Numerical Relativity and Applications in Cosmology

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Structures and Dynamics in Cosmology

Sanya, Hainan, 10 Jan 2025

Overview

- Introduction
- The Ancient World: The Birth of Numerical Relativity
- From the Dark ages to the Renaissance
- Towards the Holy Grail
- The gold rush years
- What's different in cosmological NR?
- NR results in cosmology

  Early Universe, post-BBN, Late Universe

1. Introduction, Motivation

Task: Solve this!

It's simple but it isn't easy…

How do we get the metric?

- The metric must obey $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$
- Ricci tensor, Einstein tensor, matter tensor

$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$

$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{\mu}{}_{\mu}$ "Trace reverse Ricci"

$T_{\alpha\beta}$ "Matter"

Λ "Cosmological constant"

- Solutions: Easy! Take metric $g_{\alpha\beta}$
- Calculate $\Rightarrow G_{\alpha\beta}$
- Use that for $\Rightarrow T_{\alpha\beta}$

- Physically meaningful solutions: That's the hard part!

Solving Einstein's Eqs.: The toolbox

- **Analytic solutions**
  - Symmetry assumptions

    Schwarzschild, Kerr, FLRW, Vaidya, Tangherlini, Myers-Perry, ...

- **Perturbation theory**
  - Assume solution is close to a known "background" $g^{(0)}_{\alpha\beta}$
  - Expand $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots \quad \Rightarrow$ linear system

    Regge-Wheeler-Zerilli-Moncrief, Teukolsky, QNMs, EOB, ...

- **Post-Newtonian theory**
  - Assume small velocities $\Rightarrow$ Expansion in $\frac{v}{c}$
  - N^{th} order expressions for GWs, momenta, orbits, ...

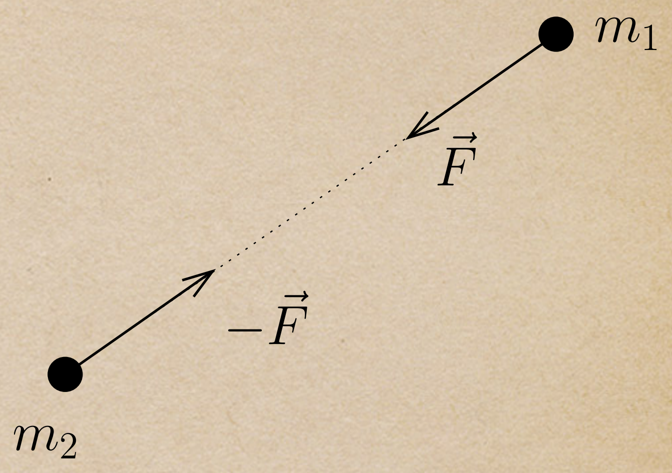
    Blanchet, Buonanno, Damour, Kidder, Schäfer, Will, ...

- **Numerical Relativity**

The Newtonian 2-body problem

- Eqs. of motion

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = \vec{F} = -G\frac{m_1 m_2}{r^2}\hat{\vec{r}} = -m_2\frac{d^2\vec{r}_2}{dt^2}$$



- Solution: Kepler ellipses, parabolic, hyperolic

$$r = \frac{r_0}{1+\epsilon\cos\theta}$$

- What is the equivalent in GR?
  - No point particles in GR $\rightarrow$ Black holes!
  - Systems typically are dissipative $\rightarrow$ Gravitational waves

- $\boxed{\text{The Holy Grail of numerical relativity: Inspiral of BH binary}}$

- History: e.g. US CQG 1411.3997

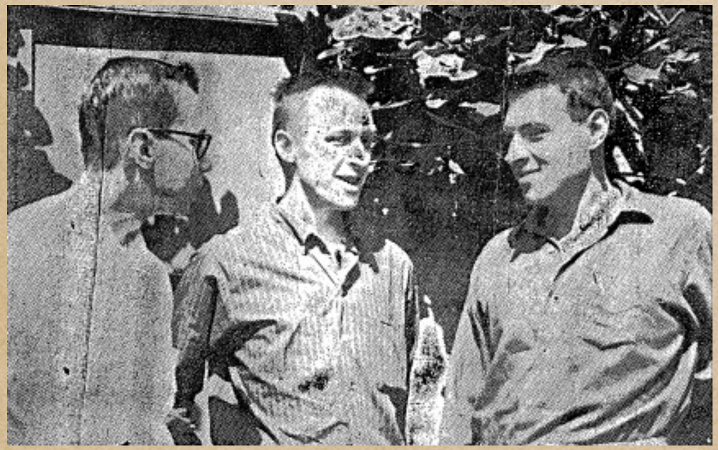
Challenges in GR

- Covariance of the Einstein equations
  - Space and time on equal footing: How to evolve? Time?
  - Well posedness? Suitability for numerical methods?
- Meaning of the solutions; cf. Schwarzschild solution or GWs
  - Gauge invariant diagnostics
  - Definition of observables
- No a-priori spacetime “stage”. Coordinates are evolved.
- Singularities
- Computational costs: 3D effect
- Numerical stability

The ancient world:
The Birth of Numerical Relativity

Foundations and the first steps

- The Cauchy problem of the Einstein equations locally has a unique solution   Choquet-Bruhat   Acta Math. 1952
- Characteristic formulation   Bondi, Sachs   Proc.Roy.Soc. 1962
- Canonical 3+1 or ADM formulation of the Einstein equations
  Arnowitt, Deser, Misner (1962)   gr-qc/0405109
- First numerical relativity simulations: $\lesssim 100$ time steps
  Hahn & Lindquist   Ann.Phys. 1964
- 1D Gravitational collapse   May & White   PR 1966



AMD

Y Choquet-Bruhat

H Bondi

R K Sachs

The 1970s

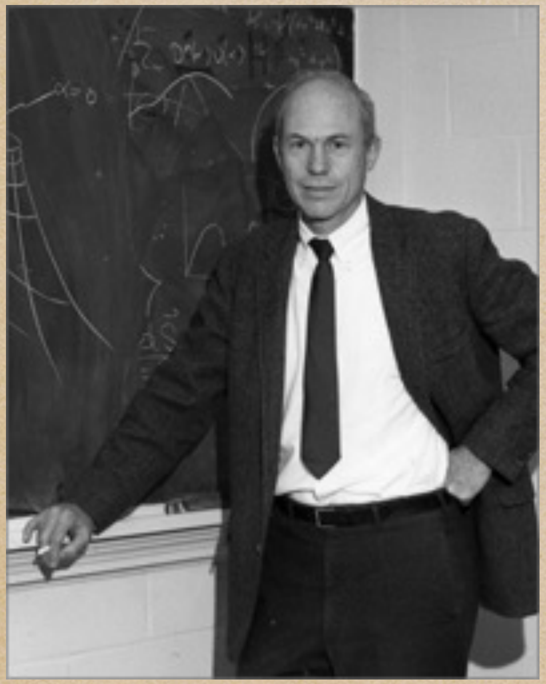
- Reinvestigation initiated by B DeWitt

  → PhD theses by A Cavez (1971), L Smarr (1975), K R Eppley (1975)

- 300 x Flops relative to Hahn & Lindquist

- ADM equations, 2D code, Misner (1960) initial data

  → single BHs, head-on collisions



B DeWitt

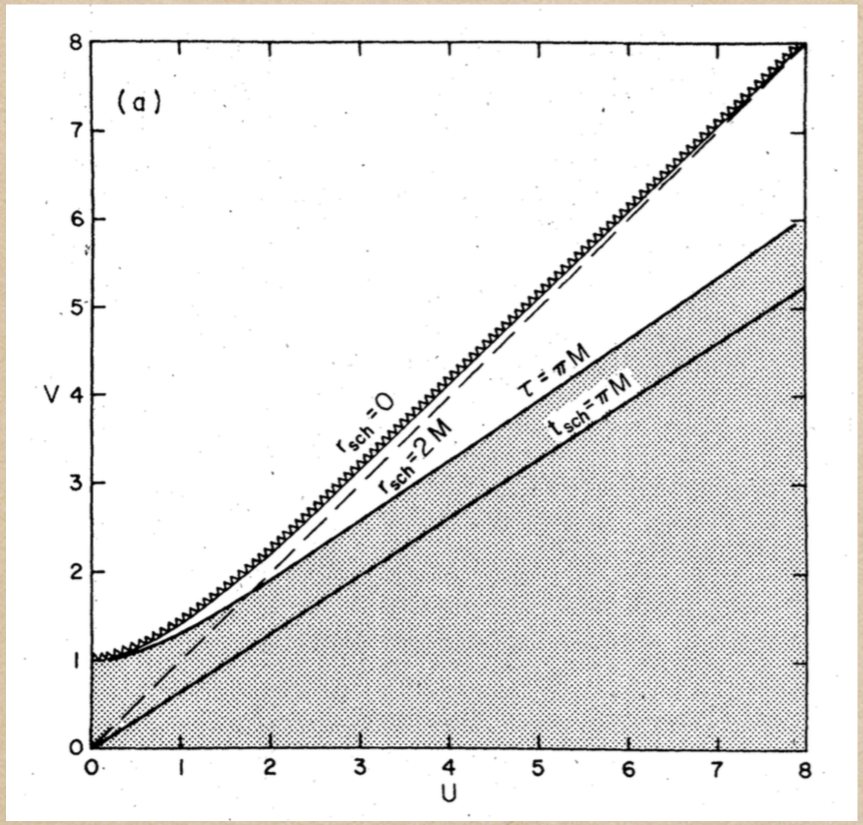
L Smarr

J W York Jr.

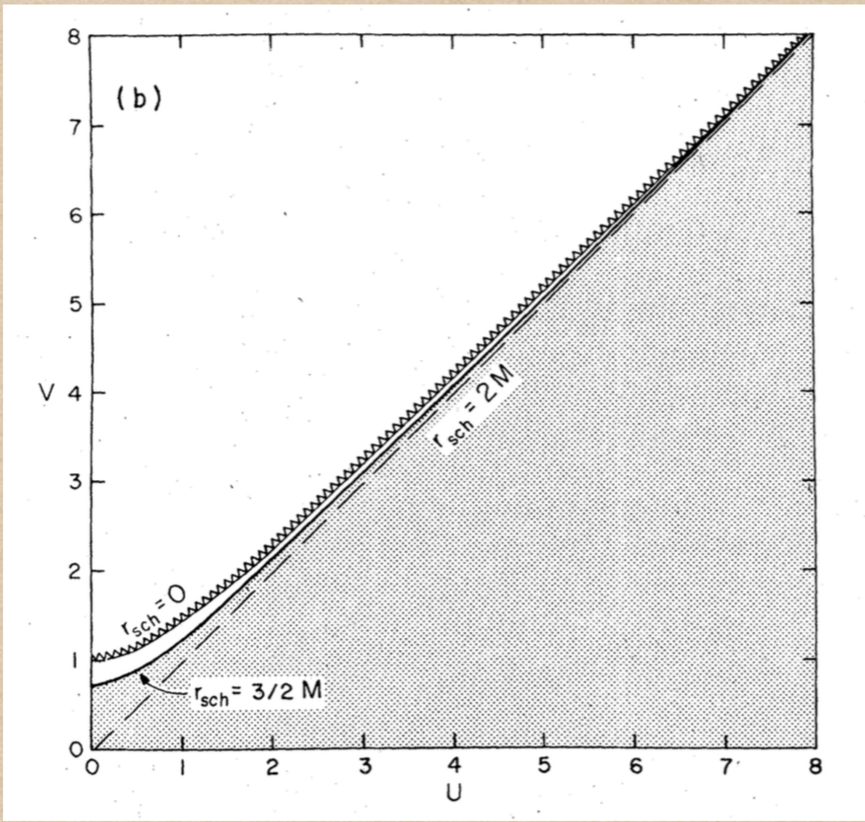
The 1970s

- Singularity avoiding slicing Smarr & York PRD 1978

  Schwarzschild-Kruskal evolved with



geodesic slicing



maximal slicing

- York formulation of the 3+1 equations

  York 1979 in "Sources of Gravitational Radiation" Ed. L Smarr

The 3+1 decomposition

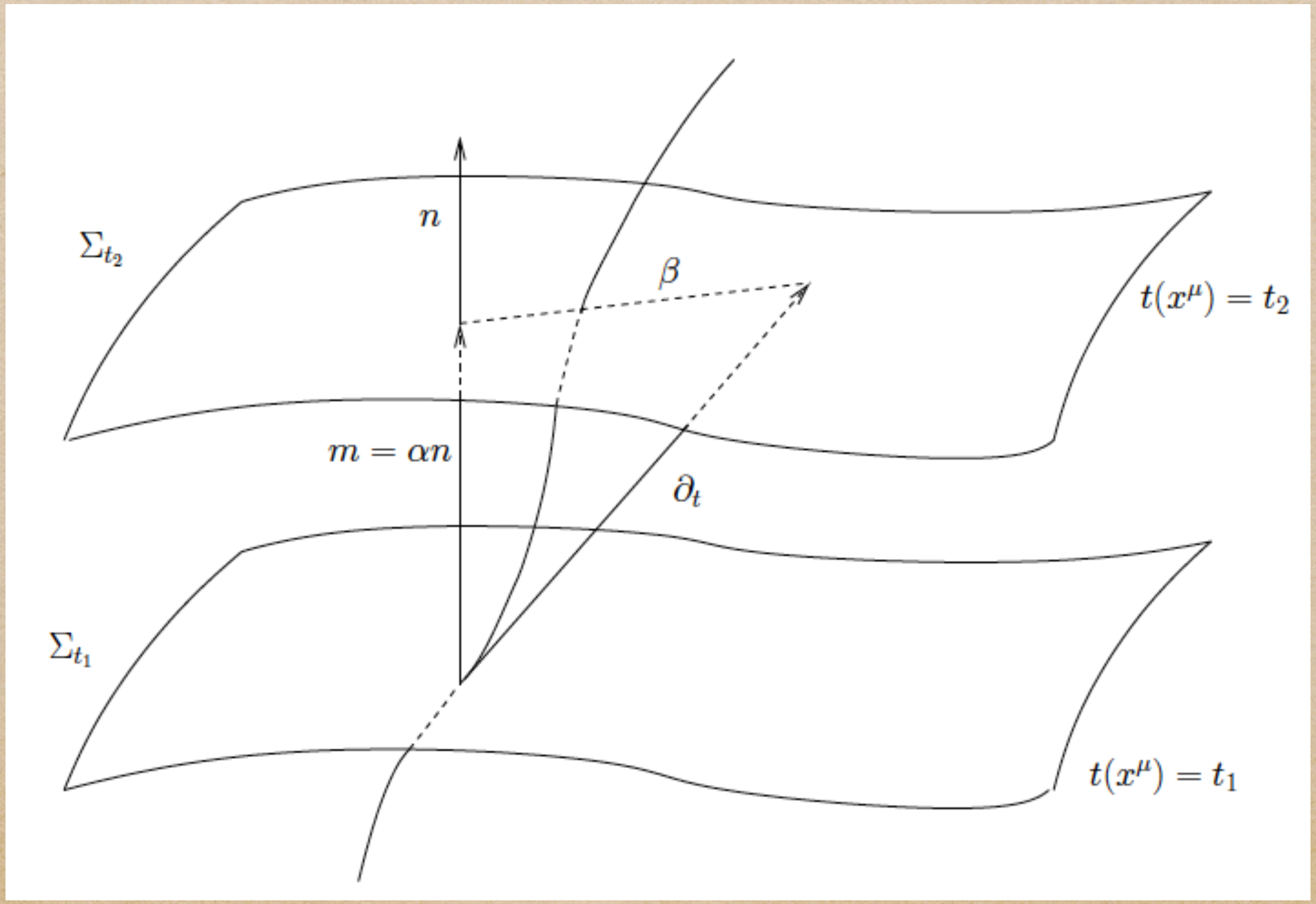
- Let $(\mathcal{M}, g)$ be a Spacetime

  = Manifold with metric of signature $-+++$

- Assume $\exists$ smooth $t : \mathcal{M} \mapsto \mathbb{R}$

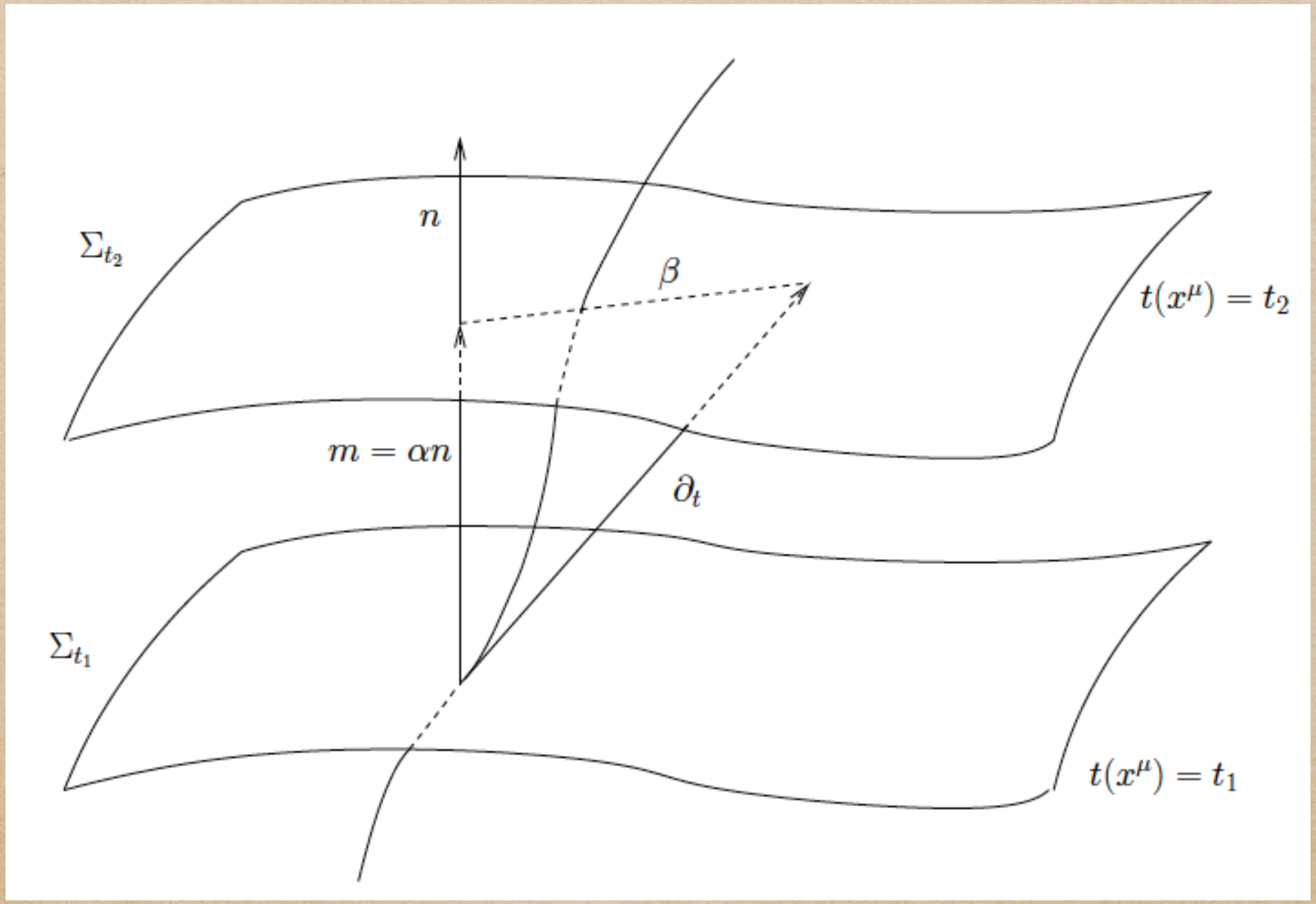
  with timelike gradient $\mathbf{d}t \neq 0$ and level surfaces

$$\forall_{t_1 \in \mathbb{R}} \ \Sigma_{t_1} = \{p \in \mathcal{M} \ : \ t(p) = t_1\} \ , \ \ \Sigma_{t_1} \cap \Sigma_{t_2} = \emptyset \ \Leftrightarrow \ t_1 \neq t_2$$



The 3+1 decomposition

- 1-Form: $\mathbf{d}t\,;$ vector: $\dfrac{\partial}{\partial t} =: \partial_t \quad \Rightarrow \quad \langle \mathbf{d}t, \partial_t \rangle = 1$
- Timelike normal $n_\mu := -\alpha(\mathbf{d}t)_\mu$
- Spatial projector $\perp^\alpha{}_\mu = \delta^\alpha{}_\mu + n^\alpha n_\mu$
- Adapted coordinates $(t,\, x^i)\,,\quad x^i$ label points inside Σ_t



(D-1)+1 decomposition of the metric

- In adapted coordinates, we write the spacetime metric

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^2 + \beta_m \beta^m & \beta_j \\ \hline \beta_i & \gamma_{ij} \end{array}\right)$$

$$\Leftrightarrow \qquad g^{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \hline \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{array}\right)$$

$$\Leftrightarrow \qquad ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- Gauge variables: Lapse α , shift β^i
- For tensors tangent in all components to Σ_t we lower indices with γ_{ij} : $S^i{}_{jk} = \gamma_{jm} S^{im}{}_k$, etc.
- Details e.g. in Gourgoulhon gr-qc/0703035

Decomposition of the Einstein eqs.

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

$$\Leftrightarrow \quad R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \right) + \Lambda g_{\alpha\beta}$$

- Energy momentum tensor

$$\rho := T_{\mu\nu} n^\mu n^\nu \,,$$

$$j_\alpha := -\perp^\mu{}_\alpha T_{\mu\nu} n^\nu \,,$$

$$S_{\alpha\beta} := \perp T_{\alpha\beta} \,, \quad S = \gamma^{\mu\nu} S_{\mu\nu} \,,$$

$$T_{\alpha\beta} = S_{\alpha\beta} + n_\alpha j_\beta + n_\beta j_\alpha + \rho n_\alpha n_\beta \,, \quad T = S - \rho \,.$$

- Lie derivative

$$\mathcal{L}_{\boldsymbol{m}} K_{ij} = \partial_t K_{ij} - \beta^m \partial_m K_{ij} - K_{mj} \partial_i \beta^m - K_{im} \partial_j \beta^m$$

$$\mathcal{L}_{\boldsymbol{m}} \gamma_{ij} = \partial_t \gamma_{ij} - \beta^m \partial_m \gamma_{ij} - \gamma_{mj} \partial_i \beta^m - \gamma_{im} \partial_j \beta^m$$

The ADM version of the Einstein eqs.

- Introduction of the extrinsic curvature:

$$\boxed{\mathcal{L}_{\boldsymbol{m}}\gamma_{ij} = -2\alpha K_{ij}}$$

- $\perp^{\mu}{}_{\alpha}\perp^{\nu}_{\beta}$ projection

$$\boxed{\mathcal{L}_{\boldsymbol{m}}K_{ij} = -D_i D_j \alpha + \alpha(\mathcal{R}_{ij} + KK_{ij} - 2K_{im}K^{m}{}_{j}) + 8\pi\alpha\left(\frac{S-\rho}{D-2}\gamma_{ij} - S_{ij}\right) - \frac{2}{D-2}\Lambda\gamma_{ij}}$$

"Evolution equations"

- $n^{\mu}n^{\nu}$ projection

$$\boxed{\mathcal{R} + K^2 - K^{mn}K_{mn} = 2\Lambda + 16\pi\rho}$$

"Hamiltonian constraint"

- $\perp^{\mu}{}_{\alpha}n^{\nu}$ projection

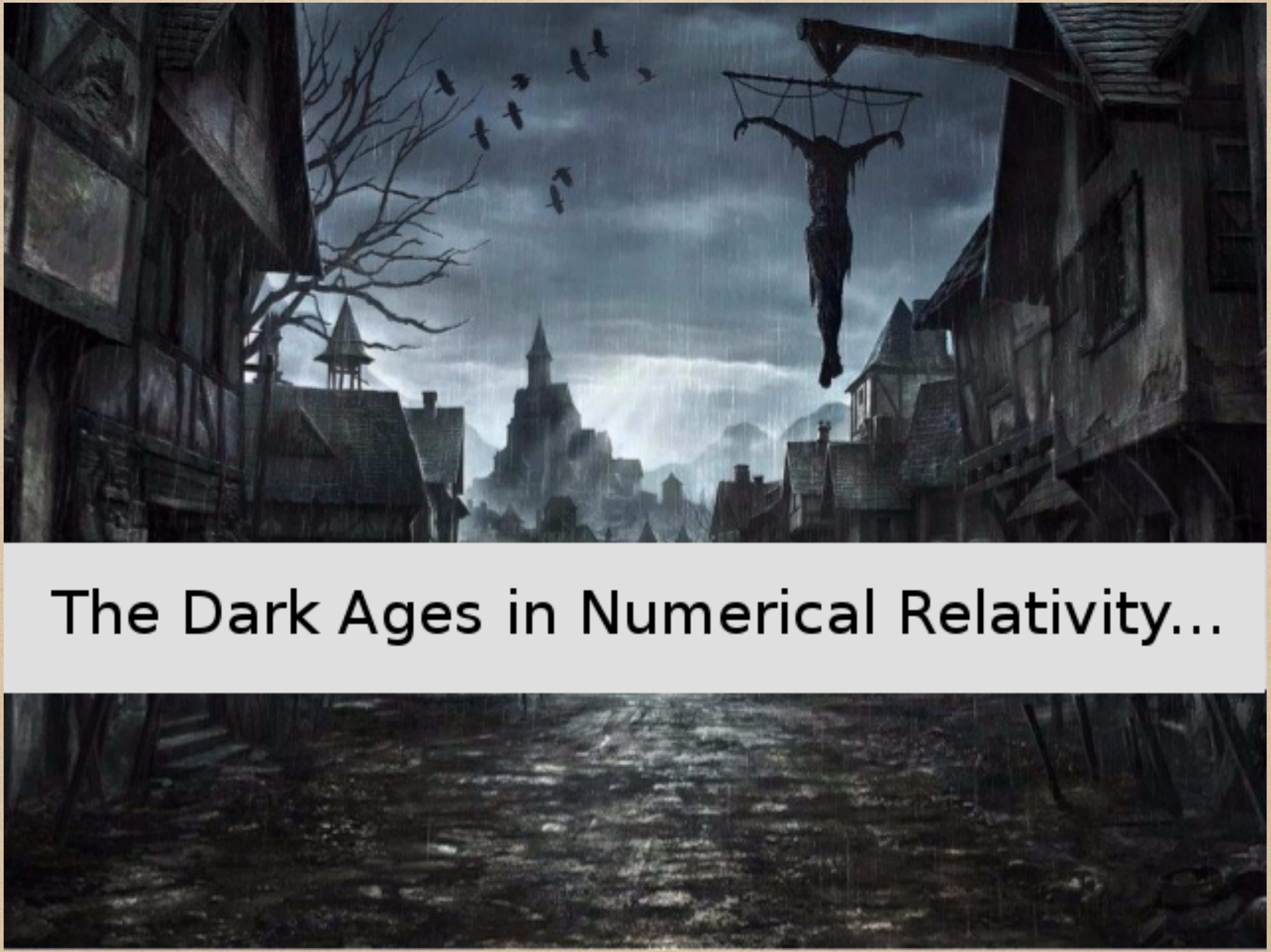
$$\boxed{D_i K - D_m K^{m}_{i} = -8\pi j_i}$$

"Momentum constraints"

- Backbone of numerical relativity for ~20 years

2. From the dark ages to the Renaissance

The 1980s



The 1990s

"Binary Black Hole Grand Challenge Project"

- 40 researchers in 10 institutions

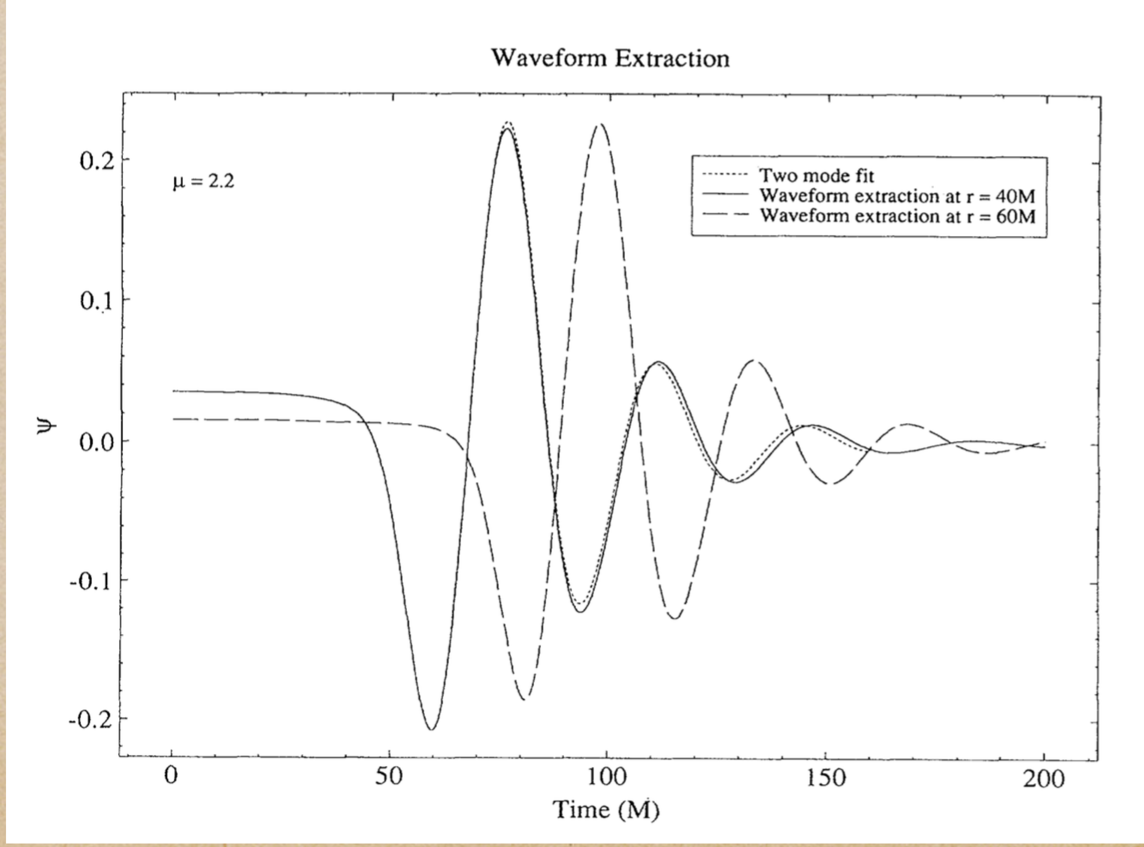
  Austin, Cornell, Illinois, North Carolina, Northwestern, Penn State, Pittsburgh, Syracuse

- Goal: Simulate BH binary inspiral, compute GW signals

- ADM formalism, axisymmetry, head-on collisions
  - $l = 2, \; l = 4$ waveforms
  - Horizon calculations
  - Unequal masses

Anninos et al

PRL 1993, PRD 1995+

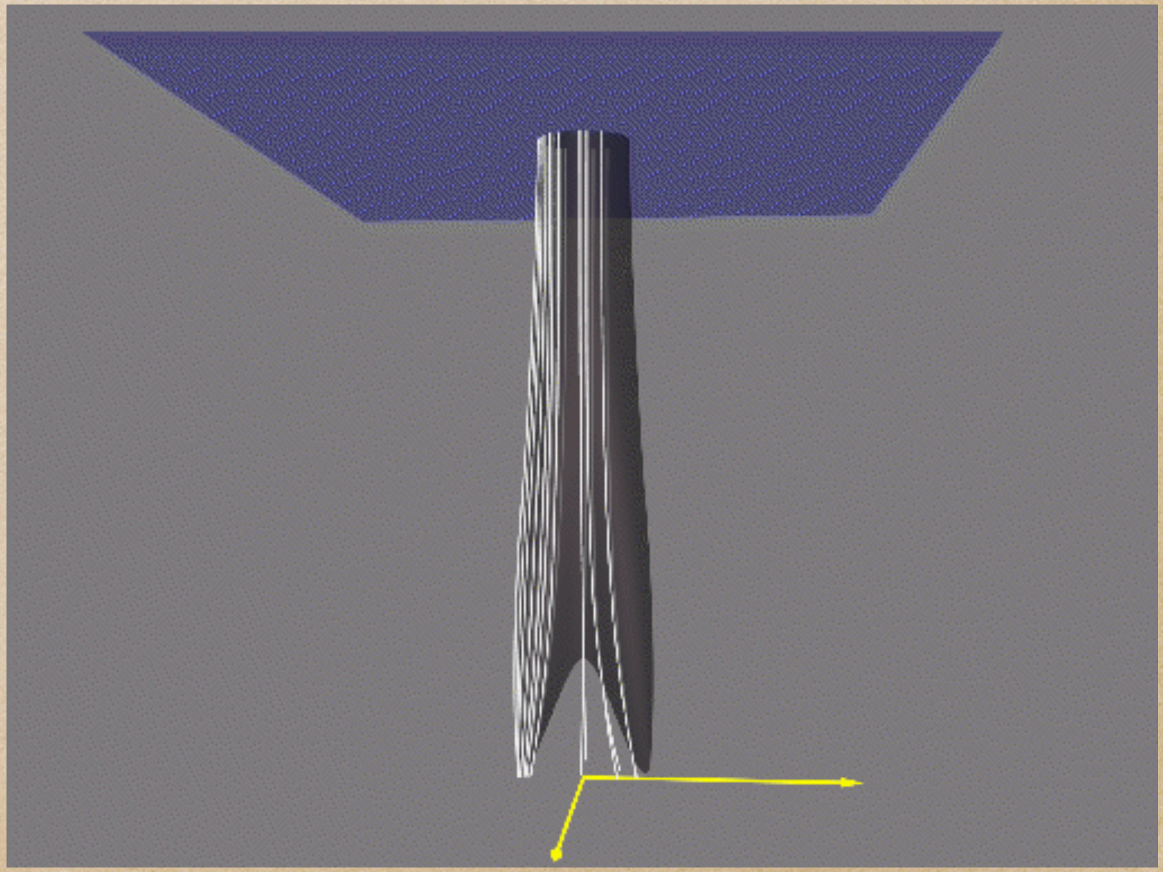


The 1990s

"Binary Black Hole Grand Challenge Project" (continued)

- First 3+1 dimensional BH simulations "G-code"
  - Single Schwarzschild BH stable up to $t \approx 50\ M$
  - Single moving BH stable up to $t \approx 60\ M$
  - Comparisons with axisymmetric simulations

Anninos et al PRD 1995



- Event horizon
  "pair of pants"
- Problem: long-term stability!
  $\rightarrow$ Well-posedness studies

Well posedness

From G Papallo, PhD thesis Cambridge, 2018

- Consider linear const.coeff. PDE system $A\partial_t u + P^i \partial_i u + Cu = 0$

Fourier trafo $\tilde{u}(t,k) = \frac{1}{\sqrt{2\pi}^n} \int u(t,x) e^{-ik_i x^i}\, d^n x$

$$\Rightarrow \partial_t \tilde{u} - i\mathcal{M}\,\tilde{u} = 0\,, \qquad \mathcal{M}(k) = A^{-1}(-P^i k_i + iC)$$

Solution $\tilde{u}(t,k) = e^{i\mathcal{M}(k)t}\,\tilde{u}(0,k)$

$$\Rightarrow u(t,x) = \frac{1}{\sqrt{2\pi}^n} \int e^{ik_i x^i}\, e^{i\mathcal{M}t}\,\tilde{u}(0,k)\, d^n k$$

- May not converge if integrand fails to decay fast with $k = \sqrt{k_i k^i}$
- ADM equations are only weakly hyperbolic
- Strong or symmetric hyperbolicity with new formulations:

BSSN, CCZ4, GHG

The post-Grand Challenge era

- Many smaller groups explored the problem independently
- Critical collapse: Collapse of spherically symmetric scalar field.

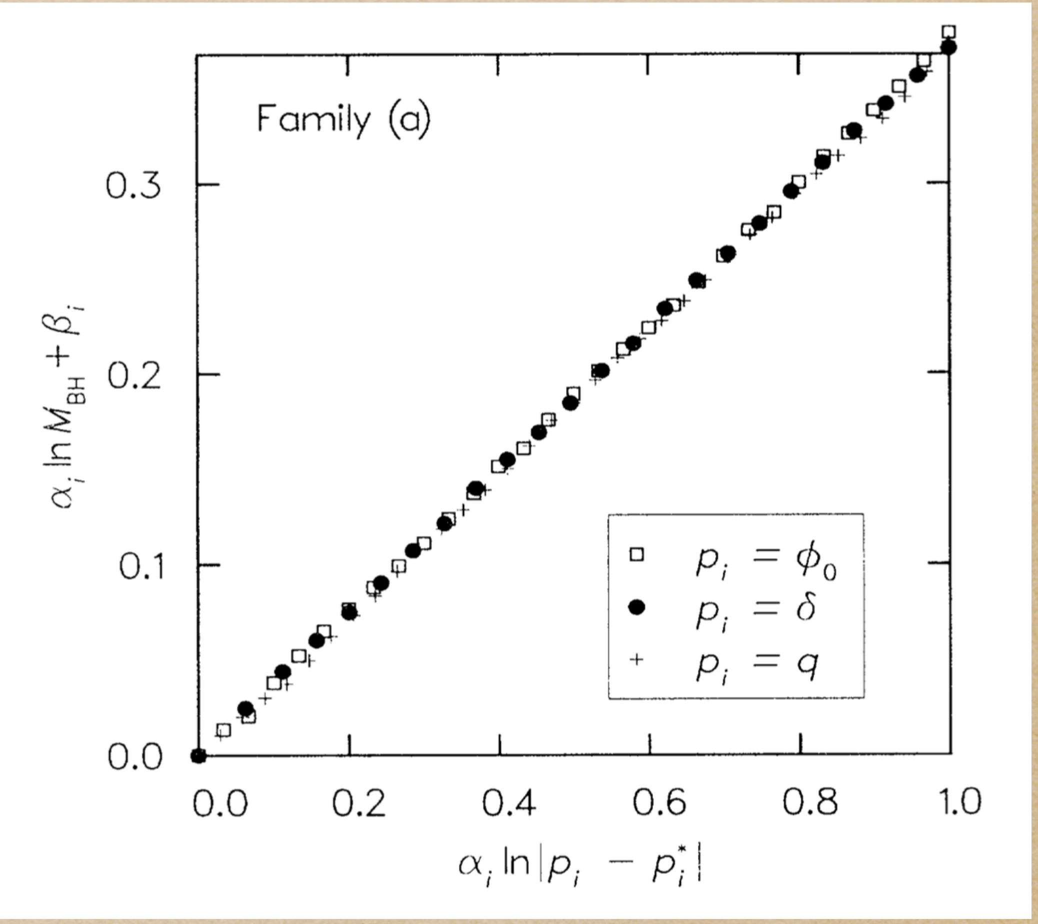
  BH formation or dispersal; Mesh refinement! Choptuik PRL 1993
- 1st Mesh refinement for 3+1 BHs

  Brügmann PRD 1996
- 1st long-term stable BH sim.

  (characteristic code)

  Gómez et al PRL 1998
- 1st BH Grazing collision

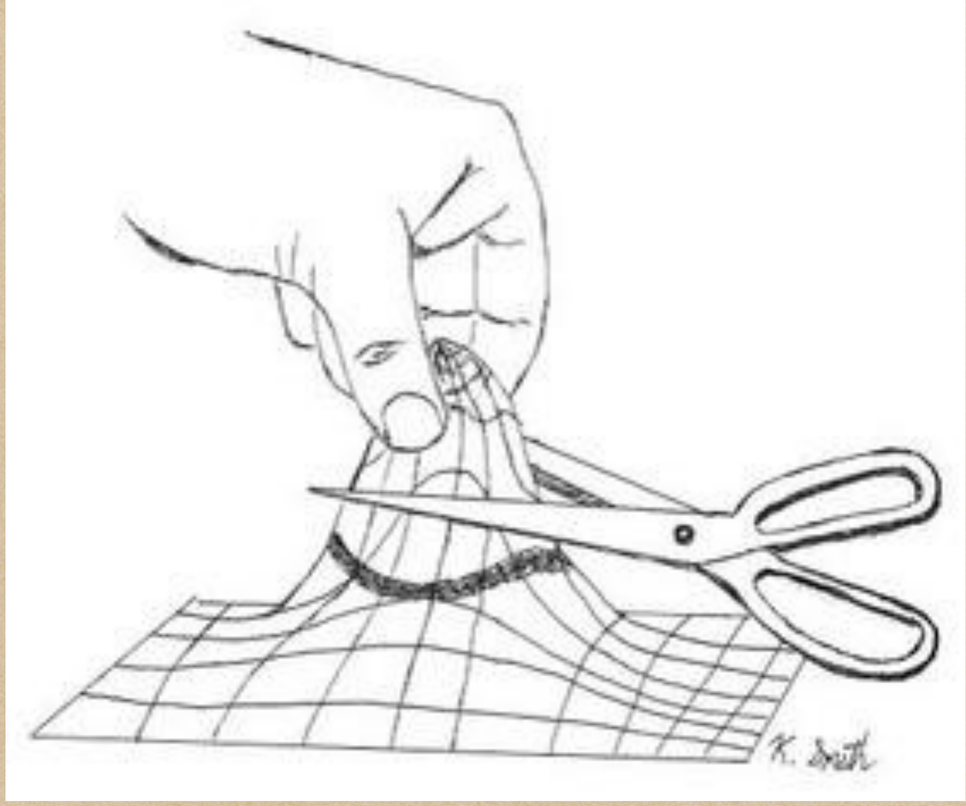
  Brandt et al PRL 2000
- Release of `Cactus 1.0`

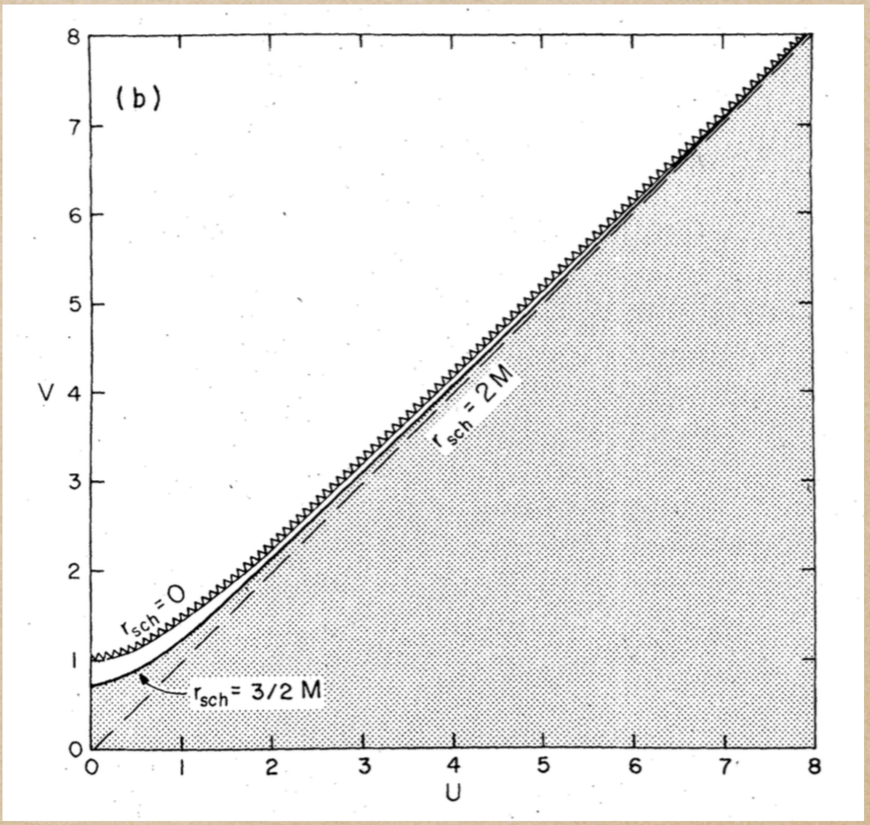


Towards the Holy Grail

Remaining problems

- Formulations: Well-posedness required
- Gauge conditions: Avoid coordinate singularities
- Physical singularities: Excision or good slicing





The generalized harmonic gauge (GHG)

- Harmonic gauge: Choose coordinates such that

$$\Box x^\alpha = \nabla^\mu \nabla_\mu x^\alpha = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0$$

- 4 dimensional Einstein eqs. in harmonic gauge:

$$R_{\alpha\beta} = -\frac{1}{2} g^{\mu\nu}\partial_\mu \partial_\nu g_{\alpha\beta} + \ldots$$

principle part of wave equation $\Rightarrow$ Manifestly hyperbolic!

- Problem: Start with a hyper surface $t = \mathrm{const}$

Does t remain timelike?

- Goal: Generalize the harmonic gauge

Garfinkle PRD gr-qc/0110013; Pretorius CQG gr-qc/0407110;

Lindblom et al CQG gr-qc/0512093

$\rightarrow$ Source function $H^\alpha = \nabla^\mu \nabla_\mu x^\alpha = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu}$

The generalized harmonic equations

- Any spacetime in any coordinates can be formulated in GH form!

  Problem: find the corresponding H^α

- Promote the H^α to evolution variables

- Einstein equations in GH form:

$$\boxed{\begin{aligned}\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = &-\partial_\nu g_{\mu(\alpha}\partial_{\beta)}g^{\mu\nu} - \partial_{(\alpha}H_{\beta)} + H_\mu\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\nu\alpha}\Gamma^\nu_{\mu\beta} \\ &-\frac{2}{3}\Lambda g_{\alpha\beta} - 8\pi\left(T_{\mu\nu} - \frac{1}{2}T\,g_{\alpha\beta}\right).\end{aligned}}$$

  with constraints $\boxed{\mathcal{C}^\alpha = H^\alpha - \Box x^\alpha = 0}$

- Still has principle part of the wave equation!!! Manifestly hyperbolic

  Friedrich Comm.Math.Phys. 1985; Garfinkle gr-qc/0110013;

  Pretorius gr-qc/0407110

Initial data: Analytic data

- Schwarzschild, Kerr, Tangherlini, Myers-Perry,...

  e.g. Schwarzschild in isotropic coordinates

$$ds^2 = -\left(\frac{2r-M}{2r+M}\right)^2 dt^2 + \left(1+\frac{M}{2r}\right)^4 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

  Time symmetric initial data with n BHs:

  Brill & Lindquist PR 1963, Misner PR 1960

- Problem: Find initial data for dynamic systems
- Goals: 1) Solve constraints

  2) Realistic snapshot of physical system
- This is mostly done using the ADM 3+1 split

Initial data and conformal decomposition

- Recall: we need to satisfy the constraints

$$\mathcal{R} + K^2 - K^{mn}K_{mn} = 2\Lambda + 16\pi\rho \qquad\qquad D_i K - D_m K^m_i = -8\pi j_i$$

- Conformal metric $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$

Lichnerowicz J.Math.Pures Appl. 1944; York PRL 1971, PRL 1972

- Conformal traceless split of the extrinsic curvature

$$K_{ij} = A_{ij} + \frac{1}{3} K \gamma_{ij} \,,$$

$$A^{ij} = \psi^{-10} \bar{A}^{ij} \quad \Leftrightarrow \quad A_{ij} = \psi^{-2} \bar{A}_{ij}$$

A Lichnerowicz

Bowen-York data

- By further splitting $\bar{A}_{ij}$ into a longitudinal and a transverse traceless part, the momentum constraints simplify substantially

  Cook LRR gr-qc/0007085

- Further assume: Vacuum, $K = 0\,, \quad \bar{\gamma}_{ij} = f_{ij}\,, \quad \lim_{r\to\infty} \psi = 0\,,$
  where f_{ij} is the flat metric in arbitrary coords.

  In words: Traceless E.Curv., conformal flatness, asymptotic flatness

- Then there exists an analytic solution to the momentum constraints

$$\bar{A}_{ij} = \frac{3}{2r^2}\left[P_i n_j + P_j n_i - (f_{ij} - n_i n_j)P^k n_k\right] \qquad P^k = \text{Momentum}$$

$$+ \frac{3}{r^3}\left(\epsilon_{kil} S^l n^k n_j + \epsilon_{kjl} S^l n^k n_i\right)\,, \qquad S^k = \text{Spin}$$

  where r is a coordinate radius and $n^i = \dfrac{x^i}{r}$

  Bowen & York PRD 1980

Puncture data

Brandt & Brügmann PRL gr-qc/9703066

- The Hamiltonian constraint is then given by

$$\bar{\nabla}^2\psi + \frac{1}{8}\psi^{-7}\bar{A}_{mn}\bar{A}^{mn} = 0$$

- Ansatz for conformal factor $\psi = \psi_{\mathrm{BL}} + u$

where $\psi_{\mathrm{BL}} = \sum_{a=1}^{N} \frac{m_a}{|\vec{r} - \vec{r}_a|}$ is the Brill-Lindquist conformal factor,

i.e. the solution for $\bar{A}_{ij} = 0$.

- There then exist unique $\mathcal{C}^2$ solutions u to the Hamiltonian constr.

- The Hamiltonian constraint in this form is particularly suitable for numerical solution.

  E.g. Ansorg, Brügmann & Tichy gr-qc/0404056

Beyond conformally flat initial data

- Problem: Conformally flat data limits spins to $S/M^2 \lesssim 0.928$

  Dain et al. PRD gr-qc/0201062

- Similar problems arise for large linear momenta

- Solution: Non-conformally flat initial data

  - Superpose Kerr-Schild data Lovelace et al. PRD 0805.4192

    Solve constraints with Conformal Thin Sandwich approach

    York PRL 82 (1999) 1350

  - Superpose boosted conformal Kerr BHs; attenuation functions

    Zlochower et al. PRD 1706.01980, Ruchlin et al. PRD 1410.8607

    Evolve with CCZ4 (constraint damping variant of BSSN)

    Alic et al. PRD 1106.2254, Hilditch et al. PRD 1212.2901

The BSSNOK system

- Goal: Modify ADM eqs. to get a strongly hyperbolic system

  Nakamura et al PTPS 1987, Shibata & Nakamura PRD 1995,

  Baumgarte & Shapiro gr-qc/9810065

- Use (i) conformal decomposition, (ii) trace split, (iii) aux. variables

$$\boxed{\begin{aligned}
&\gamma := \det \gamma_{ij}\,, \qquad \chi := \gamma^{-1/3}\,, \qquad K = \gamma^{mn} K_{mn}\,,\\
&\tilde{\gamma}_{ij} := \chi \gamma_{ij} \qquad \Leftrightarrow \qquad \tilde{\gamma}^{ij} = \frac{1}{\chi}\gamma^{ij}\,,\\
&\tilde{A}_{ij} := \chi\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) \qquad \Leftrightarrow \qquad K_{ij} = \frac{1}{\chi}\left(\tilde{A}_{ij} + \frac{1}{3}\tilde{\gamma}_{ij}K\right)\,,\\
&\tilde{\Gamma}^i := \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn}\,.
\end{aligned}}$$

- Auxiliary constraints

$$\tilde{\gamma} = 1, \qquad \tilde{\gamma}^{mn}\tilde{A}_{mn} = 0, \qquad \mathcal{G}^i \equiv \tilde{\Gamma}^i - \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn} = 0\,.$$

The BSSN equations

$$\mathcal{H} := \mathcal{R} + \frac{2}{3}K^2 - \tilde{A}^{mn}\tilde{A}_{mn} - 16\pi\rho - 2\Lambda = 0\,,$$

$$\mathcal{M}_i := \tilde{\gamma}^{mn}\tilde{D}_m\tilde{A}_{ni} - \frac{2}{3}\partial_i K - \frac{3}{2}\tilde{A}^m{}_i\frac{\partial_m\chi}{\chi} - 8\pi j_i = 0\,,$$

$$\partial_t\chi = \beta^m\partial_m\chi + \frac{2}{3}\chi(\alpha K - \partial_m\beta^m)\,,$$

$$\partial_t\tilde{\gamma}_{ij} = \beta^m\partial_m\tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{\gamma}_{ij}\partial_m\beta^m - 2\alpha\tilde{A}_{ij}\,,$$

$$\partial_t K = \beta^m\partial_m K - \chi\tilde{\gamma}^{mn}D_m D_n\alpha + \alpha\tilde{A}^{mn}\tilde{A}_{mn} + \frac{1}{3}\alpha K^2 + 4\pi\alpha(S+\rho) - \alpha\Lambda\,,$$

$$\partial_t\tilde{A}_{ij} = \beta^m\partial_m\tilde{A}_{ij} + 2\tilde{A}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{A}_{ij}\partial_m\beta^m + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{im}\tilde{A}^m{}_j$$
$$+ \chi(\alpha\mathcal{R}_{ij} - D_i D_j\alpha - 8\pi\alpha S_{ij})^{\mathrm{TF}}\,,$$

$$\partial_t\tilde{\Gamma}^i = \beta^m\partial_m\tilde{\Gamma}^i + \frac{2}{3}\tilde{\Gamma}^i\partial_m\beta^m - \tilde{\Gamma}^m\partial_m\beta^i + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + \frac{1}{3}\tilde{\gamma}^{im}\partial_m\partial_n\beta^n$$
$$- \tilde{A}^{im}\left(3\alpha\frac{\partial_m\chi}{\chi} + 2\partial_m\alpha\right) + 2\alpha\tilde{\Gamma}^i_{mn}\tilde{A}^{mn} - \frac{4}{3}\alpha\tilde{\gamma}^{im}\partial_m K - 16\pi\frac{\alpha}{\chi}j^i - \sigma\mathcal{G}^i\partial_m\beta^m\,.$$

- Note: there exist slight variations of the exact equations

The BSSN equations

- Auxiliary expressions we have used:

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} - \frac{1}{2\chi}(\delta^i{}_k\,\partial_j\chi + \delta^i{}_j\,\partial_k\chi - \tilde{\gamma}_{jk}\tilde{\gamma}^{im}\partial_m\chi)$$

$$\mathcal{R}_{ij} = \tilde{R}_{ij} + \mathcal{R}^{\chi}_{ij}\,,$$

$$\mathcal{R}^{\chi}_{ij} = \frac{\tilde{\gamma}_{ij}}{2\chi}\left(\tilde{\gamma}^{mn}\tilde{D}_m\tilde{D}_n\chi - \frac{3}{2\chi}\tilde{\gamma}^{mn}\partial_m\chi\,\partial_n\chi\right) + \frac{1}{2\chi}\left(\tilde{D}_i\tilde{D}_j\chi - \frac{1}{2}\partial_i\chi\,\partial_j\chi\right)\,,$$

$$\tilde{R}_{ij} = -\frac{1}{2}\tilde{\gamma}^{mn}\partial_m\partial_n\tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i}\partial_{j)}\tilde{\Gamma}^m + \tilde{\gamma}^m\tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn}\left[2\tilde{\Gamma}^k_{m(i}\tilde{\Gamma}_{j)kn} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{kjn}\right]\,,$$

$$D_iD_j\alpha = \tilde{D}_i\tilde{D}_j\alpha + \frac{1}{\chi}\partial_{(i}\chi\,\partial_{j)}\alpha - \frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_m\partial_n\alpha\,.$$

Beyond BSSN

- BSSN has a zero speed mode in the constraint-subsystem;
  May result in large constraint violations
- BSSN does not have systematic constraint damping
- This can be implemented by considering Generalized Einstein Eqs.
  Bona et al. PRD gr-qc/0302083 "Z4" system
- Conformal version of Z4: Very like BSSN but has constraint damping
  Alic et al. PRD 1106.2254, Hilditch et al. PRD 1212.2901
- Also allows for constraint preserving boundary conditions
  Bona et al. CQG gr-qc/0411110, Ruiz et al. PRD 1010.0523

Progress accelerates: The early 2000s

- BSSN found empirically. Then strong hyperbolicity shown

  Gundlach & Martín-García 2004

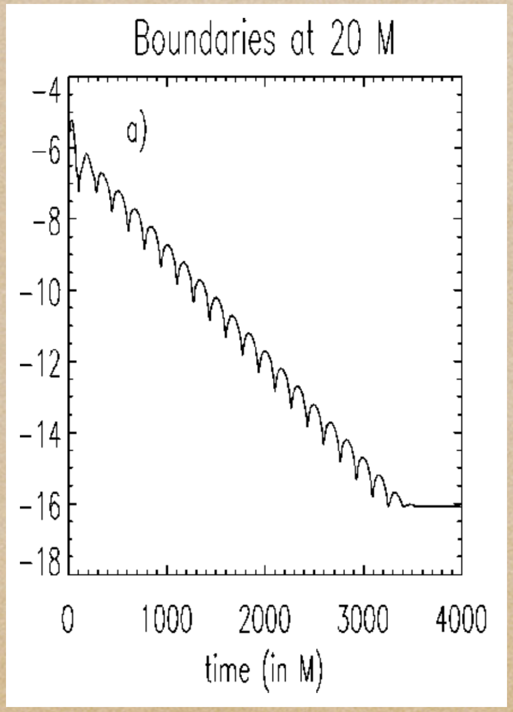
- 1st stable 3+1 evolution of Schwarzschild "simple excision"

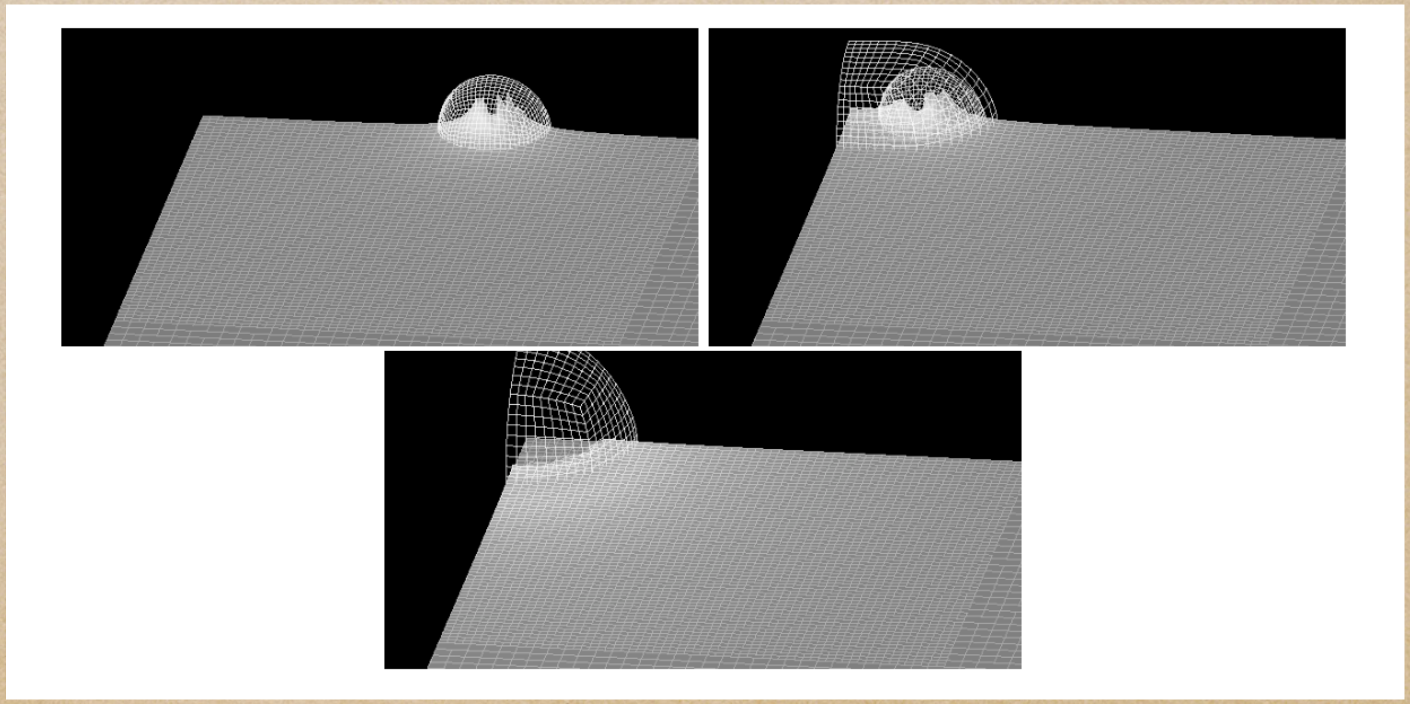
  Alcubierre & Brügmann gr-qc/0008067

- Stable head-on collisions

  US et al gr-qc/0503071, Fiske et al gr-qc/0503100

- BH binary orbit Brügmann et al gr-qc/0312112





Missing pieces I: Constraint damping in GHG

- One can show: GHG constraints related to ADM constraints

$$\mathcal{C}^\alpha = 0, \quad \partial_t \mathcal{C}^\alpha = 0 \;\; \text{at} \;\; t = 0 \quad \Rightarrow \quad \mathcal{H} = 0\,, \quad \mathcal{M}_i = 0$$

- Bianchi identies imply evolution of the $\mathcal{C}^\alpha$:

$$\Box \mathcal{C}_\alpha = -\mathcal{C}^\mu \nabla_{(\mu} \mathcal{C}_{\alpha)} - \mathcal{C}^\mu \left[8\pi \left(T_{\mu\alpha} - \frac{1}{2} T\, g_{\mu\alpha} \right) + \Lambda g_{\mu\alpha} \right] .$$

- In practice: Numerical violations of $\mathcal{C}^\mu = 0 \;\; \Rightarrow$ unstable modes!

- Solution: Add constraint damping terms

$$\boxed{\begin{aligned} \frac{1}{2} \partial_\mu \partial_\nu g_{\alpha\beta} = &- \partial_\nu g_{\mu(\alpha} \partial_{\beta)} g^{\mu\nu} - \partial_{(\alpha} H_{\beta)} + H_\mu \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\nu\alpha} \Gamma^\nu_{\mu\beta} \\ &- \Lambda g_{\alpha\beta} - 8\pi \left(T_{\mu\nu} - \frac{1}{2} T\, g_{\alpha\beta} \right) - \kappa \left[2 n_{(\alpha} \mathcal{C}_{\beta)} - \lambda g_{\alpha\beta} n^\mu \mathcal{C}_\mu \right] \end{aligned}}$$

Gundlach et al CQG (2005)

- E.g. Pretorius PRL gr-qc/0507014 uses $\kappa = 1.25/m\,, \quad \lambda = 1$

Missing pieces II: Gauge in BSSN

- Recall: Einstein's equations say nothing about $\alpha\,,\ \ \beta^i$
- Any choice of lapse and shift gives a solution to Einstein's eqs.
- This is the coordinate or gauge freedom of GR
- If the physics do not depend on $\alpha\,,\ \ \beta^i$, then why bother?
- Answer: The performance of the numerics DO depend very sensitively on the gauge!

Ingredients for good gauge

- Singularity avoidance
- Avoid slice stretching
- Aim for stationarity in a co-moving frame
- Well-posedness of the system of PDEs
- Generalize “good” gauge, e.g. harmonic
- Lots of good luck!

Bona et al PRL (1995)

Alcubierre et al PRD gr-qc/0206072

Alcubierre CQG gr-qc/0210050

Garfinkle PRD gr-qc/0110013

Moving puncture gauge

- Moving punctures is one of the NR breakthrough methods

Baker et al PRL gr-qc/0511103; Campanelli et al PRL gr-qc/0511048

- Gauge played a key role
- Variant of $1 + \log$ slicing and $\Gamma-$driver shift

Alcubierre et al PRD gr-qc/0206072

- Now in use as $\partial_t \alpha = \beta^m \partial_m \alpha - 2\alpha K$

and $$\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} B^i$$

$$\partial_t B^i = \beta^m \partial_m B^i + \partial_t \tilde{\Gamma}^i - \beta^m \partial_m \tilde{\Gamma}^i - \eta B^i$$

or $$\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} \tilde{\Gamma}^i - \eta \beta^i$$

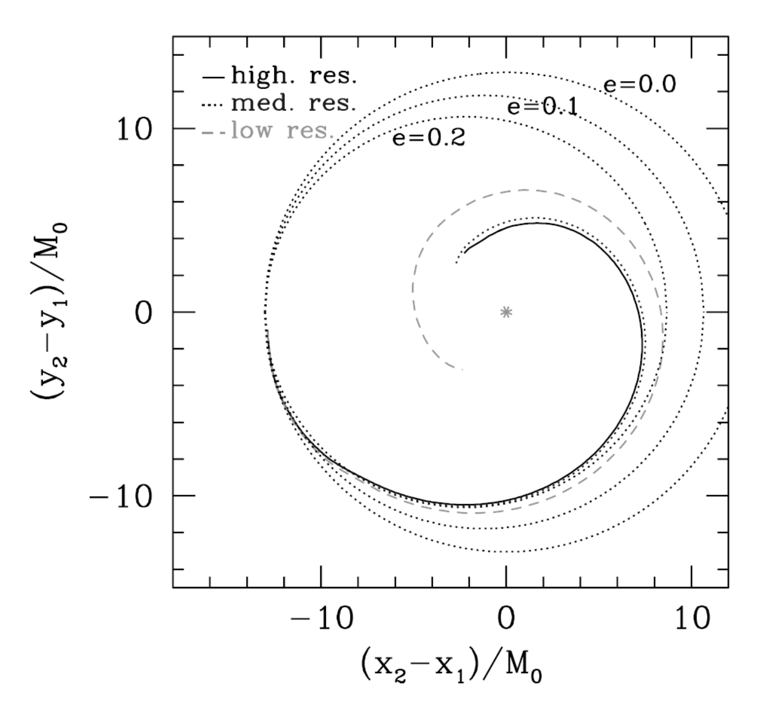
e.g. van Meter et al PRD gr-qc/0605030

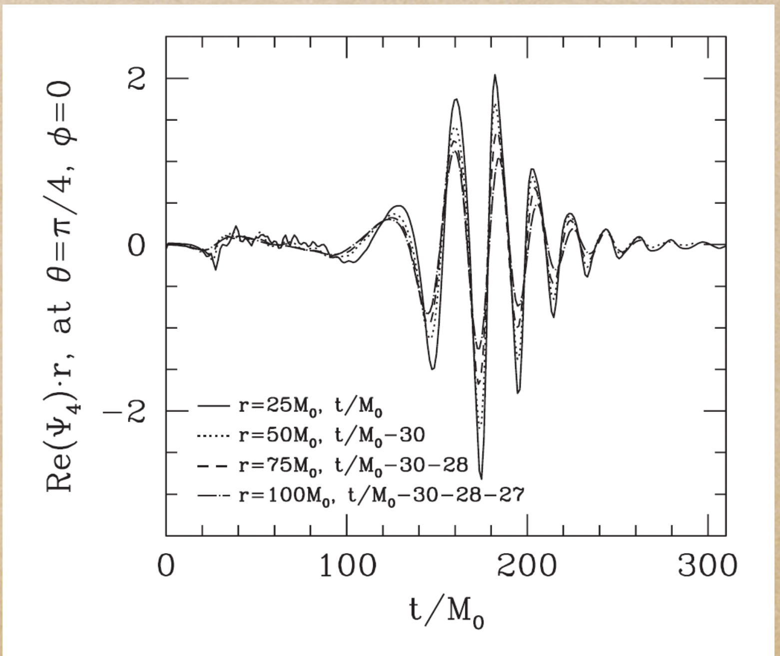
The 2005 breakthroughs

- First simulation of orbiting BBH through merger

  Pretorius PRL gr-qc/0507014

  - GHG
  - Initial data: Scalar field
  - BH excision
  - Radiated energy $\sim 3\,\%\; M$
  - Eccentricity $e \sim 0 \ldots 0.2$





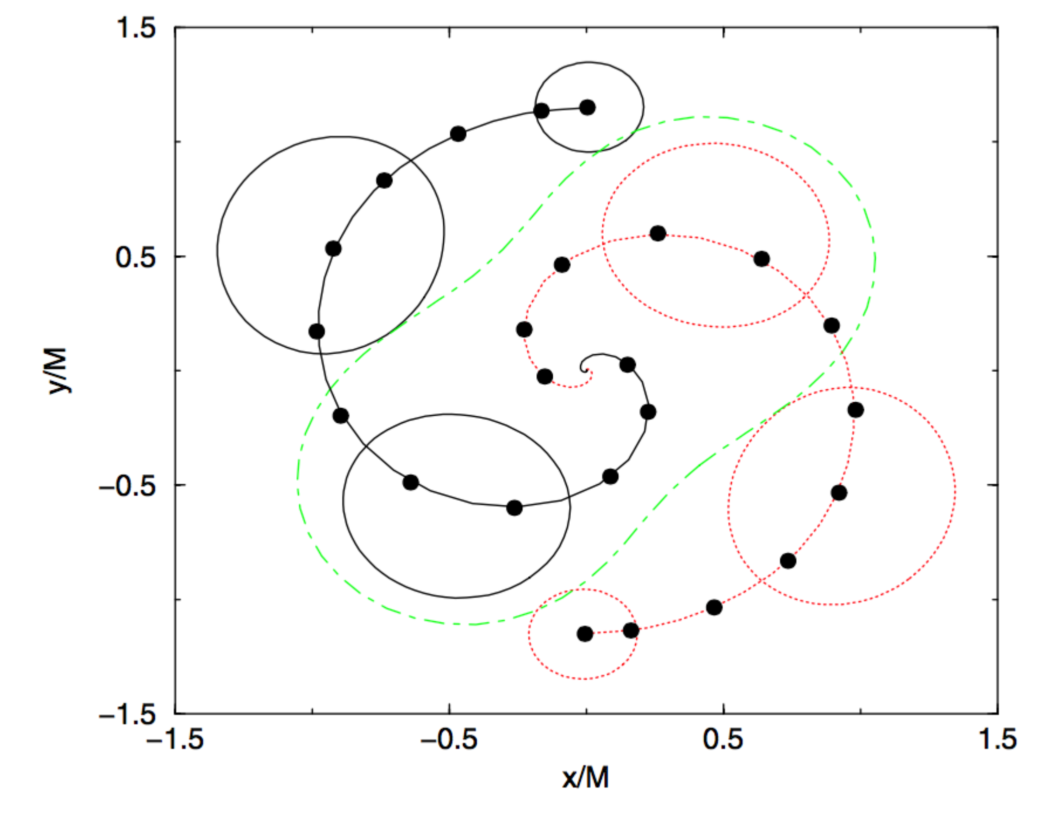
- Presented at Banff conference

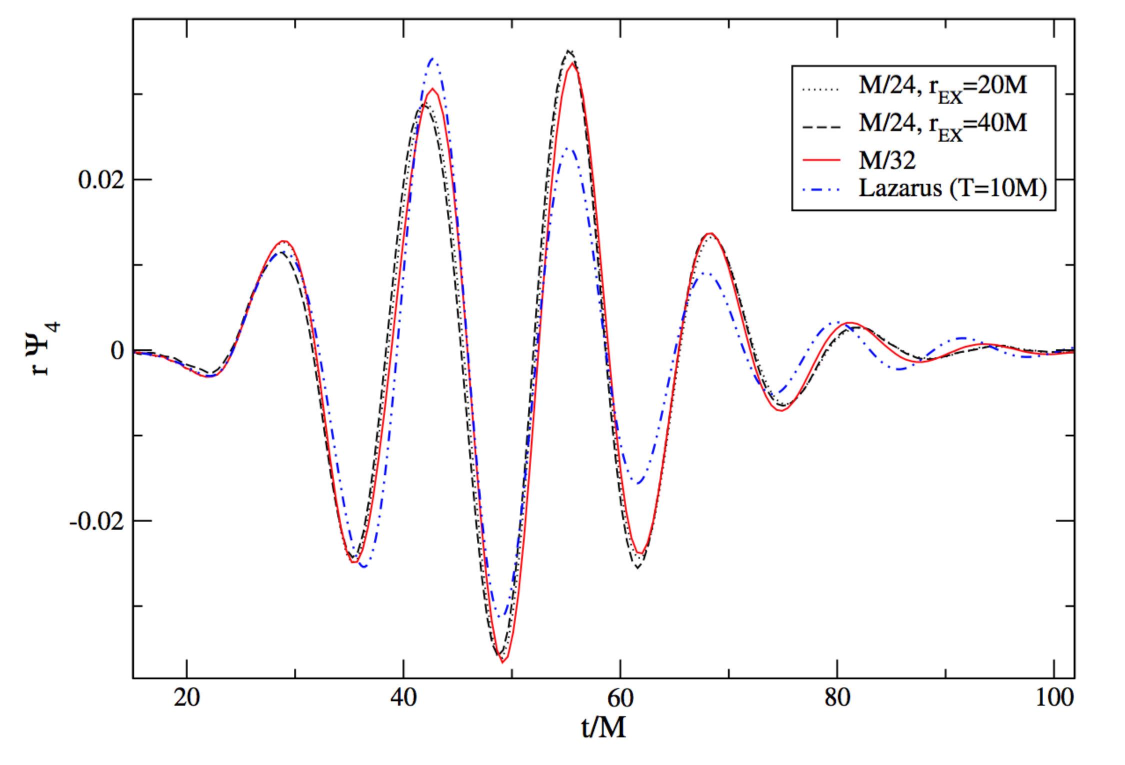
The 2005 breakthroughs

- Moving puncture breakthrough by Brownsville and Goddard groups

  Campanelli et al PRL gr-qc/0511048; Baker et al PRL gr-qc/0511103

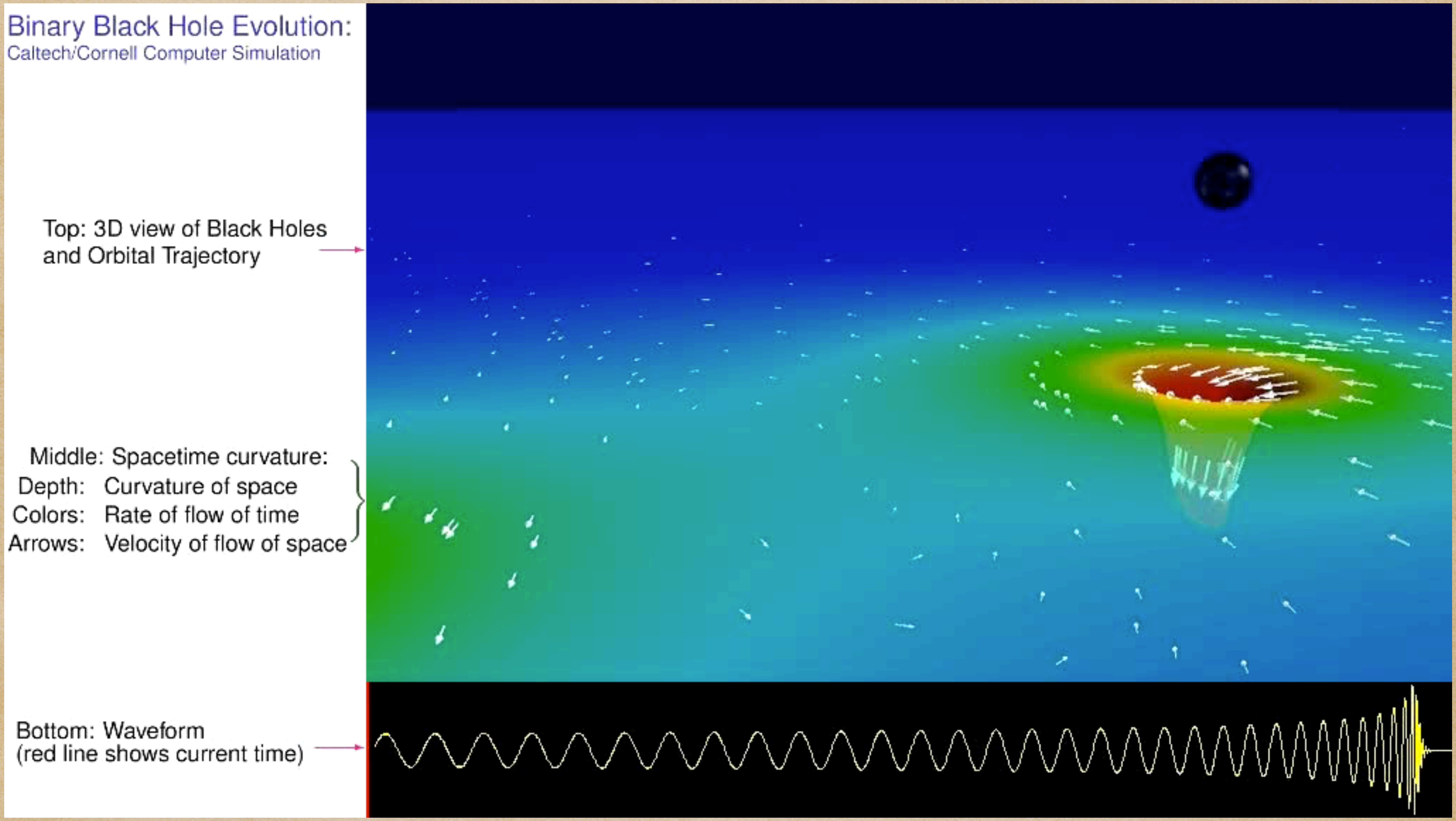
  - BSSN
  - Bowen-York initial data
  - Moving puncture gauge
  - Radiated energy $\sim 3\,\%\,M$





The goldrush years

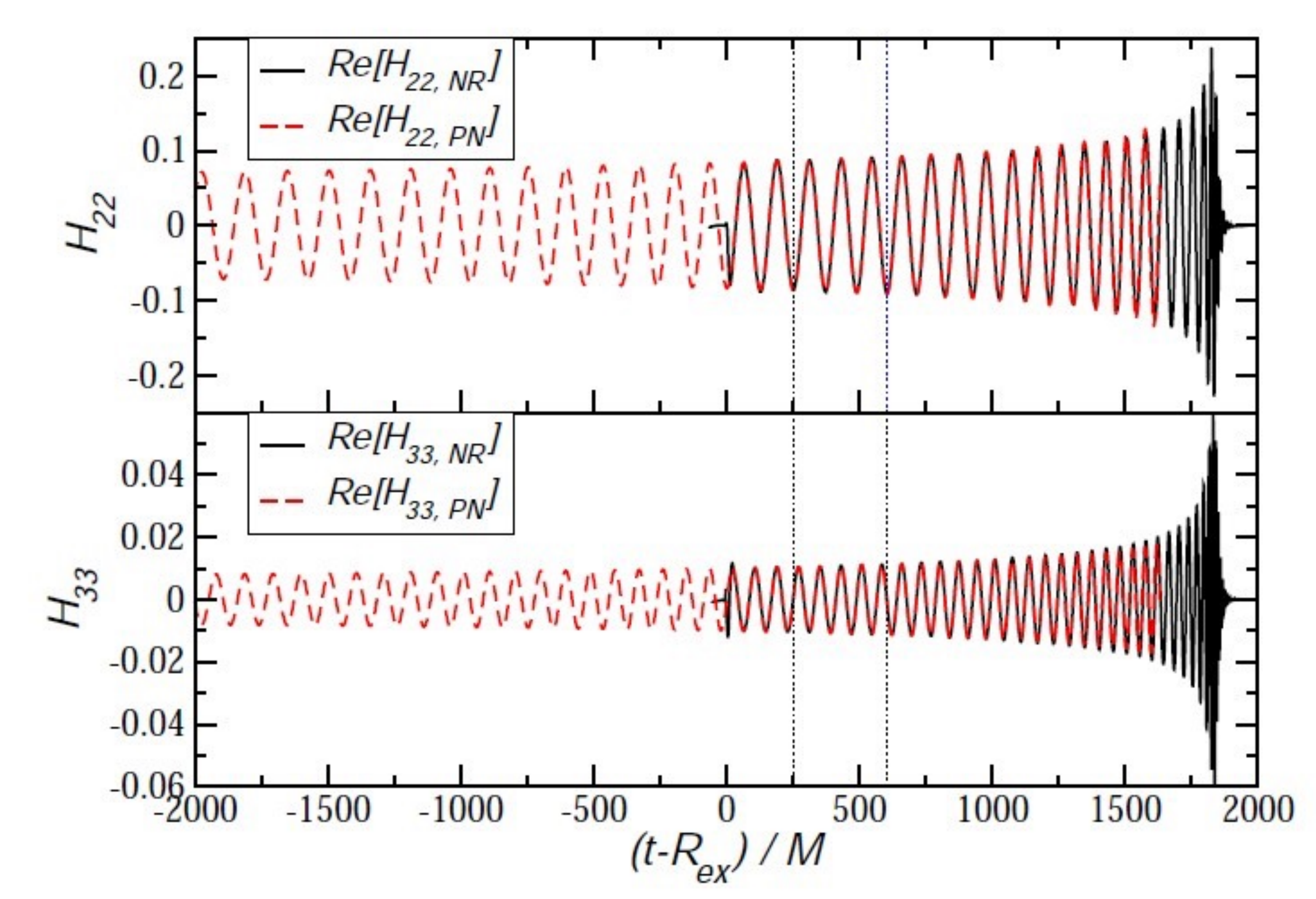
Anatomy of a BHB inspired



Thanks to Caltech-Cornell groups

Hybrid waveforms and catalogs

- Stitch together PN and NR waveforms



US et al CQG 2011

- Mass produce waveforms; Hinder et al CQG 1307.5307;
  Mroué et al PRL 1004.4697

Gravitational recoil

- Anisotropic GW emission $\Rightarrow$ recoil of remnant BH

Bonnor & Rotenburg Proc.R.Soc.Lond.A. (1961);

Peres PR (1962); Bekenstein ApJ (1973)

- Escape velocities:

Globular clusters	$\sim 30\ \mathrm{km/s}$
dSph	$20 \ldots 100\ \mathrm{km/s}$
dE	$100 \ldots 300\ \mathrm{km/s}$
Giant galaxies	$\sim 1\,000\ \mathrm{km/s}$

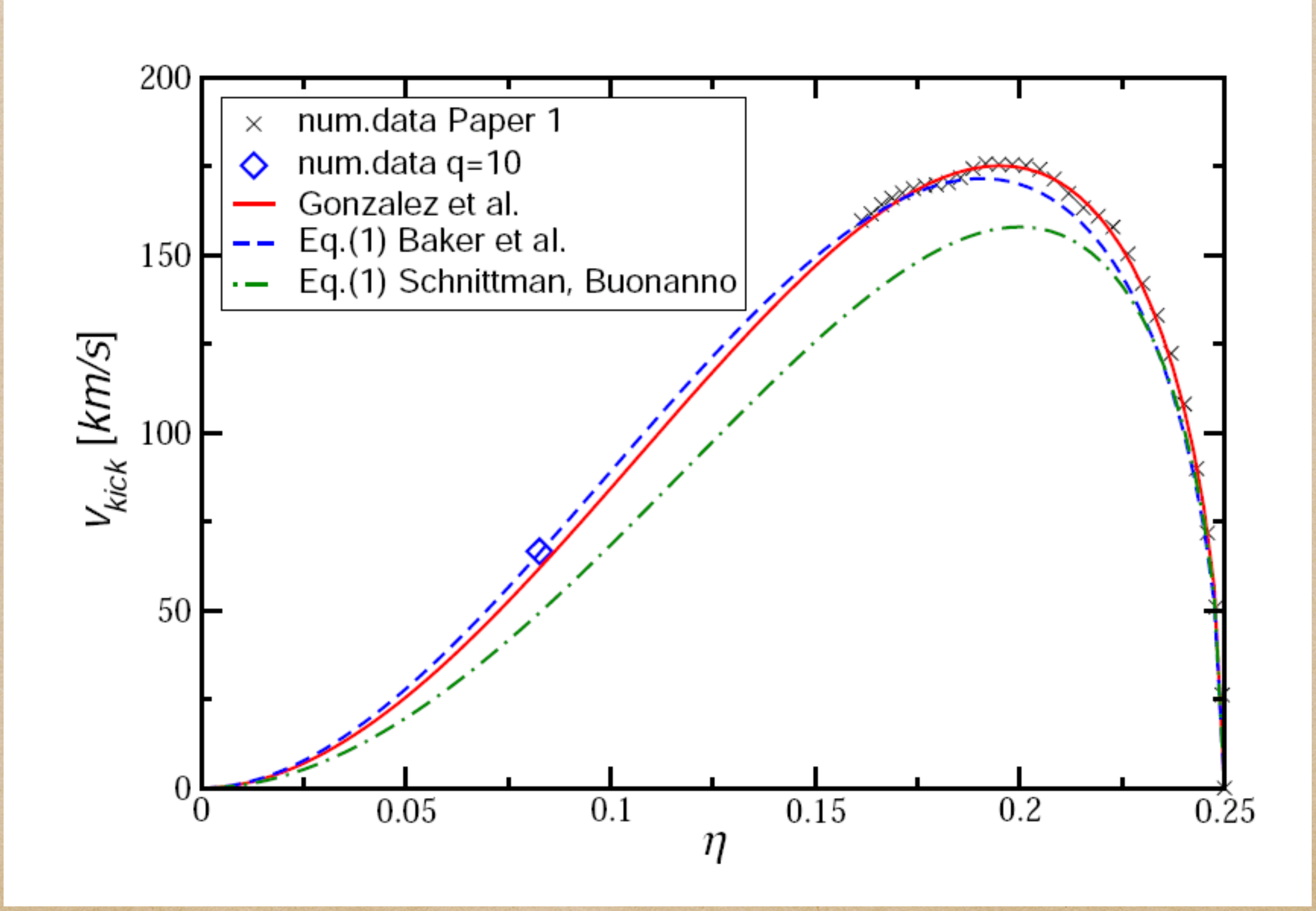
- Ejection/displacement of BHs affects
  - Growth history of SMBHs
  - BH populations, IMBHs
  - galaxy structure
  - observational "footprints"

Komossa Adv.Astron. 1202.1977

Kicks from non-spinning BH binaries

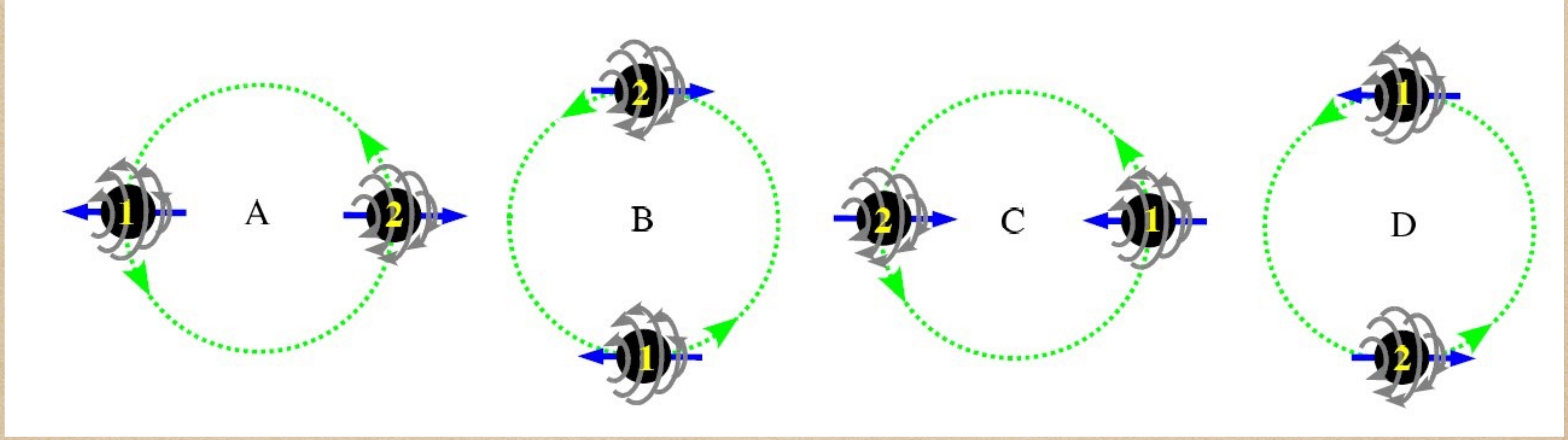
- Maximal kick: $\sim 180\ \mathrm{km/s}$ pretty harmless!

González et al PRL gr-qc/0610154



Spinning BHs: Superkicks

- Superkick configurations; Kidder gr-qc/9506022; Pretorius 0710.1338



- Kicks up to $v_{\rm max} \approx 4\,000$ km/s

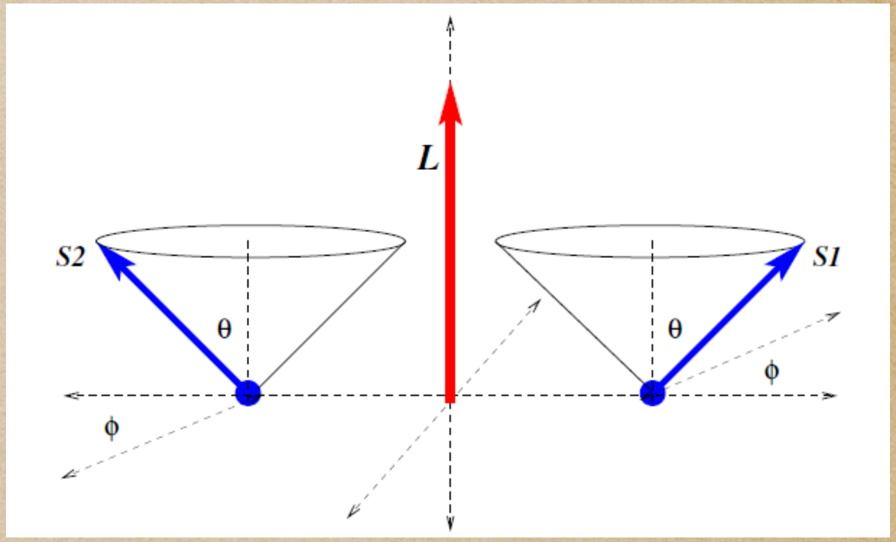
González et al PRL gr-qc/0702052

Campanelli et al PRL gr-qc/0702133

- Yet larger kicks for partially aligned spins

$v_{\rm max} \approx 5\,000$ km/s

Lousto et al PRL 1108.2009



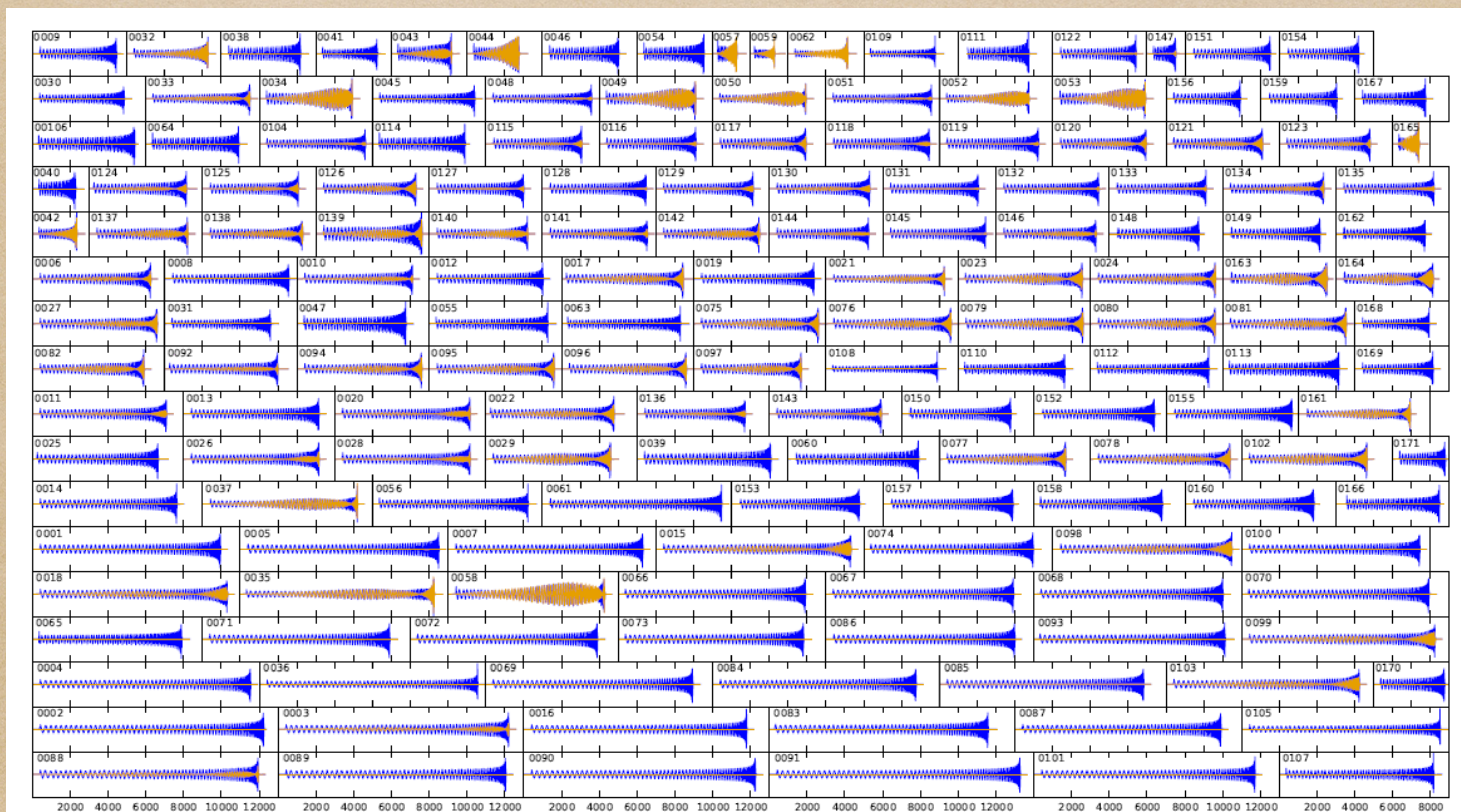
Towards new horizons

Tools of mass production

- Explore seven-dim. parameter space. E.g. SpEC catalogue:

171 waveforms: $m_1/m_2 \leq 8$ up to 34 orbits

Mroué et al PRL 1304.6077



Neutron star binaries

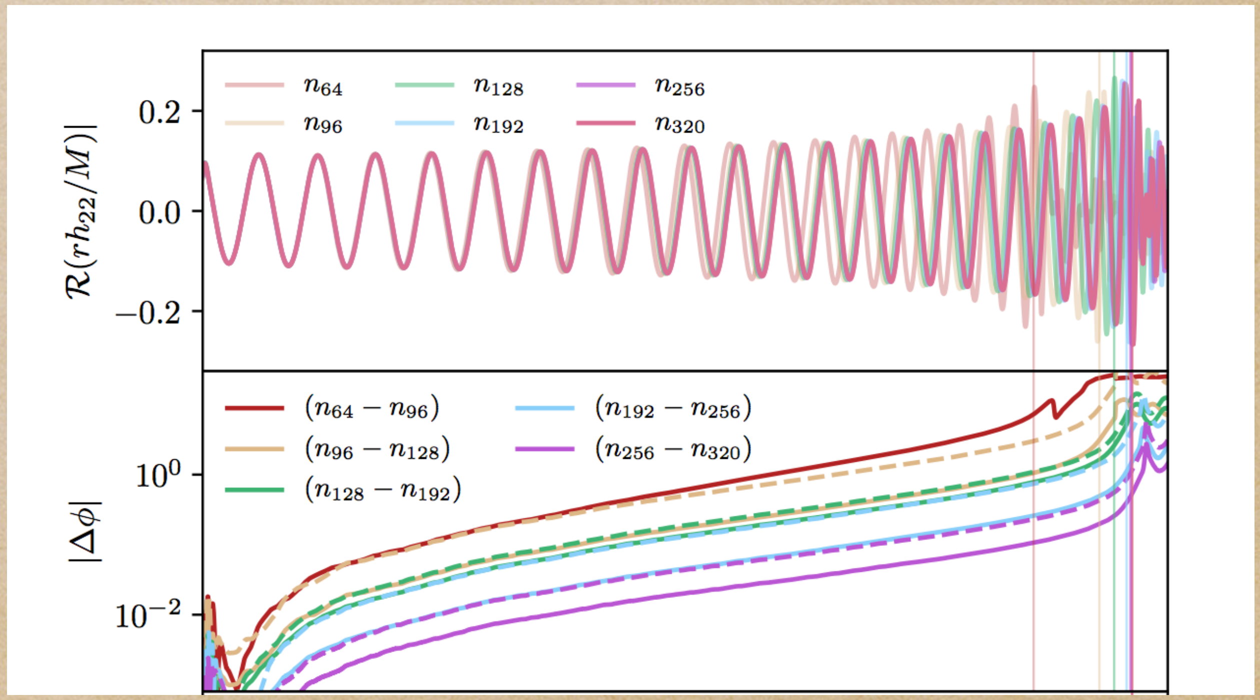
- Other challenges: No spacetime singularities, but shocks!
  - HRSC treatment needs flux conservative Eqs. $\partial_t \boldsymbol{u} + \partial_i \boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{s}$
  - Solutions not unique $\rightarrow$ entropy conditions!

- The first NS binary inspirals preceded the BH breakthrough!

Shibata+ PRD 2003, Marronetti+ PRL 2004, Miller+ PRD 2004

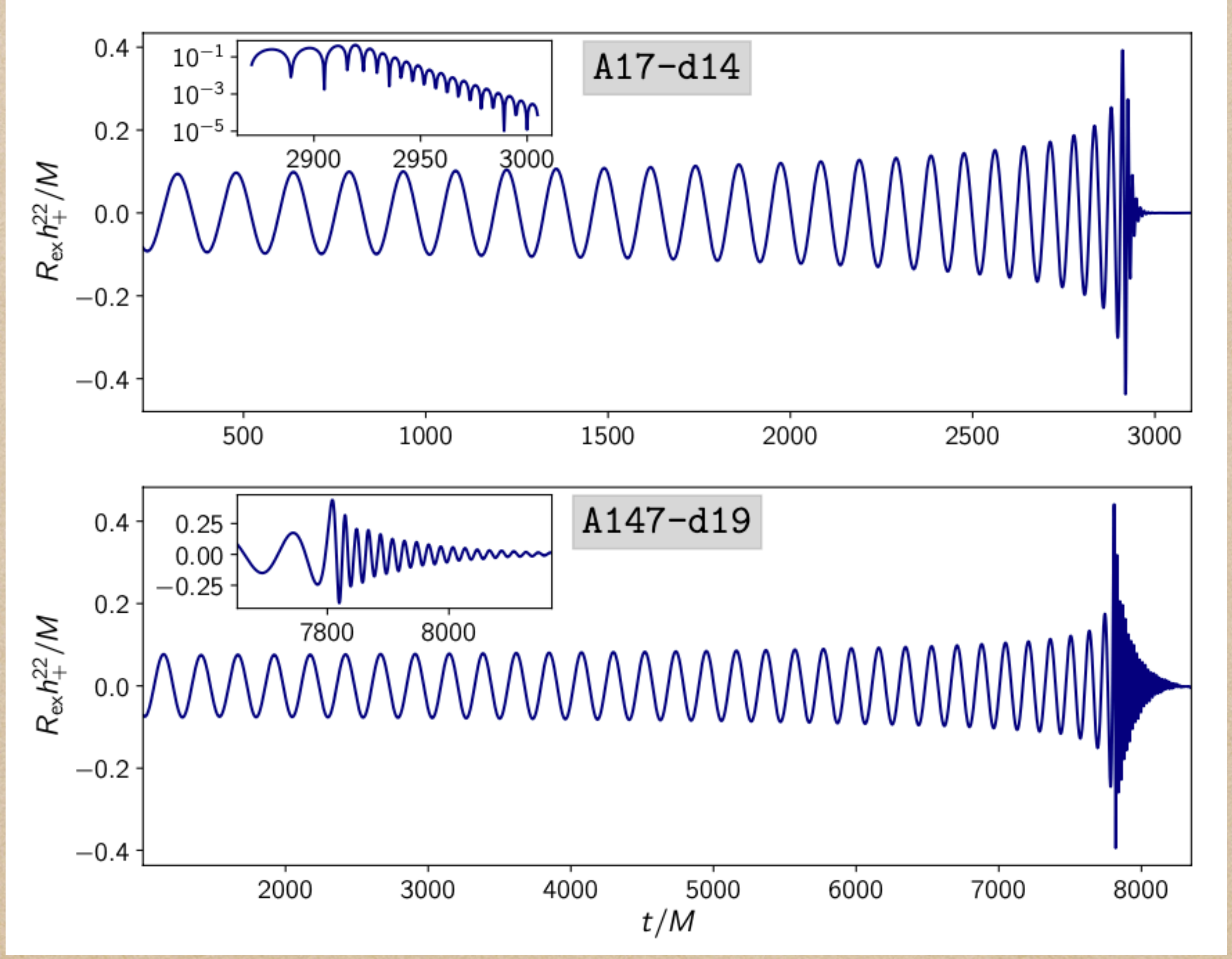
- Template constructions: e.g. Dietrich et al 1905.06011

`NRTidalv2` approximant



BS binaries

- Phase error $\approx 0.1 \ldots 0.2$
- Amplitude error $\lesssim 3\,\%$
- Eccentricity $\approx 0.002 \ldots 0.005$



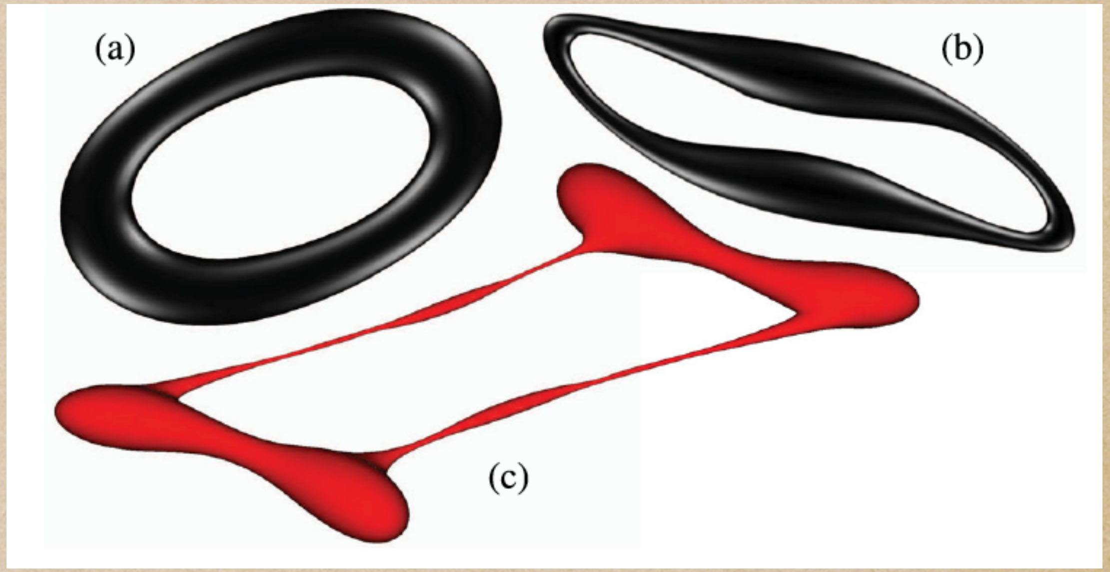
Evstafyeva et al PRL 2024

Cosmic censorship in D=5

Figueras, Kunesch & Tunyasuvunakool PRL 1512.04532

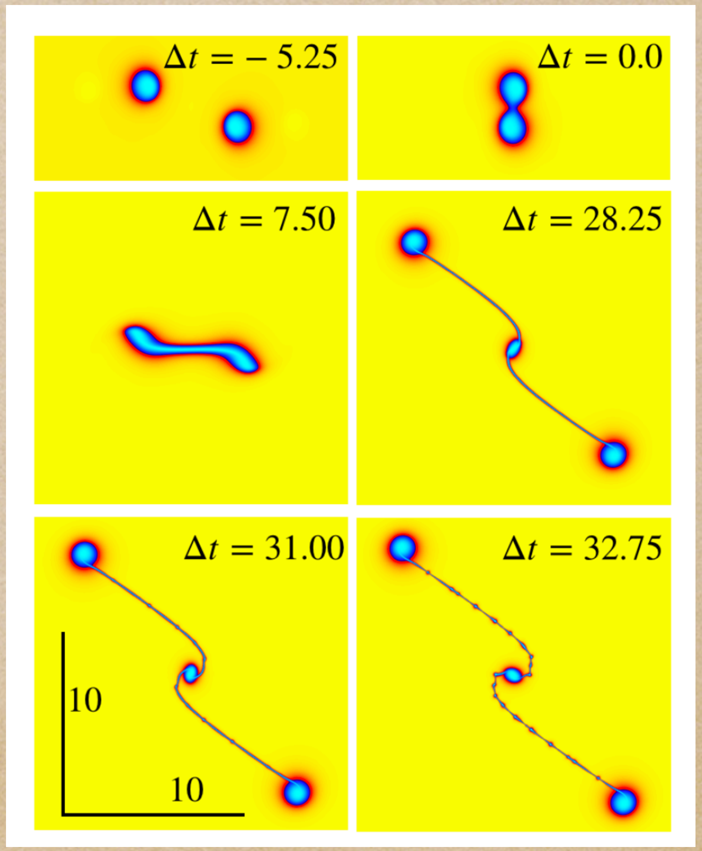
- 5D simulations (mod.cartoon)
- Conformal Z4 system
- Black ring: assympt.flat!
- Gregory-Laflamme instability

  $\Rightarrow$ Violation of CC!



Andrade, Figueras & US JHEP 2022

- 7D BH collisions
- No finetuning!



What's different in Cosmology NR?

NR specific to cosmology

Much from recent review Aurrekoetxea, Clough, Lim. 2409.01939

- Outer boundary conditions (OBCs)
  - No asymptotic flatness!
  - Instead use periodic, de Sitter, Sommerfeld on de Sitter
- Initial data
  - Conformal decomposition works as before
  - Additional complication: non-fluid matter, OBCs

    $\Rightarrow$ Possible non-uniqueness of conformal factor

    CTTK: solve for K instead of ψ but price: solve $\mathcal{M}_i = 0$

    Aurrekoetxea, Clough & Lim 2023 CQG
  - Matter may not admit conformally flat slices
  - Lack of generic solvers. Exception: `FLRWsolver`

Macpherson et al, PRD 2017

NR specific to cosmology

- Gauge
  - Geodesic may work for not too strong fields
  - Moving-punctures still a good workhorse, but $K < 0$

    may cause divergent lapse $\rightarrow$ modifications like:

    $\partial_t \alpha = -2\alpha(K - \langle K \rangle)$ Giblin & Tishue PRD 2019

- Diagnostics
  - Very different! Often use statistical or averaged quantities

    $\rightarrow$ lack of rigorous definitions, e.g. what spatial slice to choose
  - GW energy density

$$\rho_{\mathrm{GW}} = \frac{1}{32\pi G}\langle \dot{h}_{ij}^{\mathrm{TT}} \dot{h}^{ij\mathrm{TT}} \rangle \,, \qquad \ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{1}{a^2}\nabla^2 h_{ij} = 16\pi G S_{ij}$$

$$h_{ij}^{\mathrm{TT}}(t,x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \Lambda_{ij,kl}(\hat{\mathbf{k}}) h_{lm}(\mathbf{t},\hat{\mathbf{k}})$$

Results: Very early Universe

Cosmological singularities

- Approach singularity through backward time evolutions
- BKL conjecture Belinsky, Khalatnikov, Lifshitz Adv.Phys. 1970
  - Singularity is spacelike, local, oscillatory

    $\rightarrow$ points decouple Misner's Mixmaster cosmology

    $\rightarrow$ transition through Kasner spacetimes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{2p_1}\mathrm{d}x^2 + t^{2p_2}\mathrm{d}y^2 + t^{2p_3}\mathrm{d}z^2\,, \quad p_1 = \mathrm{const}, \ \sum p_i = \sum p_i^2 = 1$$

  - Transition of Kasner exponents chaotic, but well defined

    Barrow PRL 1981

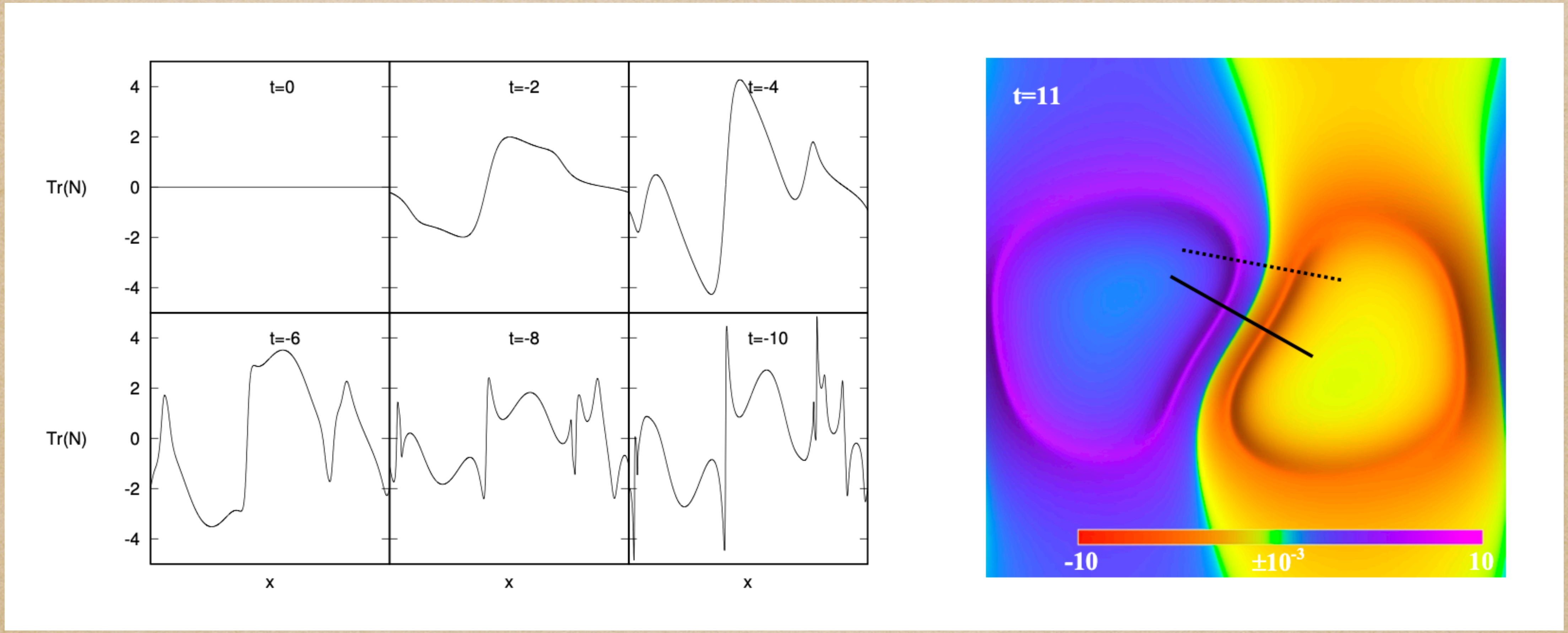
  - Transition driven by dominant ${}^{(3)}R$
- NR provides supporting evidence for BKL
- NR also identified new feature spikes i.e.

  neighbouring points have different Kasner exponents…

Cosmological singularities

- … Example for evolution of spikes in 1+1, 2+1

Garfinkle & Pretorius. PRD 2020



- Impact of spikes on dynamics decreases very close to the singularity
- Side effect: Cosmology inspired new techniques in NR, e.g. GHG

Inflation

- Large- and small-field models
- Large-field: may back react on Hubble scale $\rightarrow$ NR needed
- Initial-condition problem
  - NR simulations support that inflation persists for generic ini data

    e.g. Kurki-Suonio, Laguna, Matzner. PRD 1993
  - Simulations with BH formation may give GW dominated epoch

    Shibata et al. PRD 1994
  - Inflation operates even if BHs form East et al JCAP 2016
  - Maximum mass of BHs formed is bounded Clough et al JCAP 2017
  - Low-scale models less robust to inhomogeneities (weaker homogeneous "background") Aurrekoetxea et al. JCAP 2020
  - Plenty more; see Aurrekoetxea, Clough, Lim. 2409.01939

Ekpyrotic and bouncing models

- Ekpyrotic models: big bang = collision of branes in higher D bulk.

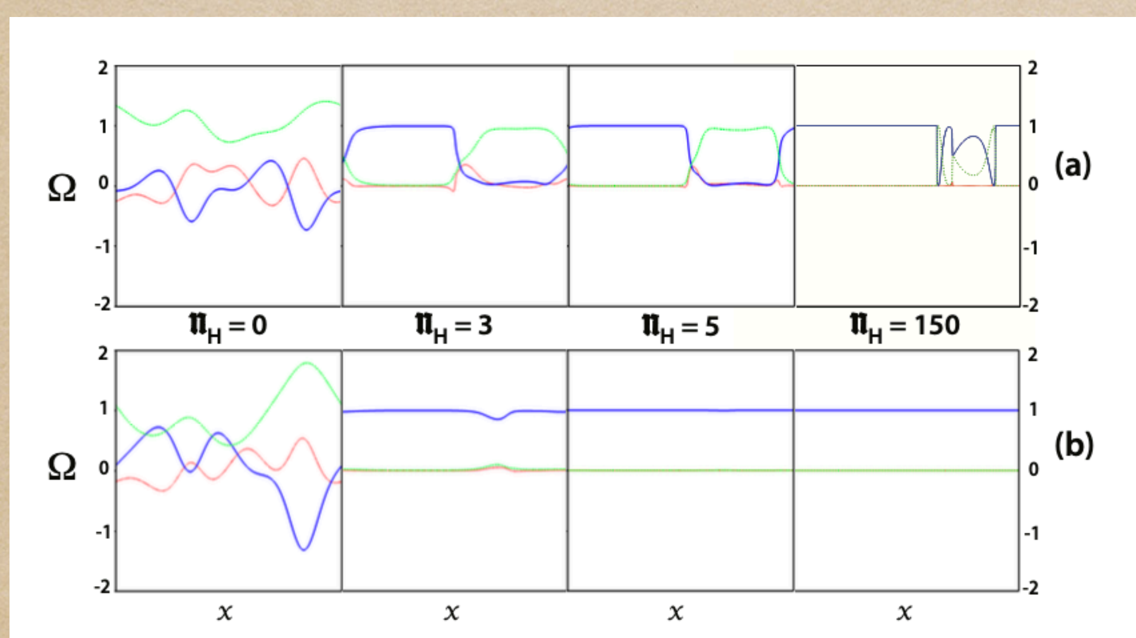
  $\Rightarrow$ Our 3D Universe then bounces from contraction to expansion.

  Khoury et al PRD 2001

- Generalized to non-singular bouncing models

  Ijjas & Steinhardt Phys.Lett.B 2019, 2022

- The slow contraction phase smoothens out inhomogeneities in the matter (non-zero), curvature and shear energy density



e.g. Cook et al. Phys.Lett.B 2020

Results: Post-BBN

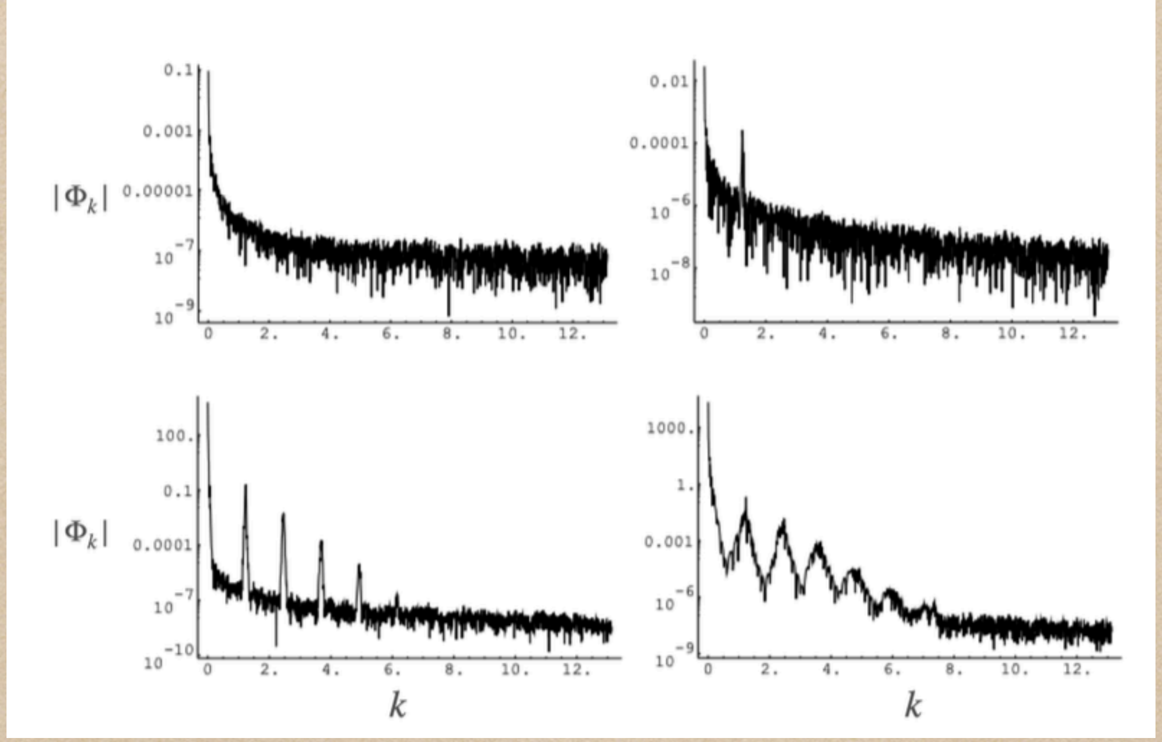
Late and post inflation

- From BBN on we know the Universe's evolution on large scales
- Preheating: Explosive particle production via parametric resonance
  (non-perturbative) during inflation
- Reheating: Conversion of inflaton to heat, i.e. a
  scalar- to radiation-dominated Universe
- Non-perturbative effects during reheating $\to$ ECOs, GWs
- During ensuing cooling: Phase transitions, cosmic strings, PBHs etc.
- Observational imprints on GWs, CMB

Preheating

- 1+1 simulations:
  - long wavelengths $\gg H$ not parametrically amplified
  - Non-linear effects broaden the amplified λ range

    Parry & Easther PRD 1999, 2000



  - Power spectrum grows in time for most k
- Also found in 3+1 simulations; e.g. Bastero-Gil et al PRL 2010

Topological defects

- Phase transitions $\rightarrow$ topological defects

  e.g. Shellard & Vilenkin Cambridge University Press (2000)

- Breaking of parity $\rightarrow$ 2D domain walls

  1+1: Blanco-Pillado et al JCAP 2019, Giombi & Hindmarsh JCAP 2024

  inhomogeneities in 3+1 $\rightarrow$ bubbles: Lin et al PRD 2022, Phys.Lett.B 2024

  may seed single and binary BHs

  Huang & Piao PRD 2024, Yuwen et al. 2406.05838

- Breaking of $U(1)$ symmetry $\rightarrow$ Cosmic strings

  Hindmarsh & Kibble Rept.Prog.Phys. 1995, Copeland & Kibble Pr.Roy.Soc. 2010

- Challenging to resolve in expanding Universe!
- GWs either as stochastic background or individual sources
- First GW templates in asymptotic flatness

  e.g. Helfer at al. PRD 2019, Aurrekoetxea et al PRL 2024

Primordial BHs

- Primary formation: collapse of over-densities at end of inflation

  → wide mass range

- Analytic estimates: minimal over density $\delta_c \approx 1/3$

  Carr & Hawking MNRAS 1974, ApJ 1975

  This is above observed (CMB) → PBHs only in tail of distribution

- PBH distribution sensitive to δ_c . Goal for NR: find δ_c !
- 1+1 codes, following May & White PRD 1966
  - Pressure impedes PBH formation and growth

    Nazhedin et al Sovast 1978
  - Softer EOSs → more massive PBHs Novikov & Polnarev AZH 1980
  - Characteristic codes found $\delta_c \approx 0.7$ Niemeyer & Jedamzik PRD 1999
  - Likely due to unphysical mode in initial data. Later work found

    $\delta_c = 0.3 \ldots 0.5$ e.g. Green et al, Hawke & Stewart, Miller & Rezzolla

Primordial BHs

- Do critical phenomena persist in asymptotically FLRW?

  $M_{\rm bh} = K(\delta - \delta_c)^\gamma, \quad \gamma \approx 0.37$ Choptuik PRL 1993

  - Near critical phenomena applies to PBH formation with $\gamma \approx 0.36$

    Niemeyer & Jedamzik PRD 1999

  - The relation breaks for small $\delta - \delta_c \;\Rightarrow$ non-zero minimal $M_{\rm bh}$

    Also FLRW $\Rightarrow$ lower γ and larger δ_c

    Hawke & Stewart CQG 2002, Musco et al CQG 2005

- 3+1 simulations relatively new, Yoo et al PRL 2013,

  Okawa et al PRD 2014, Uehara et al 2401.06329

- PBH spins relatively small $j \lesssim 0.1$ and decreasing at accretion

  Jong et al JCAP 2022, Aurrekoetxea et al PRL 2024

Results: Late Universe

The pre-NR era

- Simplifications
  - Matter non-relativistic and collisionless
  - Background FLRW + Newtonian gravitational perturbations
  - Dark matter = massive point particles; "Poisson-Vlasov" limit

→ N-body codes. **Gadget-4** Springel et al MNRAS 2021,

RAMSES Teyssier A&A 2002

- Motivation for NR
  - Small-scale inhomogeneities which may impact on large scales, e.g. negative pressure contribution
  - Non-linear effects that may source higher-order correlators; very timely for new surveys
- Hybrid codes between Poisson-Vlasov and NR

gevolution Adamek et al Nat.Phys. 2016, **GRAMSES** Barrera & Li JCAP 2020

NR simulations of late cosmology

- First NR codes: **CosmoGRaPH** Giblin, Mertens, Starkman PRL 2016, PRD 2016

  ETK-McLachlan Bentivegna & Bruni. PRL 2016

  - Matter = dust of density ρ
  - geodesic slicing
  - Volume averaging via $\langle X \rangle_S = \dfrac{\int_S X\sqrt{\gamma}\mathrm{d}^3x}{\int_S \sqrt{\gamma}\mathrm{d}^3x}$

- Density perturbations $\dfrac{\delta\rho}{\rho} \approx 0.3 \Rightarrow$ Expansion within $0.3\,\%$ of FLRW,

  i.e. spacetimes stay close to FLRW Giblin et al PRL 2016

- Turn around density (collapse vs. expansion): $\dfrac{\delta\rho_{OD}}{\rho} = 1.055$

  in good agreement with semi-analytic results

  Munoz & Bruni. PRD 2023, CQG 2023

NR simulations of late cosmology

- New code: **`ETK-GRHydro`** Macpherson, Lasky, Price PRD 2017
  - More general matter
  - New **`FLRWSolver`** for initial data

  Confirmed previous NR results

- Evolutions of matter-dominated Universe

  Macpherson et al PRD 2018, ApJL 2018, ApJ 2024 :

  Large-scale homogeneity and isotropy maintained, but on scales

  $r_D \leq 180\, h^{-1}\,\mathrm{Mpc}$ inhomogeneities up to $\sim 6\,\%$

  $\rightarrow$ Change in H_0 only at sub-percent, but could be larger for observers close to inhomogeneities

- Code comparison: good agreement with **`gevolution, GRAMSES`**

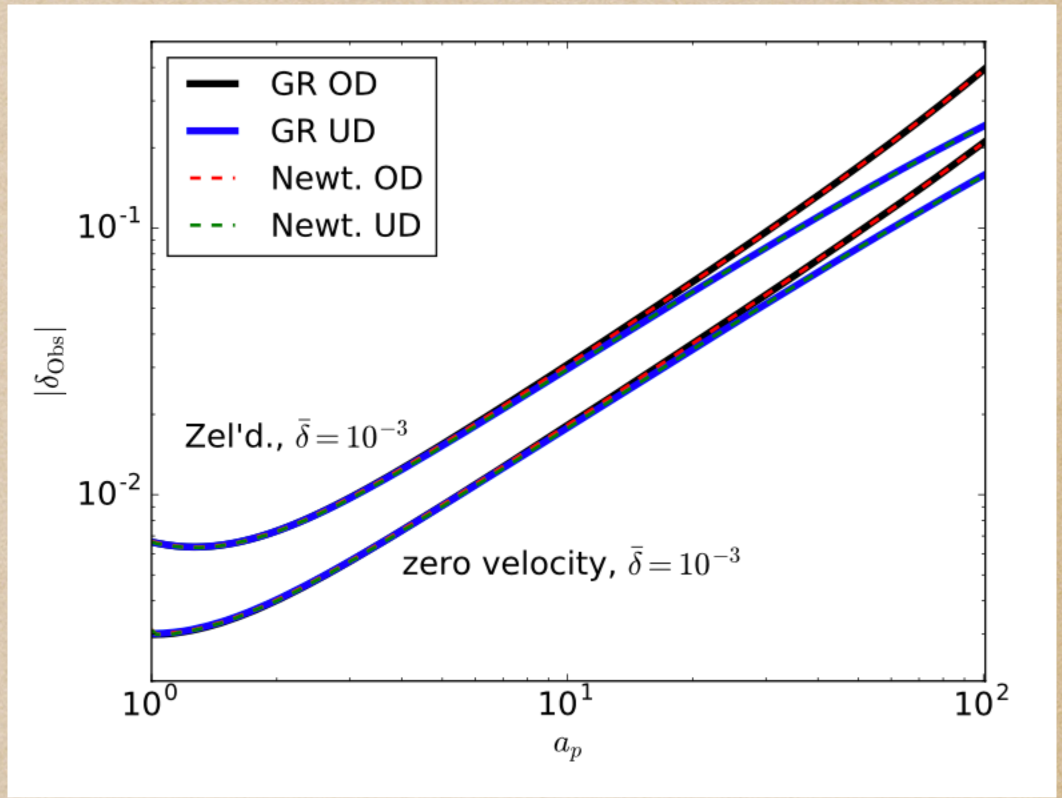
  Adamek et al. CQG 2020

NR vs. N-body comparisons

Direct comparison of NR vs N-body simulations East et al PRD 2018

- Sub-percent agreement as long as Newtonian potential weak

  E.g. density contrast (max over-, underdensity) seen by co-moving observer



- Challenge: Capture virialization in conservative hydro

  → Einstein-Vlasov formalism; average sum of particle contributions

  Pretorius & East PRD 2018

- Einstein-Vlasov agrees with Poisson-Vlasov at $<1\,\%$ provided Newtonian potential weak $\Phi_N \leq 0.1$

Cosmological observables in the non-linear regime

- In general, Poisson-Vlasov works very well: short-scale phenomena have little effect on large scales
- NR still has a role as cross check!
- E.g. Weak-lensing power spectrum
  - Born approximation: light propagates on background geodesics
  - post-Born effects leads to disagreement

    Pratten & Lewis. JCAP 2016, Marozzi et al. JCAP 2016
  - → Use ray tracing Giblin et al. PRD 2017

    → % discrepancy NR vs perturbation theory at scales $\sim 100\,\mathrm{Mpc}$

    See also Macpherson JCAP 2023, Grasso & Villa CQG 2022
- Compute H_0 and its higher moments in anisotropic spacetimes

  → $\mathcal{O}(1)\,\%$ variations depending on sky coverage

  Macpherson & Heinesen. PRD 2021

Conclusions

Conclusions

- NR has become a mature field for binary simulations
- Still, there are open challenges to improve accuracy and efficiency
- Open technical challenges in cosmology
  - Beware gauge conditions for expansion effects $K < 0$
  - Improve boundary conditions
  - Find rigorous diagnostics
- Open physics questions
  - Structure of singularities (spikes!)
  - Modeling and GW emission of topological defects
  - Non-linear effects on direct observables

    e.g. Hubble constant (Hubble controversy), lensing spectra, etc.

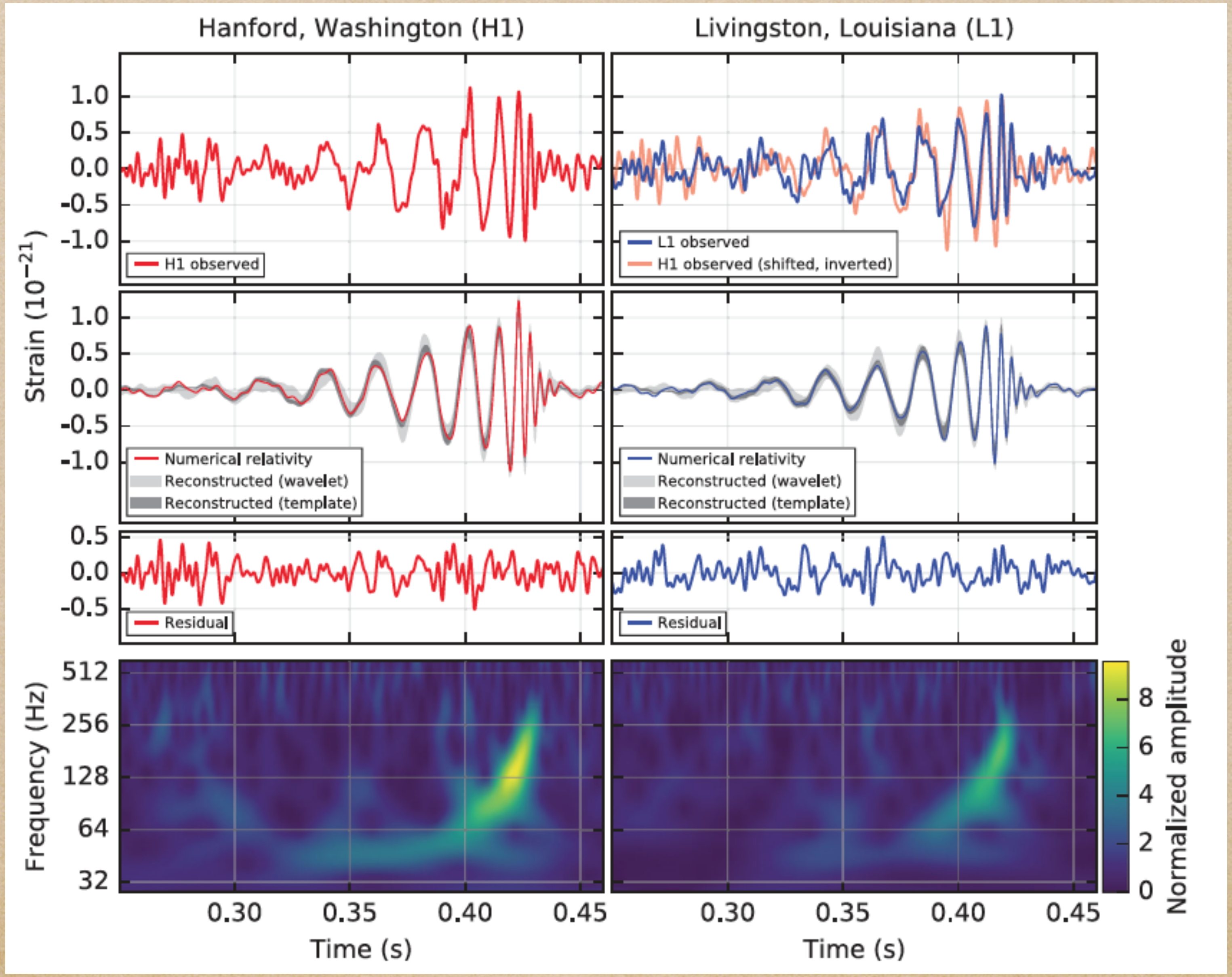
Bonus material

LIGO's detection of GW150914

- Sep 14, 2015 at 09:50:45 UTC: SNR ~ 24

  Abbott et al. PRL 1602.03837, Abbott et al. 1606.01210

- BBH inspiral, merger and ringdown: $m_1 = 35^{+5}_{-3}\, m_{\odot}\,, \quad m_2 = 30^{+3}_{-4}\, M_{\odot}$



A: Well-posedness

Well posedness

From G Papallo, PhD thesis Cambridge, 2018

- Consider linear const.coeff. PDE system $A\partial_t u + P^i \partial_i u + Cu = 0$

  Fourier trafo $\tilde{u}(t,k) = \frac{1}{\sqrt{2\pi}^n} \int u(t,x) e^{-ik_i x^i} \, d^n x$

  $$\Rightarrow \partial_t \tilde{u} - i\mathcal{M}\,\tilde{u} = 0\,, \qquad \mathcal{M}(k) = A^{-1}(-P^i k_i + iC)$$

  Solution $\tilde{u}(t,k) = e^{i\mathcal{M}(k)t}\,\tilde{u}(0,k)$

  $$\Rightarrow u(t,x) = \frac{1}{\sqrt{2\pi}^n} \int e^{ik_i x^i}\, e^{i\mathcal{M}t}\,\tilde{u}(0,k)\, d^n k$$

- May not converge if integrand fails to decay fast with $k = \sqrt{k_i k^i}$
- Convergence guaranteed if $||e^{i\mathcal{M}(k)t}|| \leq f(t)\,,$ f continuous
- L2 norm: $\Rightarrow ||u(t,.)|| \leq f(t)\,||u(0,.)||\,, \qquad ||.||$

Well posedness

- Consider two solutions $u_1\,,\ u_2$ with initial data $u_1^0\,,\ u_2^0$

  $\Rightarrow ||u_1 - u_2|| \leq f(t)\,||u_1^0 - u_2^0||$

- Define $\hat{t} = |k|\,t\,,\quad \hat{k}_i = \frac{k_i}{|k|}$ and take limit $|k| \to \infty\,,\quad \hat{t} = \text{const}$

- Sufficient condition for convergence becomes $\boxed{||e^{iM(\hat{k}_i)\hat{t}}|| \leq f(0)}$, $(\dagger)$

  where $M = -A^{-1}P^i k_i$ = "Principal Part"

- Let v be Eigenvector of M with Eigenvalue $\lambda_1 + i\lambda_2$

- $(\dagger)$ satisfied iff $\lambda_2 \geq 0$ for all $\hat{k}_i$

- But $M =$ real matrix $\Rightarrow \lambda_1 - i\lambda_2$ is also an Eigenvalue $\Rightarrow \lambda_2 = 0$

> - **Def.:** Our PDE is weakly hyperbolic if for any k_i with $k_i k^i = 1$ all Eigenvalues of $M(k_i)$ are real.

Well posedness

- Failure of weak hyperbolically leads to an exponentially growing integrand in our formal solution $u(t,x) = \frac{1}{\sqrt{2\pi}^n} \int e^{k_i x^i} \, e^{i\mathcal{M}t} \, \tilde{u}(0,k) \, d^n k$
- Is weak hyperbolicity enough? Answer: Jordan normal form $J(\hat{k}_i)$

  with $M(\hat{k}_i) = S(\hat{k}_i) \, J(\hat{k}_i) \, S^{-1}(\hat{k}_i)$

  $$\Rightarrow e^{iM(k_i)\hat{t}} = S(\hat{k}_i) \, e^{iJ(\hat{k}_i)\hat{t}} \, S^{-1}(\hat{k}_i)$$

- Suppose M is not diagonalizable $\Rightarrow J$ contains $n \times n$ Jordan block.

  Say, we have a 2×2 block associated with Eigenvalue λ

  So $J_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \Rightarrow e^{iJ_2\hat{t}} = e^{i\lambda\hat{t}} \begin{pmatrix} 1 & i\hat{t} \\ 0 & 1 \end{pmatrix}$

- This violates $||e^{iM(\hat{k}_i)\hat{t}}|| \leq f(0)$

Well posedness

- In general $n \times n$ Jordan blocks give terms $\hat{t}^p\,, \quad 1 \leq p \leq n$
- Using $\hat{t} = |k|\,t$ we thus get terms $\propto |k|^p$ and no bound

  $||u(t,.)|| \leq f(t)\,||u(0,.)||$ exists.

  So we need $M(k_i)$ to be diagnolizable.

Def.: Our PDE is strongly hyperbolic if for any k_i with $k_i k^i = 1$ all Eigenvalues of $M(k_i)$ are real and $M(k_i)$ is diagonalizable.

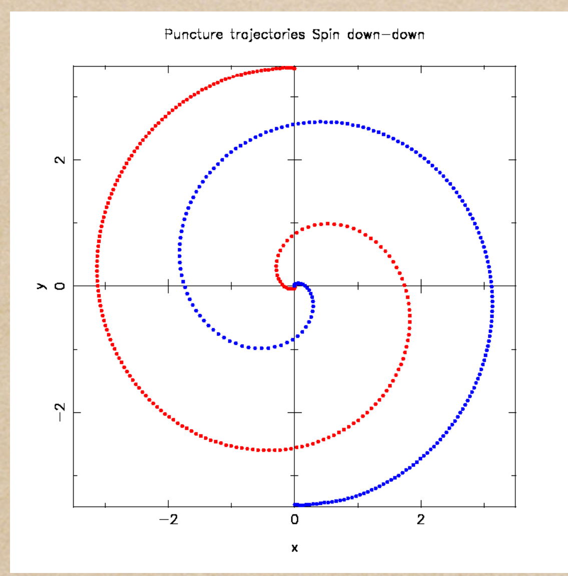
- The ADM equations are, in general, only weakly hyperbolic, but not strongly hyperbolic.

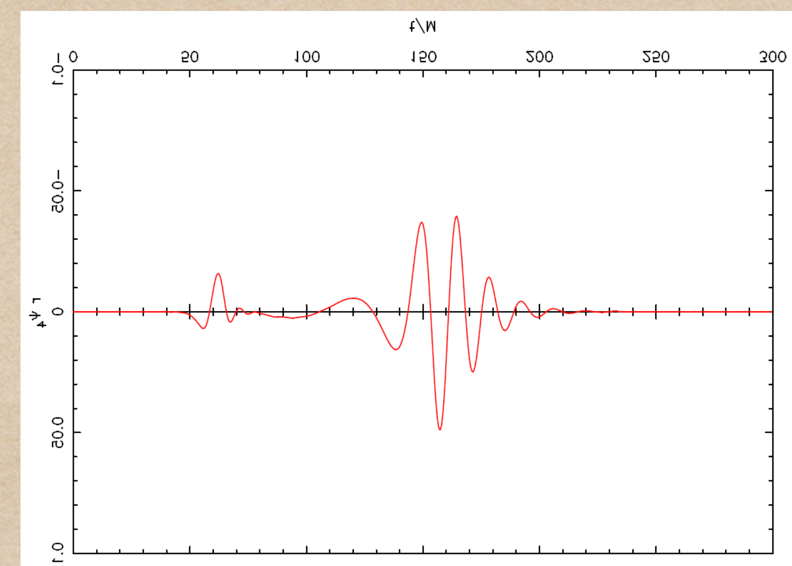
B: More results on compact objects

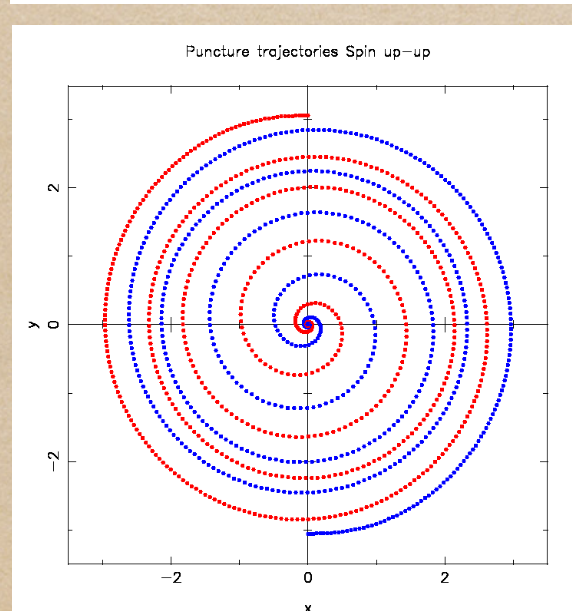
Spinning BHs: The orbital hangup

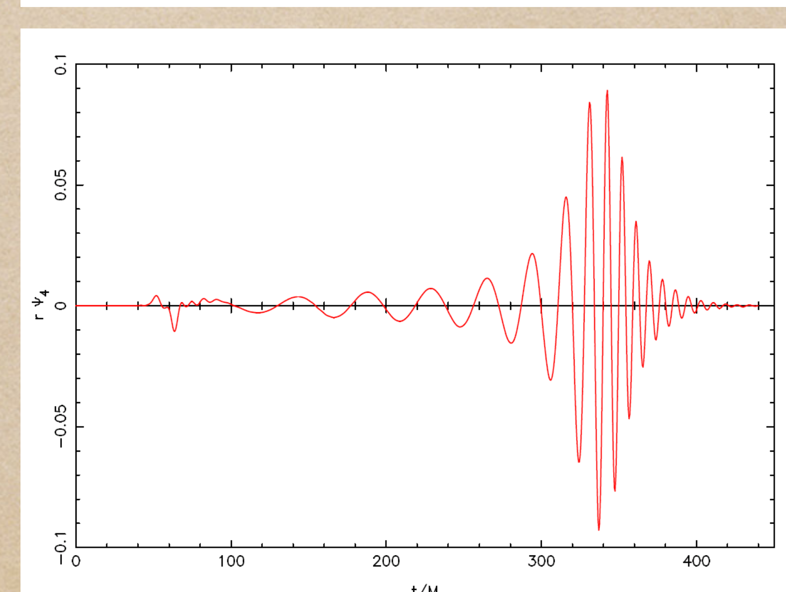
$\uparrow\Big\uparrow\uparrow$ Spins parallel to $\boldsymbol{L}$ $\Rightarrow$ more orbits, $E_{\rm rad}$, $J_{\rm rad}$ larger

$\downarrow\Big\uparrow\downarrow$ Spins anti-par. to $\boldsymbol{L}$ $\Rightarrow$ fewer orbits, $E_{\rm rad}$, $J_{\rm rad}$ smaller









Campanelli et al PRD gr-qc/0604012

Template construction

- Phenomenological waveform models
  - Model phase, amplitude with simple funcs. $\rightarrow$ Model parameters
  - Create map between physical and model parameters

    Ajith et al CQG 0704.3764, PRD 0710.2335, PRL 0909.2867;

    Khan et al PRD 1508.07253; London et al. 1708.00404
- Effective-one-body (EOB) models
  - Particle in effective metric, PN, ringdown model

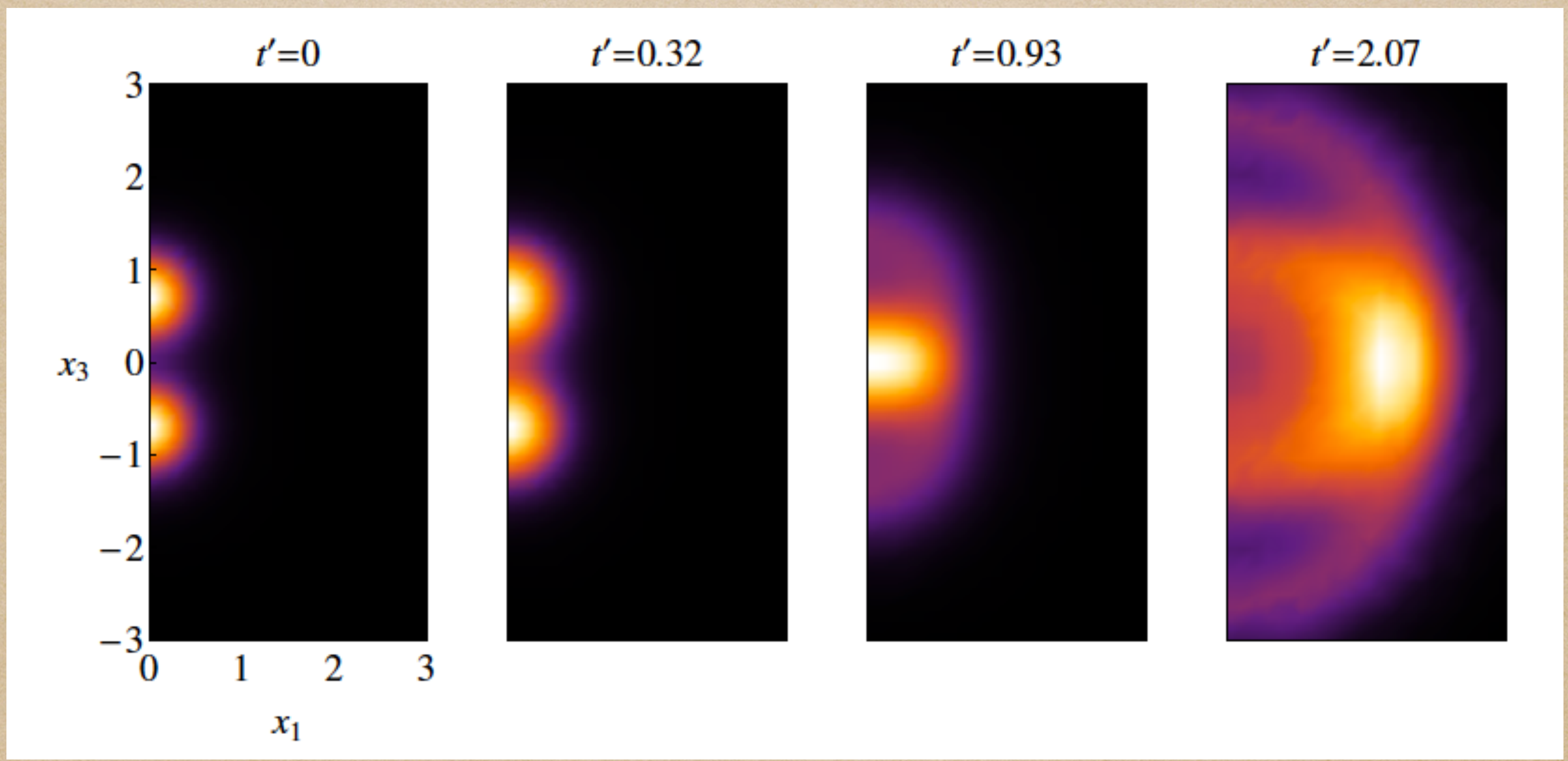
    Buonanno & Damour PRD gr-qc/9811091, PRD gr-qc/0001013
  - Resum PN, calibrate pseudo-PN parameters using NR; see e.g.:

    Pan+ PRD 1106.1021, PRD 1307.6232; Tarachini+ PRD 1311.2544;

    Damour & Nagar PRD 1406.6913; Bohe et al. 1611.03703
- Reduced order methods; e.g. Lackey et al 1610.04742, 1812.08643

Cauchy evolutions in 4+1 asympt. AdS

- First BBH collision in asymptotically AdS
- Qualitative picture: similar to shock wave collisions
- Future goals: Relax symmetry, use BBHs with boost



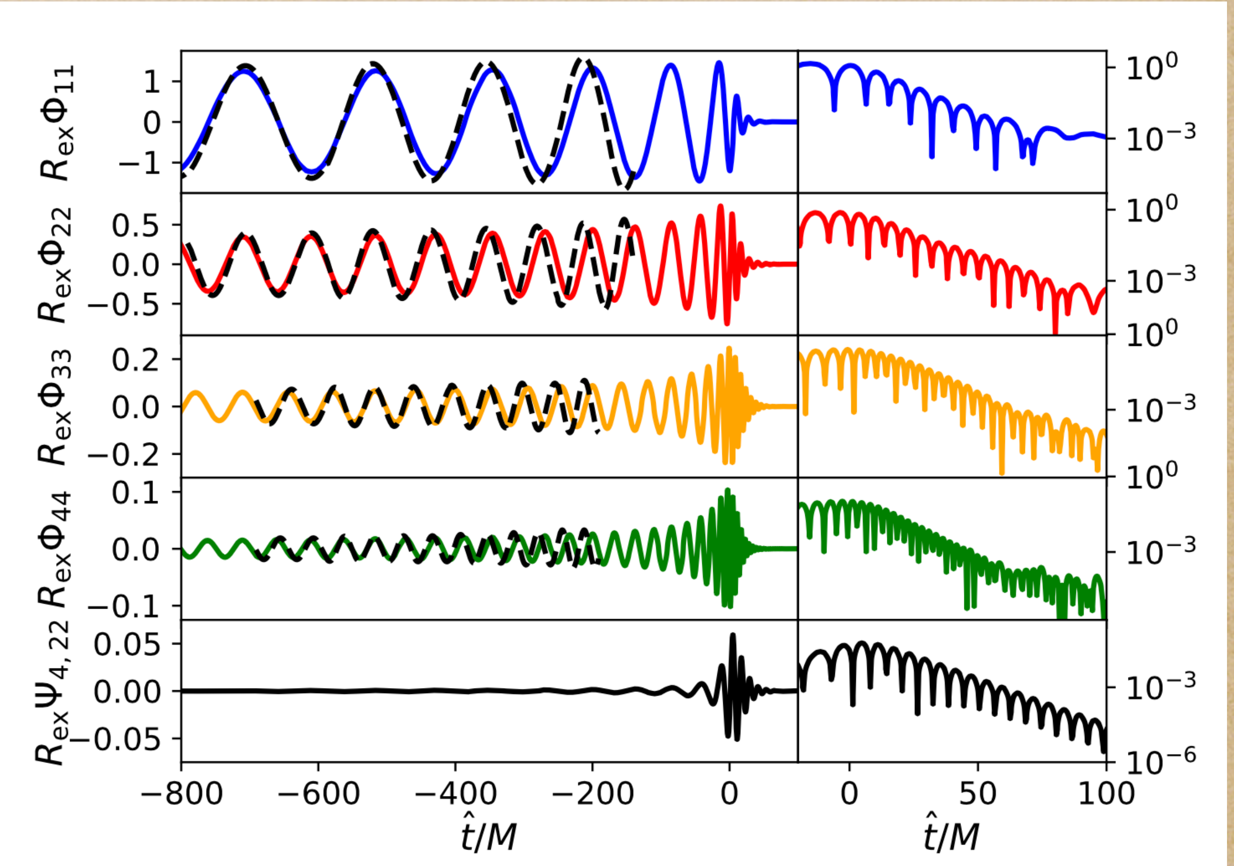
Bantilan et al PRD 1201.2132, PRL 1410.4799

BH binaries in modified theories of gravity

- Well posedness unclear $\rightarrow$ Use effective field theory approach
- Perturbative expansion of the theory about GR
- BH binary in scalar Gauss-Bonnet gravity

Witek et al PRD 1810.05177

Similar in dynamic



(a) $q = 1/2$

- Chern-Simons theory

Okounkova et al PRD 1705.0792

Major open challenges

The future...

- Waveform catalogs for binaries containing BHs, NSs
  - Precessing BHBs, high-mass ratios
  - NSNS, BHNS systems

- Waveform predictions for compact objects in modified gravity
- Model GW signatures of dark matter candidates
- Exotic compact objects
- NR in cosmology
- Applications in AdS/CFT
- Critical collapse in >1 dimensions
- Higher dimensional GR; is D=4 special?