

A Brief History of Numerical Relativity

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X Amazonian Workshop on Gravity and Analogue Models
Federal University of Pará, 3 Jul 2026



Overview

- Introduction
- The Ancient World: The Birth of Numerical Relativity
- From the Dark ages to the Renaissance
- Towards the Holy Grail
- The gold rush years
- What's different in cosmological NR?
- NR results in cosmology
Early Universe, post-BBN, Late Universe

1. Introduction, Motivation

Task: Solve this!



It's simple but it isn't easy...

How do we get the metric?

- The metric must obey $R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}$
- Ricci tensor, Einstein tensor, matter tensor

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$$

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R^{\mu}{}_{\mu}$$

“Trace reverse Ricci”

$$T_{\alpha\beta}$$

“Matter”

$$\Lambda$$

“Cosmological constant”

- Solutions: Easy! Take metric $g_{\alpha\beta}$
- Calculate $\Rightarrow G_{\alpha\beta}$
- Use that for $\Rightarrow T_{\alpha\beta}$
- Physically meaningful solutions: That’s the hard part!

Solving Einstein's Eqs.: The toolbox

- **Analytic solutions**

- Symmetry assumptions

Schwarzschild, Kerr, FLRW, Vaidya, Tangherlini, Myers-Perry, ...

- **Perturbation theory**

- Assume solution is close to a known "background" $g_{\alpha\beta}^{(0)}$

- Expand $g_{\alpha\beta} = g_{\alpha\beta}^{(0)} + \epsilon h_{\alpha\beta}^{(1)} + \epsilon^2 h_{\alpha\beta}^{(2)} + \dots \Rightarrow$ linear system

Regge-Wheeler-Zerilli-Moncrief, Teukolsky, QNMs, EOB, ...

- **Post-Newtonian theory**

- Assume small velocities $\Rightarrow$ Expansion in $\frac{v}{c}$

- N^{th} order expressions for GWs, momenta, orbits, ...

Blanchet, Buonanno, Damour, Kidder, Schäfer, Will, ...

- **Numerical Relativity**

The Newtonian 2-body problem

- Eqs. of motion

$$m_1 \frac{d^2 \vec{r}_1}{dt^2} = \vec{F} = -G \frac{m_1 m_2}{r^2} \hat{r} = -m_2 \frac{d^2 \vec{r}_2}{dt^2}$$

- Solution: Kepler ellipses, parabolic, hyperbolic

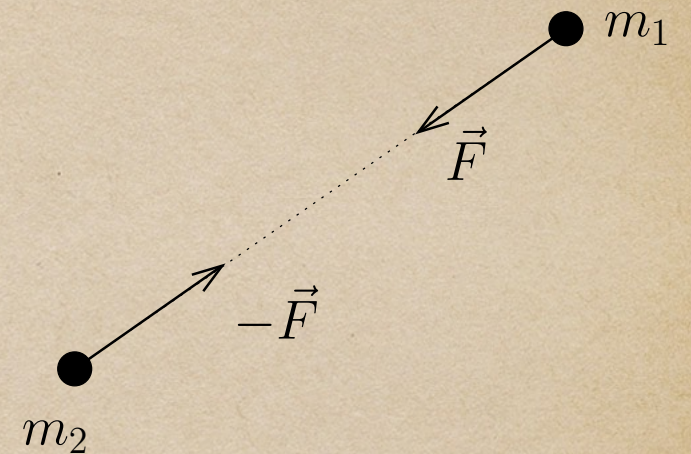
$$r = \frac{r_0}{1 + \epsilon \cos \theta}$$

- What is the equivalent in GR?

- No point particles in GR $\rightarrow$ Black holes!
- Systems typically are dissipative $\rightarrow$ Gravitational waves

- The Holy Grail of numerical relativity: Inspiral of BH binary

- History: e.g. US CQG 1411.3997



Challenges in GR

- Covariance of the Einstein equations
 - Space and time on equal footing: How to evolve? Time?
 - Well posedness? Suitability for numerical methods?
- Meaning of the solutions; cf. Schwarzschild solution or GWs
 - Gauge invariant diagnostics
 - Definition of observables
- No a-priori spacetime “stage”. Coordinates are evolved.
- Singularities
- Computational costs: 3D effect
- Numerical stability

**The Ancient World:
The Birth of Numerical Relativity**

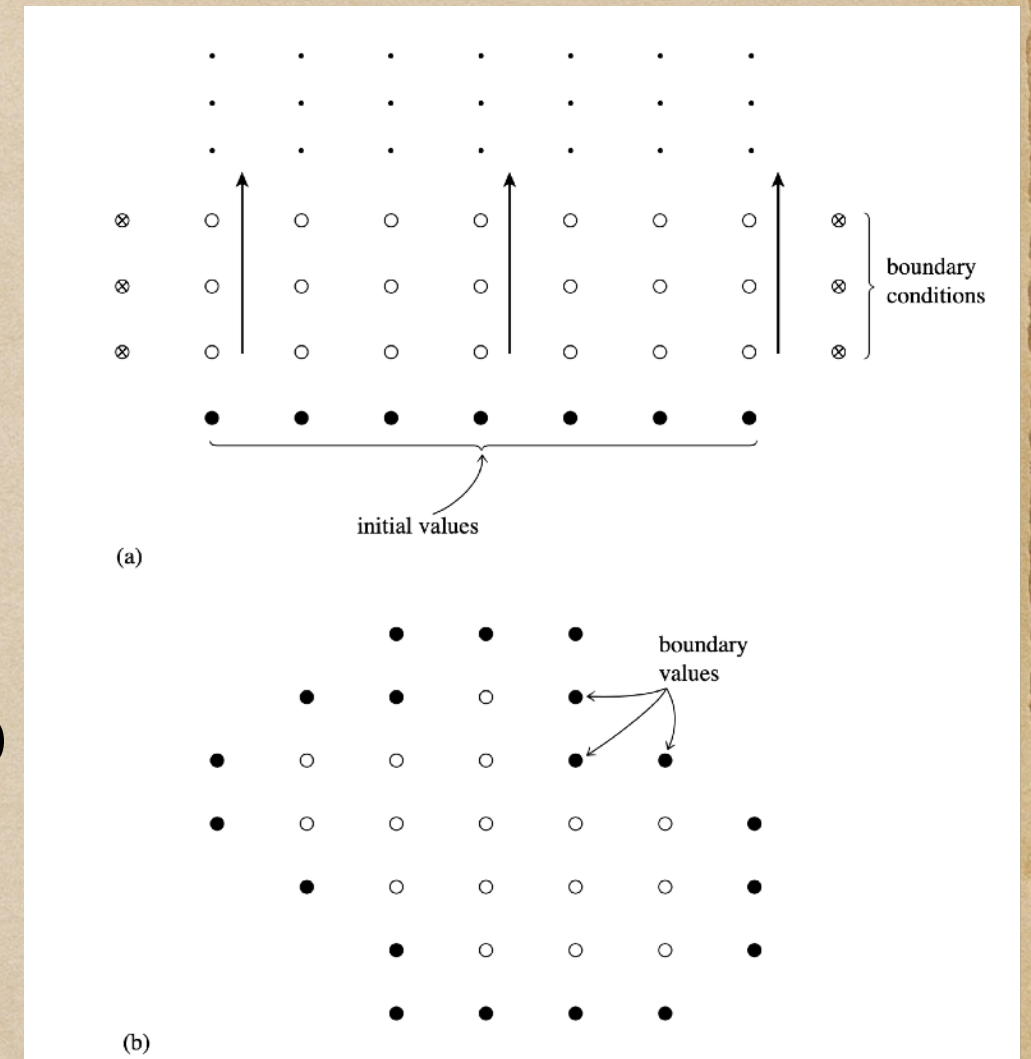
Initial-Value vs Boundary-Value Problems

- 2nd-order linear PDE for $f: \mathbb{R}^N \rightarrow \mathbb{R}$

$$A_{mn}(x_i)\partial_m\partial_nf + b_m(x_i)\partial_m f + c(x_i) + d(x_i) = 0$$

Consider EWs λ_i of A_{mn}

- Elliptic: all λ_i same sign
- Parabolic: $\lambda_1, \dots, \lambda_{N-1}$ same sign, $\lambda_N = 0$
- Hyperbolic: $\lambda_1, \dots, \lambda_{N-1}$ same sign, λ_N opposite sign



Numerical Recipes

- Hyperbolic, Parabolic $\rightarrow$ Initial-Value Problem, "time" evolution
- Elliptic $\rightarrow$ Boundary-Value Problem

The structure of the Einstein Equations

- Consider foliation of a (globally hyperbolic) spacetime:

$$\|\mathbf{d}t\|^2 < 0$$

t labels hypersurface Σ_t

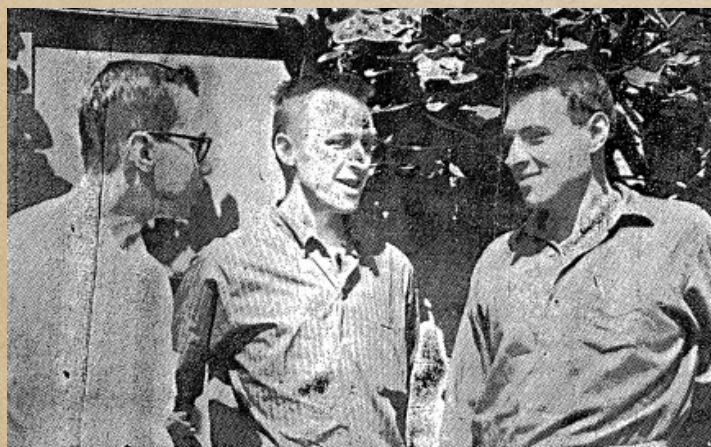
x^i labels points inside Σ_t



- Einstein Equations in vacuum: $R_{\alpha\beta} = 0$
 - No terms $\partial_0^2 g_{0\alpha}$ $\rightarrow$ Gauge
 - 10 equations for 6 unknowns $\partial_0^2 g_{ij}$ $\rightarrow$ Constraints
- Constraints preserved under evolution due to Bianchi Identities
- Summary: 6 Hyperbolic, 4 Elliptic Equations

Foundations and the first steps

- The Cauchy problem of the Einstein equations locally has a unique solution Choquet-Bruhat Acta Math. 1952
- Characteristic formulation Bondi, Sachs Proc.Roy.Soc. 1962
- Canonical 3+1 or ADM formulation of the Einstein equations
Arnowitt, Deser, Misner (1962) gr-qc/0405109
- First numerical relativity simulations: $\lesssim 100$ time steps
Hahn & Lindquist Ann.Phys. 1964
- 1D Gravitational collapse May & White PR 1966



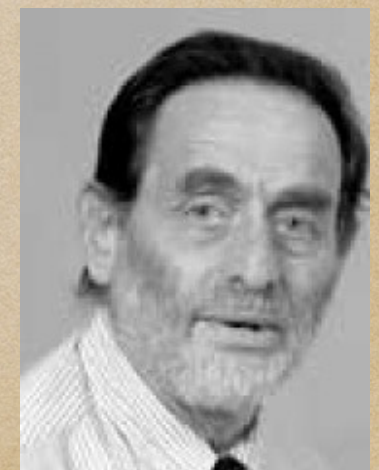
AMD



Y Choquet-Bruhat



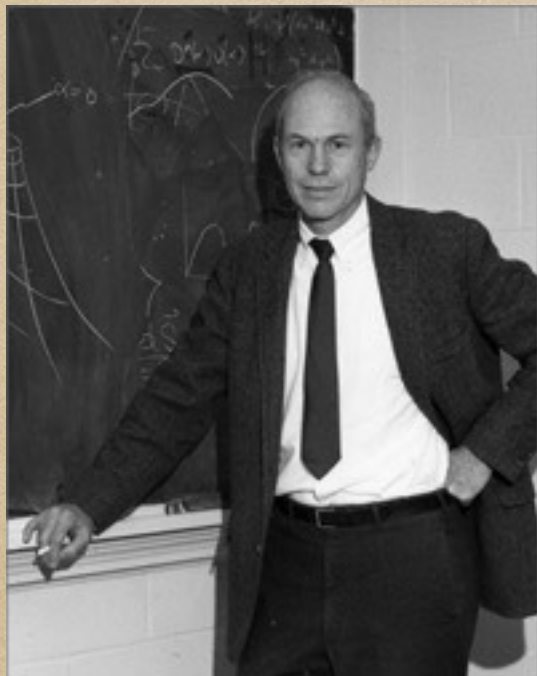
H Bondi



R K Sachs

The 1970s

- Reinvestigation initiated by B DeWitt
 - PhD theses by A Cavez (1971), L Smarr (1975), K R Eppley (1975)
- 300 x Flops relative to Hahn & Lindquist
- ADM equations, 2D code, Misner (1960) initial data
 - single BHs, head-on collisions



B DeWitt



L Smarr



J W York Jr.

3+1 decomposition of the metric

- In adapted coordinates, we write the spacetime metric

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^2 + \beta_m \beta^m & \beta_j \\ \hline \beta_i & \gamma_{ij} \end{array} \right)$$

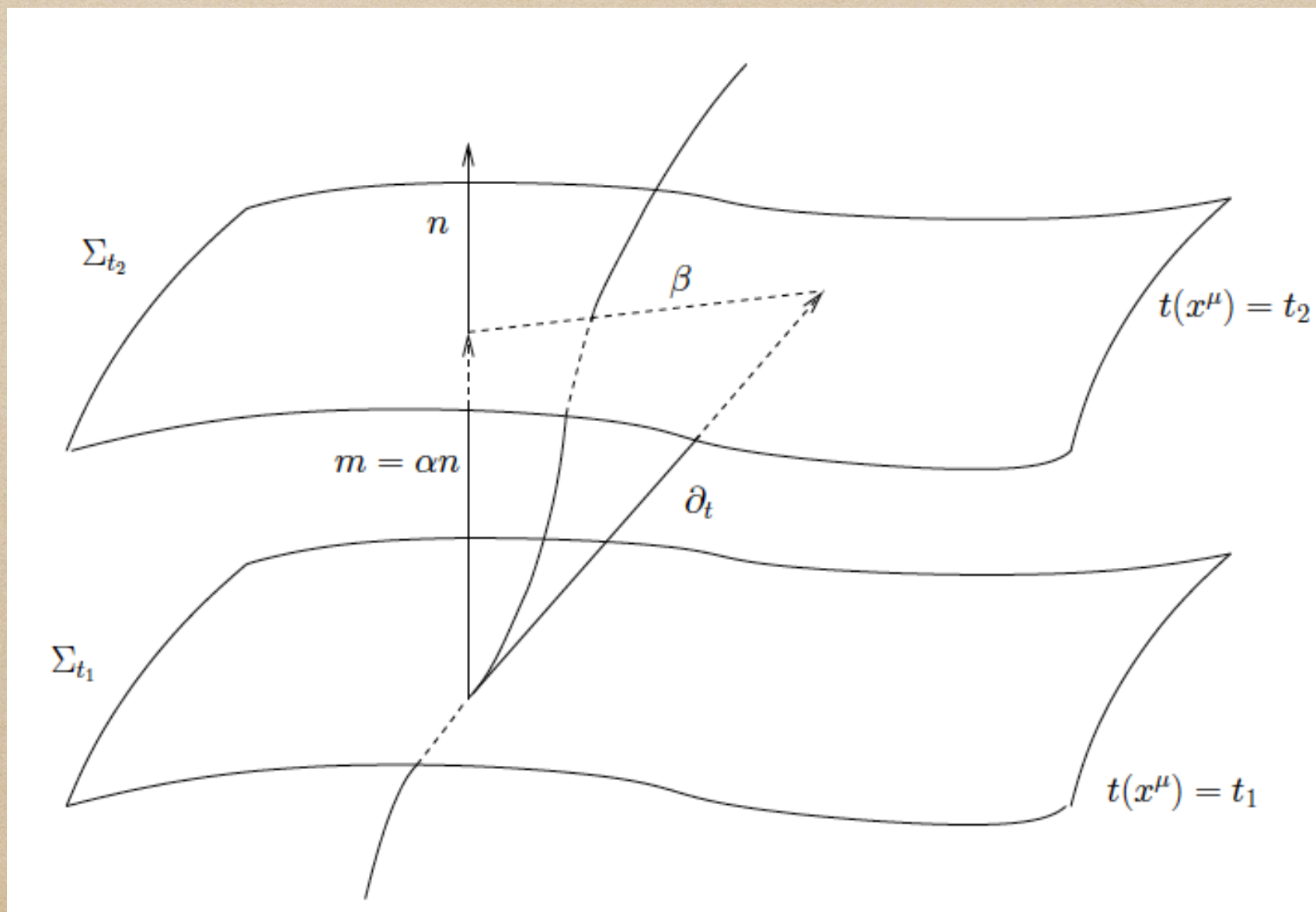
$$\Leftrightarrow g^{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^{-2} & \alpha^{-2} \beta^j \\ \hline \alpha^{-2} \beta^i & \gamma^{ij} - \alpha^{-2} \beta^i \beta^j \end{array} \right)$$

$$\Leftrightarrow ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- Gauge variables: Lapse α , shift β^i
- Details e.g. in Gourgoulhon gr-qc/0703035 Σ_t

The 3+1 decomposition

- 1-Form: $\mathbf{dt}$; vector: $\frac{\partial}{\partial t} =: \partial_t \Rightarrow \langle \mathbf{dt}, \partial_t \rangle = 1$
- Timelike normal $n^\mu = \frac{1}{\alpha} (1, -\beta^m) \Leftrightarrow \partial_t = \alpha \mathbf{n} + \boldsymbol{\beta}$



The ADM version of the Einstein eqs.

- Introduction of the extrinsic curvature: $m := \alpha n$,

$$\mathcal{L}_m \gamma_{ij} = -2\alpha K_{ij}$$

- Spatial Projections of the Einstein Equations

$$\mathcal{L}_m K_{ij} = -D_i D_j \alpha + \alpha(\mathcal{R}_{ij} + K K_{ij} - 2K_{im} K^m_j) + 8\pi\alpha \left(\frac{S - \rho}{D - 2} \gamma_{ij} - S_{ij} \right) - \frac{2}{D - 2} \Lambda \gamma_{ij}$$

“Evolution equations”

- Timelike Projection of the Einstein Equations

$$\mathcal{R} + K^2 - K^{mn} K_{mn} = 2\Lambda + 16\pi\rho$$

“Hamiltonian constraint”

- Mixed Space-Time projection

$$D_i K - D_m K^m_i = -8\pi j_i$$

“Momentum constraints”

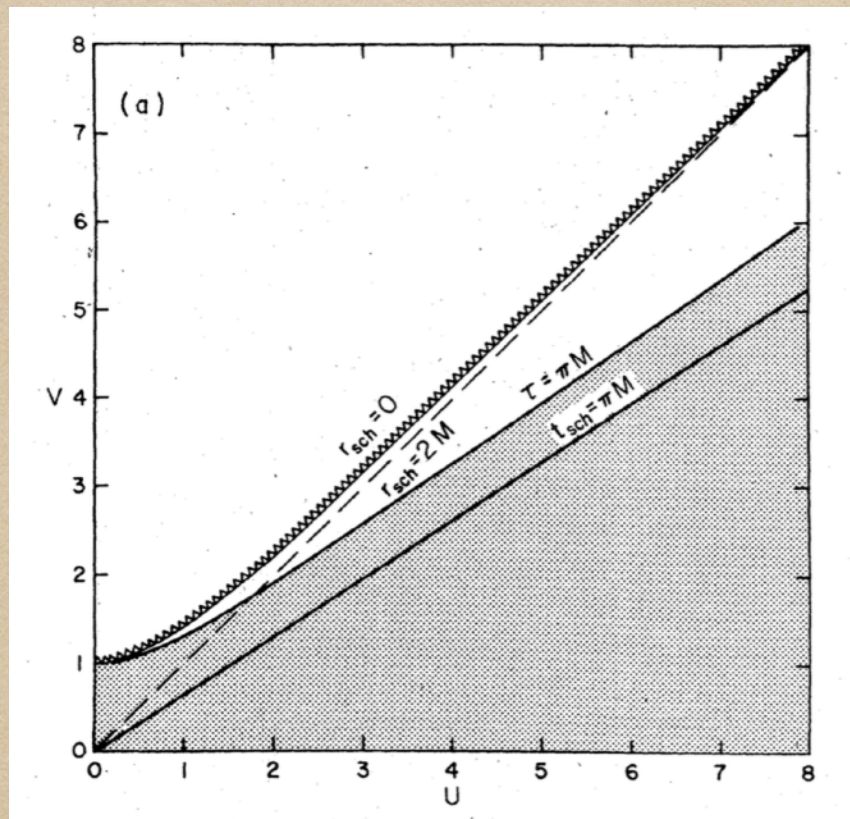
- Energy-Momentum Tensor: $T_{\mu\nu} = \rho n_\mu n_\nu + 2j_{(\mu} n_{\nu)} + S_{\mu\nu}$

- Backbone of numerical relativity for ~ 20 years

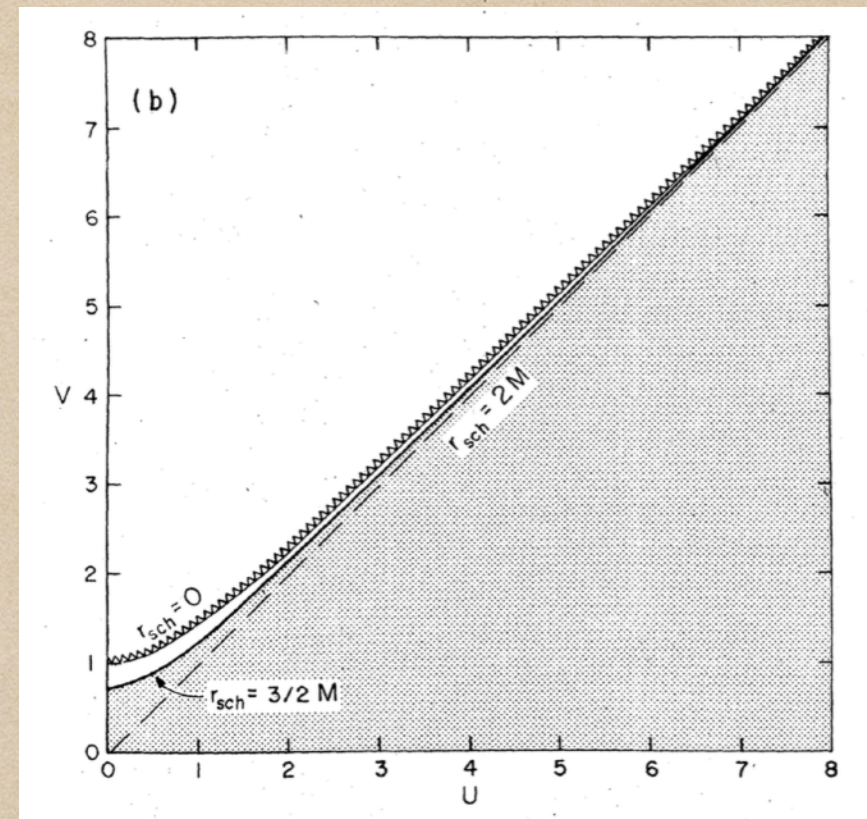
The 1970s

- Singularity avoiding slicing Smarr & York PRD 1978

Schwarzschild-Kruskal evolved with



geodesic slicing



maximal slicing

- York formulation of the 3+1 equations

York 1979 in "Sources of Gravitational Radiation" Ed. L Smarr

- Black-hole initial data Bowen & York 1980 (see below)

2. From the dark ages to the Renaissance

The 1980s



The Dark Ages in Numerical Relativity...



The 1990s

"Binary Black Hole Grand Challenge Project"

- 40 researchers in 10 institutions

Austin, Cornell, Illinois, North Carolina, Northwestern, Penn State, Pittsburgh, Syracuse

- Goal: Simulate BH binary inspiral, compute GW signals

- ADM formalism, axisymmetry, head-on collisions

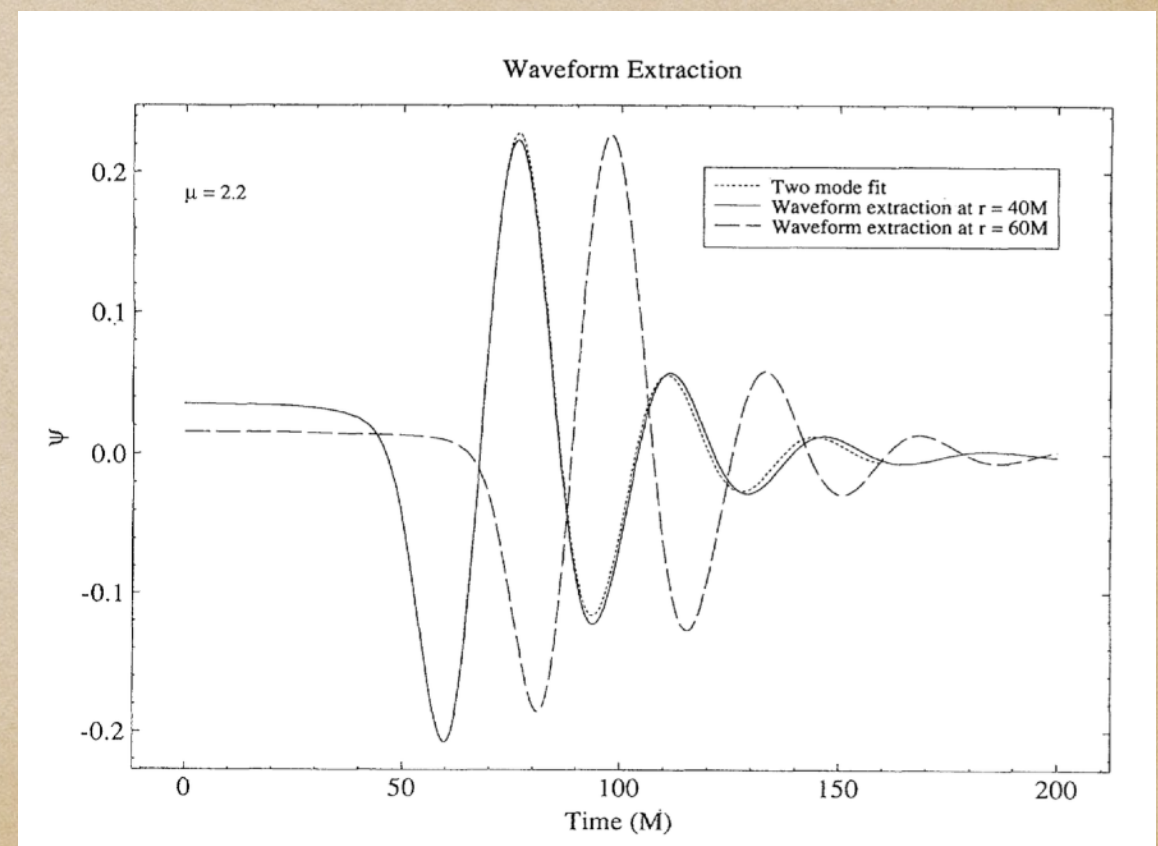
- $l = 2, l = 4$ waveforms

- Horizon calculations

- Unequal masses

Anninos et al

PRL 1993, PRD 1995+



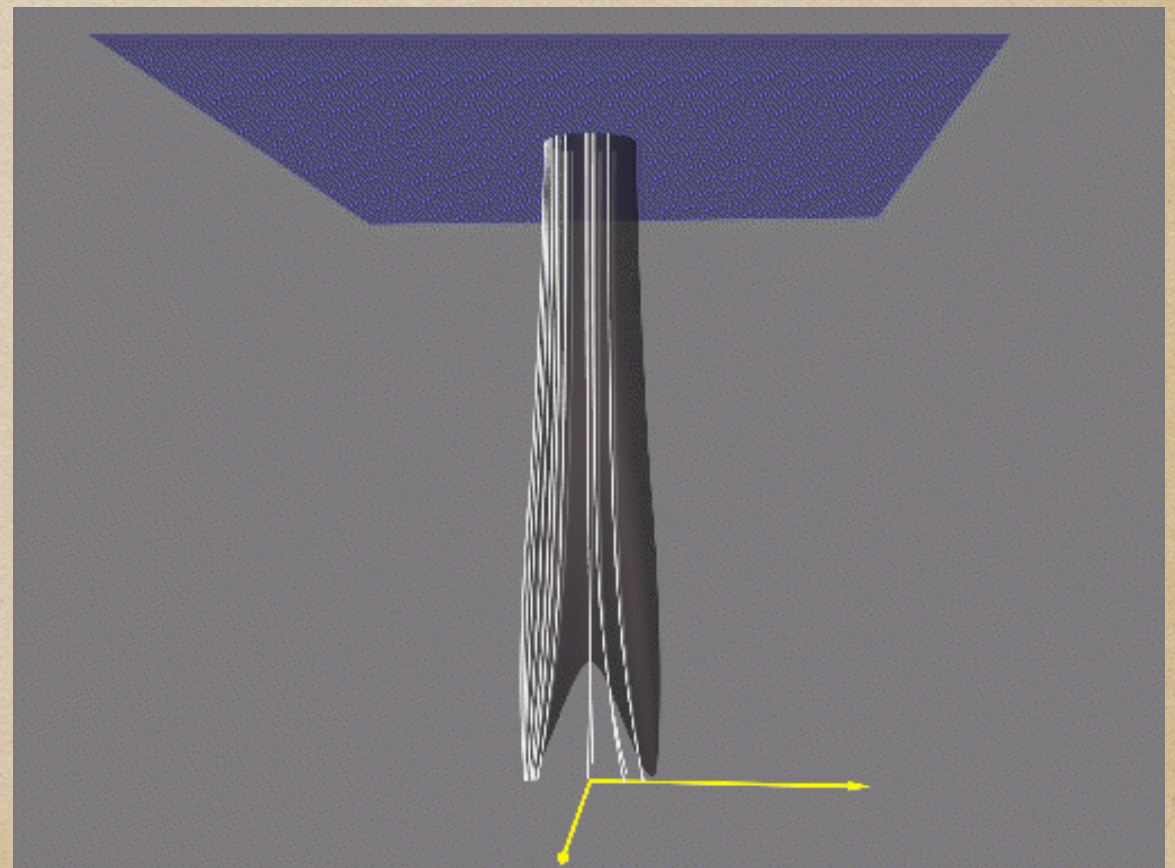
The 1990s

“Binary Black Hole Grand Challenge Project” (continued)

- First 3+1 dimensional BH simulations “G-code”
 - Single Schwarzschild BH stable up to $t \approx 50 M$
 - Single moving BH stable up to $t \approx 60 M$
 - Comparisons with axisymmetric simulations

Anninos et al PRD 1995

- Event horizon
“pair of pants”
- Problem: long-term stability!
→ Well-posedness studies
- Community toolkits: Cactus



Well posedness

From G Papallo, PhD thesis Cambridge, 2018

- Consider linear const.coeff. PDE system $A\partial_t u + P^i \partial_i u + Cu = 0$

Fourier trafo $\tilde{u}(t, k) = \frac{1}{\sqrt{2\pi}^n} \int u(t, x) e^{-ik_i x^i} d^n x$

$$\Rightarrow \partial_t \tilde{u} - i\mathcal{M} \tilde{u} = 0, \quad \mathcal{M}(k) = A^{-1}(-P^i k_i + iC)$$

Solution $\tilde{u}(t, k) = e^{i\mathcal{M}(k)t} \tilde{u}(0, k)$

$$\Rightarrow u(t, x) = \frac{1}{\sqrt{2\pi}^n} \int e^{ik_i x^i} e^{i\mathcal{M}t} \tilde{u}(0, k) d^n k$$

- May not converge if integrand fails to decay fast with $k = \sqrt{k_i k^i}$
- ADM equations are only weakly hyperbolic
- Strong or symmetric hyperbolicity with new formulations:

BSSN, CCZ4, GHG

The post-Grand Challenge era

- Many smaller groups explored the problem independently
- Critical collapse: Collapse of spherically symmetric scalar field.
BH formation or dispersal; Mesh refinement! Choptuik PRL 1993

- 1st Mesh refinement for 3+1 BHs

Brügmann PRD 1996

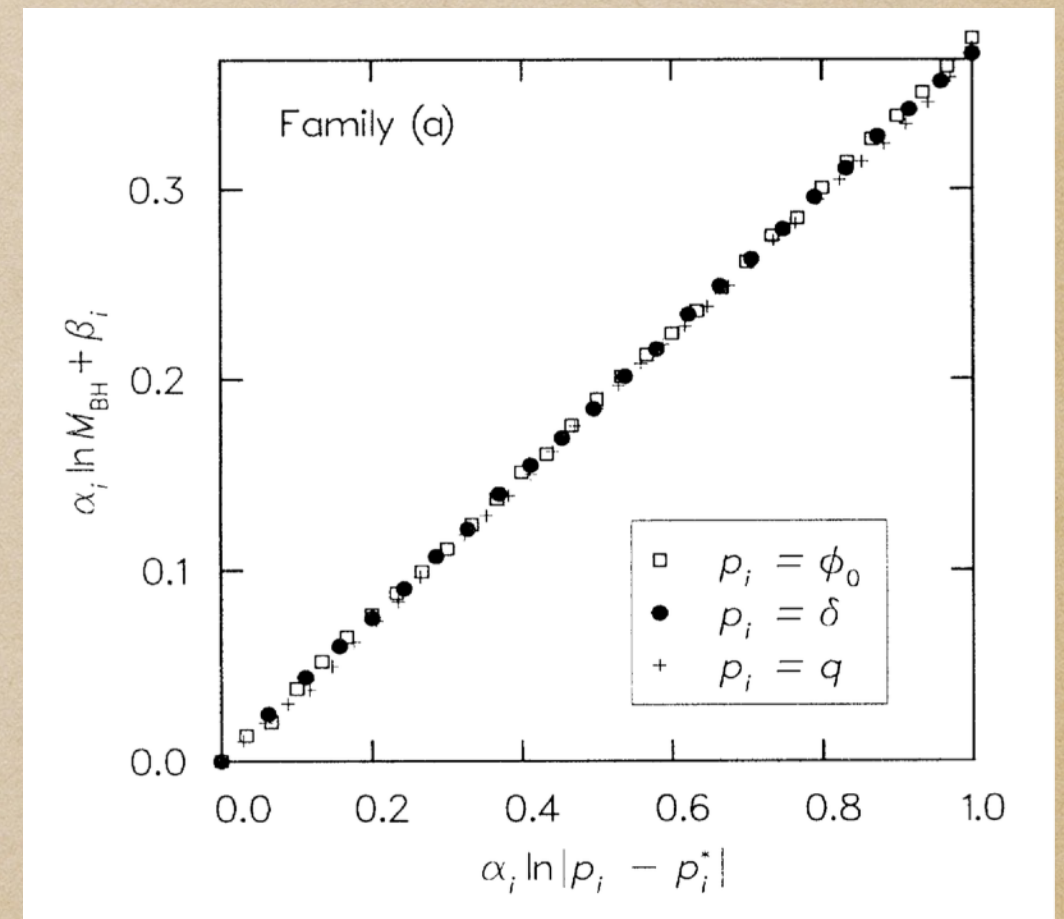
- 1st long-term stable BH sim.
(characteristic code)

Gómez et al PRL 1998

- 1st BH Grazing collision

Brandt et al PRL 2000

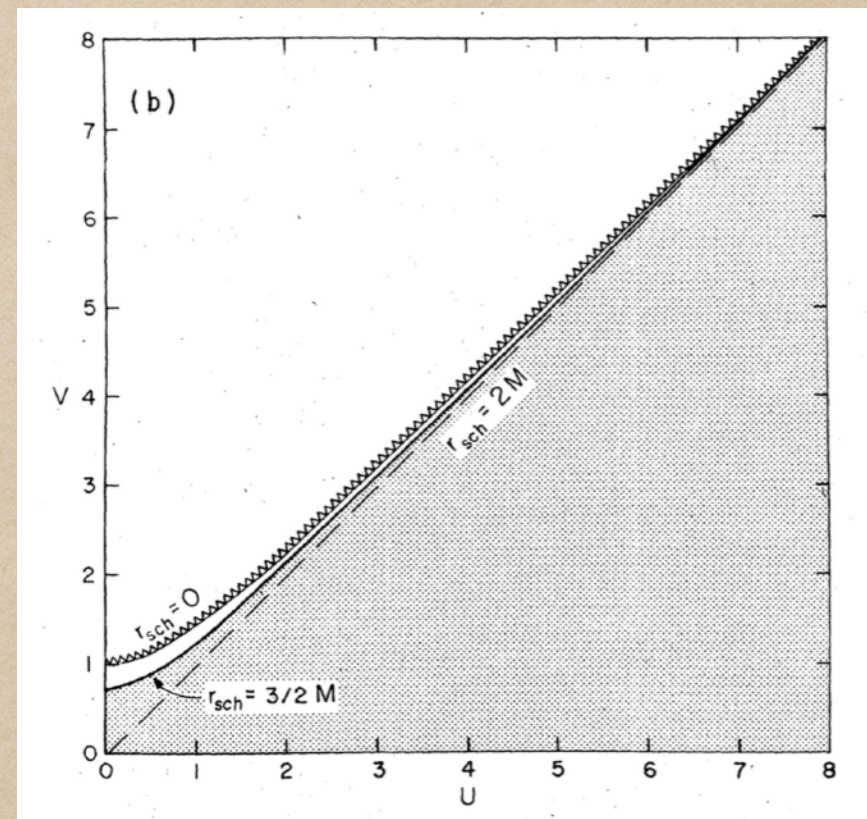
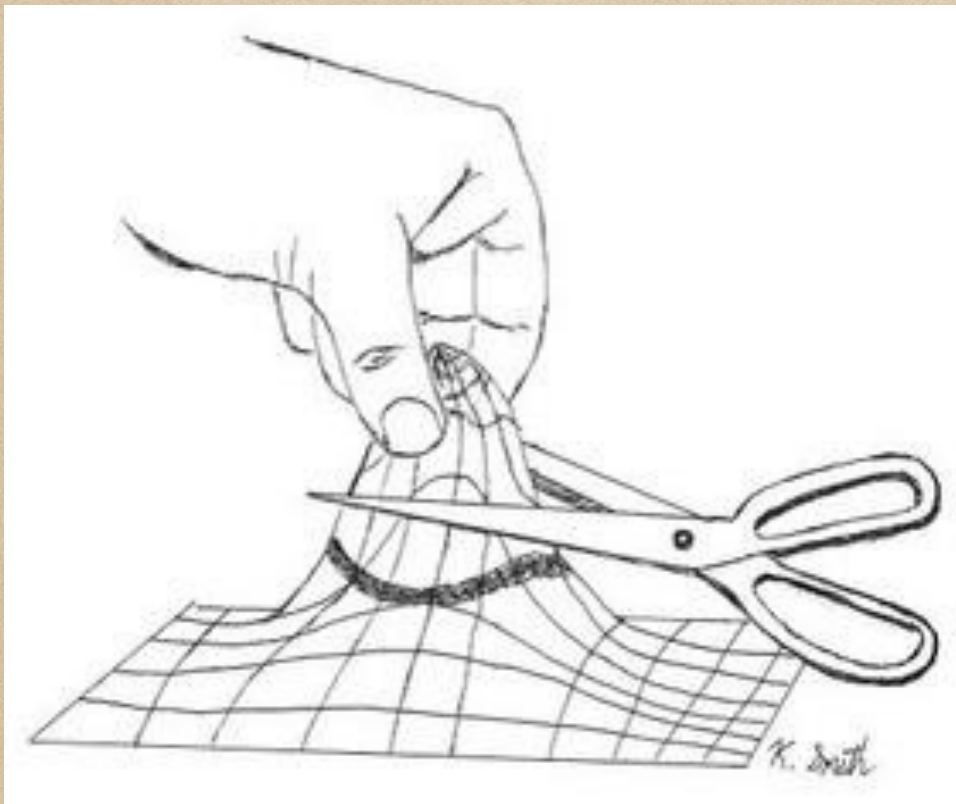
- Release of Cactus 1.0



Towards the Holy Grail

Remaining problems

- Formulations: Well-posedness required
- Gauge conditions: Avoid coordinate singularities
- Physical singularities: Excision or good slicing



The generalized harmonic gauge (GHG)

- Harmonic gauge: Choose coordinates such that

$$\square x^\alpha = \nabla^\mu \nabla_\mu x^\alpha = -g^{\mu\nu} \Gamma_{\mu\nu}^\alpha = 0$$

- 4 dimensional Einstein eqs. in harmonic gauge:

$$R_{\alpha\beta} = -\frac{1}{2}g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} + \dots$$

principle part of wave equation $\Rightarrow$ Manifestly hyperbolic!

- Problem: Start with a hyper surface $t = \text{const}$

Does t remain timelike?

- Goal: Generalize the harmonic gauge

Garfinkle PRD gr-qc/0110013; Pretorius CQG gr-qc/0407110;

Lindblom et al CQG gr-qc/0512093

$\rightarrow$ Source function $H^\alpha = \nabla^\mu \nabla_\mu x^\alpha = -g^{\mu\nu} \Gamma_{\mu\nu}^\alpha$

The generalized harmonic equations

- Any spacetime in any coordinates can be formulated in GH form!

Problem: find the corresponding H^α

- Promote the H^α to evolution variables

- Einstein equations in GH form:

$$\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = -\partial_\nu g_{\mu(\alpha}\partial_{\beta)}g^{\mu\nu} - \partial_{(\alpha}H_{\beta)} + H_\mu\Gamma_{\alpha\beta}^\mu - \Gamma_{\nu\alpha}^\mu\Gamma_{\mu\beta}^\nu - \frac{2}{3}\Lambda g_{\alpha\beta} - 8\pi\left(T_{\mu\nu} - \frac{1}{2}Tg_{\alpha\beta}\right).$$

with constraints

$$\mathcal{C}^\alpha = H^\alpha - \square x^\alpha = 0$$

- Still has principle part of the wave equation!!! Manifestly hyperbolic
Friedrich Comm.Math.Phys. 1985; Garfinkle gr-qc/0110013;
Pretorius gr-qc/0407110

The BSSNOK system

- Goal: Modify ADM eqs. to get a strongly hyperbolic system
Nakamura et al PTPS 1987, Shibata & Nakamura PRD 1995,
Baumgarte & Shapiro gr-qc/9810065
- Use (i) conformal decomposition, (ii) trace split, (iii) aux. variables

$$\begin{aligned}\gamma &:= \det \gamma_{ij}, & \chi &:= \gamma^{-1/3}, & K &= \gamma^{mn} K_{mn}, \\ \tilde{\gamma}_{ij} &:= \chi \gamma_{ij} & \Leftrightarrow & \tilde{\gamma}^{ij} = \frac{1}{\chi} \gamma^{ij}, \\ \tilde{A}_{ij} &:= \chi \left(K_{ij} - \frac{1}{3} \gamma_{ij} K \right) & \Leftrightarrow & K_{ij} = \frac{1}{\chi} \left(\tilde{A}_{ij} + \frac{1}{3} \tilde{\gamma}_{ij} K \right), \\ \tilde{\Gamma}^i &:= \tilde{\gamma}^{mn} \tilde{\Gamma}_{mn}^i.\end{aligned}$$

- Auxiliary constraints

$$\tilde{\gamma} = 1, \quad \tilde{\gamma}^{mn} \tilde{A}_{mn} = 0, \quad \mathcal{G}^i \equiv \tilde{\Gamma}^i - \tilde{\gamma}^{mn} \tilde{\Gamma}_{mn}^i = 0.$$

The BSSN equations

$$\mathcal{H} := \mathcal{R} + \frac{2}{3}K^2 - \tilde{A}^{mn}\tilde{A}_{mn} - 16\pi\rho - 2\Lambda = 0,$$

$$\mathcal{M}_i := \tilde{\gamma}^{mn}\tilde{D}_m\tilde{A}_{ni} - \frac{2}{3}\partial_i K - \frac{3}{2}\tilde{A}^m{}_i\frac{\partial_m\chi}{\chi} - 8\pi j_i = 0,$$

$$\partial_t\chi = \beta^m\partial_m\chi + \frac{2}{3}\chi(\alpha K - \partial_m\beta^m),$$

$$\partial_t\tilde{\gamma}_{ij} = \beta^m\partial_m\tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{\gamma}_{ij}\partial_m\beta^m - 2\alpha\tilde{A}_{ij},$$

$$\partial_t K = \beta^m\partial_m K - \chi\tilde{\gamma}^{mn}D_m D_n\alpha + \alpha\tilde{A}^{mn}\tilde{A}_{mn} + \frac{1}{3}\alpha K^2 + 4\pi\alpha(S + \rho) - \alpha\Lambda,$$

$$\begin{aligned}\partial_t\tilde{A}_{ij} = & \beta^m\partial_m\tilde{A}_{ij} + 2\tilde{A}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{A}_{ij}\partial_m\beta^m + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{im}\tilde{A}^m{}_j \\ & + \chi(\alpha\mathcal{R}_{ij} - D_i D_j\alpha - 8\pi\alpha S_{ij})^{\text{TF}},\end{aligned}$$

$$\begin{aligned}\partial_t\tilde{\Gamma}^i = & \beta^m\partial_m\tilde{\Gamma}^i + \frac{2}{3}\tilde{\Gamma}^i\partial_m\beta^m - \tilde{\Gamma}^m\partial_m\beta^i + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + \frac{1}{3}\tilde{\gamma}^{im}\partial_m\partial_n\beta^n \\ & - \tilde{A}^{im}\left(3\alpha\frac{\partial_m\chi}{\chi} + 2\partial_m\alpha\right) + 2\alpha\tilde{\Gamma}^i{}_{mn}\tilde{A}^{mn} - \frac{4}{3}\alpha\tilde{\gamma}^{im}\partial_m K - 16\pi\frac{\alpha}{\chi}j^i - \sigma\mathcal{G}^i\partial_m\beta^m.\end{aligned}$$

- Note: there exist slight variations of the exact equations

The BSSN equations

- Auxiliary expressions we have used:

$$\Gamma_{jk}^i = \tilde{\Gamma}_{jk}^i - \frac{1}{2\chi} (\delta^i_k \partial_j \chi + \delta^i_j \partial_k \chi - \tilde{\gamma}_{jk} \tilde{\gamma}^{im} \partial_m \chi)$$

$$\mathcal{R}_{ij} = \tilde{R}_{ij} + \mathcal{R}_{ij}^\chi,$$

$$\mathcal{R}_{ij}^\chi = \frac{\tilde{\gamma}_{ij}}{2\chi} \left(\tilde{\gamma}^{mn} \tilde{D}_m \tilde{D}_n \chi - \frac{3}{2\chi} \tilde{\gamma}^{mn} \partial_m \chi \partial_n \chi \right) + \frac{1}{2\chi} \left(\tilde{D}_i \tilde{D}_j \chi - \frac{1}{2} \partial_i \chi \partial_j \chi \right),$$

$$\tilde{R}_{ij} = -\frac{1}{2} \tilde{\gamma}^{mn} \partial_m \partial_n \tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i} \partial_{j)} \tilde{\Gamma}^m + \tilde{\gamma}^m \tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn} \left[2\tilde{\Gamma}_{m(i}^k \tilde{\Gamma}_{j)kn} + \tilde{\Gamma}_{im}^k \tilde{\Gamma}_{kjn} \right],$$

$$D_i D_j \alpha = \tilde{D}_i \tilde{D}_j \alpha + \frac{1}{\chi} \partial_{(i} \chi \partial_{j)} \alpha - \frac{1}{2\chi} \tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \partial_m \partial_n \alpha.$$

Beyond BSSN

- BSSN has a zero speed mode in the constraint-subsystem;
May result in large constraint violations
- BSSN does not have systematic constraint damping
- This can be implemented by considering Generalized Einstein Eqs.
Bona et al. PRD gr-qc/0302083 "Z4" system
- Conformal version of Z4: Very like BSSN but has constraint damping
Alic et al. PRD 1106.2254, Hilditch et al. PRD 1212.2901
- Also allows for constraint preserving boundary conditions
Bona et al. CQG gr-qc/0411110, Ruiz et al. PRD 1010.0523

Intermission: Initial Data

Initial data: Analytic data

- Schwarzschild, Kerr, Tangherlini, Myers-Perry,...

e.g. Schwarzschild in isotropic coordinates

$$ds^2 = - \left(\frac{2r - M}{2r + M} \right)^2 dt^2 + \left(1 + \frac{M}{2r} \right)^4 [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)]$$

Time symmetric initial data with n BHs:

Brill & Lindquist PR 1963, Misner PR 1960

- Problem: Find initial data for dynamic systems

- Goals: 1) Solve constraints

2) Realistic snapshot of physical system

- This is mostly done using the ADM 3+1 split

Initial data and conformal decomposition

- Recall: we need to satisfy the constraints

$$\mathcal{R} + K^2 - K^{mn}K_{mn} = 2\Lambda + 16\pi\rho$$

$$D_i K - D_m K_i^m = -8\pi j_i$$

- Conformal metric $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$

Lichnerowicz J.Math.Pures Appl. 1944; York PRL 1971, PRL 1972

- Conformal traceless split of the extrinsic curvature

$$K_{ij} = A_{ij} + \frac{1}{3}K \gamma_{ij} ,$$

$$A^{ij} = \psi^{-10} \bar{A}^{ij} \Leftrightarrow A_{ij} = \psi^{-2} \bar{A}_{ij}$$



A Lichnerowicz

Bowen-York data

- By further splitting $\bar{A}_{ij}$ into a longitudinal and a transverse traceless part, the momentum constraints simplify substantially

Cook LRR gr-qc/0007085

- Further assume: Vacuum, $K = 0$, $\bar{\gamma}_{ij} = f_{ij}$, $\lim_{r \rightarrow \infty} \psi = 0$, where f_{ij} is the flat metric in arbitrary coords.

In words: Traceless E.Curv., conformal flatness, asymptotic flatness

- Then there exists an analytic solution to the momentum constraints

$$\bar{A}_{ij} = \frac{3}{2r^2} [P_i n_j + P_j n_i - (f_{ij} - n_i n_j) P^k n_k] + \frac{3}{r^3} (\epsilon_{kil} S^l n^k n_j + \epsilon_{kjl} S^l n^k n_i),$$

$P^k = \text{Momentum}$

$S^k = \text{Spin}$

where r is a coordinate radius and $n^i = \frac{x^i}{r}$

Bowen & York PRD 1980

Puncture data

Brandt & Brügmann PRL gr-qc/9703066

- The Hamiltonian constraint is then given by

$$\bar{\nabla}^2 \psi + \frac{1}{8} \psi^{-7} \bar{A}_{mn} \bar{A}^{mn} = 0$$

- Ansatz for conformal factor $\psi = \psi_{\text{BL}} + u$

where $\psi_{\text{BL}} = \sum_{a=1}^N \frac{m_a}{|\vec{r} - \vec{r}_a|}$ is the Brill-Lindquist conformal factor,
i.e. the solution for $\bar{A}_{ij} = 0$.

- There then exist unique $\mathcal{C}^2$ solutions u to the Hamiltonian constr.
- The Hamiltonian constraint in this form is particularly suitable for numerical solution.

E.g. Ansorg, Brügmann & Tichy gr-qc/0404056

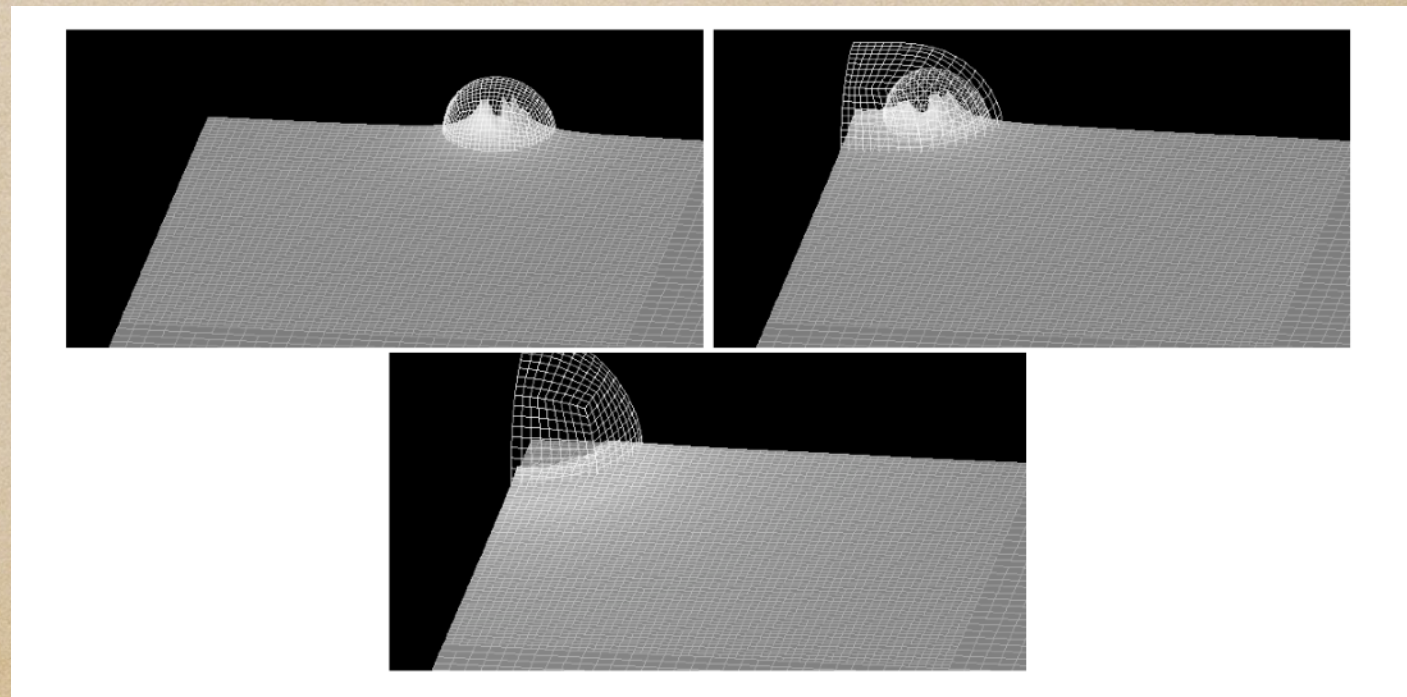
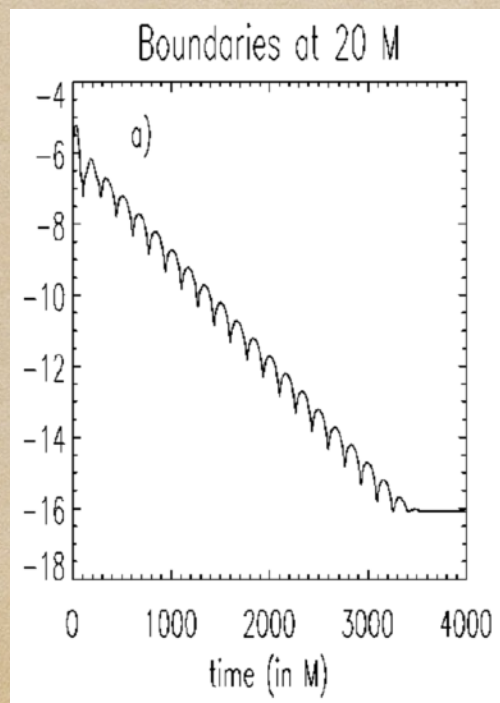
Beyond conformally flat initial data

- Problem: Conformally flat data limits spins to $S/M^2 \lesssim 0.928$
Dain et al. PRD gr-qc/0201062
- Similar problems arise for large linear momenta
- Solution: Non-conformally flat initial data
 - Superpose Kerr-Schild data Lovelace et al. PRD 0805.4192
Solve constraints with Conformal Thin Sandwich approach
York PRL 82 (1999) 1350
 - Superpose boosted conformal Kerr BHs; attenuation functions
Zlochower et al. PRD 1706.01980, Ruchlin et al. PRD 1410.8607
Evolve with CCZ4 (constraint damping variant of BSSN)
Alic et al. PRD 1106.2254, Hilditch et al. PRD 1212.2901

Back to Time Evolutions...

Progress accelerates: The early 2000s

- BSSN found empirically. Then strong hyperbolicity shown
Gundlach & Martín-García 2004
- 1st stable 3+1 evolution of Schwarzschild “simple excision”
Alcubierre & Brügmann gr-qc/0008067
- Stable head-on collisions
US et al gr-qc/0503071, Fiske et al gr-qc/0503100
- BH binary orbit Brügmann et al gr-qc/0312112



Missing pieces I: Constraint damping in GHG

- One can show: GHG constraints related to ADM constraints

$$\mathcal{C}^\alpha = 0, \quad \partial_t \mathcal{C}^\alpha = 0 \quad \text{at } t = 0 \quad \Rightarrow \quad \mathcal{H} = 0, \quad \mathcal{M}_i = 0$$

- Bianchi identities imply evolution of the $\mathcal{C}^\alpha$:

$$\square \mathcal{C}_\alpha = -\mathcal{C}^\mu \nabla_{(\mu} \mathcal{C}_{\alpha)} - \mathcal{C}^\mu \left[8\pi \left(T_{\mu\alpha} - \frac{1}{2} T g_{\mu\alpha} \right) + \Lambda g_{\mu\alpha} \right] .$$

- In practice: Numerical violations of $\mathcal{C}^\mu = 0 \Rightarrow$ unstable modes!

- Solution: Add constraint damping terms

$$\begin{aligned} \frac{1}{2} \partial_\mu \partial_\nu g_{\alpha\beta} = & -\partial_\nu g_{\mu(\alpha} \partial_{\beta)} g^{\mu\nu} - \partial_{(\alpha} H_{\beta)} + H_\mu \Gamma_{\alpha\beta}^\mu - \Gamma_{\nu\alpha}^\mu \Gamma_{\mu\beta}^\nu \\ & - \Lambda g_{\alpha\beta} - 8\pi \left(T_{\mu\nu} - \frac{1}{2} T g_{\alpha\beta} \right) - \kappa [2n_{(\alpha} \mathcal{C}_{\beta)} - \lambda g_{\alpha\beta} n^\mu \mathcal{C}_\mu] \end{aligned}$$

Gundlach et al CQG (2005)

- E.g. Pretorius PRL gr-qc/0507014 uses $\kappa = 1.25/m$, $\lambda = 1$

Missing pieces II: Gauge in BSSN

- Recall: Einstein's equations say nothing about α , β^i
- Any choice of lapse and shift gives a solution to Einstein's eqs.
- This is the coordinate or gauge freedom of GR
- If the physics do not depend on α , β^i , then why bother?
- Answer: The performance of the numerics DO depend very sensitively on the gauge!

Ingredients for good gauge

- Singularity avoidance
- Avoid slice stretching
- Aim for stationarity in a co-moving frame
- Well-posedness of the system of PDEs
- Generalize “good” gauge, e.g. harmonic
- Lots of good luck!

Bona et al PRL (1995)

Alcubierre et al PRD gr-qc/0206072

Alcubierre CQG gr-qc/0210050

Garfinkle PRD gr-qc/0110013

Moving puncture gauge

- Moving punctures is one of the NR breakthrough methods

Baker et al PRL gr-qc/0511103; Campanelli et al PRL gr-qc/0511048

- Gauge played a key role

- Variant of $1 + \log$ slicing and Γ -driver shift

Alcubierre et al PRD gr-qc/0206072

- Now in use as $\partial_t \alpha = \beta^m \partial_m \alpha - 2\alpha K$

and
$$\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} B^i$$

$$\partial_t B^i = \beta^m \partial_m B^i + \partial_t \tilde{\Gamma}^i - \beta^m \partial_m \tilde{\Gamma}^i - \eta B^i$$

or
$$\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} \tilde{\Gamma}^i - \eta \beta^i$$

e.g. van Meter et al PRD gr-qc/0605030

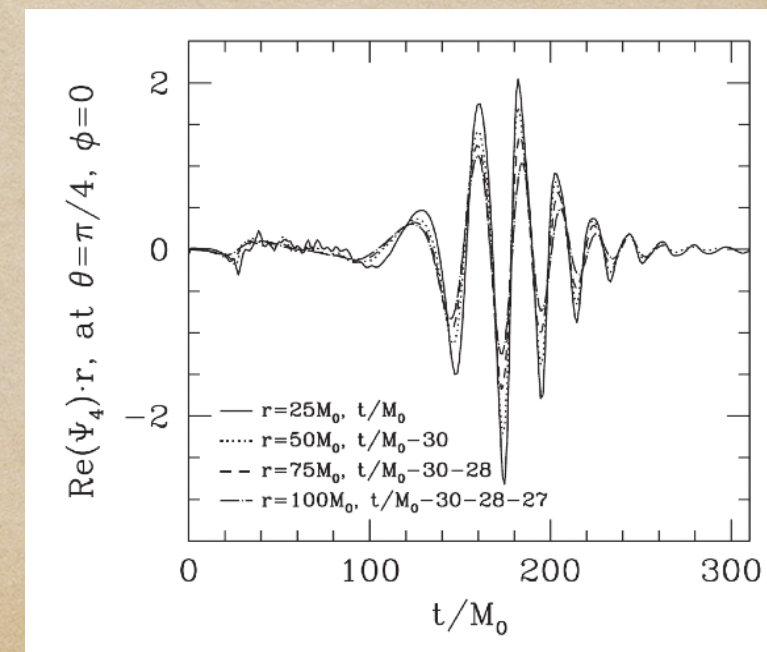
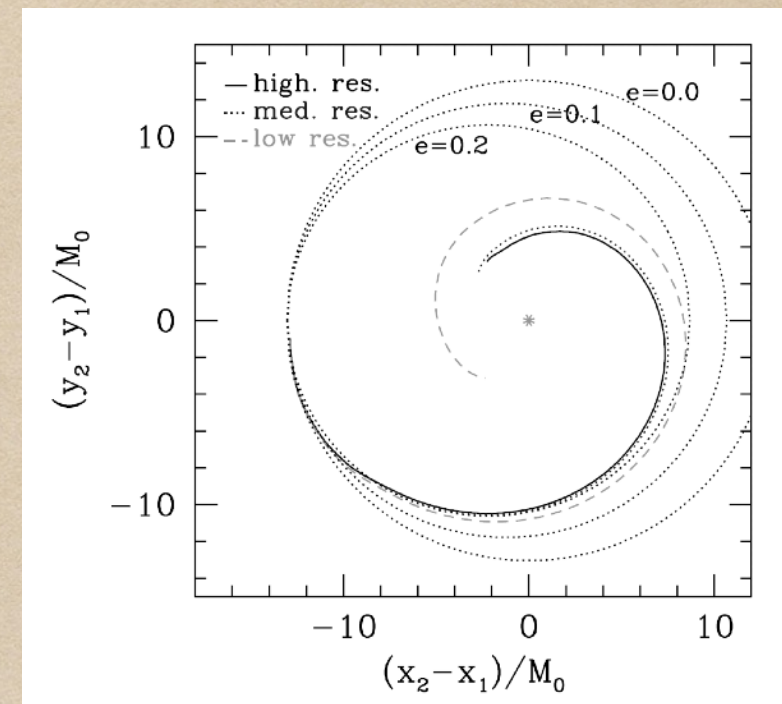
The 2005 breakthroughs

- First simulation of orbiting BBH through merger

Pretorius PRL gr-qc/0507014

- GHG
- Initial data: Scalar field
- BH excision
- Radiated energy $\sim 3\% M$
- Eccentricity $e \sim 0 \dots 0.2$

- Presented at Banff conference

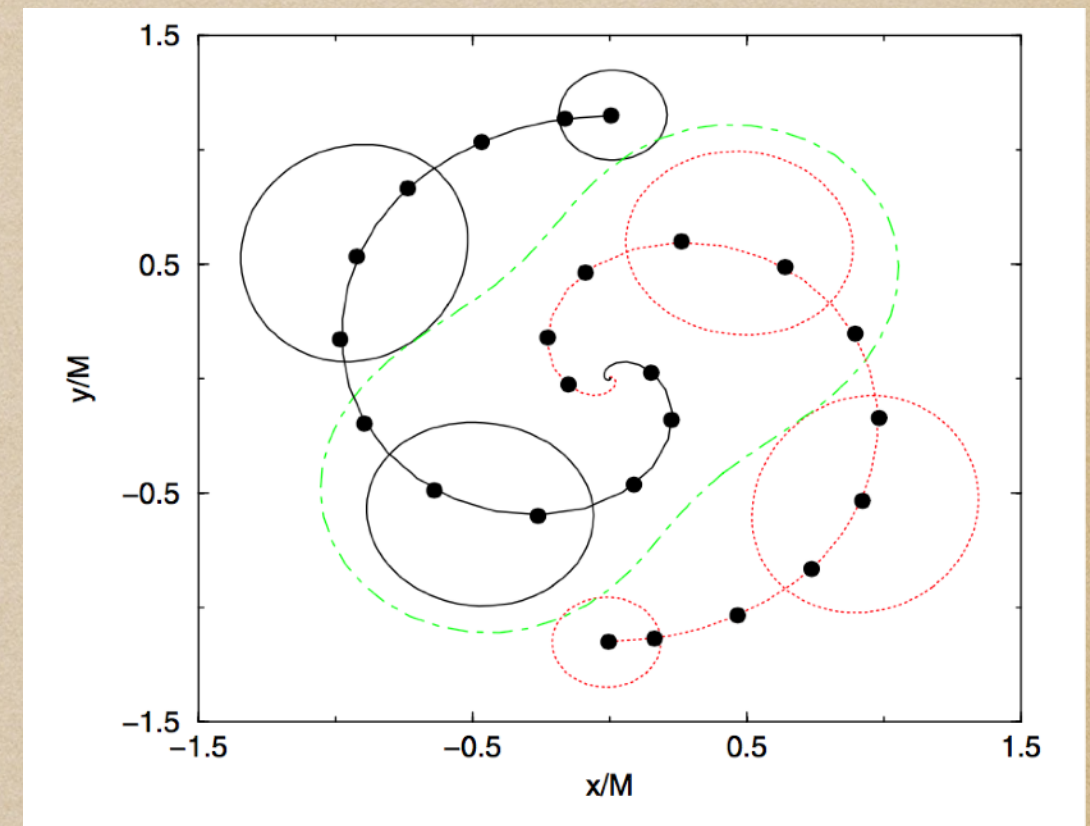
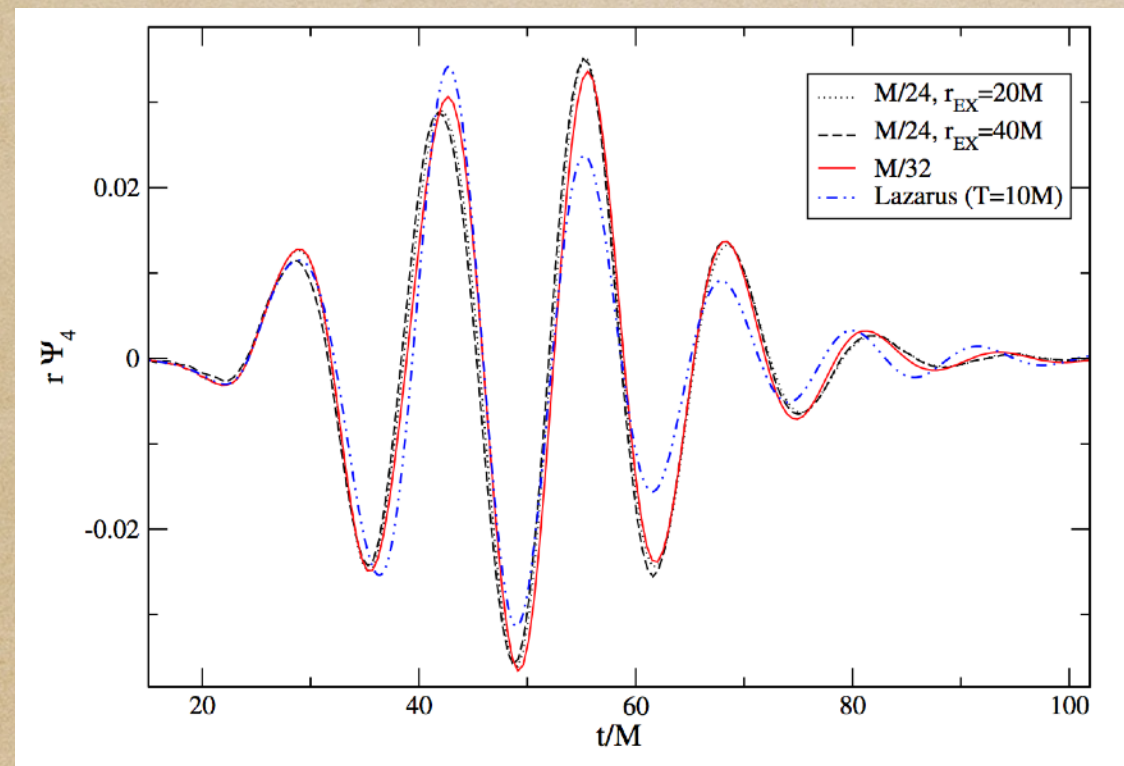


The 2005 breakthroughs

- Moving puncture breakthrough by Brownsville and Goddard groups

Campanelli et al PRL gr-qc/0511048; Baker et al PRL gr-qc/0511103

- BSSN
- Bowen-York initial data
- Moving puncture gauge
- Radiated energy $\sim 3\% M$



The goldrush years

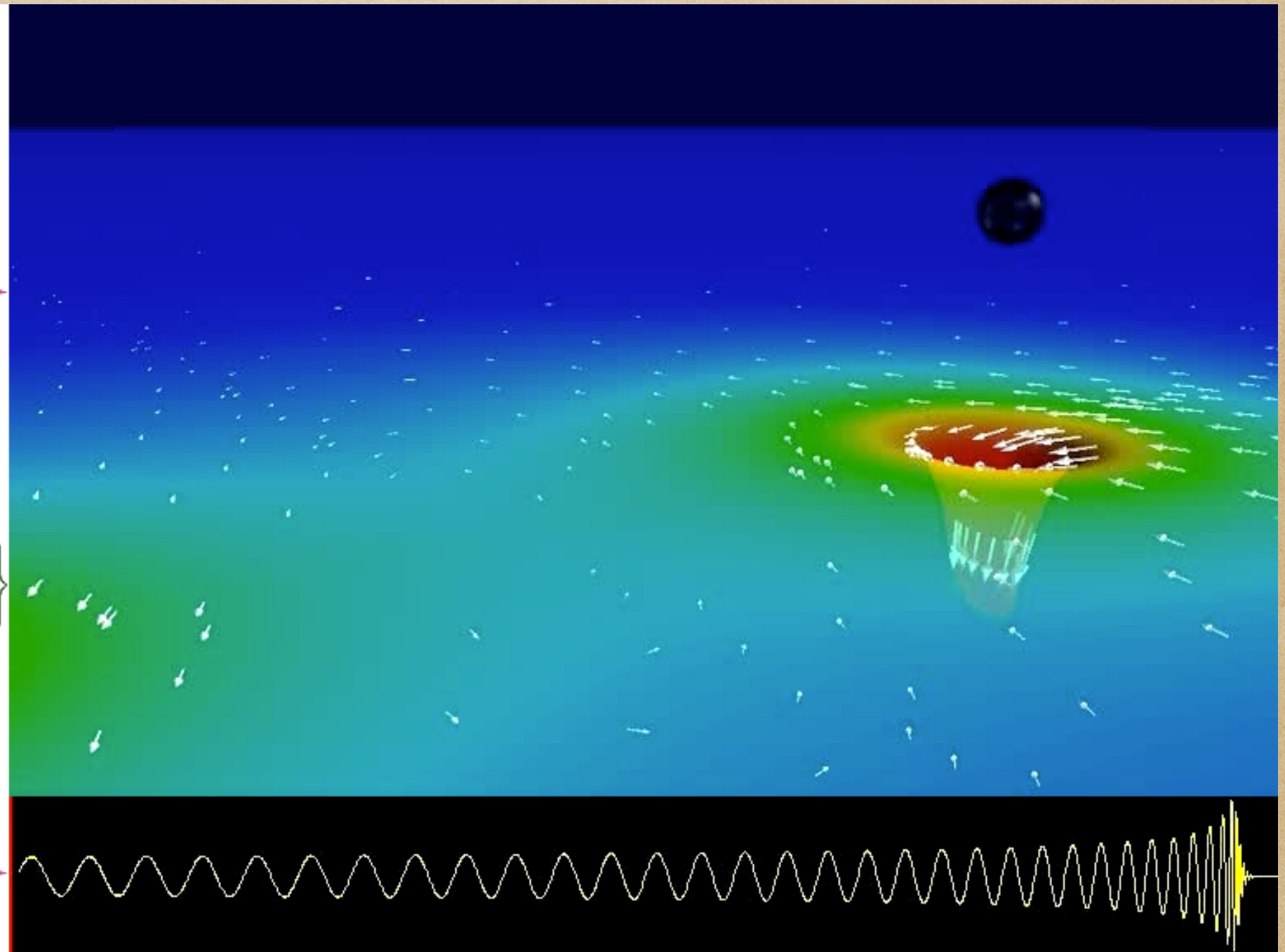
Anatomy of a BHB inspired

Binary Black Hole Evolution:
Caltech/Cornell Computer Simulation

Top: 3D view of Black Holes
and Orbital Trajectory

Middle: Spacetime curvature:
Depth: Curvature of space
Colors: Rate of flow of time
Arrows: Velocity of flow of space

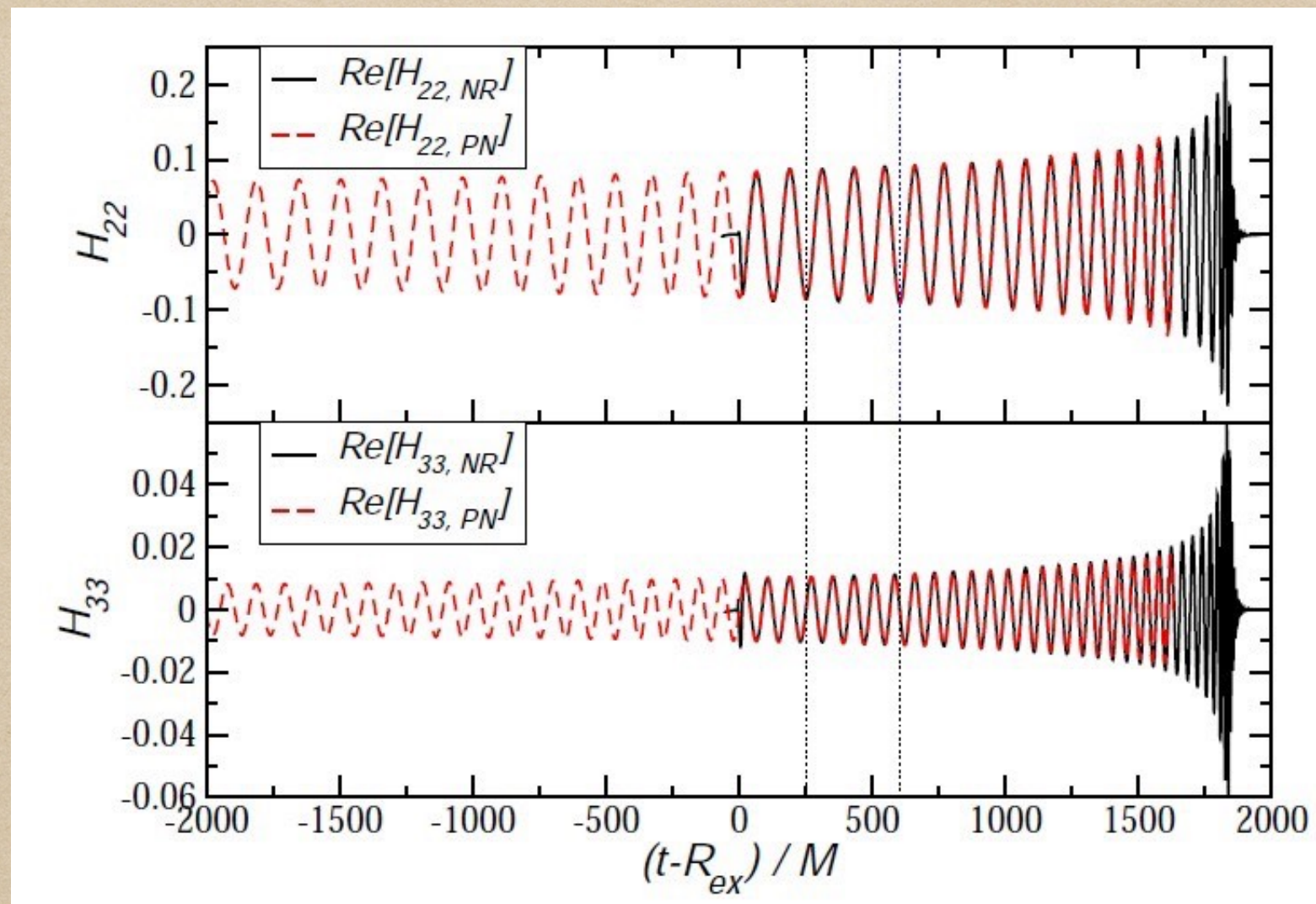
Bottom: Waveform
(red line shows current time)



Thanks to Caltech-Cornell groups

Hybrid waveforms and catalogs

- Stitch together PN and NR waveforms



US et al CQG 2011

- Mass produce waveforms; Hinder et al CQG 1307.5307;
Mroué et al PRL 1004.4697

Gravitational recoil

- Anisotropic GW emission $\Rightarrow$ recoil of remnant BH

Bonnor & Rotenberg Proc.R.Soc.Lond.A. (1961);

Peres PR (1962); Bekenstein ApJ (1973)

- Escape velocities: Globular clusters ~ 30 km/s
dSph $20 \dots 100$ km/s
dE $100 \dots 300$ km/s
Giant galaxies $\sim 1\,000$ km/s

- Ejection/displacement of BHs affects

- Growth history of SMBHs
- BH populations, IMBHs
- galaxy structure
- observational "footprints"

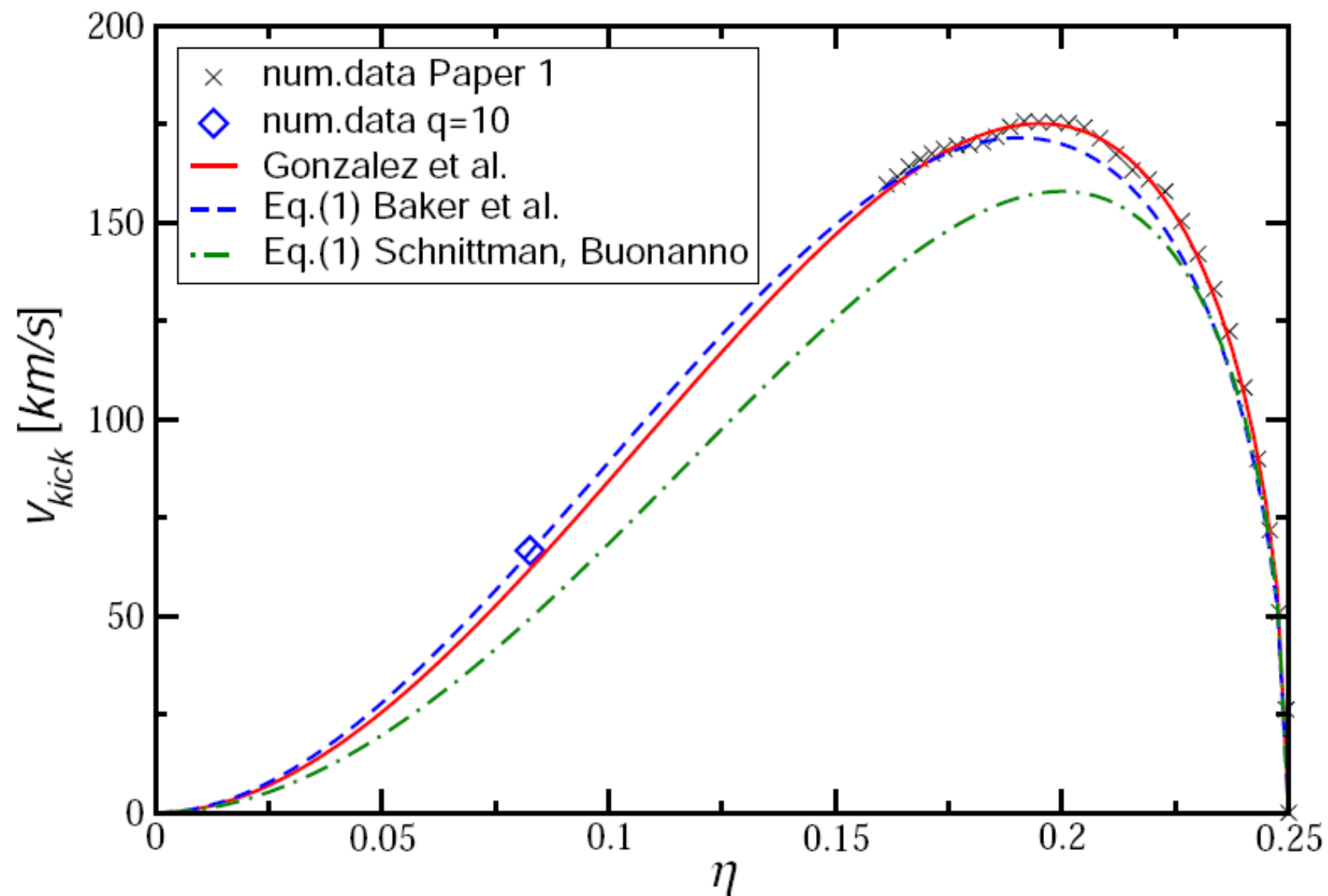
Komossa Adv.Astron. 1202.1977



Kicks from non-spinning BH binaries

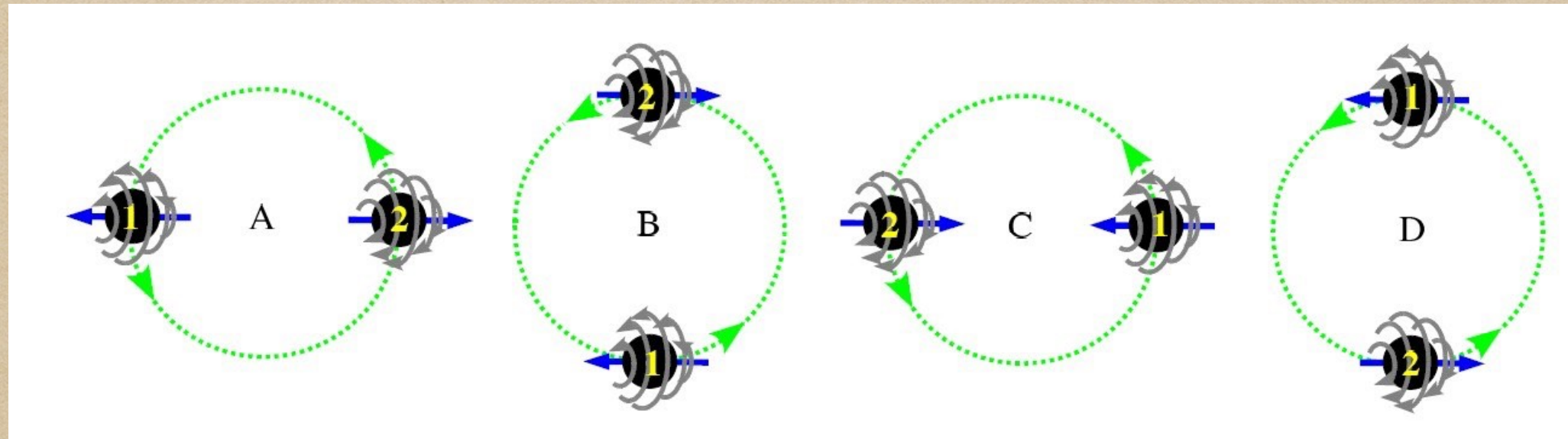
- Maximal kick: ~ 180 km/s pretty harmless!

González et al PRL gr-qc/0610154

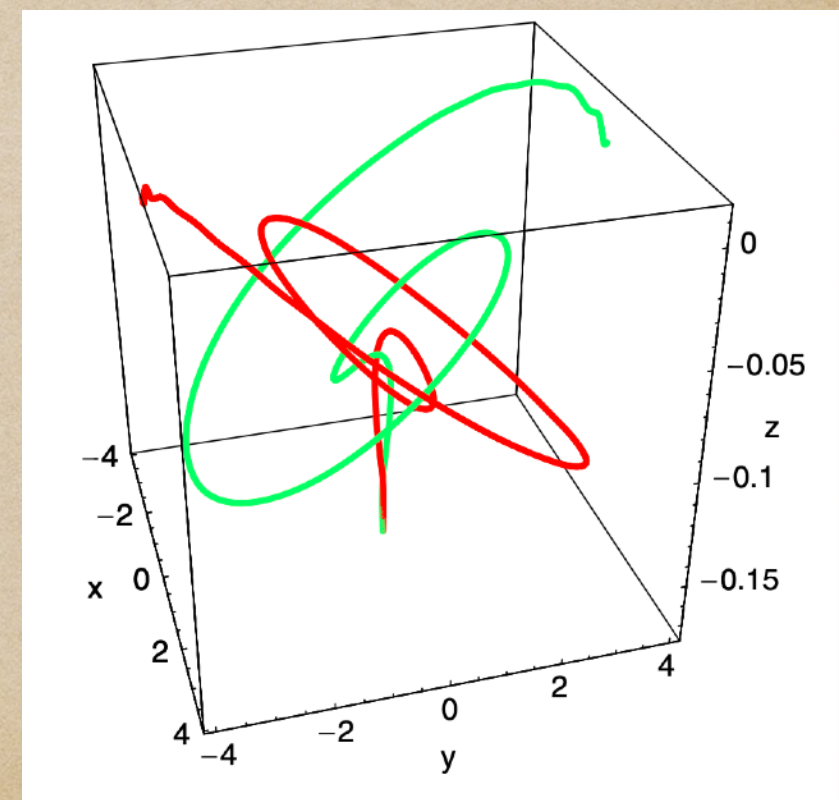


Spinning BHs: Superkicks

- Superkick configurations; Kidder gr-qc/9506022; Pretorius 0710.1338



- Kicks up to $v_{\text{max}} \approx 4\,000\text{ km/s}$
González et al PRL gr-qc/0702052
Campanelli et al PRL gr-qc/0702133
- Yet larger kicks for partially aligned spins
 $v_{\text{max}} \approx 5\,000\text{ km/s}$
Lousto et al PRL 1108.2009

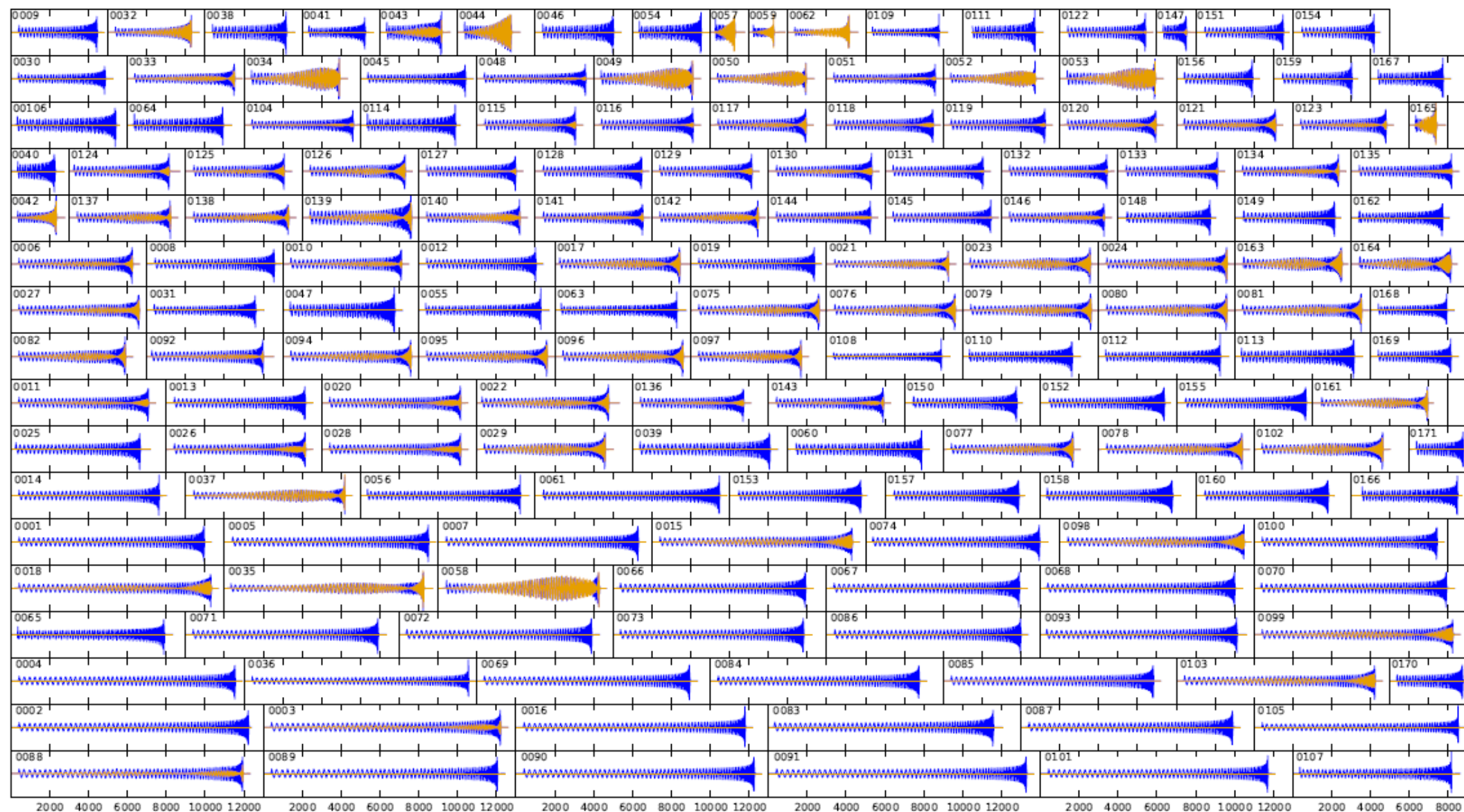


Towards new horizons

Tools of mass production

- Explore seven-dim. parameter space. E.g. SpEC catalogue:
171 waveforms: $m_1/m_2 \leq 8$ up to 34 orbits

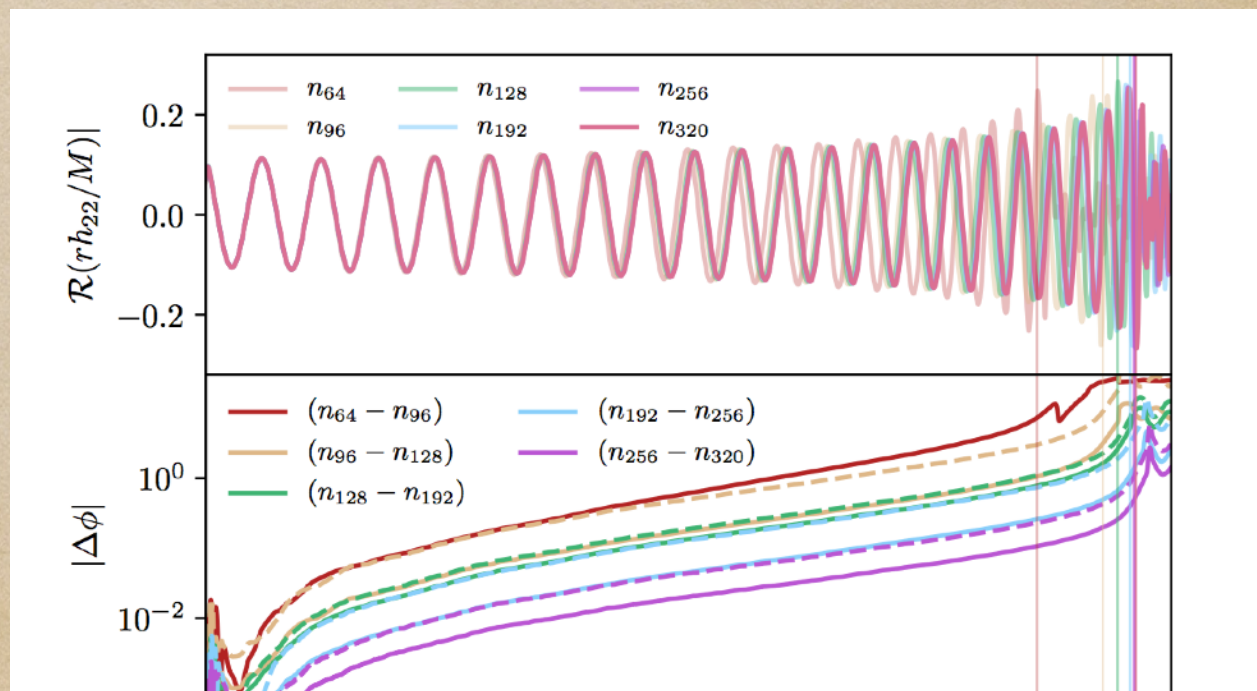
Mroué et al PRL 1304.6077



Neutron star binaries

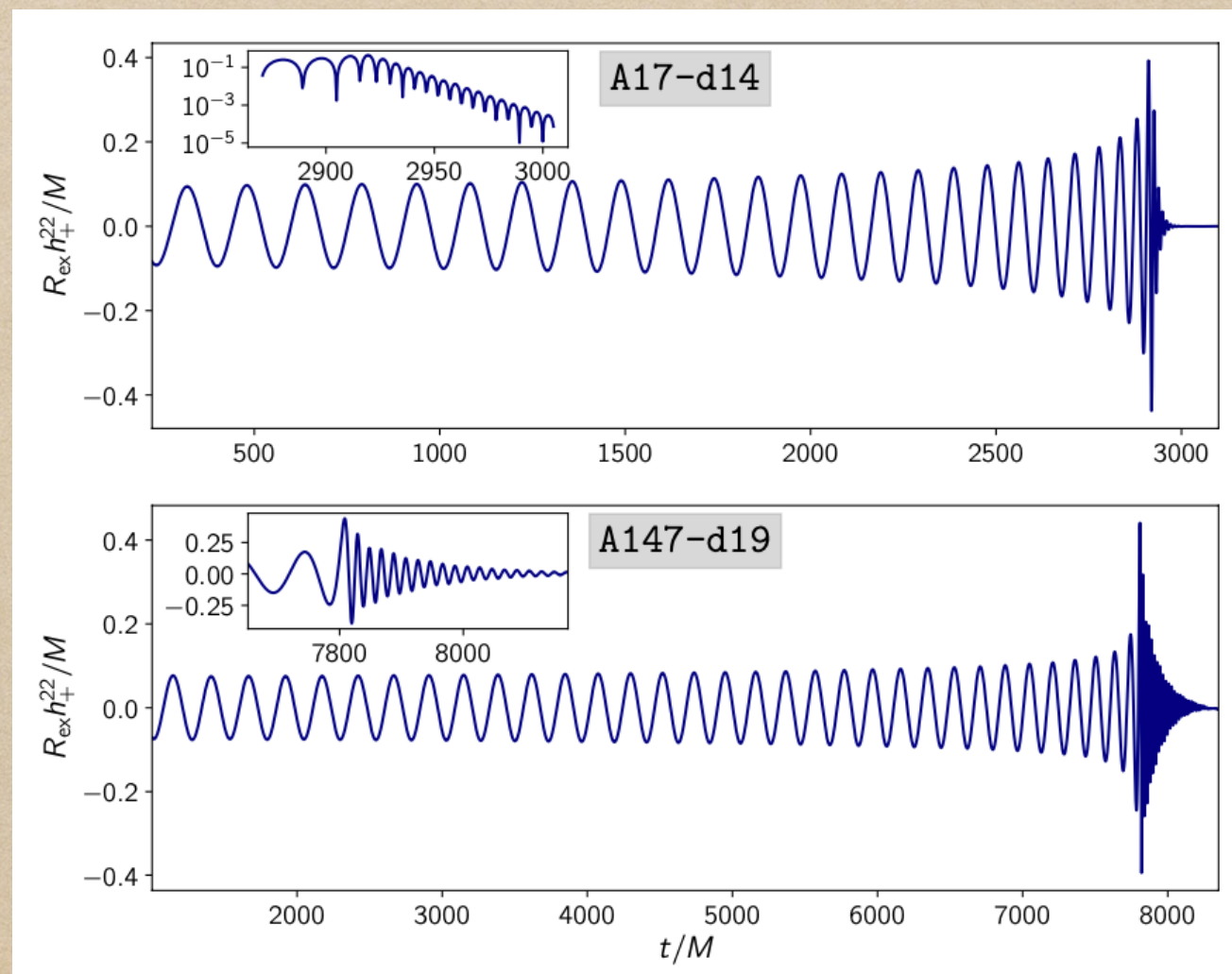
- Other challenges: No spacetime singularities, but shocks!
- HRSC treatment needs flux conservative Eqs. $\partial_t \mathbf{u} + \partial_i \mathbf{f}(\mathbf{u}) = \mathbf{s}$
- Solutions not unique $\rightarrow$ entropy conditions!
- The first NS binary inspirals preceded the BH breakthrough!
Shibata+ PRD 2003, Marronetti+ PRL 2004, Miller+ PRD 2004
- Template constructions: e.g. Dietrich et al 1905.06011

NRTidalv2 approximant



BS binaries

- Phase error $\approx 0.1 \dots 0.2$
- Amplitude error $\lesssim 3\%$
- Eccentricity $\approx 0.002 \dots 0.005$

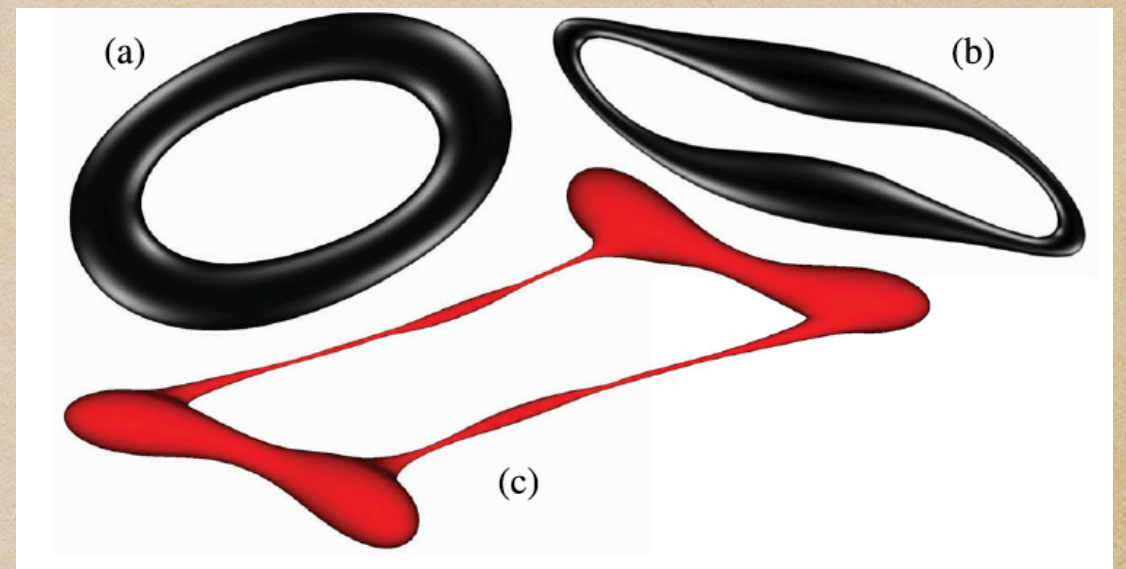


Evstafyeva et al PRL 2024

Cosmic censorship in D=5

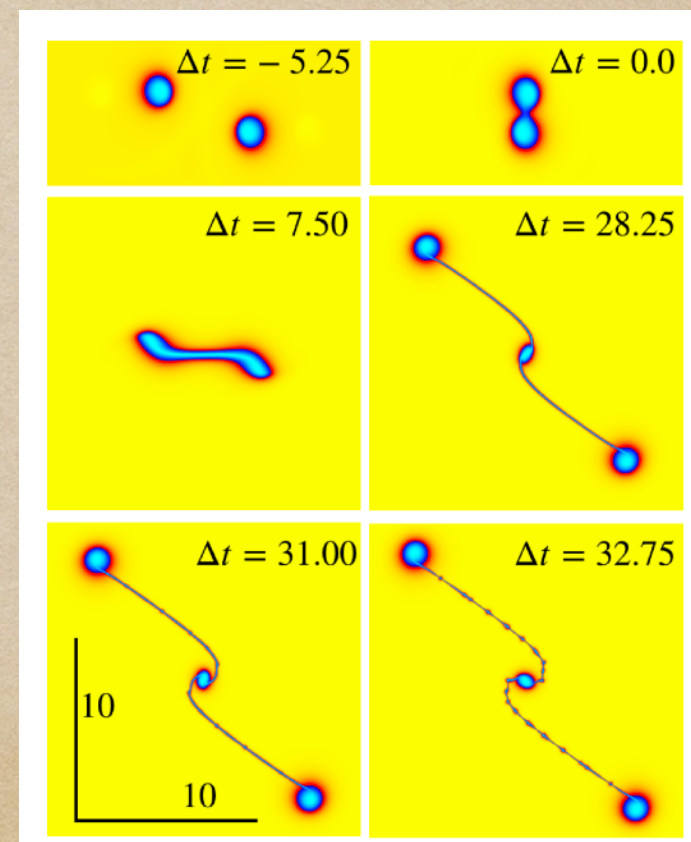
Figueras, Kunesch & Tunyasuvunakool PRL 1512.04532

- 5D simulations (mod.cartoon)
- Conformal Z4 system
- Black ring: *assympt.flat!*
- Gregory-Laflamme instability
⇒ Violation of CC!



Andrade, Figueras & US JHEP 2022

- 7D BH collisions
- No finetuning!



Cosmological singularities

- Approach singularity through backward time evolutions
- BKL conjecture Belinsky, Khalatnikov, Lifshitz Adv.Phys. 1970

- Singularity is spacelike, local, oscillatory

- points decouple Misner's Mixmaster cosmology

- transition through Kasner spacetimes

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2, \quad p_1 = \text{const}, \quad \sum p_i = \sum p_i^2 = 1$$

- Transition of Kasner exponents chaotic, but well defined

- Barrow PRL 1981

- Transition driven by dominant $^{(3)}R$

- NR provides supporting evidence for BKL

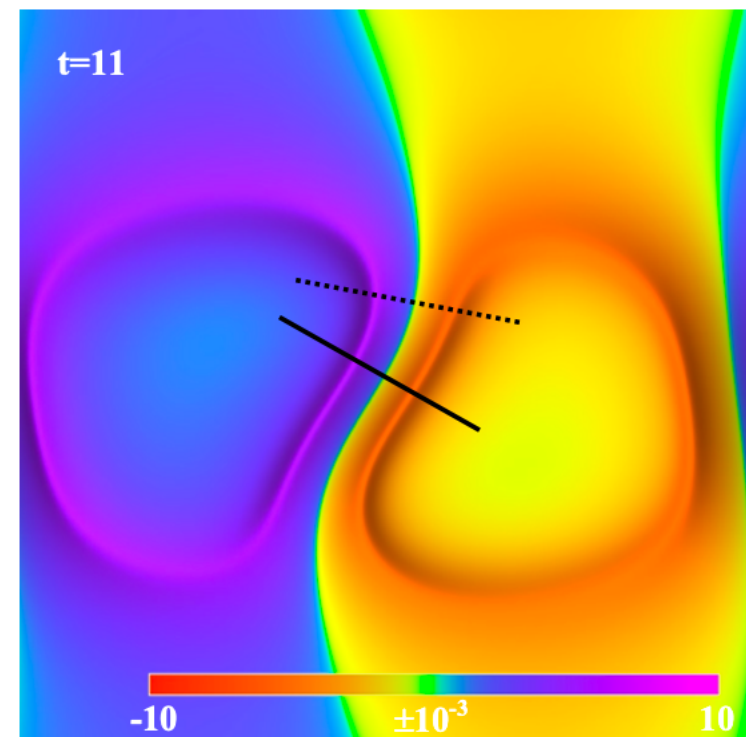
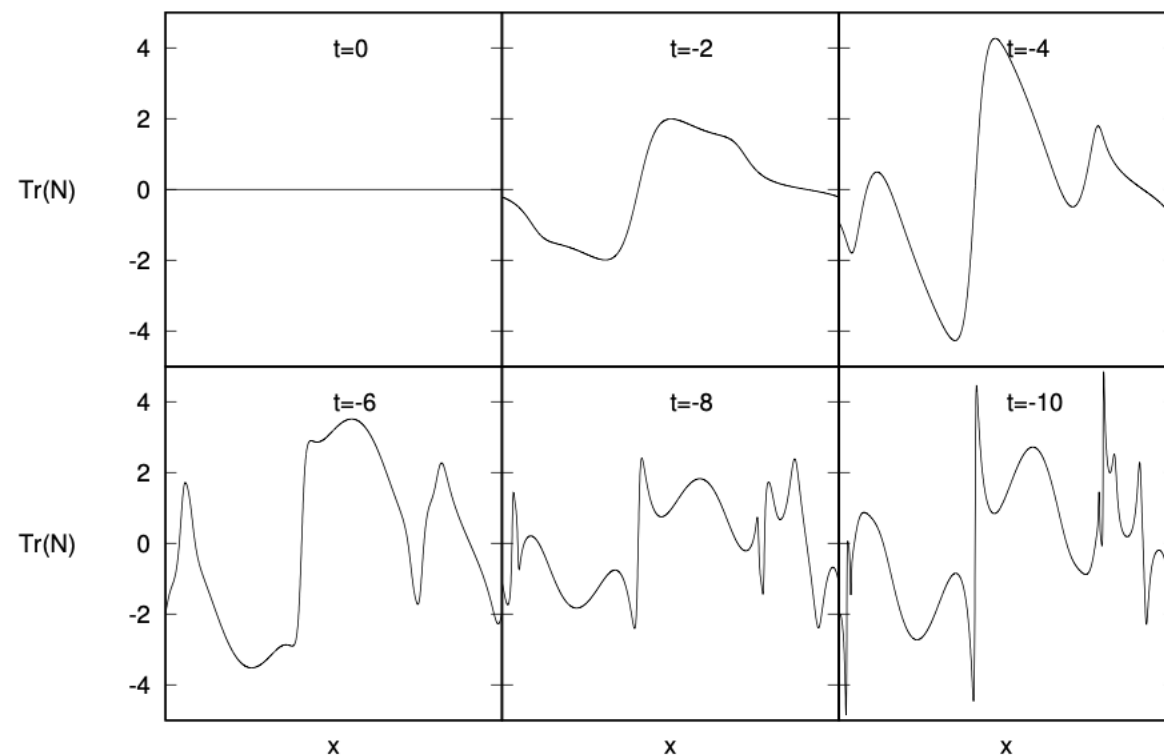
- NR also identified new feature spikes i.e.

- neighbouring points have different Kasner exponents...

Cosmological singularities

- ... Example for evolution of spikes in 1+1, 2+1

Garfinkle & Pretorius. PRD 2020



- Impact of spikes on dynamics decreases very close to the singularity
- Side effect: Cosmology inspired new techniques in NR, e.g. GHG

The future...

- Waveform catalogs for binaries containing BHs, NSs
 - Precessing BHBs, high-mass ratios, eccentric binaries
 - NSNS, BHNS systems
- Waveform predictions for compact objects in modified gravity
- Model GW signatures of dark matter candidates
- Exotic compact objects
- NR in cosmology
- Applications in AdS/CFT
- Critical collapse in >1 dimensions
- Higher dimensional GR; is $D=4$ special?