D20b

MATHEMATICAL TRIPOS PART II *Dr M Wingate*

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Statistical Physics - Examples II *Lent 2007*

1. Making use of the fact that the free energy $F(T,V,N)$ of a thermodynamic system must be extensive, explain why

$$F \;=\; V\left(\frac{\partial F}{\partial V}\right)_{T,N} \;+\; N\left(\frac{\partial F}{\partial N}\right)_{T,V} \quad .$$

Use this fact to show that the Gibbs free energy G can be expressed as $G = \mu N$. Explain why this result was to be expected from the scaling behavior of G.

2. A gas consists of four types of particle and the average number of each type is fixed, so that when $\ln W$ is maximised additional Lagrange multipliers are needed. Show that

$$dE \;=\; TdS \;-\; pdV \;+\; \sum_{\alpha=1}^{4} \mu_\alpha dN_\alpha \quad .$$

Suppose instead there is a chemical reaction such that when a particle of type 1 collides with a particle of type 2 they can transform into a particle of type 3 and one of type 4, and vice versa:

$$1 \;+\; 2 \;\leftrightarrow\; 3 \;+\; 4 \quad .$$

Deduce that one fewer Lagrange multiplier is needed, and show that the expression for dE may be retained provided that the constraint $\mu_1 + \mu_2 \;=\; \mu_3 + \mu_4$ is applied.

3. Show that the grand partition function $\mathcal{Z}$ is related to the canonical partition Z_N for N particles by

$$\mathcal{Z}(\mu,V,T) = \sum_{N=0}^{\infty} \lambda^N Z_N(V,T) \quad , \quad \lambda = e^{\beta\mu} \quad .$$

Show that

$$\langle N\rangle = \lambda\frac{\partial}{\partial\lambda}\ln\mathcal{Z} \quad , \quad (\Delta N)^2 = \left(\lambda\frac{\partial}{\partial\lambda}\right)^2 \ln\mathcal{Z} \quad .$$

If $Z_N = z^N/N!$ show that $\mathcal{Z}(\lambda,V,T) = e^{\lambda z(V,T)}$. For this case, show also that

$$\frac{\Delta N}{\langle N\rangle} \;=\; \frac{1}{\langle N\rangle^{1/2}} \quad .$$

4. A system $\mathcal{S}$ has two 1-particle energy levels with energies 0 and ϵ. These can be occupied by fermions from a particle and heat bath with temperature T and chemical potential μ. The fermions are non-interacting. Show that there are four possible microstates, and show that the grand partition function is

$$\mathcal{Z}(\mu,V,T) \;=\; 1 \;+\; \lambda \;+\; \lambda e^{-\epsilon/kT} \;+\; \lambda^2 e^{-\epsilon/kT} \quad , \quad \text{where } \lambda = e^{\mu/kT} \,.$$

Evaluate the average occupation number of the state of energy ϵ, and show that this is compatible with the result of the calculation of the average energy of the system using the Fermi-Dirac distribution.

How could you take account of fermion interactions?

5. In an ideal Fermi gas the average occupation numbers of the single particle states (labelled by r) are $\bar{n}_r$. Show that the expression for the entropy

$$S = -k \sum_r [(1-\bar{n}_r)\ln(1-\bar{n}_r) + \bar{n}_r \ln \bar{n}_r]$$

agrees with the result

$$S = \frac{\partial}{\partial T}(kT \ln \mathcal{Z})_{\mu,V} .$$

Find the corresponding expression for an ideal Bose gas. (It involves some sign changes.)

6. With $\bar{n}_r$ as in question 5 show that $(\Delta n_r)^2 = \bar{n}_r(1-\bar{n}_r)$ for the ideal Fermi gas. Comment on this, especially for very low T. What is the corresponding result for an ideal Bose gas?

7. Consider a gas of non-interacting ultra-relativistic electrons, whose mass may be neglected. Find an integral for the grand potential Ω. Show that $3pV = E$. Show that at $T = 0, \quad pV^{4/3} = \text{const}$. Also show that $E = 3NkT$ in the non-degenerate limit (high T) and note the equation of state.

8. By considering the total energy and total number of photons in black body radiation, find the mean photon energy as a function of temperature. By considering the Planck distribution, find the most likely energy of a photon in black body radiation at temperature T (the mode of the distribution).

9. A black body at temperature T absorbs all the radiation that falls on it, and emits radiation at the rate $\mathcal{E} = \sigma T^4$ per unit area, where σ is Stefan's constant.

A black, perfectly conducting spherical planet orbits, at a distance of 1.5×10^8km, a star of radius 7×10^5km. The star radiates like a black body of temperature 6000K. Estimate the temperature of the sphere.

10. Consider a region of volume V in the cosmos containing black body radiation of temperature T. Suppose the cosmos expands (slowly) by a scale factor α, so that the wavevector $\vec{k}$ and angular frequency ω of each electromagnetic radiation mode are rescaled by $1/\alpha$. Explain why you should expect the mean number of photons in each mode not to change. Show that the Planck distribution is valid after the expansion provided the temperature is also rescaled by $1/\alpha$.

Verify, from the formula for the entropy of black body radiation, that the entropy in the expanded volume is the same as the original entropy, thus confirming the adiabatic character of the expansion.

[This explains why the microwave background radiation in the cosmos still has a black body spectrum, even though it has not been in thermal equilibrium with matter since very early in the universe's history.]