D20c

MATHEMATICAL TRIPOS PART II *Dr M Wingate*

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Statistical Physics - Examples III *Lent 2007*

1. Consider a quantum harmonic oscillator of angular frequency ω, and at temperature T, and discard the zero point energy, so the energy levels are $\varepsilon_n = n\hbar\omega$. Show that this system is thermodynamically equivalent to the Planck oscillator, which has just a single energy level $\hbar\omega$ that can be occupied by any number of particles n, in contact with a heat and particle reservoir. What is the temperature and chemical potential of the reservoir?

The vibrational modes of a violin string can be regarded as a set of Planck oscillators with frequencies $\omega, 2\omega, 3\omega, \ldots$. Find the expectation value of the thermodynamic energy of the string. Does the sum converge?

2. Let

$$I_n = \int_0^\infty \frac{x^n}{e^x - 1}\, dx\,,$$

where $n > 0$. By expanding $(e^x - 1)^{-1}$ as $e^{-x} + e^{-2x} + e^{-3x} + \cdots$, show that $I_n = n!\zeta(n+1)$, where $\zeta(n) = \sum_{r=1}^\infty r^{-n}$ is the Riemann zeta function. Assume (or prove, using a contour integral method) that if the Taylor expansion of $\pi z \cot \pi z$ about the origin is $\sum\limits_{m=0}^{\infty} a_m z^m$ then

$$a_{2m} = -2\sum_{r=1}^{\infty} r^{-2m} \qquad (m \geq 1).$$

Hence find $\zeta(4)$, and the value of I_3.

3. In 1912 Debye derived a formula for the specific heat of a solid of N atoms occupying volume V. He viewed the solid as a continuous medium in which there exist elastic waves, with $g(\omega)d\omega$ wave modes in the frequency range ω to $\omega + d\omega$, where

$$g(\omega) = \frac{3\omega^2 V}{2\pi^2 c_s^3}\,.$$

(The factor 3 arises from the simplification in which the two transverse modes and the one longitudinal wave mode of each frequency are assumed to have the same speed c_s.) Debye also assumed that the spectrum of ω was cut off at a maximum value $\omega_{\max}$ so chosen that the total number of modes is equal to $3N$, the number appropriate to a solid of N atoms, and that the state of thermal equilibrium of the solid was governed by the analogue of the Planck distribution.

Show that it follows that the total energy of the solid is given by

$$E = 3NkT\, D(T_D/T)\,, \quad kT_D = \hbar\omega_{\max}\,, \quad D(z) = \frac{3}{z^3}\int_0^z \frac{x^3\, dx}{e^x - 1}\,.$$

Show that the heat capacity of the solid is given by (Dulong and Petit's) classical result $3Nk$ for $T \gg T_D$, and is proportional to T^3 for $T \ll T_D$.

4. Consider the condensed state of an ideal gas of N Bose particles in a cubic box of side L at a temperature T, where $0 < T < T_c$. Show that $T_c \propto N^{2/3}/L^2$ and that the 1-particle energy levels $\varepsilon_n \propto L^{-2}$. Show that the mean occupancy of the first few excited 1-particle states is large, but not as large as $O(N)$.

5. As a simple model of a semiconductor, suppose that there are N bound electron states per unit volume, having energy $-\Delta\varepsilon$, which are filled at zero temperature. At non-zero

temperature some electrons are excited into the 'conduction band' which is a set of states of continuous positive energies ε, where the density of states is given by $g(\varepsilon)d\varepsilon = A\sqrt{\varepsilon}d\varepsilon$ where A is a constant. Show that at temperature T the number n of excited electrons is determined by the pair of equations

$$n = \frac{N}{e^{(\mu+\Delta\varepsilon)/kT}+1} = \int_0^\infty \frac{g(\varepsilon)\,d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}\,.$$

Show also that, if $n \ll N$ and T is so small that $e^{\mu/kT} \ll 1$,

$$2\mu \approx -\Delta\varepsilon + kT\ln\left[\frac{2N}{A\sqrt{\pi(kT)^3}}\right].$$

6. Consider an almost degenerate Fermi gas of electrons at low temperature. Using the formula for N and the expansion of the Fermi-Dirac function $f_{\frac{3}{2}}(e^\xi)$ for $\xi \gg 0$, show that

$$\xi = \frac{\varepsilon_F}{kT}\left(1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2+\cdots\right).$$

Deduce that the energy of the electron gas (formula in terms of $f_{\frac{5}{2}}(e^\xi)$) is

$$E = \frac{3}{5}N\varepsilon_F\left(1+\frac{5\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2+\cdots\right).$$

7. A classical system has Hamiltonian

$$H = \sum_{i=1}^N\left[\frac{p_i^2}{2m}+\lambda\,q_i^4\right] \quad ,$$

where p_i is the momentum conjugate to q_i and λ is a positive constant. Find the specific heat of the system. (Answer: $\frac{3}{4}k$ per particle.)

8. Consider a perfect classical gas of diatomic molecules for which each molecule has a magnetic moment m aligned along its axis. Let there be a magnetic field B, so that each molecule has a potential energy $-mB\cos\theta$ (θ being the angle between the axis of the molecule and the magnetic field). Show that the rotational part of the partition function is $Z_{\rm rot} = (z_{\rm rot})^N$ where

$$z_{\rm rot} = \left[\frac{8\pi^2 I}{h^2 mB\beta^2}\right]\sinh(m\beta B) \quad .$$

Show that the average value of the total magnetisation is $kT\frac{\partial}{\partial B}\ln Z_{\rm rot}$. Evaluate this and sketch its dependence upon $m\beta B$. Show that, for large $m\beta B$ (*i.e.* for $kT \ll mB$ at fixed B or $mB \gg kT$ at fixed T), the average value of the potential energy is $NkT - NmB(1+2e^{-2m\beta B}+\ldots)$.

9. A classical, non-perfect gas of N monatomic molecules has Hamiltonian

$$H = \sum_{r=1}^N\left[\frac{1}{2m}\mathbf{p}_r^2 + \sum_{s=r+1}^N \phi(R_{rs})\right]$$

where $R_{rs} = |\mathbf{x}_r - \mathbf{x}_s|$. Show that the partition function $Z = Z_0 K$ where Z_0 is the partition function for a perfect gas and

$$V^N K = \int\prod_{r=1}^N d^3\mathbf{x}_r\,\exp\left[-\beta\sum_{s=r+1}^N\phi(R_{rs})\right].$$

Let $\lambda_{rs} = \exp[-\beta\phi(R_{rs})]-1$ and work to first order in the λ's. Thus get $K \approx 1+\frac{N^2}{2V}f(T)$ where $f(T) = \int d^3\mathbf{y}\,[e^{-\beta\phi(y)}-1]$, $y = |\mathbf{y}|$. Deduce that the pressure $P \approx \frac{NkT}{V}[1-\frac{N}{2V}f(T)]$. Comment on the connection between the sign of ϕ and the sign of f.