Excited-state spectroscopy of triply-bottom baryons

[arXiv:1202.1312]

Stefan Meinel

The College of William & Mary

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Introduction

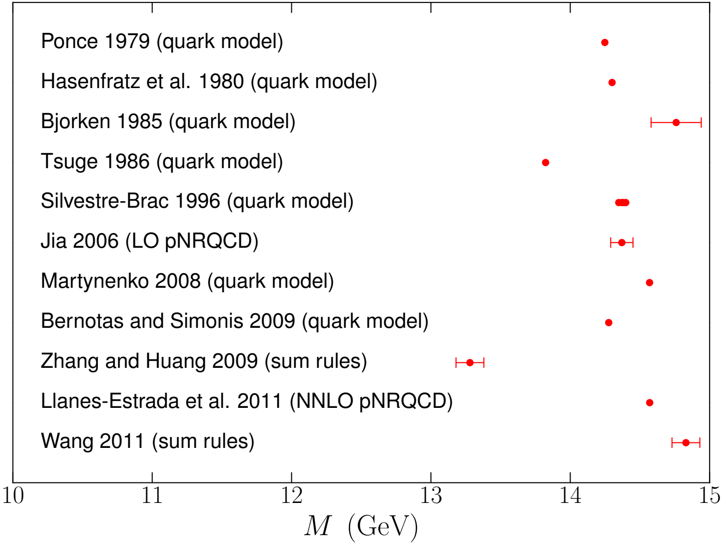
> appreciate that. A new dimension for the future would be the understanding of baryons containing only heavy quarks; the mesonic analogue after all has been of crucial significance. Just as anticipated observations of the $t\bar{t}$ mesons are especially interesting to the QCD theorist, so also would be the observations of the properties of ttt baryons. That would appear to be out of the question experimentally for the foreseeable future. But already the ssc system (observed!) begins to enter the purely heavy quark world-in the same sense that it is not totally foolish to regard the ϕ as "strange-onium". Why not search for the ccs, or bcs?
>
> It is our purpose here to discuss what might be possible. We shall do this not by trying to make a general survey (that is a big job) but, just for fun, to concentrate on the single case of the triply charmed $(ccc)^{++}$ baryon. It has that esthetic quality possessed by the Ω^- and would be wonderful to observe. It may be beyond the fringe to take that possibility seriously. But better to try to reach too far than not far enough. And by studying the prospects for (ccc) we learn something about what is between us and it. That is interesting too.

J. D. Bjorken, *"Is ccc a new deal for baryon spectroscopy?"*
1st International Conference on Hadron Spectroscopy, 1985

Introduction

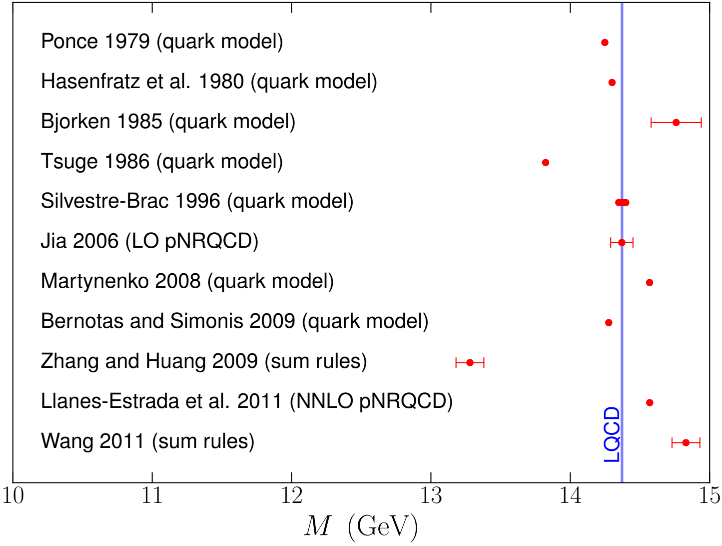
Here we shall concentrate on the *bbb* case.

Introduction: Ω_{bbb} ground-state mass



Introduction: Ω_{bbb} ground-state mass

Lattice QCD: $M_{\Omega_{bbb}} = 14.371 \pm 0.004_{\text{stat}} \pm 0.011_{\text{syst}}$ GeV [Meinel 2010]



Introduction: Ω_{bbb} excited states

Silvestre-Brac, 1996:
potential model with
two-body interactions,

$$V_{ij}^{q\bar{q}}(r) = -\frac{\kappa(1-\exp(-r/r_c))}{r} + \lambda r^p - \Lambda + \frac{2\pi}{3m_i m_j}\kappa'(1-\exp(-r/r_c))\frac{\exp(-r^2/r_0^2)}{\pi^{3/2}r_0^3}\vec{\sigma}_i\vec{\sigma}_j$$

Note: no spin-orbit
or tensor interactions

L and S separately conserved

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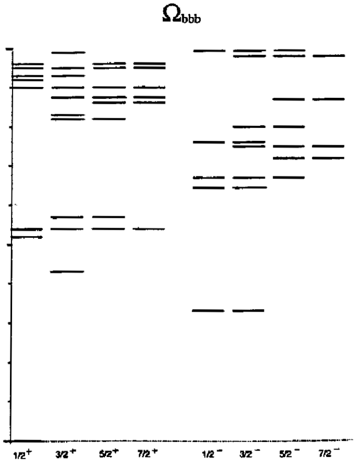

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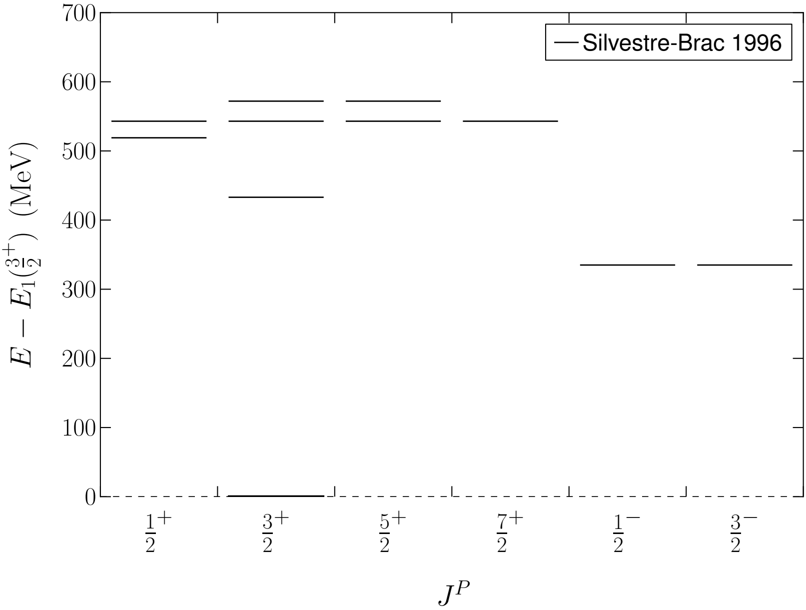
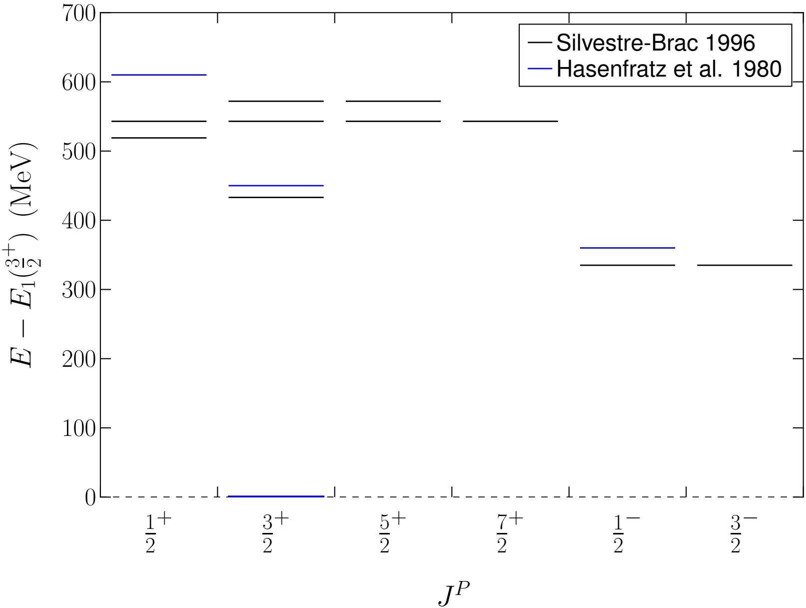
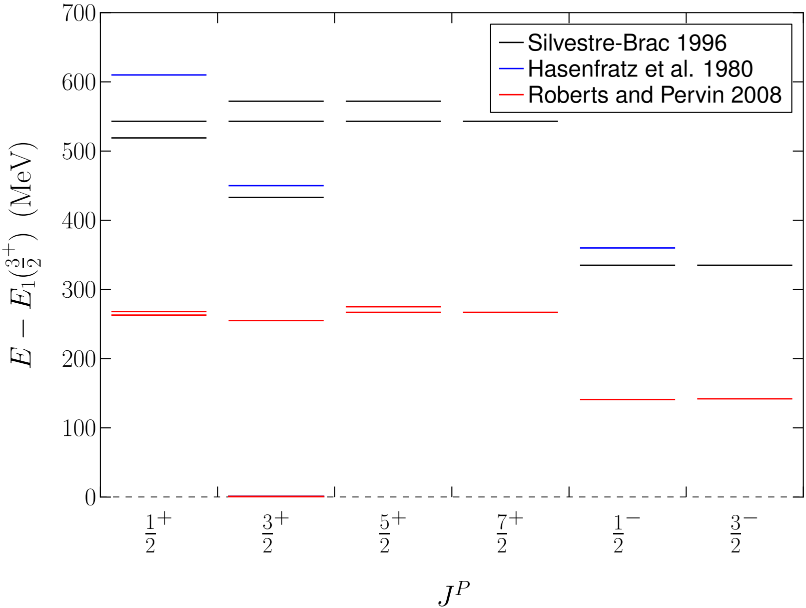


Introduction: Ω_{bbb} excited states



Introduction: Ω_{bbb} excited states



Outline

- Construction of bbb interpolating fields
- Lattice actions and parameters
- Fitting of the two-point functions and identifying J
- Chiral extrapolation and final results
- Contributions of individual NRQCD operators to the energy splittings

Construction of *bbb* interpolating fields

Spectrum from Euclidean two-point functions

$$\left\langle \sum_{\mathbf{x}} \Omega_\Gamma(\mathbf{x},t)\,\Omega^\dagger_{\Gamma'}(0) \right\rangle = \sum_n \langle 0|\boldsymbol{\Omega}_\Gamma|n\rangle\langle n|\boldsymbol{\Omega}^\dagger_{\Gamma'}|0\rangle\, e^{-E_n\,t}$$

where the sum goes over a complete set of zero-momentum energy eigenstates, and $\langle \ldots \rangle$ denotes the Euclidean path integral

$$\langle \ldots \rangle \equiv \frac{1}{Z}\int \mathcal{D}[U]\mathcal{D}[\Psi]\mathcal{D}[\overline{\Psi}]\;(\ldots)\,e^{-S[U,\Psi,\overline{\Psi}]}.$$

$\rightarrow$ Need to construct suitable Ω_Γ's from quark fields and gauge links.

Construction of *bbb* interpolating fields

Method for constructing Ω_Γ is based on [Edwards et al., 2011].

Step 1: construct operators with definite continuum J, (i.e., definite $SU(2)$ irrep) using covariant derivatives. Important: Pauli-principle for identical particles, so keep track of permutation symmetries.

$$\left[\Omega\right]^J_m = \left[D(L)\ \ O(S)\right]^J_m$$

Step 2: subduction to definite irreps of the double-cover octahedral group ${}^2\mathrm{O}$

$$\left[\Omega\right]^J_{{}^n\Lambda,r} = \sum_m \mathcal{S}^{J,m}_{{}^n\Lambda,r}\ \left[\Omega\right]^J_m.$$

Construction of *bbb* interpolating fields

- color "wave function": ϵ_{abc}, totally antisymmetric.
- thus, (space)$\times$(spin) "wave function" must be totally symmetric under quark permutations.
- construct (space)$\times$(spin) part first, suppress color indices.

Construction of *bbb* interpolating fields: spin

We will use lattice NRQCD for the *b* quarks $\rightarrow$ Pauli spinors $\tilde{\psi}_\alpha$ with $\alpha = \uparrow, \downarrow$.

Basic spin combinations: totally symmetric (S),

$$
\begin{aligned}
O_{\mathsf{S}}(\tfrac{3}{2}, +\tfrac{3}{2}) &= \tilde{\psi}_\uparrow \tilde{\psi}_\uparrow \tilde{\psi}_\uparrow, \\
O_{\mathsf{S}}(\tfrac{3}{2}, +\tfrac{1}{2}) &= \frac{1}{\sqrt{3}} \Big(\tilde{\psi}_\uparrow \tilde{\psi}_\uparrow \tilde{\psi}_\downarrow + \tilde{\psi}_\uparrow \tilde{\psi}_\downarrow \tilde{\psi}_\uparrow + \tilde{\psi}_\downarrow \tilde{\psi}_\uparrow \tilde{\psi}_\uparrow \Big), \\
O_{\mathsf{S}}(\tfrac{3}{2}, -\tfrac{1}{2}) &= \frac{1}{\sqrt{3}} \Big(\tilde{\psi}_\downarrow \tilde{\psi}_\downarrow \tilde{\psi}_\uparrow + \tilde{\psi}_\downarrow \tilde{\psi}_\uparrow \tilde{\psi}_\downarrow + \tilde{\psi}_\uparrow \tilde{\psi}_\downarrow \tilde{\psi}_\downarrow \Big), \\
O_{\mathsf{S}}(\tfrac{3}{2}, -\tfrac{3}{2}) &= \tilde{\psi}_\downarrow \tilde{\psi}_\downarrow \tilde{\psi}_\downarrow,
\end{aligned}
$$

(the tilde indicates three-dimensional Gaussian smearing)

Construction of *bbb* interpolating fields: spin

mixed-symmetric (MS),

$$O_{\mathrm{MS}}(\tfrac{1}{2},+\tfrac{1}{2}) = \frac{1}{\sqrt{6}}\left(\tilde{\psi}_\uparrow\tilde{\psi}_\downarrow\tilde{\psi}_\uparrow + \tilde{\psi}_\downarrow\tilde{\psi}_\uparrow\tilde{\psi}_\uparrow - 2\tilde{\psi}_\uparrow\tilde{\psi}_\uparrow\tilde{\psi}_\downarrow\right),$$

$$O_{\mathrm{MS}}(\tfrac{1}{2},-\tfrac{1}{2}) = -\frac{1}{\sqrt{6}}\left(\tilde{\psi}_\downarrow\tilde{\psi}_\uparrow\tilde{\psi}_\downarrow + \tilde{\psi}_\uparrow\tilde{\psi}_\downarrow\tilde{\psi}_\downarrow - 2\tilde{\psi}_\downarrow\tilde{\psi}_\downarrow\tilde{\psi}_\uparrow\right),$$

and mixed-antisymmetric (MA)

$$O_{\mathrm{MA}}(\tfrac{1}{2},+\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\left(\tilde{\psi}_\uparrow\tilde{\psi}_\downarrow\tilde{\psi}_\uparrow - \tilde{\psi}_\downarrow\tilde{\psi}_\uparrow\tilde{\psi}_\uparrow\right),$$

$$O_{\mathrm{MA}}(\tfrac{1}{2},-\tfrac{1}{2}) = -\frac{1}{\sqrt{2}}\left(\tilde{\psi}_\downarrow\tilde{\psi}_\uparrow\tilde{\psi}_\downarrow - \tilde{\psi}_\uparrow\tilde{\psi}_\downarrow\tilde{\psi}_\downarrow\right).$$

Construction of *bbb* interpolating fields: derivatives

A single derivative has $L = 1$. Use spherical basis

$$\begin{aligned} D_{\pm 1} &= \pm \tfrac{i}{2}(D_x \pm iD_y), \\ D_0 &= -\tfrac{i}{\sqrt{2}} D_z. \end{aligned}$$

Mixed-symmetric and mixed-antisymmetric combinations of derivative hitting first, second, or third quark field:

$$D^{[1]}_{\mathrm{MS}}(1, m) = \frac{1}{\sqrt{6}}\left(2D^{(3)}_m - D^{(1)}_m - D^{(2)}_m\right),$$

$$D^{[1]}_{\mathrm{MA}}(1, m) = \frac{1}{\sqrt{2}}\left(D^{(1)}_m - D^{(2)}_m\right).$$

Construction of *bbb* interpolating fields: derivatives

Two derivatives with definite L and definite permutation symmetry: $1 \otimes 1 = 0 \oplus 1 \oplus 2$

$$
\begin{aligned}
D_{\mathrm{S}}^{[2]}(L,m) &= \frac{1}{\sqrt{2}} \sum_{m_1,m_2} \langle L,m|1,m_1;1,m_2\rangle \Big(+ D_{\mathrm{MS}}^{[1]}(1,m_1) D_{\mathrm{MS}}^{[1]}(1,m_2) + D_{\mathrm{MA}}^{[1]}(1,m_1) D_{\mathrm{MA}}^{[1]}(1,m_2) \Big), \\
D_{\mathrm{MS}}^{[2]}(L,m) &= \frac{1}{\sqrt{2}} \sum_{m_1,m_2} \langle L,m|1,m_1;1,m_2\rangle \Big(- D_{\mathrm{MS}}^{[1]}(1,m_1) D_{\mathrm{MS}}^{[1]}(1,m_2) + D_{\mathrm{MA}}^{[1]}(1,m_1) D_{\mathrm{MA}}^{[1]}(1,m_2) \Big), \\
D_{\mathrm{MA}}^{[2]}(L,m) &= \frac{1}{\sqrt{2}} \sum_{m_1,m_2} \langle L,m|1,m_1;1,m_2\rangle \Big(+ D_{\mathrm{MS}}^{[1]}(1,m_1) D_{\mathrm{MA}}^{[1]}(1,m_2) + D_{\mathrm{MA}}^{[1]}(1,m_1) D_{\mathrm{MS}}^{[1]}(1,m_2) \Big), \\
D_{\mathrm{A}}^{[2]}(L,m) &= \frac{1}{\sqrt{2}} \sum_{m_1,m_2} \langle L,m|1,m_1;1,m_2\rangle \Big(+ D_{\mathrm{MS}}^{[1]}(1,m_1) D_{\mathrm{MA}}^{[1]}(1,m_2) - D_{\mathrm{MA}}^{[1]}(1,m_1) D_{\mathrm{MS}}^{[1]}(1,m_2) \Big).
\end{aligned}
$$

The top three combinations give $L = 0, 2$. The last one gives $L = 1$.

(for other L values, one gets either get zero or a hybrid operator; see Dudek and Edards, 2012)

Construction of *bbb* interpolating fields: derivatives $\otimes$ spin

Now combine derivative and spin structure. Must be totally symmetric.

- no derivative: $\left[O_{\mathrm{S}}(\tfrac{3}{2})\right]_m^{J=\frac{3}{2}}$

$$= O_{\mathrm{S}}(\tfrac{3}{2}, m)$$

- one derivative: $\left[D_{\mathrm{M}}^{[1]}(1)\;\; O_{\mathrm{M}}(\tfrac{1}{2})\right]_m^{J=\frac{1}{2},\,\frac{3}{2}}$

$$= \frac{1}{\sqrt{2}} \sum_{m_1, m_2} \langle J, m | 1, m_1; \tfrac{1}{2}, m_2 \rangle$$

$$\times \Big(D_{\mathrm{MS}}^{[1]}(1, m_1)\; O_{\mathrm{MS}}(\tfrac{1}{2}, m_2) + D_{\mathrm{MA}}^{[1]}(1, m_1)\; O_{\mathrm{MA}}(\tfrac{1}{2}, m_2) \Big)$$

Construction of *bbb* interpolating fields: derivatives ⊗ spin

- two derivatives: similarly, get

$$\left[D_{\mathrm{S}}^{[2]}(0)\ \ O_{\mathrm{S}}(\tfrac{3}{2})\right]_m^{J=\frac{3}{2}}$$

$$\left[D_{\mathrm{M}}^{[2]}(0)\ \ O_{\mathrm{M}}(\tfrac{1}{2})\right]_m^{J=\frac{1}{2}}$$

$$\left[D_{\mathrm{M}}^{[2]}(2)\ \ O_{\mathrm{M}}(\tfrac{1}{2})\right]_m^{J=\frac{3}{2},\,\frac{5}{2}}$$

$$\left[D_{\mathrm{S}}^{[2]}(2)\ \ O_{\mathrm{S}}(\tfrac{3}{2})\right]_m^{J=\frac{1}{2},\,\frac{3}{2},\,\frac{5}{2},\,\frac{7}{2}}$$

In total, we got 11 different continuum interpolating fields (each of them has $2J+1$ possible polarizations). Their parity is

$$P = (-1)^{n_D}$$

where n_D is the number of derivatives.

Construction of *bbb* interpolating fields: subduction

Angular momentum in the continuum: double-cover of 3-dim. rotation group ${}^2SO(3) = SU(2)$, an infinite number of irreps labeled by J with dimension $(2J + 1)$

Irrep J	Dimension
0	1
1/2	2
1	3
3/2	4
⋮	⋮

Construction of *bbb* interpolating fields: subduction

Angular momentum on a 3-dim. cubic lattice: double-cover of octahedral group, ^2O. This has 48 elements in 8 conjugacy classes, and hence only 8 irreps.

Irrep Λ	Dimension
A_1	1
A_2	1
E	2
T_1	3
T_2	3
G_1	2
G_2	2
H	4

Construction of *bbb* interpolating fields: subduction

r-components of lattice irreps = linear combinations of different m-components

$$[\Omega]^{J}_{{}^{n}\Lambda,r} = \sum_{m} \mathcal{S}^{J,m}_{{}^{n}\Lambda,r} \, [\Omega]^{J}_{m}$$

J	Subduction
0	A_1
1	T_1
2	$E \oplus T_2$
3	$A_2 \oplus T_1 \oplus T_2$
4	$A_1 \oplus E \oplus T_1 \oplus T_2$
⋮	⋮

J	Subduction
$1/2$	G_1
$3/2$	H
$5/2$	$G_2 \oplus H$
$7/2$	$G_1 \oplus G_2 \oplus H$
$9/2$	$G_1 \oplus {}^1H \oplus {}^2H$
⋮	⋮

Construction of *bbb* interpolating fields: subduction

Subduction matrices $\mathcal{S}^{\frac{1}{2},m}_{G_1,r}$, $\mathcal{S}^{1,m}_{T_1,r}$ and $\mathcal{S}^{\frac{3}{2},m}_{H,r}$ are trivial:

$J = \frac{1}{2} \to G_1$

r \ m	$+\frac{1}{2}$	$-\frac{1}{2}$
1	1	0
2	0	1

$J = 1 \to T_1$

r \ m	$+1$	0	-1
1	1	0	0
2	0	1	0
3	0	0	1

$J = \frac{3}{2} \to H$

r \ m	$+\frac{3}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1

Construction of *bbb* interpolating fields: subduction

Example: subduction matrices $\mathcal{S}^{\frac{5}{2},m}_{H,r}$ and $\mathcal{S}^{\frac{5}{2},m}_{G_2,r}$:

$J = \frac{5}{2} \to H$

$r \backslash m$	$+\frac{5}{2}$	$+\frac{3}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$
1	0	$+\sqrt{\frac{1}{6}}$	0	0	0	$+\sqrt{\frac{5}{6}}$
2	0	0	-1	0	0	0
3	0	0	0	$+1$	0	0
4	$-\sqrt{\frac{5}{6}}$	0	0	0	$-\sqrt{\frac{1}{6}}$	0

$J = \frac{5}{2} \to G_2$

$r \backslash m$	$+\frac{5}{2}$	$+\frac{3}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{2}$
1	$+\sqrt{\frac{1}{6}}$	0	0	0	$-\sqrt{\frac{5}{6}}$	0
2	0	$-\sqrt{\frac{5}{6}}$	0	0	0	$+\sqrt{\frac{1}{6}}$

Construction of *bbb* interpolating fields: subduction

Operator(s)	Structure $\sim \left[D(L)\ \ O(S)\right]^J$
$H_g^{(1)}$	$\left[O_{\mathrm{S}}(\tfrac{3}{2})\right]^{J=\frac{3}{2}}$
$G_{1u}^{(1)}$	$\left[D_{\mathrm{M}}^{[1]}(1)\ \ O_{\mathrm{M}}(\tfrac{1}{2})\right]^{J=\frac{1}{2}}$
$H_u^{(1)}$	$\left[D_{\mathrm{M}}^{[1]}(1)\ \ O_{\mathrm{M}}(\tfrac{1}{2})\right]^{J=\frac{3}{2}}$
$H_g^{(2)}$	$\left[D_{\mathrm{S}}^{[2]}(0)\ \ O_{\mathrm{S}}(\tfrac{3}{2})\right]^{J=\frac{3}{2}}$
$G_{1g}^{(1)}$	$\left[D_{\mathrm{M}}^{[2]}(0)\ \ O_{\mathrm{M}}(\tfrac{1}{2})\right]^{J=\frac{1}{2}}$
$G_{1g}^{(2)}$	$\left[D_{\mathrm{S}}^{[2]}(2)\ \ O_{\mathrm{S}}(\tfrac{3}{2})\right]^{J=\frac{1}{2}}$
$H_g^{(3)}$	$\left[D_{\mathrm{S}}^{[2]}(2)\ \ O_{\mathrm{S}}(\tfrac{3}{2})\right]^{J=\frac{3}{2}}$
$H_g^{(4)},\ G_{2g}^{(1)}$	$\left[D_{\mathrm{S}}^{[2]}(2)\ \ O_{\mathrm{S}}(\tfrac{3}{2})\right]^{J=\frac{5}{2}}$
$H_g^{(5)},\ G_{1g}^{(3)},\ G_{2g}^{(2)}$	$\left[D_{\mathrm{S}}^{[2]}(2)\ \ O_{\mathrm{S}}(\tfrac{3}{2})\right]^{J=\frac{7}{2}}$
$H_g^{(6)}$	$\left[D_{\mathrm{M}}^{[2]}(2)\ \ O_{\mathrm{M}}(\tfrac{1}{2})\right]^{J=\frac{3}{2}}$
$H_g^{(7)},\ G_{2g}^{(3)}$	$\left[D_{\mathrm{M}}^{[2]}(2)\ \ O_{\mathrm{M}}(\tfrac{1}{2})\right]^{J=\frac{5}{2}}$

g = even parity, u = odd parity

Construction of *bbb* interpolating fields

On the lattice,

$$D_j\tilde{\psi} \to \nabla_j\tilde{\psi}(\mathbf{x},t) = \frac{1}{2a}\left(\tilde{U}_j(\mathbf{x},t)\tilde{\psi}(\mathbf{x}+a\hat{\mathbf{j}},t) - \tilde{U}_{-j}(\mathbf{x},t)\tilde{\psi}(\mathbf{x}-a\hat{\mathbf{j}},t)\right)$$

The Gaussian smearing is done using

$$\tilde{\psi} = \left(1 + \frac{r_S^2}{2n_S}\Delta^{(2)}\right)^{n_S}\psi$$

with $r_S \approx 0.14$ fm.

Construction of *bbb* interpolating fields

Using the notation $\tilde{\psi}_{a\alpha i}$, where $a = 1, 2, 3$ (color), $\alpha =\uparrow, \downarrow$ (spin), $i = 1, ..., 13$ (derivatives)

$$\begin{aligned}
\tilde{\psi}_{a\alpha 1} &= \tilde{\psi}_{a\alpha}, \\
\tilde{\psi}_{a\alpha 2} &= (\nabla_x \tilde{\psi})_{a\alpha}, \\
\tilde{\psi}_{a\alpha 3} &= (\nabla_y \tilde{\psi})_{a\alpha}, \\
\tilde{\psi}_{a\alpha 4} &= (\nabla_z \tilde{\psi})_{a\alpha}, \\
\tilde{\psi}_{a\alpha 5} &= (\nabla_x \nabla_x \tilde{\psi})_{a\alpha}, \\
\tilde{\psi}_{a\alpha 6} &= (\nabla_y \nabla_x \tilde{\psi})_{a\alpha}, \\
&\vdots \\
\tilde{\psi}_{a\alpha 13} &= (\nabla_z \nabla_z \tilde{\psi})_{a\alpha}.
\end{aligned}$$

we can write all operators as

$$\Omega_\Gamma(\mathbf{x}, t) = \Gamma_{\alpha i\, \beta j\, \gamma k}\;\; \epsilon_{abc}\;\; \tilde{\psi}_{a\alpha i}(\mathbf{x}, t)\, \tilde{\psi}_{b\beta j}(\mathbf{x}, t)\, \tilde{\psi}_{c\gamma k}(\mathbf{x}, t).$$

Lattice actions and parameters

Light quarks and gluons

RBC/UKQCD ensembles, 2+1 flavors of domain-wall fermions, Iwasaki gluon action
[Allton et al. 2007, 2008; Aoki et al. 2011]

$L^3 \times T$	β	$am_{u,d}$	am_s	am_b	a (fm)	m_π (GeV)
$24^3 \times 64$	2.13	0.005	0.04	2.487	0.1119(17)	0.3377(54)
$24^3 \times 64$	2.13	0.01	0.04	2.522	0.1139(19)	0.4194(70)
$24^3 \times 64$	2.13	0.02	0.04	2.622	0.1177(29)	0.541(14)
$24^3 \times 64$	2.13	0.03	0.04	2.691	0.1196(29)	0.641(15)
$32^3 \times 64$	2.25	0.004	0.03	1.831	0.0849(12)	0.2950(40)
$32^3 \times 64$	2.25	0.006	0.03	1.829	0.0848(17)	0.3529(69)
$32^3 \times 64$	2.25	0.008	0.03	1.864	0.0864(12)	0.3950(55)

b quarks: lattice NRQCD

[Thacker and Lepage 1991, Lepage et al. 1992]

$$S_\psi = a^3 \sum_{\mathbf{x},t} \psi^\dagger(\mathbf{x},t)\left[\psi(\mathbf{x},t) - K(t)\,\psi(\mathbf{x},t-a)\right]$$

$$K(t) = \left(1 - \frac{a\,\delta H}{2}\right)\left(1 - \frac{aH_0}{2n}\right)^n U_0^\dagger(t-a)\left(1 - \frac{aH_0}{2n}\right)^n\left(1 - \frac{a\,\delta H}{2}\right).$$

H_0 is the leading-order (order v^2) term

$$H_0 \quad = \quad -\frac{1}{2m_b}\Delta^{(2)},$$

δH contains order-v^4, order-v^6, and Symanzik-improvement terms.

For $b\bar{b}$ and bbb hadrons, one has $v^2 \sim 0.1$

b quarks: lattice NRQCD

[Thacker and Lepage 1991, Lepage et al. 1992]

$$
\begin{aligned}
\delta H &= -c_1 \frac{\left(\Delta^{(2)}\right)^2}{8m_b^3} + c_2 \frac{ig}{8m_b^2} \left(\boldsymbol{\nabla} \cdot \widetilde{\mathbf{E}} - \widetilde{\mathbf{E}} \cdot \boldsymbol{\nabla}\right) \\
&\quad - c_3 \frac{g}{8m_b^2} \boldsymbol{\sigma} \cdot \left(\widetilde{\boldsymbol{\nabla}} \times \widetilde{\mathbf{E}} - \widetilde{\mathbf{E}} \times \widetilde{\boldsymbol{\nabla}}\right) - c_4 \frac{g}{2m_b} \boldsymbol{\sigma} \cdot \widetilde{\mathbf{B}} \\
&\quad + c_5 \frac{a^2 \Delta^{(4)}}{24 m_b} - c_6 \frac{a \left(\Delta^{(2)}\right)^2}{16n\, m_b^2} \\
&\quad - c_7 \frac{g}{8m_b^3} \left\{\Delta^{(2)}, \boldsymbol{\sigma} \cdot \widetilde{\mathbf{B}}\right\} - c_8 \frac{3g}{64m_b^4} \left\{\Delta^{(2)}, \boldsymbol{\sigma} \cdot \left(\widetilde{\boldsymbol{\nabla}} \times \widetilde{\mathbf{E}} - \widetilde{\mathbf{E}} \times \widetilde{\boldsymbol{\nabla}}\right)\right\} \\
&\quad - c_9 \frac{ig^2}{8m_b^3} \boldsymbol{\sigma} \cdot (\widetilde{\mathbf{E}} \times \widetilde{\mathbf{E}}).
\end{aligned}
$$

Matching to QCD gives $c_i = 1 + \mathcal{O}(\alpha_s)$. Radiative corrections small thanks to tadpole improvement. Use tree-level for most couplings, but tune c_3 and c_4 nonperturbatively.

b quarks: lattice NRQCD

Nonperturbative tuning of c_3 and c_4 [Gray et al. 2005, Dowdall et al. 2011]): adjust so that the following two combinations of bottomonium P-wave states agree with experiment

$$-2E(\chi_{b0}) - 3E(\chi_{b1}) + 5E(\chi_{b2}) \propto c_3$$

$$-2E(\chi_{b0}) + 3E(\chi_{b1}) - E(\chi_{b2}) \propto c_4^2$$

This gives

$$c_3 = \begin{cases} 1.196 \pm 0.106, & a \approx 0.11 \ \mathrm{fm}, \\ 1.175 \pm 0.084, & a \approx 0.08 \ \mathrm{fm}, \end{cases}$$

$$c_4 = \begin{cases} 1.168 \pm 0.081, & a \approx 0.11 \ \mathrm{fm}, \\ 1.113 \pm 0.053, & a \approx 0.08 \ \mathrm{fm}. \end{cases}$$

Fitting of the two-point functions and identifying J

Matrix two-point functions

Recall that

Operator(s) $\Lambda^{(i)}$	Structure $\sim \left[D(L)\ \ O(S)\right]^J$
$H_g^{(1)}$	$\left[O_\mathrm{S}(\tfrac{3}{2})\right]^{J=\frac{3}{2}}$
$G_{1u}^{(1)}$	$\left[D_\mathrm{M}^{[1]}(1)\ \ O_\mathrm{M}(\tfrac{1}{2})\right]^{J=\frac{1}{2}}$
$H_u^{(1)}$	$\left[D_\mathrm{M}^{[1]}(1)\ \ O_\mathrm{M}(\tfrac{1}{2})\right]^{J=\frac{3}{2}}$
$H_g^{(2)}$	$\left[D_\mathrm{S}^{[2]}(0)\ \ O_\mathrm{S}(\tfrac{3}{2})\right]^{J=\frac{3}{2}}$
$G_{1g}^{(1)}$	$\left[D_\mathrm{M}^{[2]}(0)\ \ O_\mathrm{M}(\tfrac{1}{2})\right]^{J=\frac{1}{2}}$
$G_{1g}^{(2)}$	$\left[D_\mathrm{S}^{[2]}(2)\ \ O_\mathrm{S}(\tfrac{3}{2})\right]^{J=\frac{1}{2}}$
$H_g^{(3)}$	$\left[D_\mathrm{S}^{[2]}(2)\ \ O_\mathrm{S}(\tfrac{3}{2})\right]^{J=\frac{3}{2}}$
$H_g^{(4)}$, $G_{2g}^{(1)}$	$\left[D_\mathrm{S}^{[2]}(2)\ \ O_\mathrm{S}(\tfrac{3}{2})\right]^{J=\frac{5}{2}}$
$H_g^{(5)}$, $G_{1g}^{(3)}$, $G_{2g}^{(2)}$	$\left[D_\mathrm{S}^{[2]}(2)\ \ O_\mathrm{S}(\tfrac{3}{2})\right]^{J=\frac{7}{2}}$
$H_g^{(6)}$	$\left[D_\mathrm{M}^{[2]}(2)\ \ O_\mathrm{M}(\tfrac{1}{2})\right]^{J=\frac{3}{2}}$
$H_g^{(7)}$, $G_{2g}^{(3)}$	$\left[D_\mathrm{M}^{[2]}(2)\ \ O_\mathrm{M}(\tfrac{1}{2})\right]^{J=\frac{5}{2}}$

Matrix two-point functions

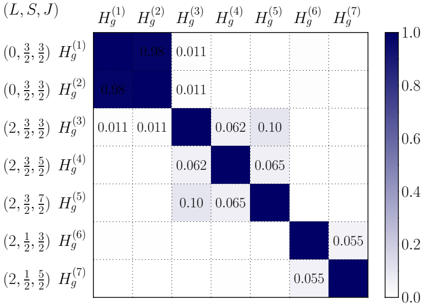
For each irrep Λ, the matrix of two-point functions, averaged over rows,

$$C_{ij}^{(\Lambda)}(t-t') = \frac{1}{\dim(\Lambda)} \sum_{r=1}^{\dim(\Lambda)} \left\langle \sum_{\mathbf{x}} \Omega_{\Lambda_r^{(i)}}(\mathbf{x}, t)\, \Omega^{\dagger}_{\Lambda_r^{(j)}}(0) \right\rangle$$

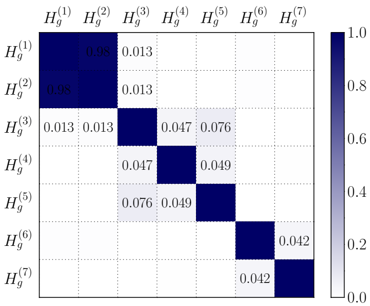
H_g : 7×7
G_{1g}: 3×3
G_{2g}: 3×3
H_u : 1×1
G_{1u}: 1×1

Matrix two-point functions at one time slice: H_g irrep

Plots of $|C_{ij}^{(\Lambda)}|/\sqrt{C_{ii}^{(\Lambda)}C_{jj}^{(\Lambda)}}$ at one time slice



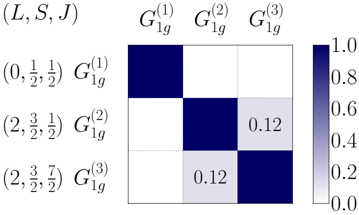
$a \approx 0.112$ fm, $(t-t')/a = 5$



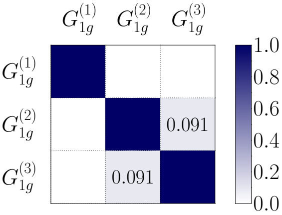
$a \approx 0.085$ fm, $(t-t')/a = 6$

Matrix two-point functions at one time slice: G_{1g} irrep

Plots of $|C_{ij}^{(\Lambda)}|/\sqrt{C_{ii}^{(\Lambda)} C_{jj}^{(\Lambda)}}$ at one time slice



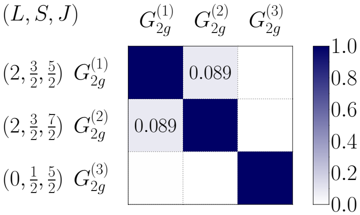
$a \approx 0.112$ fm, $(t - t')/a = 5$



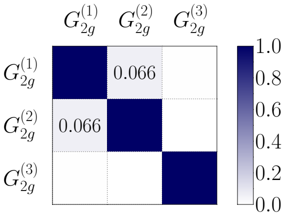
$a \approx 0.085$ fm, $(t - t')/a = 6$

Matrix two-point functions at one time slice: G_{2g} irrep

Plots of $|C_{ij}^{(\Lambda)}|/\sqrt{C_{ii}^{(\Lambda)} C_{jj}^{(\Lambda)}}$ at one time slice



$a \approx 0.112$ fm, $(t - t')/a = 5$



$a \approx 0.085$ fm, $(t - t')/a = 6$

Fitting the matrix two-point functions

For each irrep Λ, fit using

$$C_{ij}^{(\Lambda)}(t - t') = \sum_{n=1}^{N} A_{n,i}^{(\Lambda)} \, A_{n,j}^{(\Lambda)} \, e^{-E_n^{(\Lambda)}(t-t')}$$

where the number of exponentials, N, is equal to the number of interpolating fields in that irrep.

Choose $t - t' \geq t_{\mathrm{min}}$ with t_{min} large enough so that contamination from higher states is negligible.

Use values of $A_{n,j}^{(\Lambda)} = \langle 0 | \Omega_{\Lambda^{(i)}} | n(\Lambda) \rangle$ to assign a continuum angular momentum to the energy level $E_n^{(\Lambda)}$ [Edwards et al. 2011]

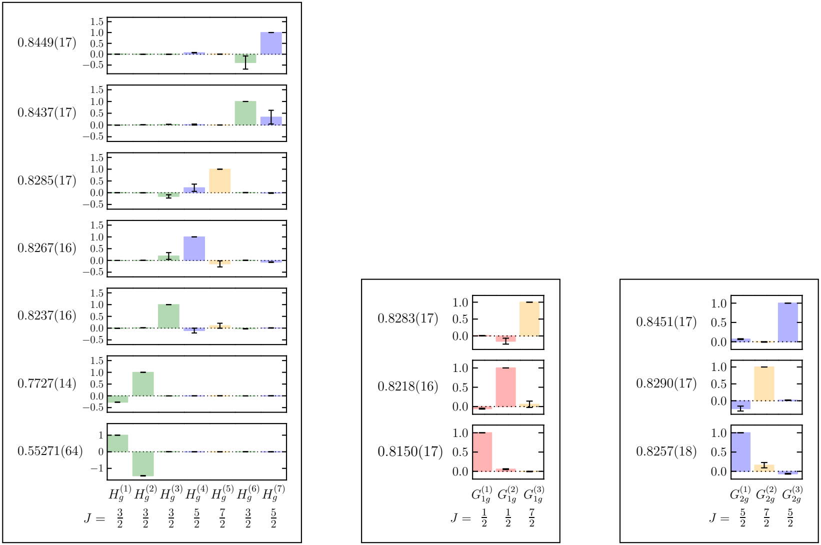
Angular momentum identification

For each energy level $E_n^{(\Lambda)}$, the operator $\Lambda^{(i)}$ with the largest relative overlap factor $A_{n,i}^{(\Lambda)}/A_i^{(\Lambda)}$ is determined.

The value of J from which this operator was subduced is then assigned to this energy level.

Angular momentum identification

Fitted energies (in lattice units) and values of $A_{n,i}^{(\Lambda)}/A_i^{(\Lambda)}$:



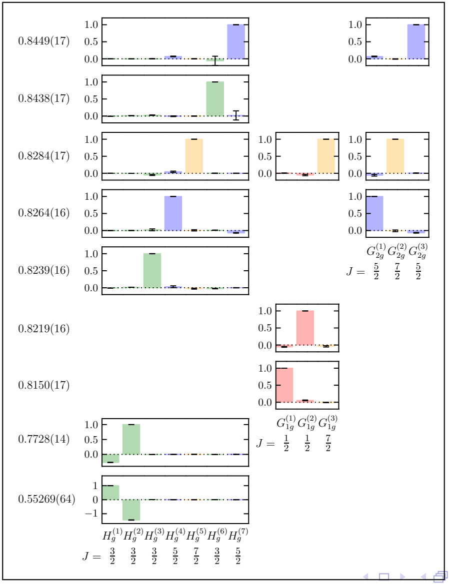
Splitting of one continuum level into different lattice irreps

Continuum energy level with $J > 3/2$ split into multiple levels on the lattice. These splittings vanish in the continuum limit, where $SU(2)$ rotational symmetry is restored.

Continuum J^P	Splitting	$a \approx 0.11$ fm	$a \approx 0.08$ fm
$\frac{5}{2}^+$	$E_4^{(H_g)} - E_1^{(G_{2g})}$	5.8(2.0) MeV	2.5(2.0) MeV
$\frac{5}{2}^+$	$E_3^{(G_{2g})} - E_7^{(H_g)}$	0.70(44) MeV	0.44(64) MeV
$\frac{7}{2}^+$	$E_2^{(G_{2g})} - E_3^{(G_{1g})}$	2.1(1.1) MeV	1.6(1.4) MeV
$\frac{7}{2}^+$	$E_5^{(H_g)} - E_3^{(G_{1g})}$	1.49(78) MeV	0.38(79) MeV
$\frac{7}{2}^+$	$E_2^{(G_{2g})} - E_5^{(H_g)}$	0.59(45) MeV	1.24(72) MeV

Coupled fit of H_g, G_{1g}, and G_{2g} two-point functions

Enforce equality of matching levels in the fit model:



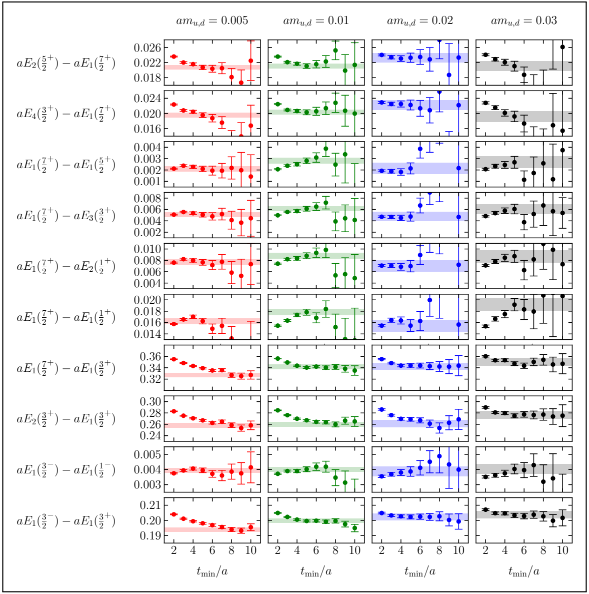
Coupled fit of H_g, G_{1g}, and G_{2g} two-point functions

Having identified the angular momenta, we can now rename the extracted energies as $E_n(J^P)$ where the subscript counts the states in each J^P channel by increasing energy

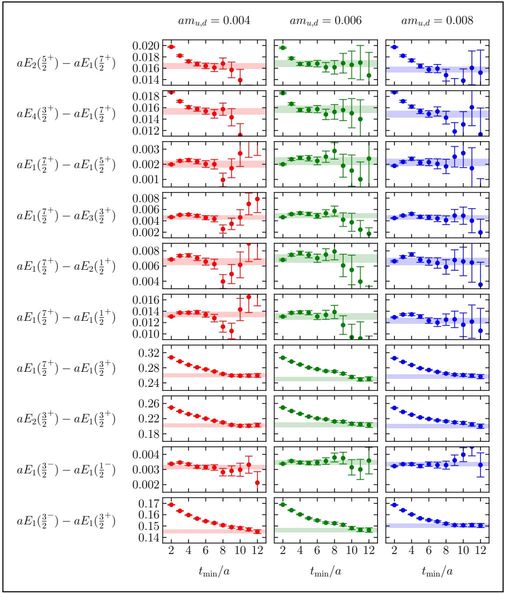
$$
\begin{aligned}
&E_1(\tfrac{1}{2}^+),\ E_2(\tfrac{1}{2}^+),\\
&E_1(\tfrac{3}{2}^+),\ E_2(\tfrac{3}{2}^+),\ E_3(\tfrac{3}{2}^+),\ E_4(\tfrac{3}{2}^+),\\
&E_1(\tfrac{5}{2}^+),\ E_2(\tfrac{5}{2}^+),\\
&E_1(\tfrac{7}{2}^+),\\
&E_1(\tfrac{1}{2}^-),\\
&E_1(\tfrac{3}{2}^-).
\end{aligned}
$$

Note: since NRQCD is used here, all energies are shifted by approx $-3m_b$. This shift cancels in energy differences.

t_{min}-dependence at $a \approx 0.11$ fm



$t_{\min}$-dependence at $a \approx 0.08$ fm



Chiral extrapolation and final results

Chiral extrapolation

Use m_π^2 as proxy for light sea-quark mass $m_{u,d}$. Let $E(m_\pi^2,\ \beta)$ denote some *bbb* energy splitting. Have data at

$$\beta_1 = 2.25: \qquad a \approx 0.08\ \text{fm}$$
$$\beta_2 = 2.13: \qquad a \approx 0.11\ \text{fm}$$

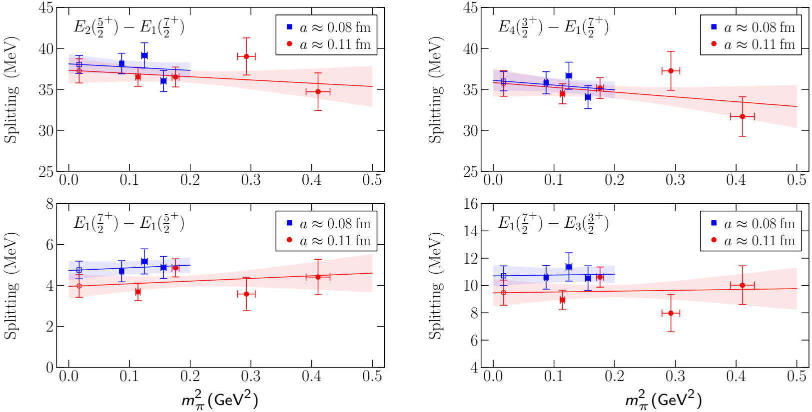
Fit data using

$$\begin{aligned} E(m_\pi^2,\ \beta_1) &= E(0, \beta_1) + A\, m_\pi^2, \\ E(m_\pi^2,\ \beta_2) &= E(0, \beta_2) + A\, m_\pi^2, \end{aligned}$$

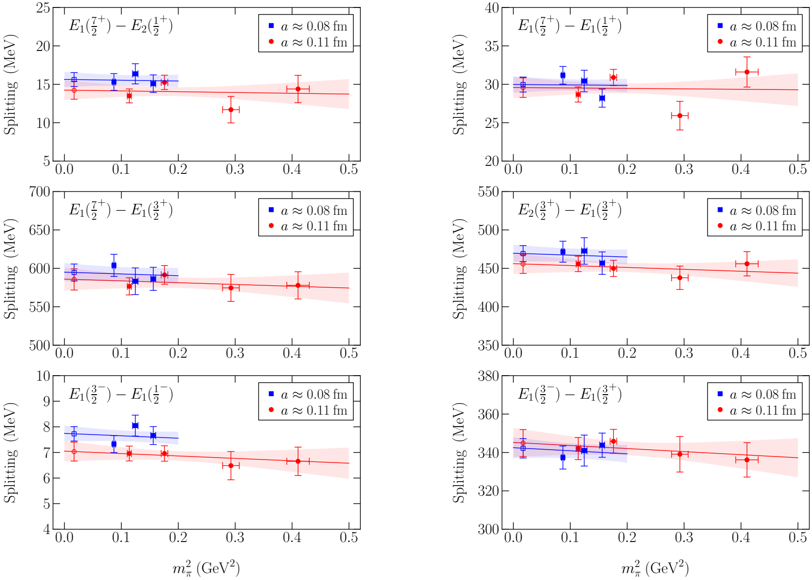
with three parameters $E(0, \beta_1)$, $E(0, \beta_2)$, and A.

No continuum extrapolation.

Chiral extrapolation



Chiral extrapolation



The bbb spectrum at $m_\pi = 138$ MeV

	$a \approx 0.11$ fm	$a \approx 0.08$ fm	Final result
$E_1(\frac{1}{2}^+) - E_1(\frac{3}{2}^+)$	556(15)	565(12)	$565 \pm 12 \pm 12$
$E_2(\frac{1}{2}^+) - E_1(\frac{3}{2}^+)$	571(15)	579(12)	$579 \pm 12 \pm 13$
$E_2(\frac{3}{2}^+) - E_1(\frac{3}{2}^+)$	456(12)	469(10)	$469 \pm 10 \pm 9$
$E_3(\frac{3}{2}^+) - E_1(\frac{3}{2}^+)$	576(15)	584(12)	$584 \pm 12 \pm 12$
$E_4(\frac{3}{2}^+) - E_1(\frac{3}{2}^+)$	621(15)	631(12)	$631 \pm 12 \pm 13$
$E_1(\frac{5}{2}^+) - E_1(\frac{3}{2}^+)$	582(14)	590(12)	$590 \pm 12 \pm 12$
$E_2(\frac{5}{2}^+) - E_1(\frac{3}{2}^+)$	623(15)	633(12)	$633 \pm 12 \pm 13$
$E_1(\frac{7}{2}^+) - E_1(\frac{3}{2}^+)$	586(14)	595(11)	$595 \pm 11 \pm 12$
$E_1(\frac{1}{2}^-) - E_1(\frac{3}{2}^+)$	337.9(7.4)	334.5(5.3)	$334.5 \pm 5.3 \pm 7.4$
$E_1(\frac{3}{2}^-) - E_1(\frac{3}{2}^+)$	344.9(7.0)	342.2(5.1)	$342.2 \pm 5.1 \pm 7.2$
$E_1(\frac{1}{2}^+) - E_1(\frac{7}{2}^+)$	−29.6(1.3)	−29.96(98)	$-29.96 \pm 0.98 \pm 0.76$
$E_2(\frac{1}{2}^+) - E_1(\frac{7}{2}^+)$	−14.2(1.2)	−15.61(91)	$-15.6 \pm 0.9 \pm 1.6$
$E_3(\frac{3}{2}^+) - E_1(\frac{7}{2}^+)$	−9.47(92)	−10.72(72)	$-10.7 \pm 0.7 \pm 1.2$
$E_4(\frac{3}{2}^+) - E_1(\frac{7}{2}^+)$	35.7(1.6)	36.0(1.2)	$36.0 \pm 1.2 \pm 1.4$
$E_1(\frac{5}{2}^+) - E_1(\frac{7}{2}^+)$	−3.98(55)	−4.76(43)	$-4.76 \pm 0.43 \pm 0.55$
$E_2(\frac{5}{2}^+) - E_1(\frac{7}{2}^+)$	37.3(1.5)	38.0(1.1)	$38.0 \pm 1.1 \pm 1.1$
$E_1(\frac{1}{2}^-) - E_1(\frac{3}{2}^-)$	−7.03(37)	−7.73(28)	$-7.73 \pm 0.28 \pm 0.90$
$E_4(\frac{3}{2}^+) - E_2(\frac{5}{2}^+)$	−1.54(59)	−2.05(46)	$-2.05 \pm 0.46 \pm 0.59$

All in MeV.

Estimates of systematic uncertainties

Computed individually for each energy splitting E using

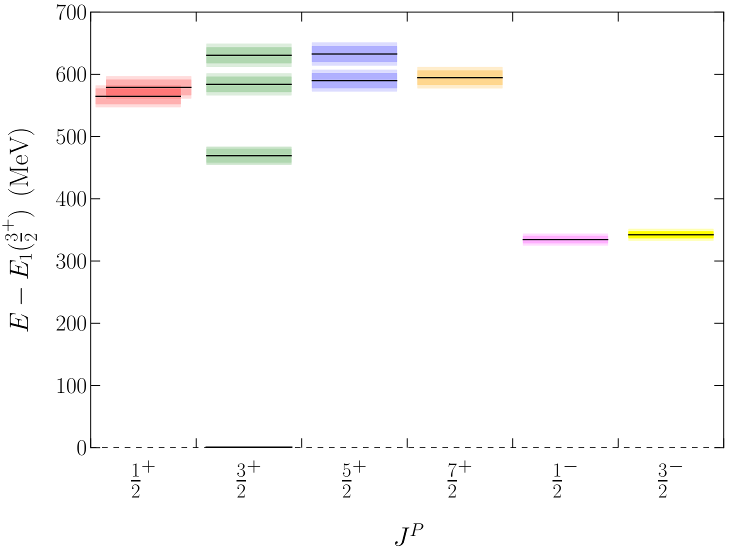
$$\sigma_E^{(\mathrm{syst})} = \left[\left(\frac{\partial E}{\partial c_3} \right)^2 \sigma_{c_3}^2 + \left(\frac{\partial E}{\partial c_4} \right)^2 \sigma_{c_4}^2 + \Big(0.02\, E_{\mathrm{SI}} \Big)^2 + \Big(0.07\, (E - E_{\mathrm{SI}}) \Big)^2 \right]^{1/2},$$

where

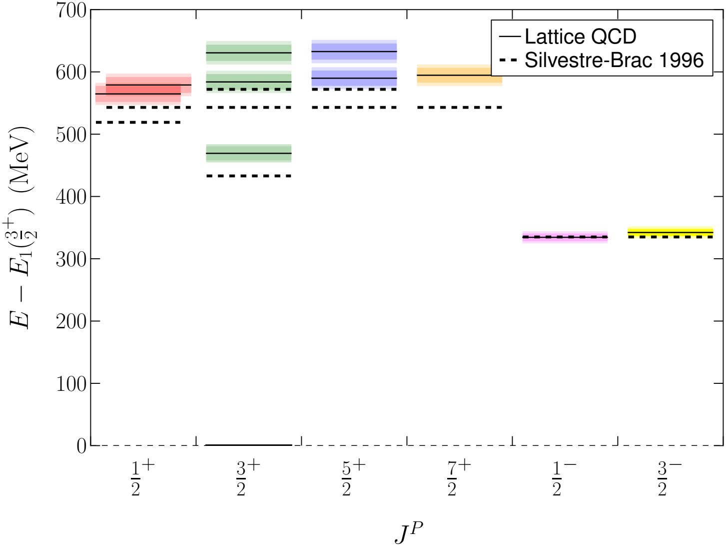
E_{SI} = spin-independent contribution, computed by setting
$c_3 = c_4 = c_7 = c_8 = c_9 = 0$.

Derivatives w.r.t c_3 and c_4 approximated using discrete differences

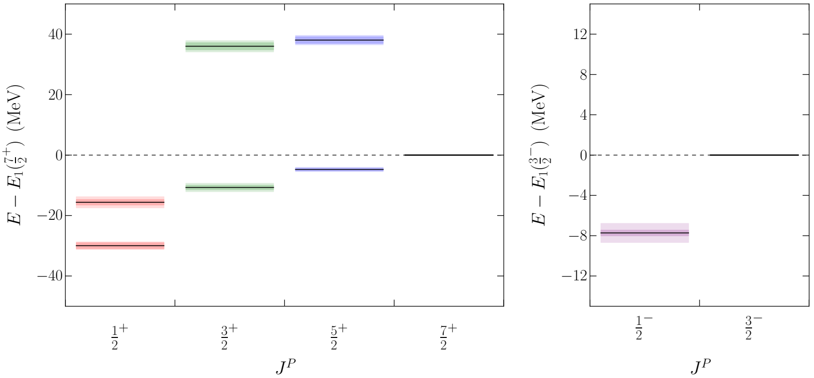
The bbb spectrum at $m_\pi = 138$ MeV



The bbb spectrum at $m_\pi = 138$ MeV



The bbb spectrum at $m_\pi = 138$ MeV



Contributions of individual NRQCD operators

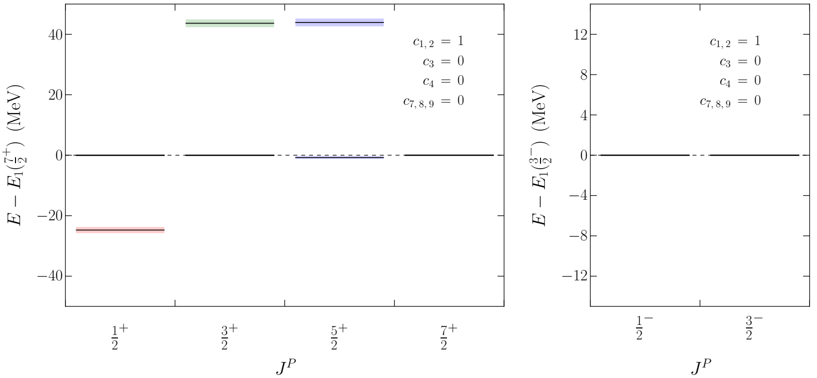
Contributions of individual NRQCD operators

Recall:

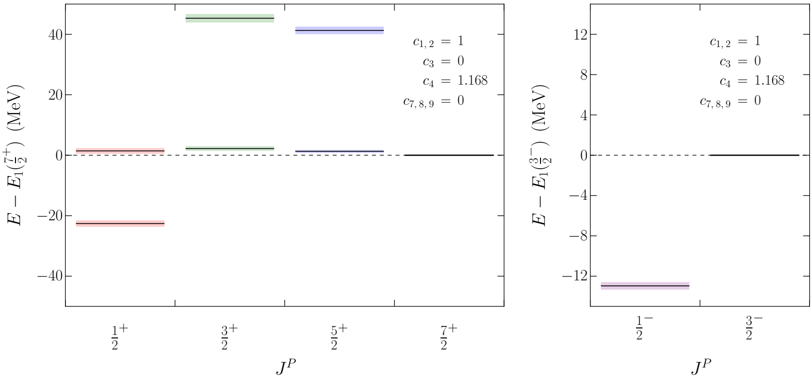
$$
\begin{aligned}
\delta H \quad = \quad & -c_1 \frac{\left(\Delta^{(2)}\right)^2}{8m_b^3} + c_2 \frac{ig}{8m_b^2}\left(\nabla\cdot\widetilde{\mathbf{E}} - \widetilde{\mathbf{E}}\cdot\nabla\right) \\
& -c_3 \frac{g}{8m_b^2}\,\boldsymbol{\sigma}\cdot\left(\widetilde{\nabla}\times\widetilde{\mathbf{E}} - \widetilde{\mathbf{E}}\times\widetilde{\nabla}\right) - c_4 \frac{g}{2m_b}\,\boldsymbol{\sigma}\cdot\widetilde{\mathbf{B}} \\
& +c_5 \frac{a^2\Delta^{(4)}}{24m_b} - c_6 \frac{a\left(\Delta^{(2)}\right)^2}{16n\, m_b^2} \\
& -c_7 \frac{g}{8m_b^3}\left\{\Delta^{(2)},\, \boldsymbol{\sigma}\cdot\widetilde{\mathbf{B}}\right\} - c_8 \frac{3g}{64m_b^4}\left\{\Delta^{(2)},\, \boldsymbol{\sigma}\cdot\left(\widetilde{\nabla}\times\widetilde{\mathbf{E}} - \widetilde{\mathbf{E}}\times\widetilde{\nabla}\right)\right\} \\
& -c_9 \frac{ig^2}{8m_b^3}\,\boldsymbol{\sigma}\cdot(\widetilde{\mathbf{E}}\times\widetilde{\mathbf{E}}).
\end{aligned}
$$

In the following, set some coefficients c_i to zero. Done on one ensembles of gauge fields only to save computer time ($a \approx 0.11$ fm, $am_{u,d} = 0.005$).

Case 1: all spin-dependent NRQCD interactions turned off

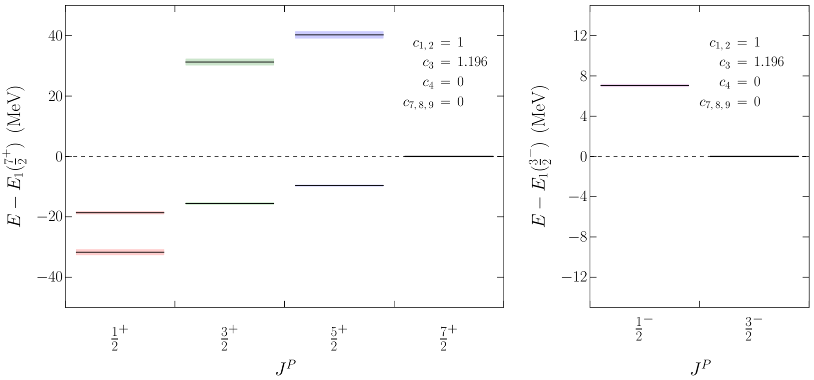


Case 2: $\boldsymbol{\sigma} \cdot \widetilde{\mathbf{B}}$ turned on



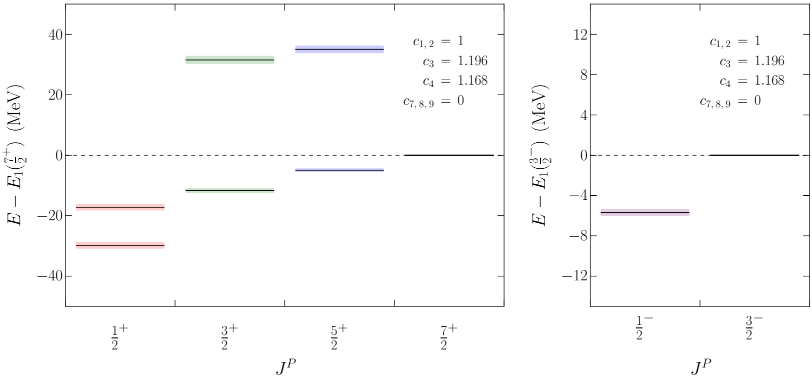
Note: potential models for baryons that include only spin-spin and tensor interactions predict $E_1(\frac{1}{2}^-) - E_1(\frac{3}{2}^-) = 0$ [Isgur and Karl 1978, Chao et al. 1980, Gromes 1982]

Case 3: $\boldsymbol{\sigma} \cdot (\widetilde{\boldsymbol{\nabla}} \times \widetilde{\mathbf{E}} - \widetilde{\mathbf{E}} \times \widetilde{\boldsymbol{\nabla}})$ turned on

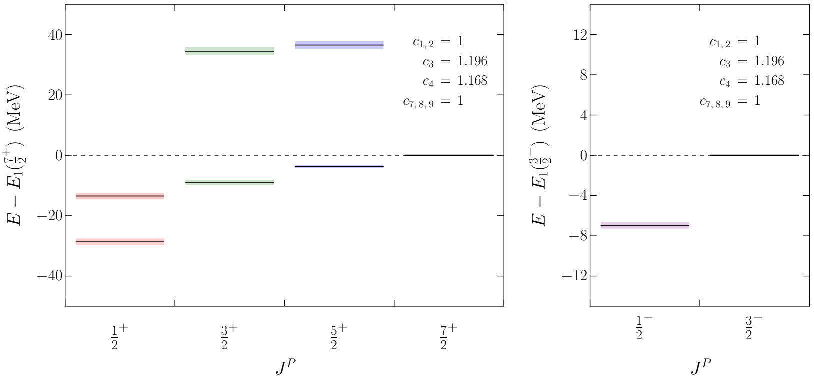


Note: splitting of levels with totally symmetric spatial wavefunctions are approximately proportional to $2\,\mathbf{L} \cdot \mathbf{S} = J(J+1) - L(L+1) - S(S+1)$, as expected for a pure spin-orbit interaction [Gromes, 1976]

Case 4: order-v^4 action



Case 4: order-v^6 action



Note: most spin splittings decreased compared to v^4,
but $E_1(\frac{1}{2}^-) - E_1(\frac{3}{2}^-) = 0$ increased

Summary

- first nonperturbative QCD calculation of the baryonic analogue of the bottomonium spectrum
- ${}^2O$-subduced interpolating fields have strong "memory" of original J (as seen for light baryons by Edwards et al.) and, even more so, of original L and S (new feature for heavy baryons)
- demonstrated improvement of rotational symmetry breaking as lattice spacing is reduced: decrease of cross-correlation between operators subduced from different J states, decrease of lattice-induced level splittings
- investigated contributions from individual NRQCD operators to spin splittings (perhaps useful input for quark models/pNRQCD?)