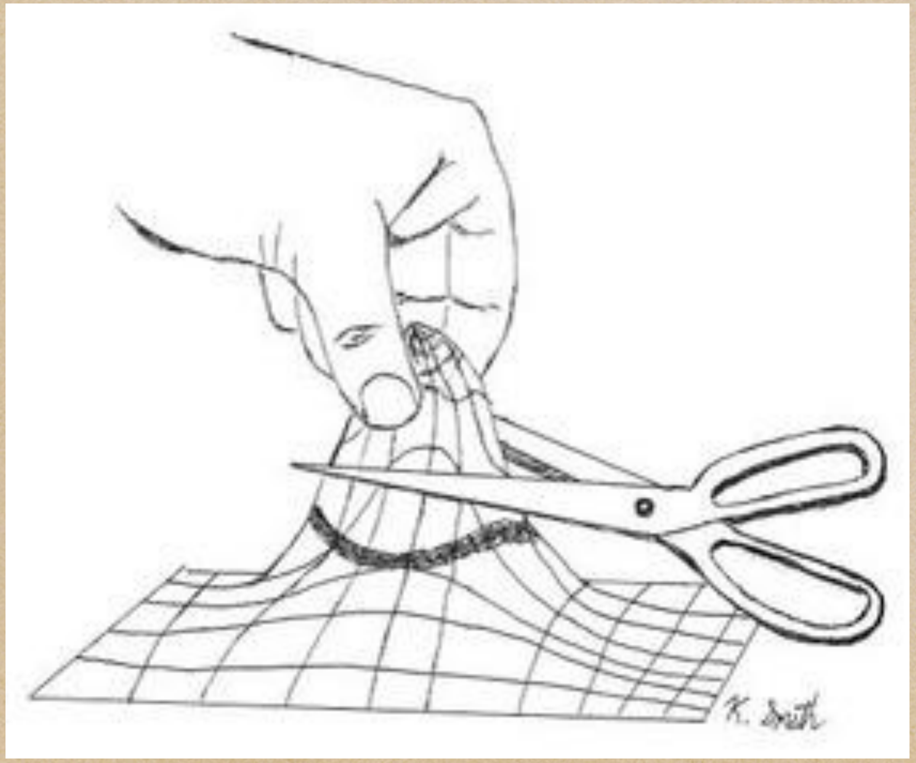
2.5 Boundaries

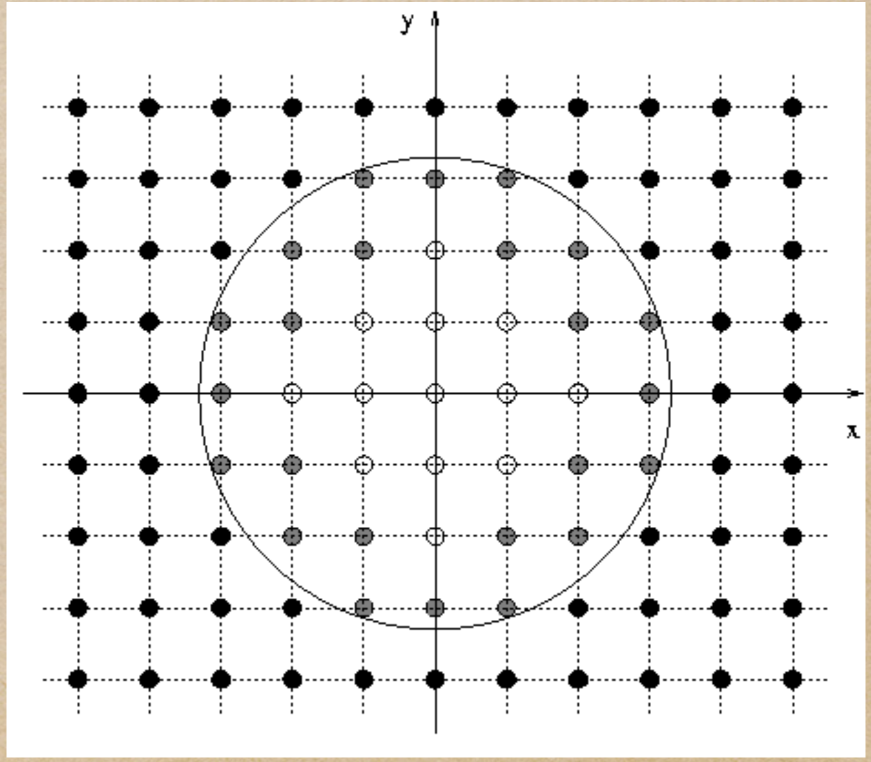
Inner boundary: Singularity treatment

- Cosmic censorship $\Rightarrow$ horizon protects outside from singularity
- Moving puncture method: "we get away with it..."

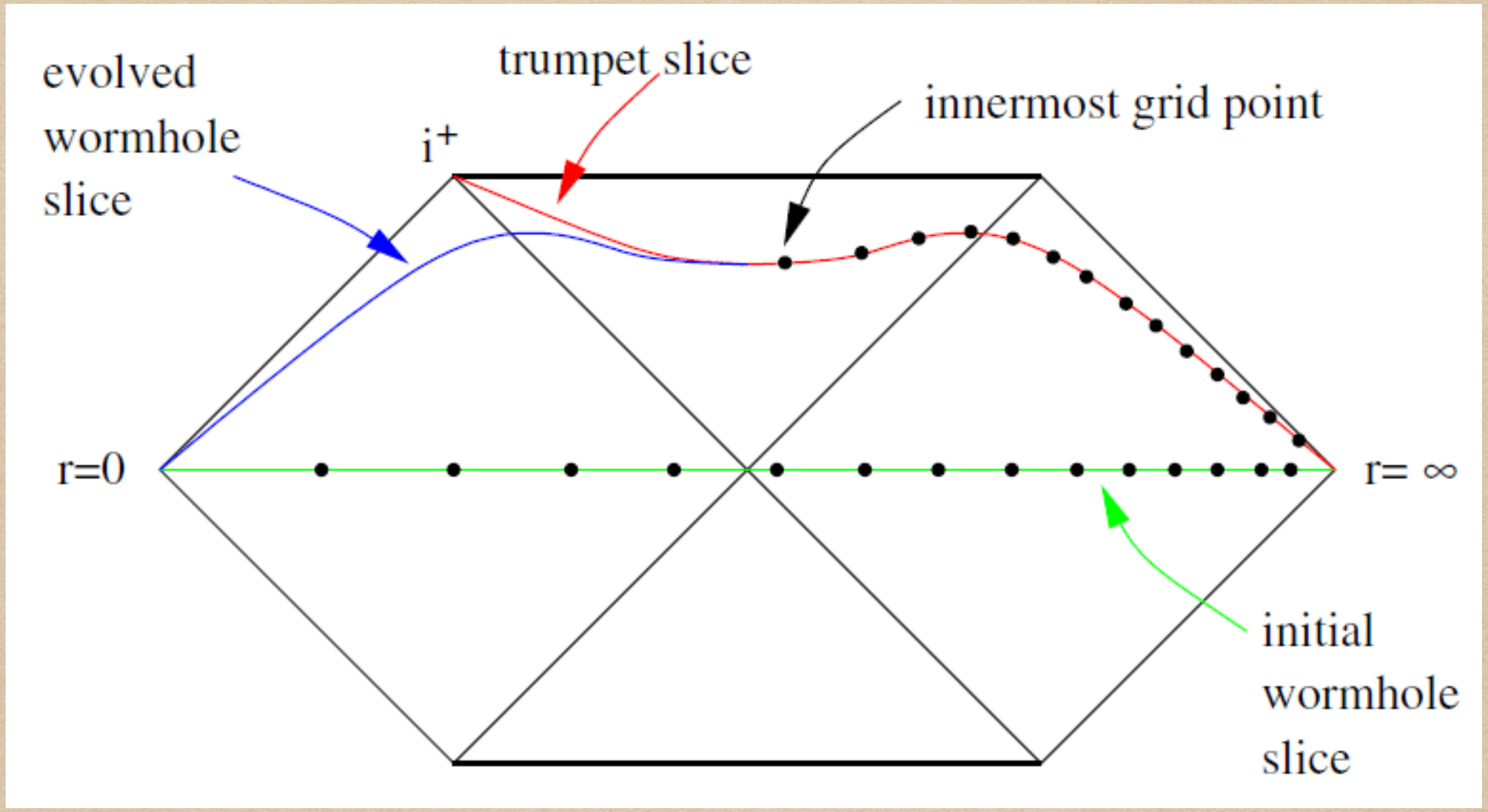
  Baker et al PRL gr-qc/0511103; Campanelli et al PRL gr-qc/0511048
- Excision: Cut out region around the singularity

  Caltech-Cornell-CITA code, Pretorius' code





Moving puncture slices: Schwarzschild



- Wormhole evolves into "Trumpet slice" $=$ stationary $1 + \log$ slice.

  Hannam et al PRL gr-qc/0606099, PRD 0804.0628

  Brown PRD 0705.1359, CQG 0705.3845

- Note: Gauge might propagate at $> c$, but: no pathologies apparent

  Moving puncture $=$ "Natural excision" Brown PRD 0908.3814

Outer boundary: Outgoing radiation

- Computational domains typically don't extend to ∞

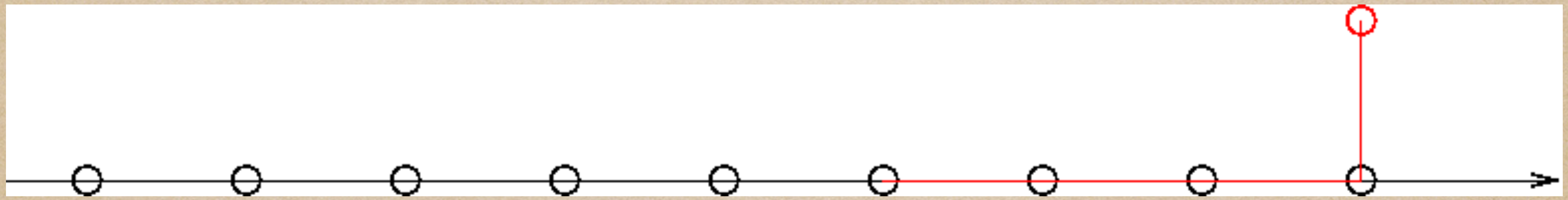
- Outgoing Sommerfeld condition:

Assume: $f = f_0 + \frac{u(t-r)}{r^n}$ where f_0 is the asymptotic value

$$\Rightarrow \partial_t u + \partial_r u = 0$$

$$\Rightarrow \partial_t f + n\frac{f - f_0}{r} + \frac{x^m}{r}\partial_m f = 0$$

- Implemented through upwinding, i.e. one sided derivatives



- This method is straightforwardly generalized to asymptotically de Sitter spacetimes Zilhão et al PRD 1204.2019

2.6 Diagnostics

The subtleties of diagnostics in GR

- Successful NR simulation $\rightarrow$ Tons of numbers for grid functions
- Typically: Spacetime metric $g_{\alpha\beta}$ and time derivative $\partial_t g_{\alpha\beta}$, or

  ADM variables $\gamma_{ij}\,,\quad K_{ij}\,,\quad \alpha\,,\quad \beta^i$
- Challenges
  - Coordinate dependence of numbers $\Rightarrow$ Gauge invariants
  - Global quantities at ∞, domain finite $\Rightarrow$ Extrapolation
  - Complexity of variables, e.g. GWs $\Rightarrow$ Spherical harmonics
  - Local quantities meaningful? $\Rightarrow$ Horizons
- AdS/CFT correspondence: Dictionary

Global quantities

- Assumptions:
  - Asymptotically, the metric is flat and time independent
  - Our expressions refer to Cartesian coordinates
- ADM mass = Total mass-energy of the spacetime

$$M_{\rm ADM} = \frac{1}{4\Omega_{D-2}G} \lim_{r\to\infty} \int_{S_r} \sqrt{\gamma}\left[\gamma^{mn}\gamma^{kl}(\partial_n\gamma_{mk} - \partial_k\gamma_{mn})\right] dS_l$$

- Linear momentum of spacetime

$$P_i = \frac{1}{2\Omega_{D-2}G} \lim_{r\to\infty} \int_{S_r} \sqrt{\gamma}\,(K^m{}_i - \delta^m{}_i\,K)\, dS_m$$

- Angular momentum in $D = 4$

$$J_i = \frac{1}{8\pi}\epsilon_{il}{}^m \lim_{r\to\infty} \int_{S_r} \sqrt{\gamma}\, x^l\,(K^n{}_m - \delta^n{}_m\,K)\, dS_n$$

- By construction, these are time independent!

Apparent horizons

- By cosmic censorship, existence of an apparent horizon implies an event horizon
- Consider outgoing null geodesics with tangent vector k^μ
- **Def.:** Expansion $\Theta := \nabla_\mu k^\mu$
- **Def.:** Apparent horizon $:=$ Outermost surface on Σ_t where $\Theta = 0$
- On a hypersurface Σ_t, the condition for $\Theta = 0$ becomes

  $\hat{D}_m s^m - K + K_{mn} s^m s^n = 0\,,$

  where $s^i =$ unit normal to the $D-2$ dimensional AH surface

  and $\hat{D}_i =$ the cov.deriv. of the metric induced on this surface;

  e.g. Thornburg PRD gr-qc/9508014

Apparent horizons

- Parametrize the horizon by $r = f(\varphi^i)$,

  where r is the radial and φ^i are angular coordinates

- Rewrite the condition $\Theta = 0$ in terms of $f(\varphi)$

  $\Rightarrow$ Elliptic equation for $f(\varphi)$

- This can be solved e.g. with Flow, Newton methods

  Thornburg PRD gr-qc/9508014; Gundlach PRD gr-qc/9707050

  Alcubierre CQG gr-qc/9809004; Schnetter CQG gr-qc/0306006

- Irreducible mass: $M_{\rm irr} = \sqrt{\frac{A_{\rm AH}}{16\pi G^2}}$

- Total BH mass: $M^2 = M_{\rm irr}^2 + \frac{S^2}{4M_{\rm irr}^2} \quad (+\, P^2)$

  where S is the spin of the BH

  Christodoulou PRL 25 1596

GW extraction: Newman Penrose Scalars

- Construct a tetrad
  - $n^\alpha =$ Timelike unit normal field
  - Spatial triad $\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}$ Gram-Schmidt orthonormalization

    E.g. starting with $u^i = [x,\, y,\, z]\,,\quad v^i = [xz,\, yz,\, -x^2 - y^2]\,,$

    $$w^i = \epsilon^i{}_{mn} v^m w^n\,.$$
  - $l^\alpha = \frac{1}{\sqrt{2}}(n^\alpha + u^\alpha)\,,\quad k^\alpha = \frac{1}{\sqrt{2}}(n^\alpha - u^\alpha)\,,\ m^\alpha = \frac{1}{\sqrt{2}}(v^\alpha + i\, w^\alpha)$

    $\Rightarrow\quad -\boldsymbol{l}\cdot\boldsymbol{k} = 1 = \boldsymbol{m}\cdot\bar{\boldsymbol{m}}$, all other products vanish
- Newman-Penrose scalar $\Psi_4 = -C_{\alpha\beta\gamma\delta} k^\alpha \bar{m}^\beta k^\gamma \bar{m}^\delta$
- In vacuum: $R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}$
- For more details, see e.g.

Nerozzi PRD gr-qc/0407013; Brügmann et al PRD gr-qc/0610128

Analysis of Ψ_4

- Multipolar decomposition: $\Psi_4 = \sum_{\ell,m} \psi_{\ell m}(t,r) Y^{-2}_{\ell m}(\theta,\phi)$

  where $\psi_{\ell m} = \int_0^{2\pi} \int_0^{\pi} \Psi_4 \overline{Y^{-2}_{\ell m}} \sin^2\theta \, d\theta \, d\phi$

- Radiated energy $\frac{dE}{dt} = \lim_{r\to\infty} \left[\frac{r^2}{16\pi} \int_\Omega \left| \int_{-\infty}^{t} \Psi_4 \, d\tilde{t} \right|^2 d\Omega \right]$

- Momentum: $\frac{dP_i}{dt} = -\lim_{r\to\infty} \left[\frac{r^2}{16\pi} \int_\Omega \ell_i \left| \int_{-\infty}^{t} \Psi_4 \, d\tilde{t} \right|^2 d\Omega \right],$

  where $\ell_i = [-\sin\theta \, \cos\phi, \; -\sin\theta \, \sin\phi, \; -\cos\theta]$

- Angular momentum:

$$\frac{dJ_z}{dt} = -\lim_{r\to\infty} \left\{ \frac{r^2}{16\pi} \mathrm{Re} \left[\int_\Omega \left(\partial_\phi \int_{-\infty}^{t} \Psi_4 d\tilde{t} \right) \left(\int_{-\infty}^{t} \int_{-\infty}^{\hat{t}} \bar{\Psi}_4 \, d\tilde{t} \, d\hat{t} \right) d\Omega \right] \right\}$$

  see e.g. Ruiz et al GRG 0707.4654

- Wave strain: $\Psi_4 = \ddot{h}_+ - i\ddot{h}_\times$

Alternative extraction methods

- Landau-Lifshitz pseudo tensor: simple but gauge dependent

  see e.g. Lovelace et al PRD 0907.0869

- Regge-Wheeler-Zerilli-Moncrief perturbation formalism:

  perturbations on Schwarzschild $\rightarrow$ gauge invariant master function

  Regge & Wheeler PR (1957); Zerilli PRL (1970);

  Moncrief Ann.Phys. (1974);

  For applications in NR see e.g. Reisswig et al PRD 1012.0595

  Sperhake et al PRD gr-qc/0503071; Rezzolla gr-qc/0302025

- Cauchy-characteristic extraction at $\mathcal{I}^+$ using a compactified

  exterior vacuum patch with characteristic coordinates: very accurate

  Reisswig et al PRL 0907.2637, CQG 0912.1285;

  Babiuc et al PRD 1011.4223

Further reading

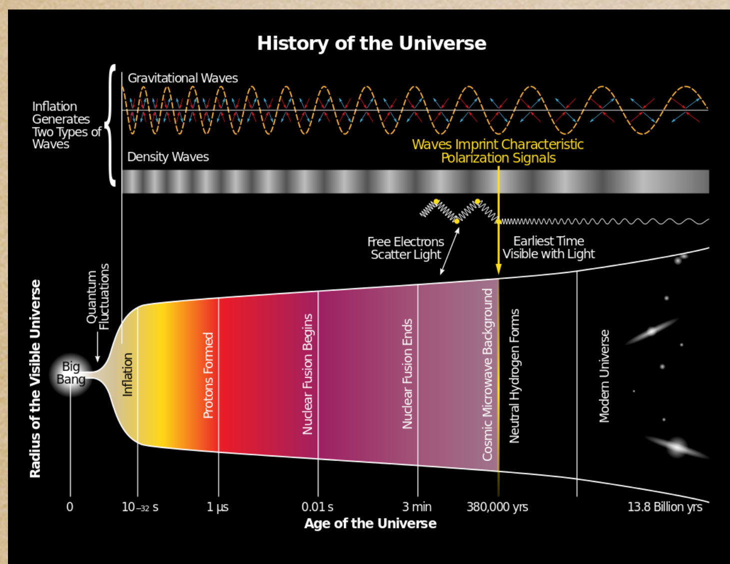
- Isolated and dynamic horizon

  Ashtekar & Krishnan LRR gr-qc/0407042

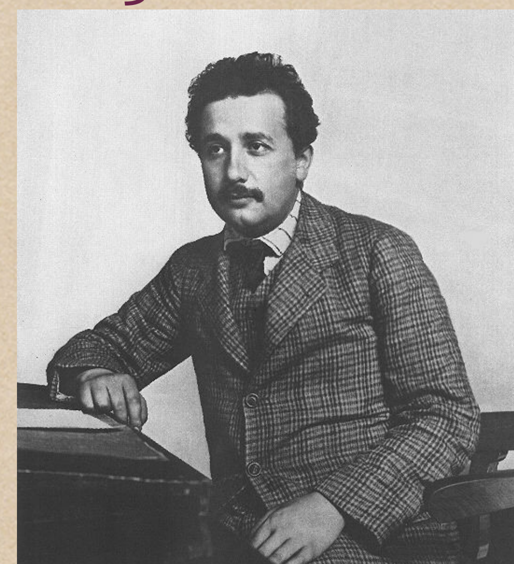
3. Results from BH simulations

Overview

Early Universe

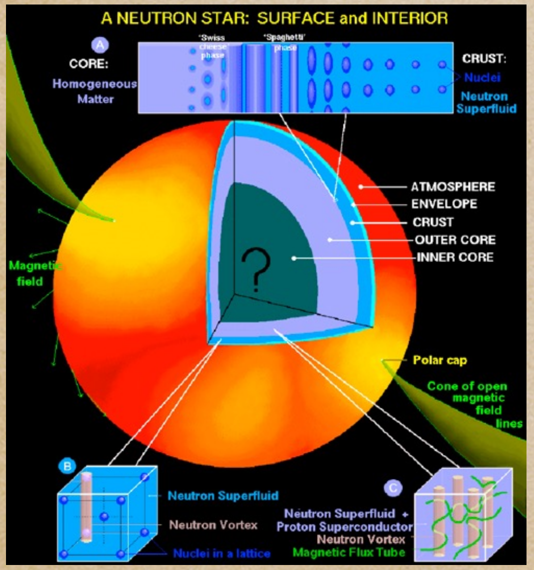


Testing Einstein's theory



Galaxy history

Equation of state



BH populations

The unknown…

Gravitational waves: weak-field solutions

- Consider small deviations from Minkowski in Cartesian coordinates

  "Background": Manifold $\mathcal{M} = \mathbb{R}^4, \quad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$

  "Perturbation": $h_{\mu\nu} = \mathcal{O}(\epsilon) \ll 1 \quad \Rightarrow \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

- Coordinate freedom: "Transverse-traceless (TT)" gauge

$$h^{\mu}{}_{\mu} = 0, \quad \partial^{\nu} h_{\mu\nu} = 0$$

- Vacuum, no cosmological constant: $T_{\mu\nu} = 0, \quad \Lambda = 0$
- Einstein's eqs.: $\Box h_{\mu\nu} = 0$
- Plane wave solution in z direction: $h_{\mu\nu} = H_{\mu\nu} e^{i k_{\sigma} x^{\sigma}}$

$$k^{\mu} = \omega(1, 0, 0, 1) \qquad H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Effect on particles

- Geodesic eq.

  Particle at rest at x^μ stays at $x^\mu = \text{const}$ in TT gauge

- Proper separation:

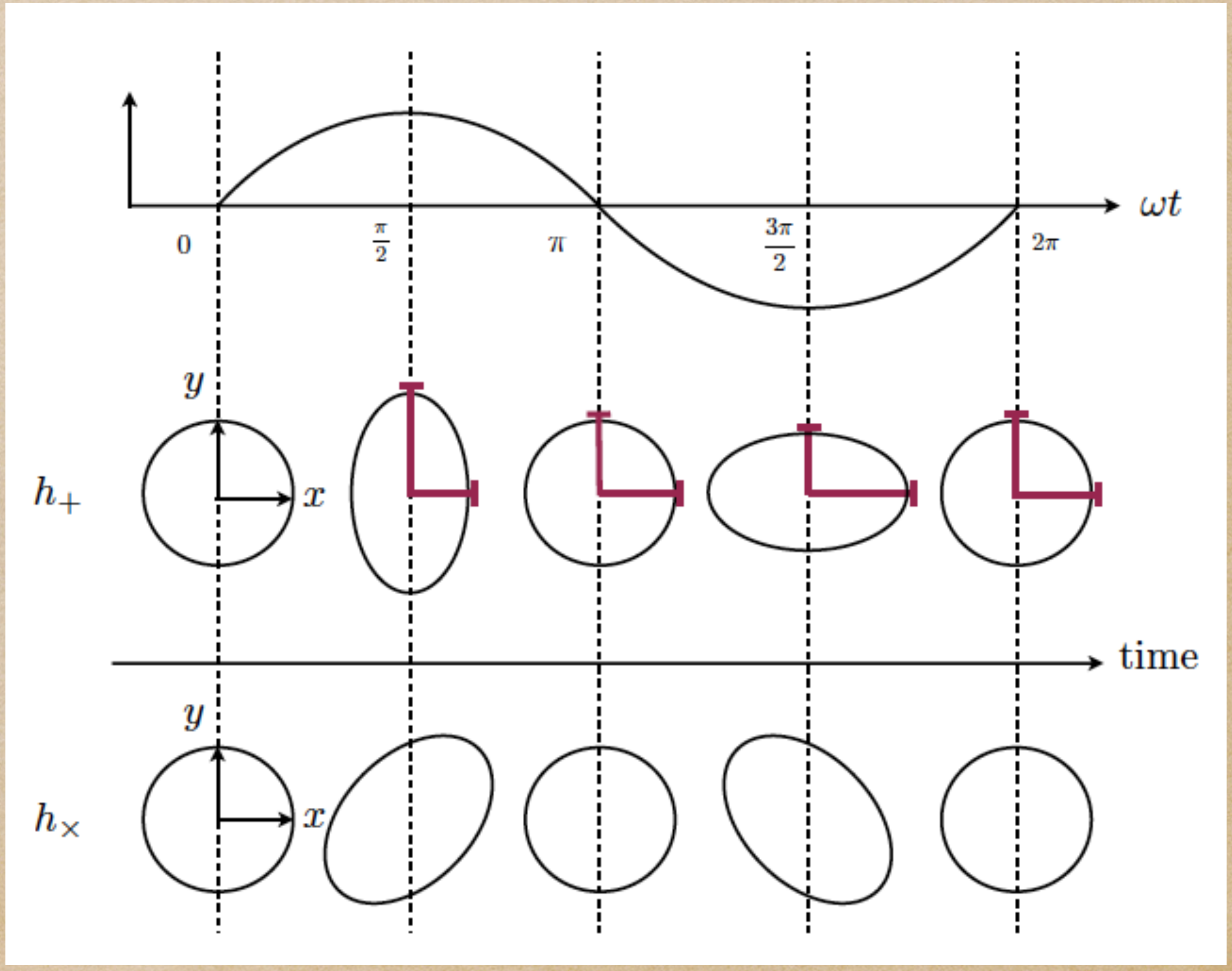
$$ds^2 = -dt^2 + (1+h_+)\,dx^2 + (1-h_+)\,dy^2 + 2h_\times\,dx\,dy + dz^2$$

- Effect on test particles:

  Mirshekari 1308.5240

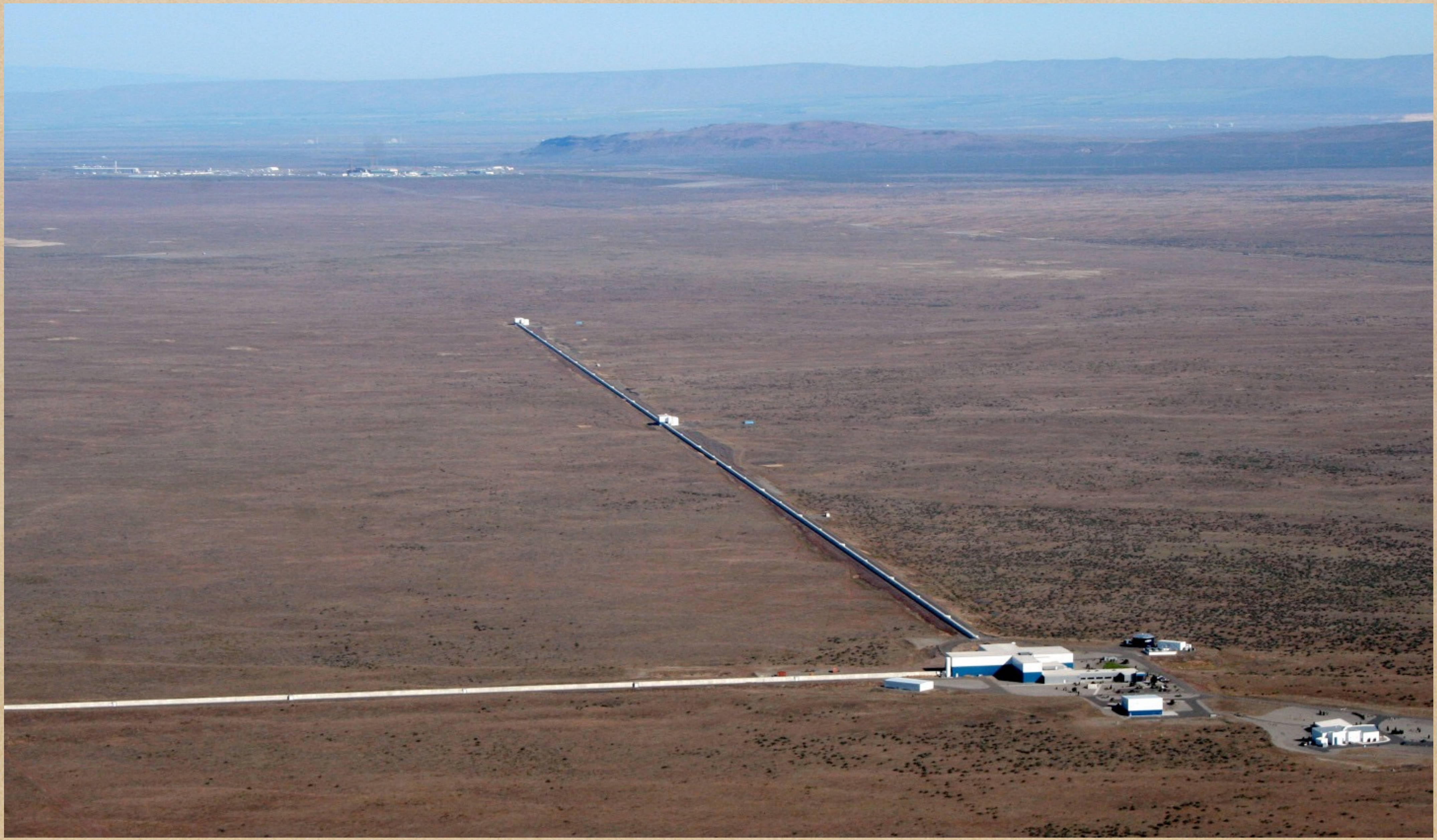
- Debate on physical reality until late 1950s

  e.g.Saulson GRG (2011)



Effect on particles

- Measure this effect; Michelson-Morley type interferometer



The gravitational wave spectrum

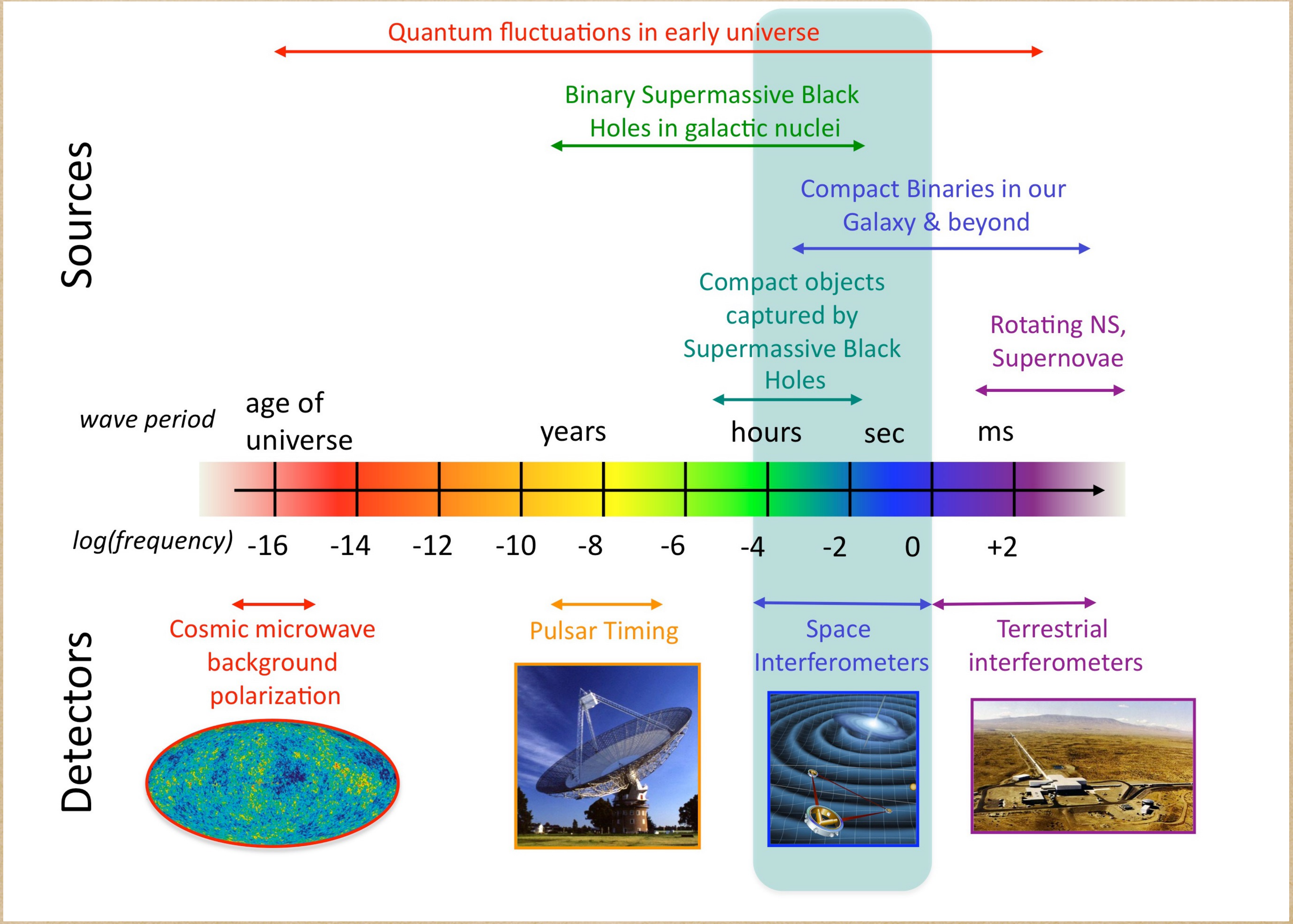
- Source types and detection strategies $\Rightarrow$ 4 regimes

Ultra low	$f \sim 10^{-18} \ldots 10^{-15}$ Hz
Very low	$f \sim 10^{-9} \ldots 10^{-6}$ Hz
Low	$f \sim 10^{-4} \ldots 10^{-1}$ Hz
High	$f \sim 10^{1} \ldots 10^{3}$ Hz

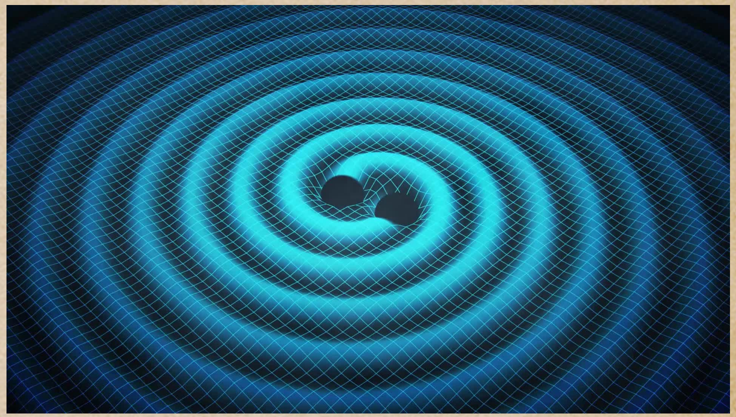
- Major sources

Ultra low:	Fluctuations in the early universe
Very low:	Supermassive BH binaries (high M, z)
Low:	SMBHs, EMRIs, Compact binaries,...
High:	Neutron star / BH binaries, supernovae,...

The gravitational wave spectrum

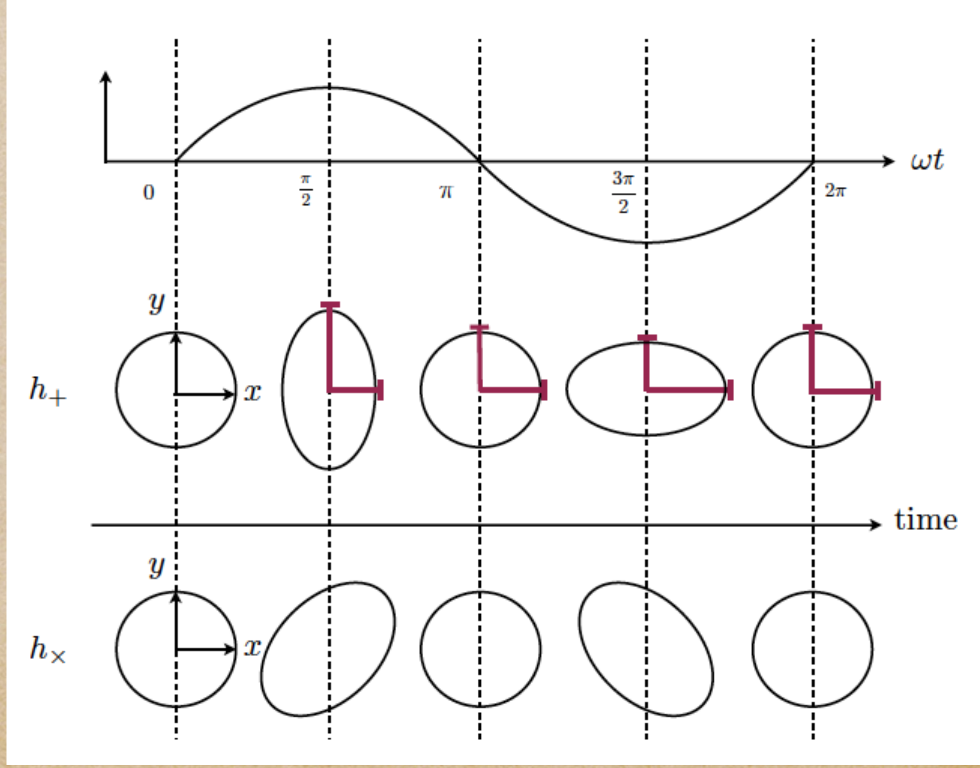


The search for GWs in the data stream



$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}\,; \qquad \frac{8\pi G}{c^4} = 2.07 \times 10^{-43} \frac{\mathrm{s}^2}{\mathrm{m\,kg}}$$

- Weak effect of matter on geometry
- GWs carry huge energy but barely interact with anything
- Induced changes in length: < atomic nucleus / km



Detection and parameter estimation

Generic transient search

- No specific waveform model
- Identify excess power in detector strain data
- Use multi detector maximum likelihood Klimenko et al. 1511.05999

Binary coalescence search

- "Matched Filtering"
- Compare data stream with GW templates ("Finger print search")
- Bayesian analysis: Prior $\longrightarrow$ Posterior

Black-hole binaries: parameters

- 8+2 Intrinsic parameters

  Masses m_1 , m_2

  Spins $\boldsymbol{S}_1$, $\boldsymbol{S}_2$

  Eccentricity (often ignored; GW emission circularizes orbit)

- 7 Extrinsic parameters

  Location: Luminosity distance D_L , Right ascension α , Declination δ

  Orientation: Inclination ι , Polarization ψ

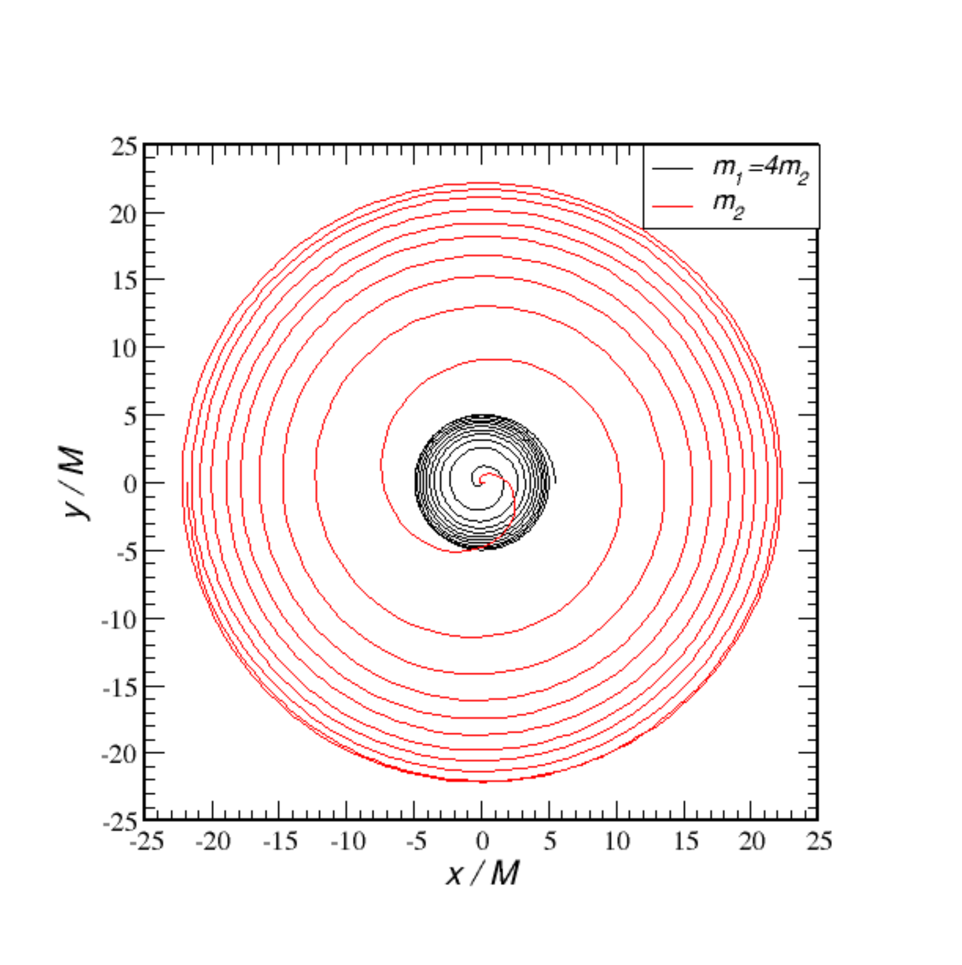
  Time t_c and Phase ϕ_c of coalescence

Binary BH trajectory and waveform

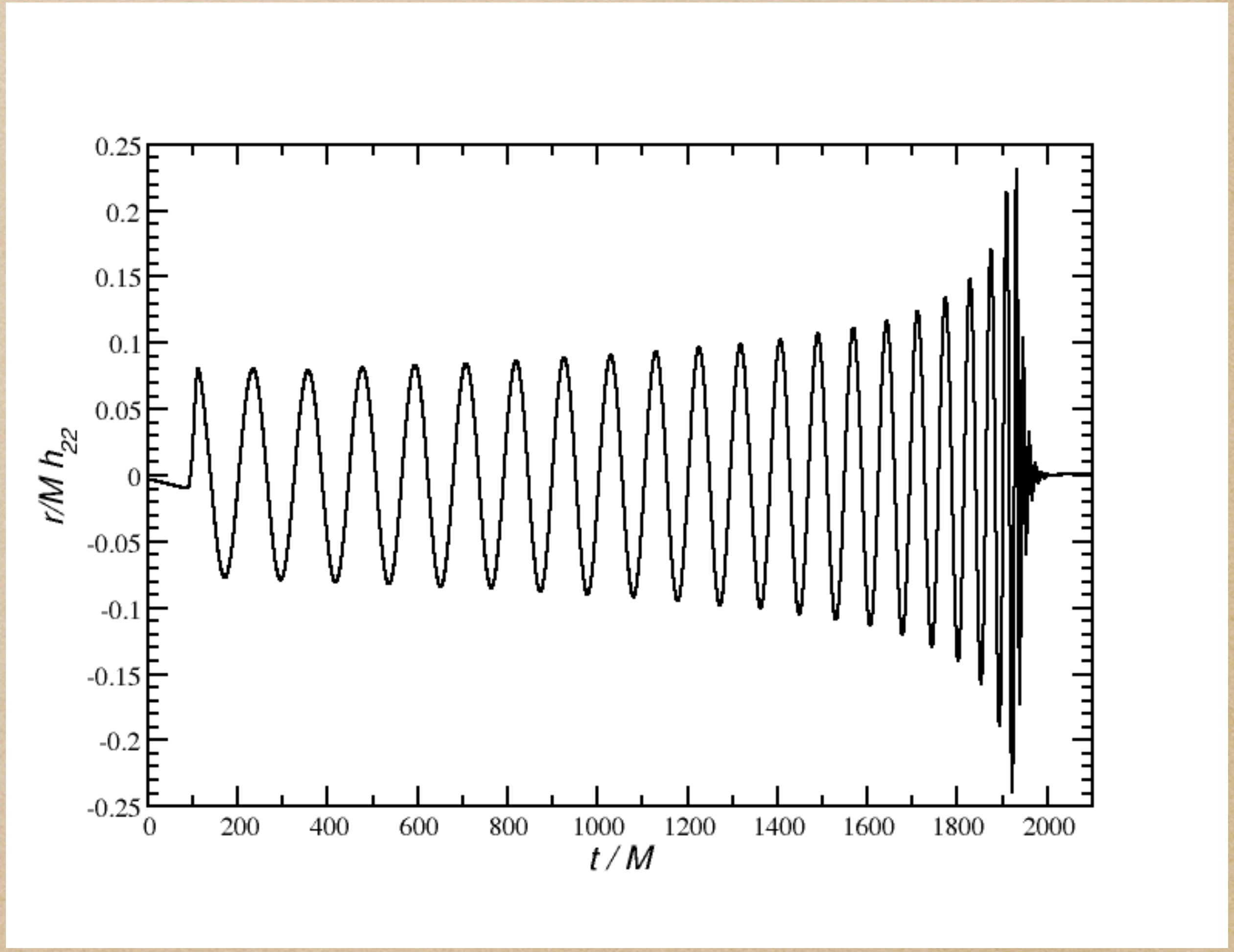
- $\frac{m_1}{m_2} = 4$ non-spinning binary; ≈ 11 orbits

  Sperhake et al CQG 1012.3173

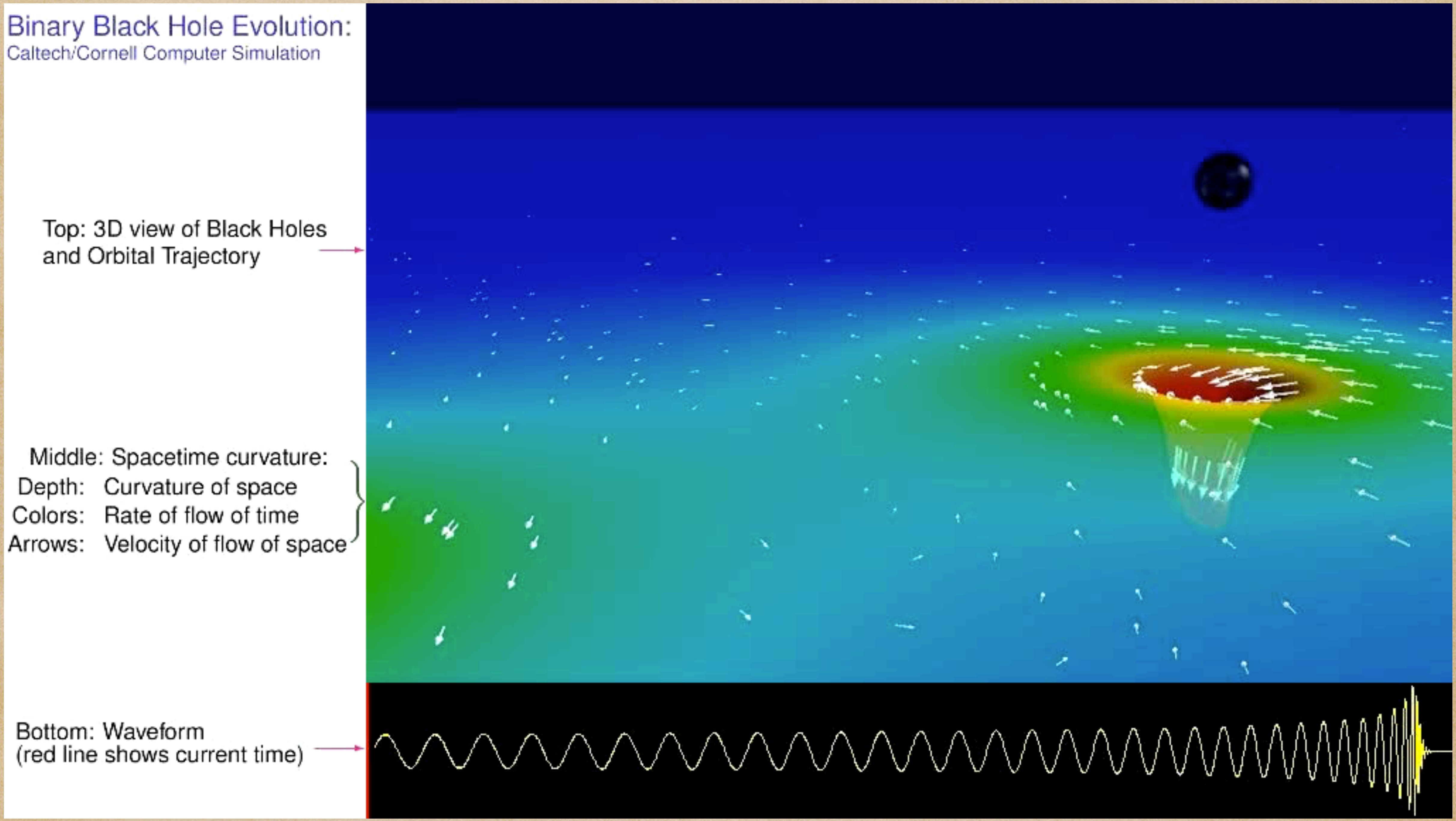
Trajectory



Quadrupole mode



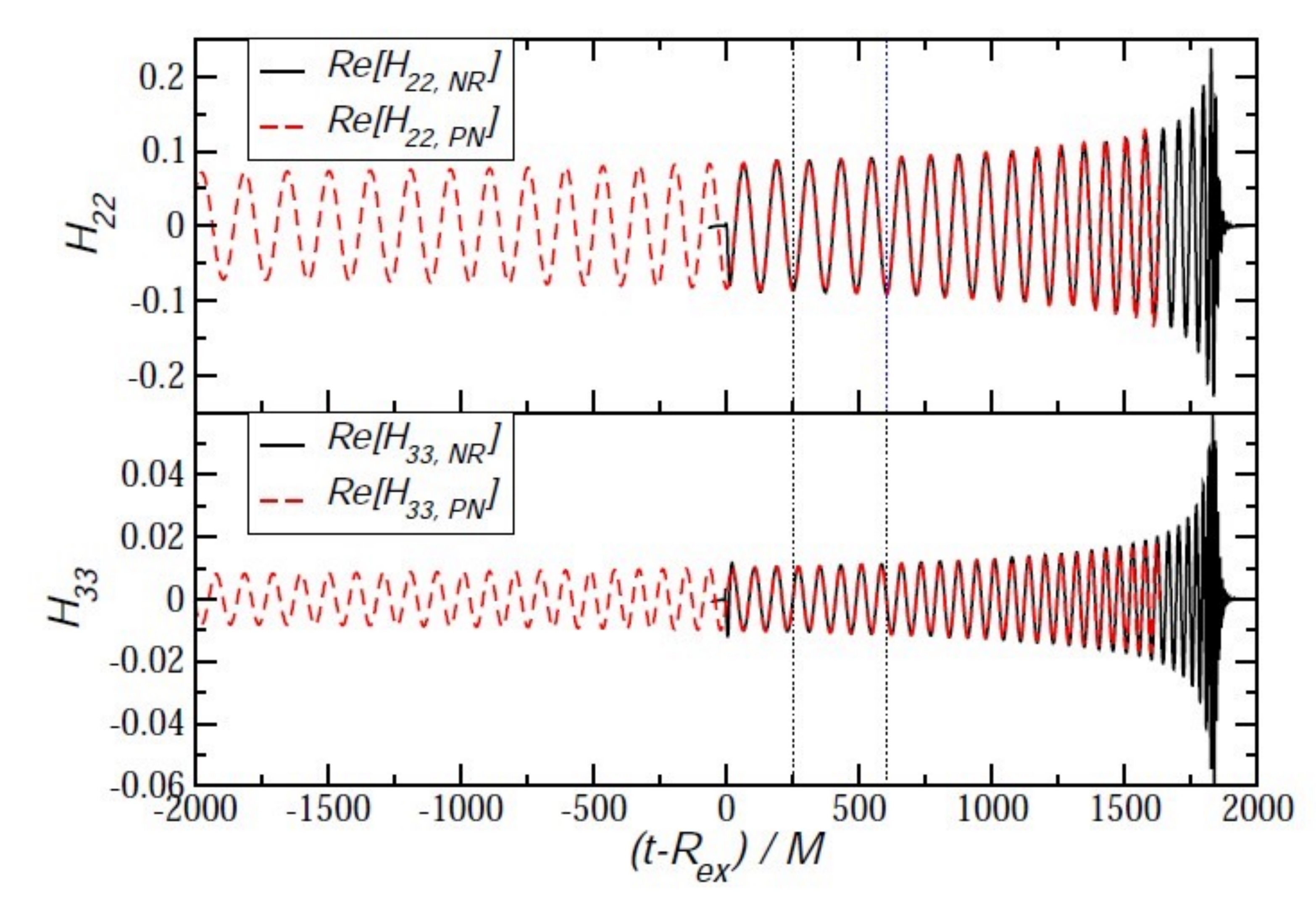
Anatomy of a BHB coalescence



Thanks to Caltech-Cornell groups

Hybrid waveforms and catalogs

- Stitch together PN and NR waveforms



Sperhake et al CQG 2011

- Mass produce waveforms; Hinder et al CQG 1307.5307;
  Mroué et al PRL 1004.4697

GW source modeling

- Key requirement for matched filtering: GW template catalog
- Model black holes in general relativity
  - Post Newtonian theory $\rightarrow$ Inspiral   Blanchet LRR-2006-4
  - Numerical relativity $\rightarrow$ final orbits, merger

    Pretorius PRL 2005, Baker et al PRL 2006, Campanelli et al PRL 2006
  - Perturbation theory $\rightarrow$ Ringdown
- Combine "NR" with "Post-Newtonian", "Effective one body" methods
- 2 families in use: Phenomenological, Effective one body
- Use reduced bases or similar to cover parameter space
- Multipolar decomposition

$$h_+ - ih_\times = \sum_{\ell m} {}_{-2}Y_{\ell m}(\theta, \phi) h_{\ell m}(t)$$

Template construction

- Phenomenological waveform models
  - Model phase, amplitude with simple funcs. $\rightarrow$ Model parameters
  - Create map between physical and model parameters
  - Time or frequency domain; see e.g.:

    Ajith et al CQG 0704.3764, PRD 0710.2335, PRL 0909.2867;

    Santamaria et al PRD 1005.3306; Khan et al PRD 1508.07253
- Effective-one-body (EOB) models
  - Particle in effective metric, PN, bringdown model

    Buonanno & Damour PRD gr-qc/9811091, PRD gr-qc/0001013
  - Resum PN, calibrate pseudo-PN parameters using NR; see e.g.:

    Buonanno+ PRD 0709.3839; Pan+ PRD 1106.1021, PRD 1307.6232;

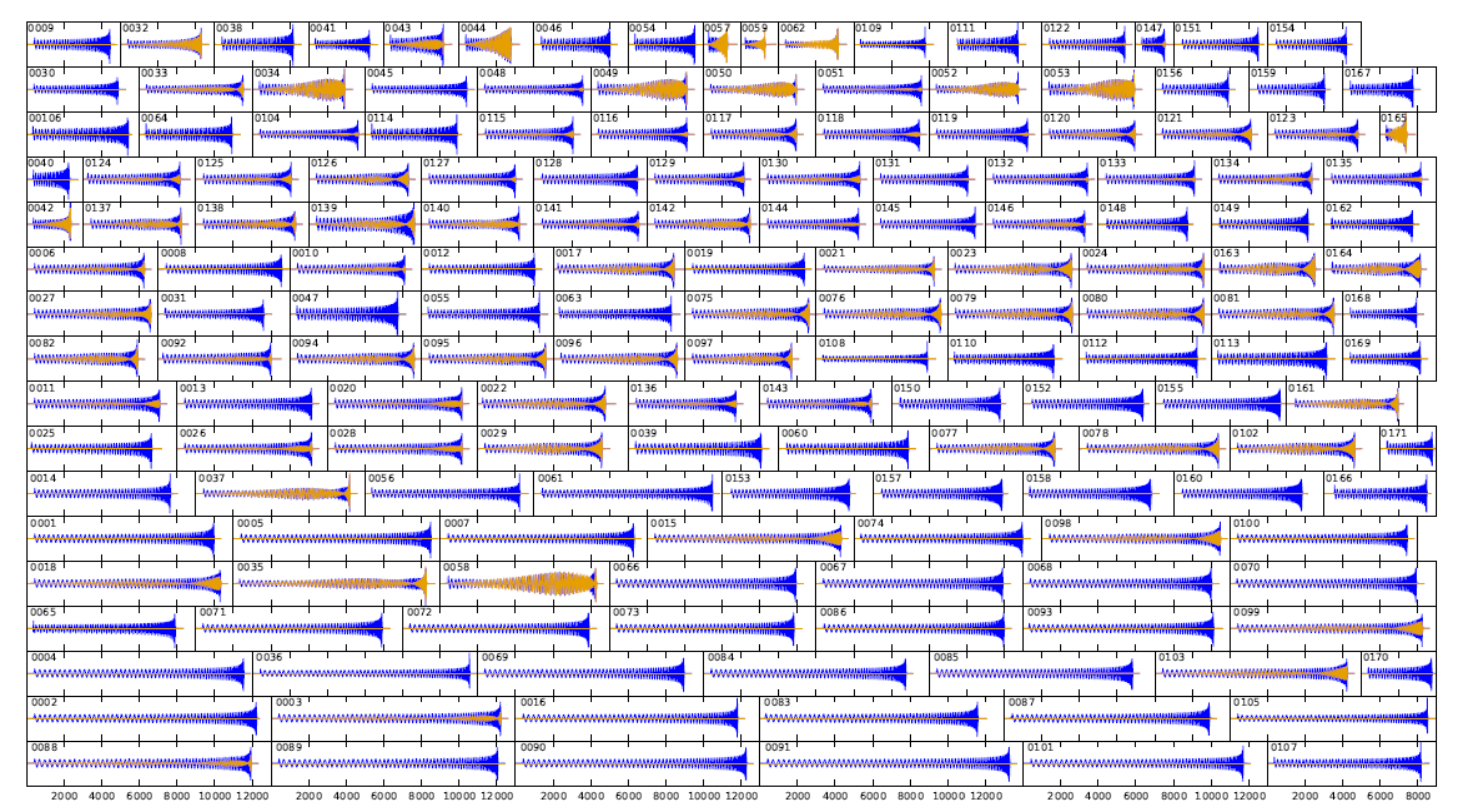
    Tarachini+ PRD 1311.2544; Damour & Nagar PRD 1406.6913

Tools of mass production

- Explore seven-dim. parameter space. E.g. SpEC catalogue:

  171 waveforms: $m_1/m_2 \leq 8$ up to 34 orbits

  Mroué et al PRL 1304.6077



Limits in parameter space

- Mass ratio: $m_1/m_2 = 100$; better waveforms needed

  Lousto & Zlochower PRL 1009.0292

- Spins: $S/M^2 = 0.994$

  Superposed Kerr-Schild data better than punctures here

  Lovelace et al CQG 1411.7297

- Length ≈ 175 orbits

  Szilágyi et al PRL 1502.04953

- Spin precession remains a considerable challenge

  e.g. Ossokine PRD 1502.01747; Hannam et al PRL 1308.3271;

  Gerosa et al PRD 1506.03492

Further reading

- Reviews of numerical relativity

  Centrella et al Rev.Mod.Phys. 1010.5260

  Pretorius 0710.1338

  Sperhake et al Comptes Rend. phys. 1107.2819

  Pfeiffer CQG 1203.5166

  Hannam CQG 0901.2931

  Cardoso et al LRR 1409.0014

  Barrack et al

4. Generation of Waveform catalogs

4.1. A brief sketch of post Newtonian Theory

The pN approximation

- Approximation of GR for weak fields and low velocities
- For bound systems: $\frac{v}{c} \sim \sqrt{\frac{GM}{c^2 r}}$
- pN order counting: Corrections of order $(v/c)^n$ are $n/2$ PN order
- For GW source modeling we really need three theories
  - pN theory for modeling the slow sources
  - post-Minkowskian theory for weak fields: expansion in G
  - Multipolar expansion for computation of GWs

  Blanchet LRR 1310.1528

  Poisson & Will "Gravity", Poisson "Introduction to pN theory"

  much of the following from Poisson and Poisson & Will...
- Three approaches: ADM Hamiltonian, **Harmonic**, Surface Integral

Landau-Lifshitz formulation of GR

- Define $\mathrm{g}^{\alpha\beta} := \sqrt{-g}\, g^{\alpha\beta}$

$$H^{\alpha\mu\beta\nu} := \mathrm{g}^{\alpha\beta}\mathrm{g}^{\mu\nu} - \mathrm{g}^{\alpha\mu}\mathrm{g}^{\beta\nu}$$

$$\Rightarrow \;\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = \frac{16\pi G}{c^4}(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{\mathrm{LL}})$$

  with $t^{\alpha\beta}_{\mathrm{LL}} \sim \partial\mathrm{g}\,\partial\mathrm{g}$

  conservation of energy momentum: $\partial_\beta \left[(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{\mathrm{LL}})\right] = 0$

- Global conservation laws: $\dfrac{dE}{dt} = -c \oint (-g) t^{0j}_{\mathrm{LL}} dS_j \,,$

$$E = \int (-g)(T^{00} + t^{00}_{\mathrm{LL}}) d^3x$$

- Likewise for linear and angular momentum

Relaxed Field equations

- Define $h^{\alpha\beta} := \eta^{\alpha\beta} - \mathrm{g}^{\alpha\beta}$
- Use harmonic gauge $\partial_\mu h^{\alpha\mu} = 0$

$$\Rightarrow \quad \Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}$$

where $\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$,

$$\tau^{\alpha\beta} = (-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\mathrm{LL}} + t^{\alpha\beta}_{\mathrm{H}}\right),$$

$$t^{\alpha\beta}_{\mathrm{H}} \sim \partial h \cdot \partial h + h\,\partial^2 h$$

- Key gain: Wave Eq. in flat spacetime!
- Conservation Eq.: $\partial_\beta \tau^{\alpha\beta} = 0 \quad \Leftrightarrow \quad \partial_\beta h^{\alpha\beta} = 0$

post Minkowskian Theory

- Solve these equations iteratively

$$h_0^{\alpha\beta} = 0\,, \qquad \Box h_{n+1}^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}[h_n]$$

- Result: a formal expansion of $h^{\alpha\beta}$ in powers of G

- Principle for solving the wave equation: Greens function!

  analogous to solving: $\Box\psi = -4\pi\mu$

  with $\psi(t, \boldsymbol{x}) = \int \frac{(t - |\boldsymbol{x} - \boldsymbol{x'}|/c, \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|} d^3x'$

  integral over past light cone of $(t, \boldsymbol{x})$

- In the near zone for slow sources: $|\boldsymbol{x} - \boldsymbol{x'}| \ll ct$

  $\Rightarrow \mu(t - |\boldsymbol{x} - \boldsymbol{x'}|/c) = \mu(t) - \frac{1}{c}\frac{\partial\mu}{\partial t}|\boldsymbol{x} - \boldsymbol{x'}| + \ldots$ expansion in v/c

The matter sources

- Note: The unknown matter terms appear in this formal solution.

  To be determined from conservation $\partial_\beta \tau^{\alpha\beta} = 0$

- Example: perfect fluid

  Define rescaled mass density: $\rho^* := \sqrt{-g}(u^0/c)\rho$

  pressure: p

  internal energy per mass: ε

  velocity: $\boldsymbol{v} := \dfrac{d\boldsymbol{x}}{dt}$

- Conservation law: $\partial_t \rho^* + \partial_j(\rho^* v^j) = 0$

- For 1pN order: Need two iterations in the relaxed field equations

  ...

The matter sources

$$\Rightarrow \ldots \Rightarrow \quad g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\psi + \frac{1}{2}\partial_t^2 X - U^2\right) + \mathcal{O}(c^{-6})$$

$$g_{0j} = -\frac{4}{c^3}U_j + \mathcal{O}(c^{-5})$$

$$g_{jk} = \delta_{jk}\left(1 + \frac{2}{c^2}U\right) + \mathcal{O}(c^{-4})$$

where $U(t,\boldsymbol{x}) = G\int \frac{\rho^*(t,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x'$

$$U^j(t,\boldsymbol{x}) = G\int \frac{\rho^*(t,\boldsymbol{x'})\, v'^j}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x'$$

$$\psi(t,\boldsymbol{x}) = G\int \frac{\rho^*(t,\boldsymbol{x'})\left(\frac{3}{2}v'^2 - U(t,\boldsymbol{x'}) + \varepsilon(t,\boldsymbol{x'})\right) + 3p(t,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x'$$

$$X(t,\boldsymbol{x}) = G\int \rho^*(t,\boldsymbol{x'})\, |\boldsymbol{x}-\boldsymbol{x'}|\, d^3x'$$

The matter sources

- The equations of motion for the matter sources come from the post-Newtonian Euler equations

$$
\begin{aligned}
\rho^* \frac{dv^j}{dt} = & -\partial_j p + \rho^* \partial_j U \\
& + \frac{1}{c^2}\left[\left(\frac{1}{2}v^2 + U + \varepsilon + \frac{p}{\rho^*}\right)\partial_j p - v^j \partial_t p\right] \\
& + \frac{1}{c^2}\rho^* \Big[(v^2 - 4U)\partial_j U - v^j(3\partial_t U + 4v^k \partial_k U) \\
& \qquad + 4\partial_t U_j + 4v^k(\partial_k U_j - \partial_j U_k) + \partial_j \psi + \frac{1}{2}\partial_t^2 \partial_j X\Big] \\
& + \mathcal{O}(c^{-4})
\end{aligned}
$$

which follows from $\nabla_\mu T^{\alpha\mu}$ using the pN metric!

The matter sources

- One obtains

mass: $$m_A = \int_A \rho^* d^3x$$

position: $$\boldsymbol{x}_A(t) = \frac{1}{m_A}\int_A \rho^* \, \boldsymbol{x} d^3x$$

velocity: $$\boldsymbol{v}_A(t) = \frac{1}{m_A}\int_A \rho^* \, \boldsymbol{v} d^3x = \frac{d\boldsymbol{x}_A}{dt}$$

acceleration: $$\boldsymbol{a}_A(t) = \frac{1}{m_A}\int_A \rho^* \, \frac{d\boldsymbol{v}}{dt} d^3x = \frac{d\boldsymbol{v}_A}{dt}$$

- Difficulty: need to insert Euler Eq. for $\frac{d\boldsymbol{v}}{dt}$ —> Iteration scheme
- One can also compute integrals for kinetic energy, grav.energy, include spins etc.

  —> Big mess! We stop here…

Gravitational wave signal

- GW signal is the transverse-traceless part of $h^{\alpha\beta}$ for $R := |\boldsymbol{x}| \to \infty$

$$h^{jk}_{\rm TT} = \left(\perp^j{}_m \perp^k{}_n - \frac{1}{2} \perp^{jk} \perp_{mn} \right) h^{mn} \,, \qquad \perp^j{}_m := \delta^j{}_m - N^j N_m \,, \qquad N^i := \frac{\boldsymbol{x}^i}{R}$$

  One can show that this indeed gives $N_j h^{jk}_{\rm TT} = 0 \,, \quad \delta_{jk} h^{jk}_{\rm TT} = 0$

- Example: With two iterations one obtains the leading-order quadrupole $h^{jk}_{\rm TT} = \dfrac{2G}{c^4} \ddot{I}^{jk}_{\rm TT}$

$$I^{jk}(t) = \int \rho^*(t, \boldsymbol{x}) \left(x^j x^k - \frac{1}{3} |\boldsymbol{x}|^2 \delta^{jk} \right) d^3x$$

- More iterations give higher pN order terms and higher-order multipoles...

Example: equal-mass binary, no spins, 1.5pN

$$h_+ = \frac{2\eta G m}{c^2 R}\beta^2\left[(1+2\pi\beta^3)H^{[0]}+\beta H^{[1/2]}+\beta^2 H^{[1]}+\beta^3 H^{[3/2]}+\cdots\right]$$

$$H^{[0]} = -(1+C^2)\cos 2\Psi,$$

$$H^{[1/2]} = -\frac{\Delta}{8}S\left[(5+C^2)\cos\Psi - 9(1+C^2)\cos 3\Psi\right]$$

$$H^{[1]}_\times = \frac{1}{3}C\left[(17-4C^2)-(13-12C^2)\eta\right]\sin 2\Psi - \frac{8}{3}(1-3\eta)S^2 C\sin 4\Psi$$

$$H^{[3/2]}_\times = \frac{\Delta}{96}SC\left[(63-5C^2)-2(23-5C^2)\eta\right]\sin\Psi$$

$$-\frac{9\Delta}{64}SC\left[(67-15C^2)-2(19-15C^2)\eta\right]\sin 3\Psi + \frac{625\Delta}{192}(1-2\eta)S^3 C\sin 5\Psi$$

$$m = M_1 + M_2, \qquad \eta = \frac{M_1 M_2}{(M_1+M_2)^2}, \qquad \Delta = \frac{M_1 - M_2}{M_1 + M_2}$$

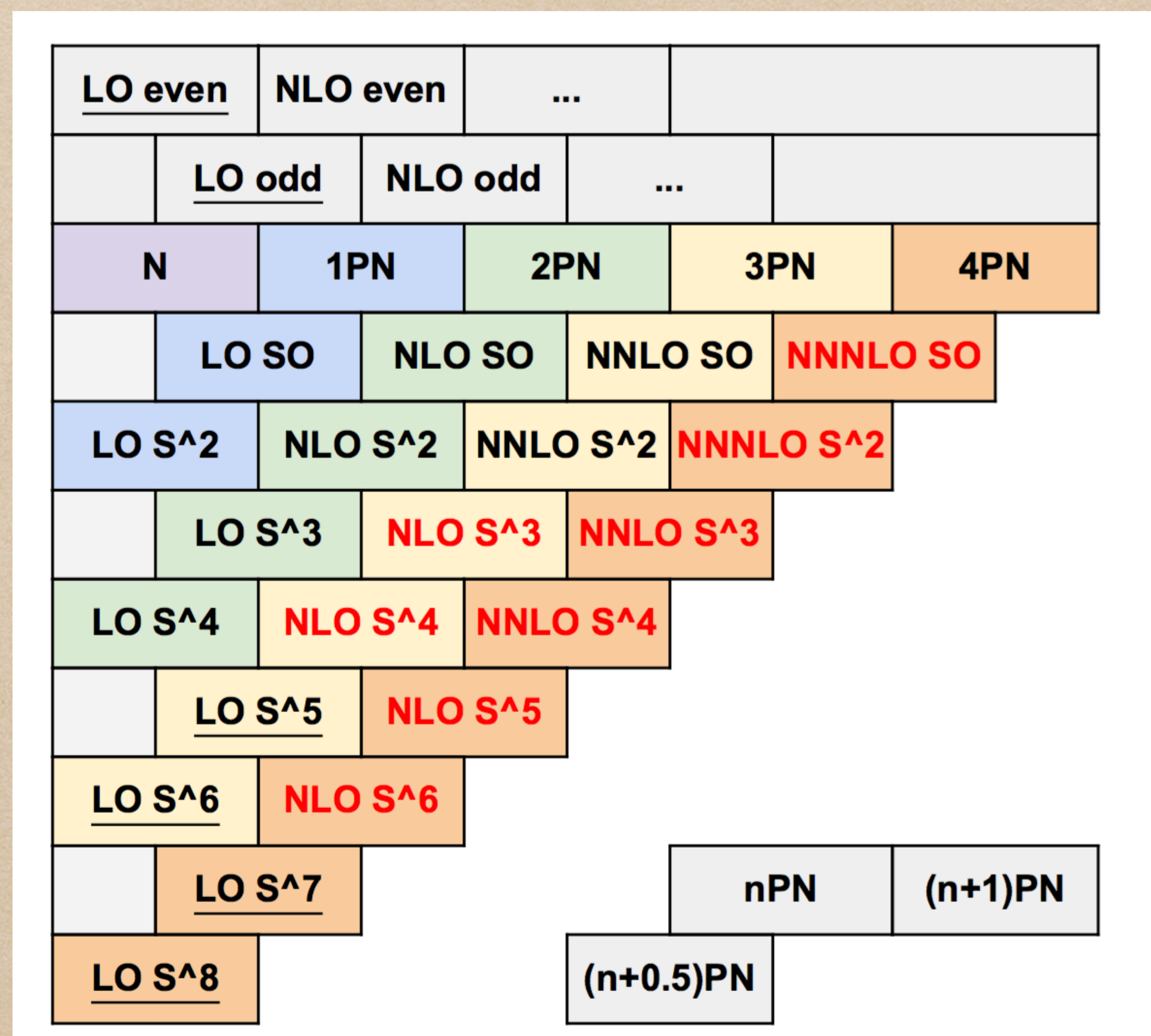
$$S = \sin\iota, \qquad C = \cos\iota, \qquad \Omega = \text{angular velocity}, \qquad \beta = \left(\frac{Gm\Omega}{c^3}\right)^{1/3} \sim v/c$$

$$\Psi = \Omega\left(t - R/c - \frac{2Gm}{c^3}\ln\frac{4\Omega R}{c}\right)$$

from Eric Poisson "Introduction to pN theory"

Addition of spin terms

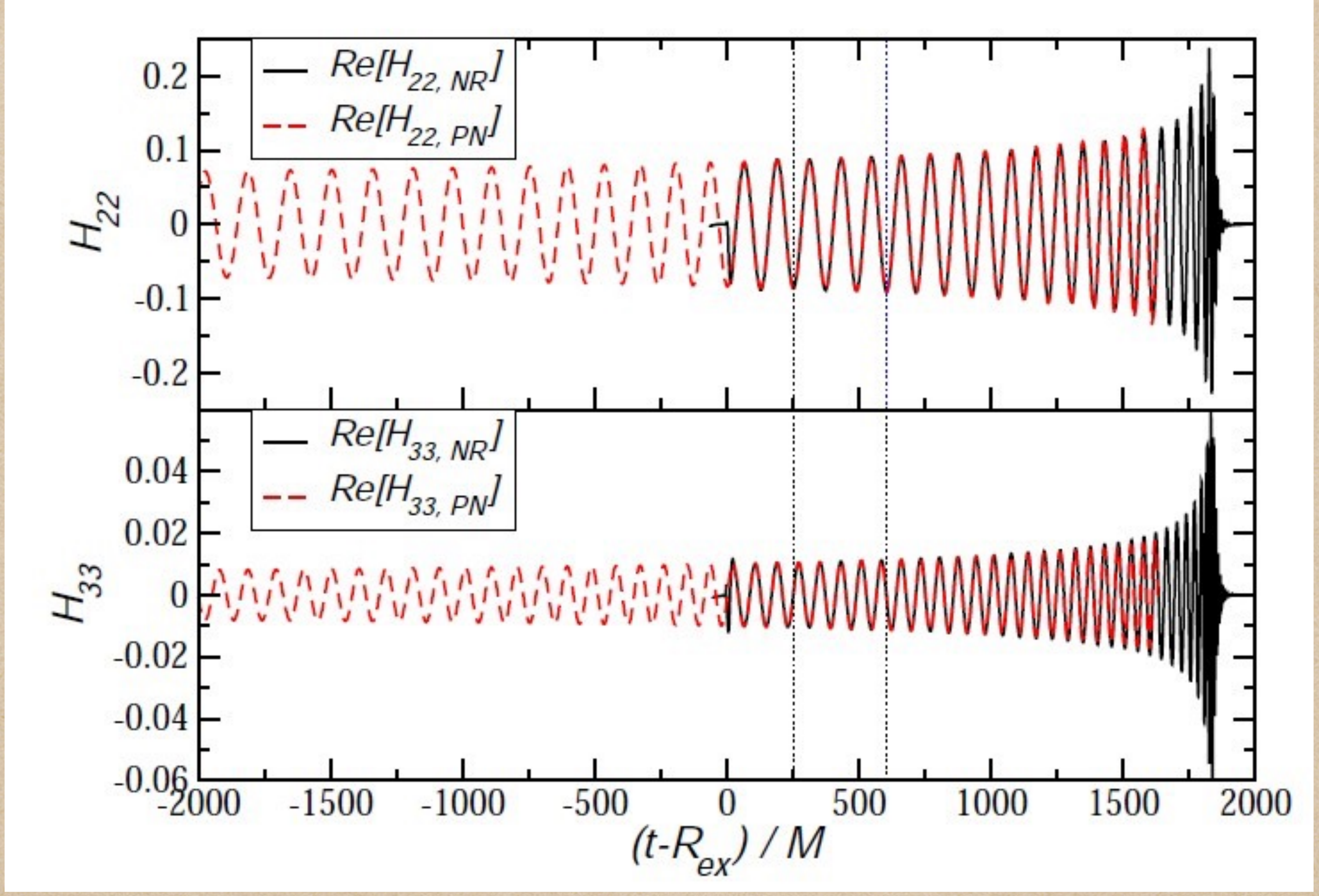
- Non-spinning binaries: up to 4PN terms
- Spins: more complicated; spin-spin, spin-orbit coupling etc.



from Vines & Steinhoff 1606.08832

Hybrid waveforms and catalogs

- Stitch together PN and NR waveforms



Sperhake et al CQG 2011

- Mass produce waveforms; Hinder et al CQG 1307.5307;
  Mroué et al PRL 1004.4697

4.2. Effective one body (EOB) waveforms

The main ingredients of EOB

Much from T.Hinderer in Barack et al. 1806.05195

- Hamiltonian for inspiral dynamics
- Prescription for computation of GWs and radiation reaction forces
- Transition from inspiral-merger to ringdown of a perturbed BH

Buonanno & Damor PRD gr-qc/9811091, gr-qc/0001013

- Consider two bodies with $m_{1,2},\ \boldsymbol{S}_{1,2}$ and separation/motion $\boldsymbol{x},\ \boldsymbol{p}$
- Map this to an effective particle with $\mu := \dfrac{m_1 m_2}{m_1 + m_2}\,,\ \boldsymbol{S}_{\rm eff}(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{s}, \boldsymbol{p})$

  moving in an effective Kerr-like spacetime with

  $g^{\rm eff}_{\alpha\beta}(M, \boldsymbol{S}_{\rm Kerr}; \nu)$ where $M := m_1 + m_2\,,\quad \nu := \dfrac{\mu}{M}$

Mapping the Hamiltonians

- For the mapping one requires
  - The test particle limit reduces to a test particle in Kerr
  - For slow motion and weak fields, the Hamiltonian reduces to the pN Hamiltonian
- Use also insight from QED and scattering theory

$$\Rightarrow \quad H = M\sqrt{1 + 2\nu\left(\frac{H_{\rm eff}}{\mu} - 1\right)}$$

- Still need $H_{\rm eff}$
  - Flavor 1: spinning test particle in Kerr recovered to linear order

    Barausse & Buonanno 0912.3517, 1107.2904; Taracchini et al 1311.2544
  - Flavor 2: Don't require this $\Rightarrow$ more compact formalism

    Damour, Nagar+ gr-qc/0103018, 1106.4349, 1406.6913
- The two flavors also differ in technical aspects: Resummation

Wave generation

- Factorized waveforms $h_{lm}(t) = h_{lm}^{N,\epsilon}\, \hat{S}_{\rm eff}^{(\epsilon)}\, T_{lm}\, e^{i\delta_{lm}}\, f_{lm}\, N_{lm}$

where: $\epsilon =$ Parity of mode
$h_{lm}^{N,\epsilon} =$ Newtonian contribution
$\hat{S}_{\rm eff}^{(\epsilon)} =$ Effective “source” term; depends on energy/ang.mom.
$T_{lm} =$ Logarithmic terms from tails
$e^{i\delta_{lm}} =$ Phase correction from sub-leading terms
$f_{lm} =$ Remaining PN terms
$N_{lm} =$ Phenomenological correction calibrated to NR

- Construct dissipative force $\boldsymbol{\mathcal{F}}$ from the waveform and add to EOMs

$$\frac{d\boldsymbol{x}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}}\,, \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{x}} + \boldsymbol{\mathcal{F}}\,,$$

$$\frac{d\boldsymbol{S}_{1,2}}{dt} = \frac{\partial H}{\partial \boldsymbol{S}_{1,2}} \times \boldsymbol{S}_{1,2}$$

- More details e.g. in reviews

Damour 0802..4047, Buonanno & Sathyprakash 1410.7832

EOB waveforms

- The inspiral plunge waveform is smoothly matched to the quasi normal ringdown: Fit to exponentially decaying sinusoids

- SEOBNRv1: Model for non-spinning binaries

  Barausse & Buonanno 1107.2904

- SEOBNRv2: Aligned spins

  Taracchini et al 1202.0790, 1311.2544

- SEOBNRv4: Improved version of those models

  Bohe et al. 1611.03703 using Damour & Nagar 1406.0401

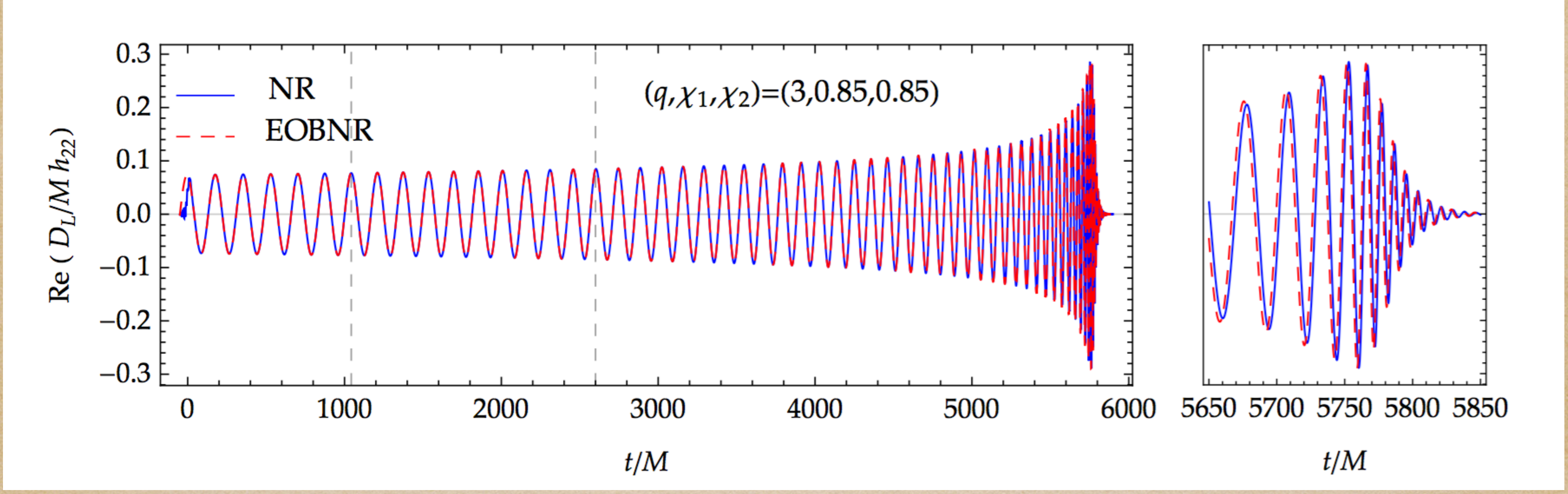
- SEOBNRv3: Transform spin-aligned model to a precessing frame

  Babak, Taracchini & Buonanno 1607.05661

Warning: No claim for completeness of references!

EOB waveforms

- Example: $q := \frac{m_2}{m_1} = \frac{1}{3}\,, \quad \chi_1 = \chi_2 = 0.85$



Bohe et al. 1611.03703

- Main problem: EOB waveforms still computationally expensive

  → Use "reduced order models" to interpolate in parameter space

Puerrer 1512.02248, Bohe et al. 1611.03703

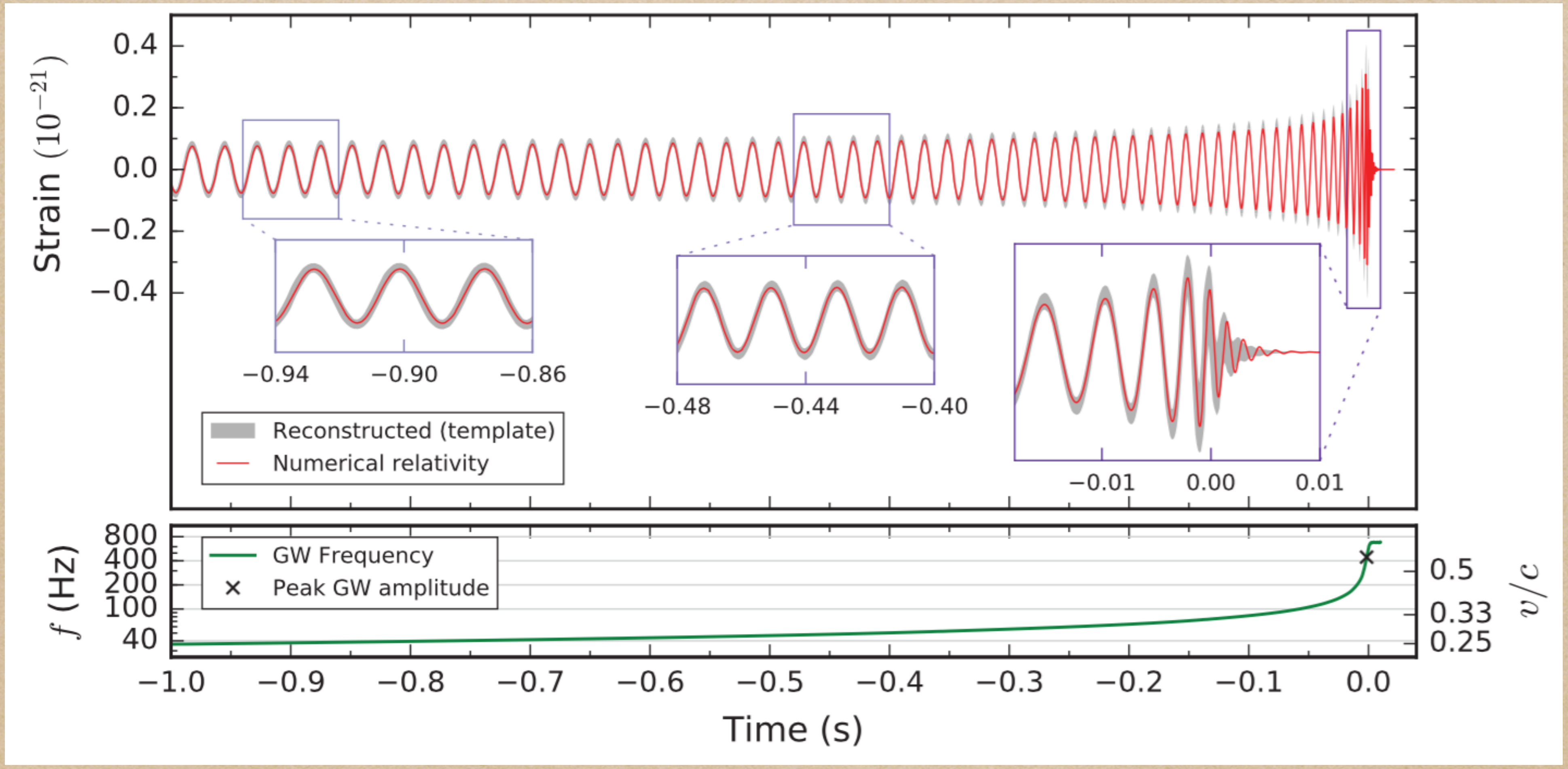
4.3. Phenomenological waveform models

Phase and amplitude

- GW strain: $\Psi_4 = \ddot{h}_+ - i\ddot{h}_\times$

  Multipolar decomposition: $h_+ - ih_\times = \sum_{\ell,m} h_{\ell m}(t,r)\, Y_{\ell m}^{-2}(\theta,\phi)$

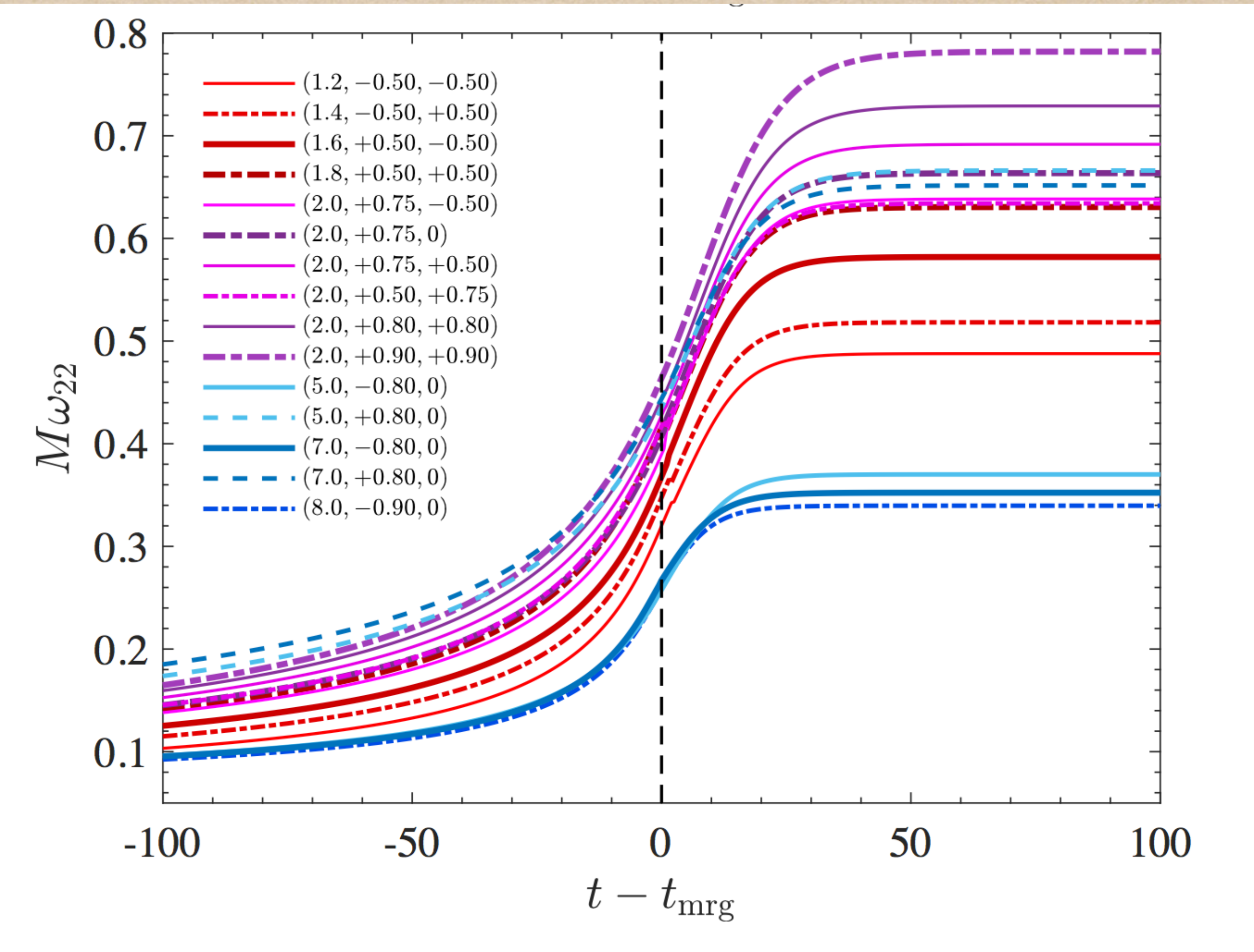
  Complex numbers: Write as amplitude, phase: $h_{\ell m} = A_{\ell m} e^{i\phi_{\ell m}}$



Abbott et al. 1606.04855

The frequency domain

- Frequency is monotonically increasing (well, almost…)



Nagar et al. 1806.01772

- one-to-one map between time and frequency
- Functions have simpler structure in the frequency domain

Phenomenological models

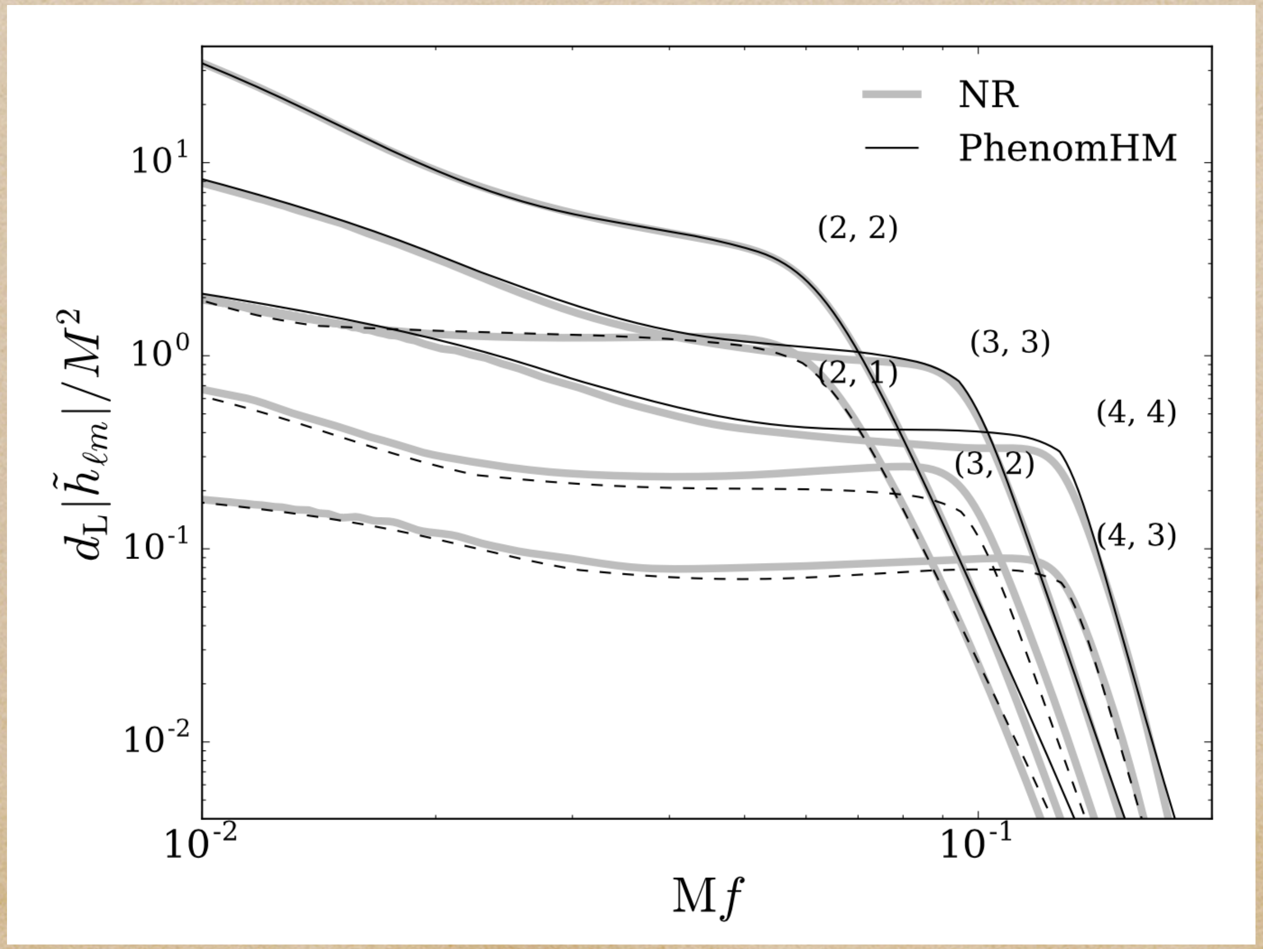
- Goal: find closed-form expressions approximating the GW signal

  in the frequency domain: $h_{\mathrm{phen}}(f; \vec{\alpha}; \vec{\beta}) = A(f; \vec{\alpha}) e^{\phi(f; \vec{\beta})}$

  $\vec{\alpha},\ \vec{\beta} =$ amplitude and phase parameters

Ajith et al. 0704.3764, 0710.2335

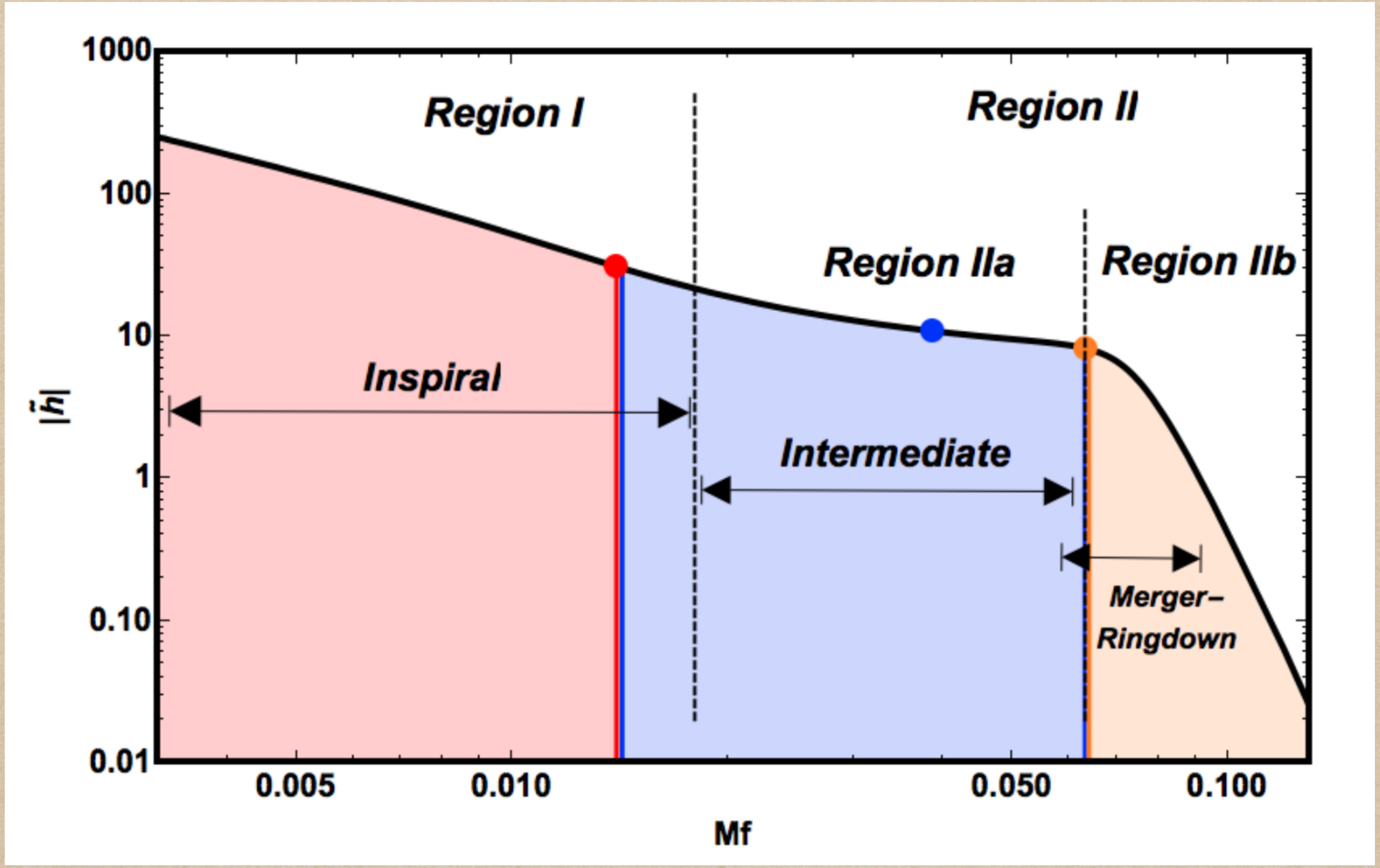
- Example: $q = \dfrac{m_2}{m_1} = \dfrac{1}{8}\,, \quad \chi_1 = 0.5$ (anti-aligned)



London et al. 1708.00404

Three regimes

- Recall 3 main regimes in binary BH coalescence



Khan et al. 1508.07253

Three regimes (cntd.)

- Use PN for inspiral: "TaylorF2" stationary-phase approximation

$$\phi_{\rm TF2} = 2\pi f t_c - \phi_c - \pi/4 + \frac{3}{128\nu}(\pi M f)^{-5/3} \sum_{i=0}^{7} c_i (\pi M f)^{i/3}$$

$$\phi = \phi_{\rm TF2} + \frac{1}{\nu}\sum_{i=0}^{6} \sigma_i f^{i/3}$$

$c_i =$ PN coefficients

$\sigma_i =$ phenomenological coefficients

- For merger-ringdown: Similarly make phenomenological Ansatz

using 17 phenom. parameters that are mapped to $(\nu, \chi_{\rm eff})$

$\chi_{\rm eff} = \frac{m_1}{M}\vec{\chi}_1 \cdot \hat{L}_N + \frac{m_2}{M}\vec{\chi}_2 \cdot \hat{L}_N$ dominant non-precessing spin contrib.

Ajith et al. 0909.2867

- Create mapping between physical and phenomenological parameters

Phenomenological models

- Phenom: non-spinning

Ajith et al. 0704.3764, 0710.2335

- PhenomB,C,D: Aligned spins

Ajith et al. 0909.2867, Santamaria et al. 1005.3306,

Husa et al. 1508.07250, Khan et al. 1508.07253

- PhenomP: Effective precession model using

$$\chi_p = \frac{1}{B_1 m_1^2} \max(B_1 S_{1,\perp}, B_2 S_{2\perp}) \,, \qquad B_1 = 2 + \frac{3m_2}{2m_1} \,, \; B_2 = 2 + \frac{3m_1}{2m_2}$$

Hannam et al. 1308.3271

- First model beyond the quadrupole

London et al. 1708.00404

Further reading

- Reviews of numerical relativity

Hannam 1312.3641

Barack et al. 1806.05195: Sec.II 6 by T.Hinderer + Refs. therein

5. What we haven’t done

Further reading

- Neutron stars

  Baumgarte & Shapiro "Numerical Relativity"

- Alternative theories of gravity

  Berti et al. 1501.07274

- More exotic sources: Boson stars, hairy BHs, Eccentricity,...

  Barack et al. 1608.05195 + References therein