Natural Sciences Tripos

Part IA Mathematics - Course A

Mathematical Methods I

Examples Sheet 1

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Michaelmas Term 2023

This sheet provides exercises covering the material contained in the first half of the Michaelmas Term. **ALL** questions should be attempted by students attending Course A lectures.

Basic Skills questions in each section should either be revision of work completed at A-level or provide an opportunity to practise simple examples of the main concepts. These questions can be attempted at anytime and should be relatively quick to complete. Numerical answers to these questions are provided at the back of this sheet.

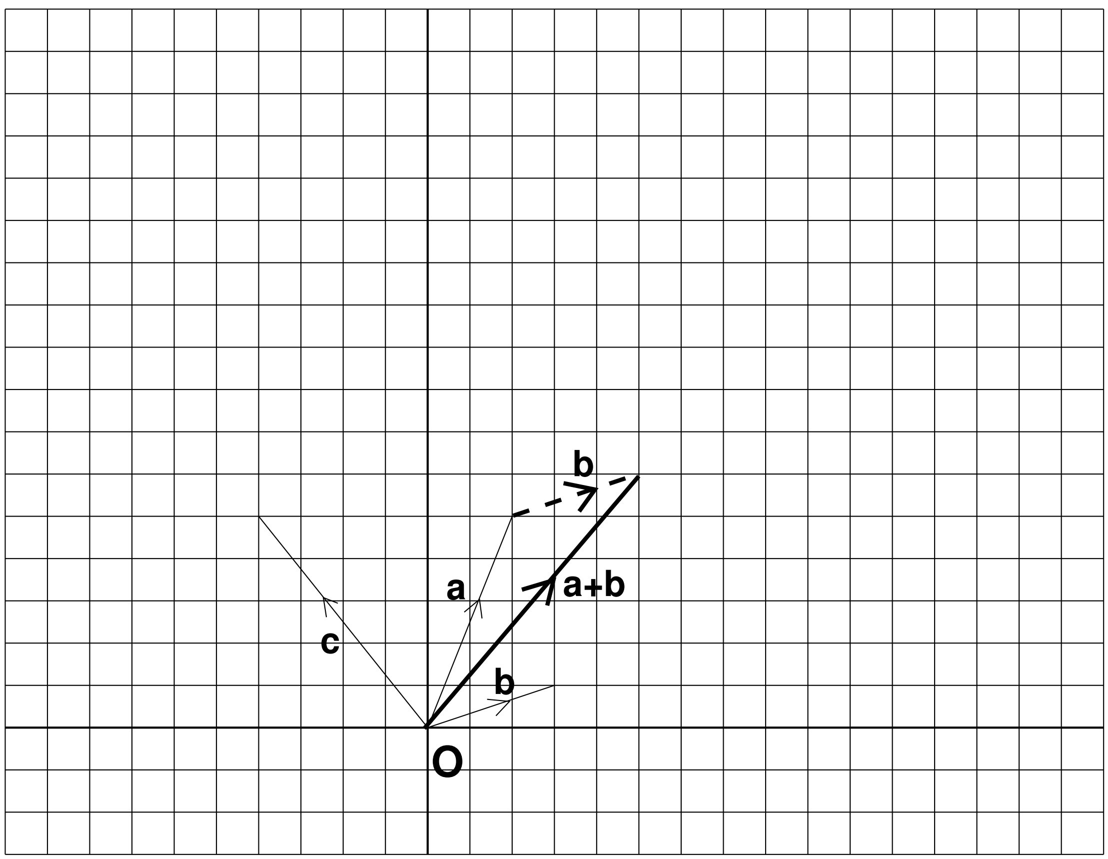
A. Vector Addition and Subtraction

Basic Skills

1. Evaluate the following vector additions and subtractions and represent all vectors in questions (a) - (g) on the graphical diagram below (question (a) has been done as an example).

   Take $\boldsymbol{a} = (2, 5)$; $\boldsymbol{b} = (3, 1)$; $\boldsymbol{c} = (-4, 5)$; $\lambda = 3$; $\mu = 0.5$:

   (a) What is $\boldsymbol{a} + \boldsymbol{b}$? (b) What is $\boldsymbol{a} - \boldsymbol{b}$?
   (c) What is $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$? (d) What is $\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c}$?
   (e) What is $\lambda\boldsymbol{a} + \boldsymbol{b}$? (f) What is $\boldsymbol{b} - \mu\boldsymbol{a}$?
   (g) What is $\boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$?



   In 3D, take $\boldsymbol{a} = (6, 3, 2)$; $\boldsymbol{b} = (0, 1, 9)$:

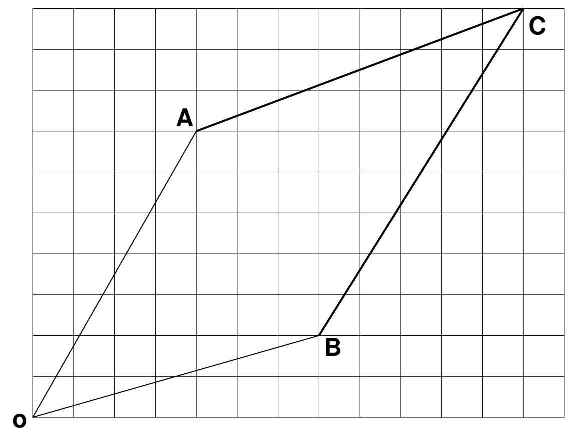
   (h) What is $\boldsymbol{a} - \boldsymbol{b}$?

   Take $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{k}}$; $\boldsymbol{b} = \hat{\boldsymbol{i}} + 4\hat{\boldsymbol{k}}$,

   (i) What is $\boldsymbol{a} + \boldsymbol{b}$?

2. If $\overrightarrow{OA} = (2, 3)$ and $\overrightarrow{AB} = (4, 6)$, calculate the vector $\overrightarrow{BO}$.

3. Using the gridded diagram below:
   (a) Work out the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$
   (b) use them to calculate $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$.
   (c) Also show that $\overrightarrow{AB} = -\overrightarrow{BA}$, $\overrightarrow{BC} = -\overrightarrow{CB}$ and $\overrightarrow{AC} = -\overrightarrow{CA}$.



4. Consider 3 points A, B and C with co-ordinates (2,3,1), (6,2,5), and (3,3,8) respectively. Which pair of points are closest together, and what is their relative displacement?

Main Questions

5. If $\overrightarrow{OA} = (1, 2, 3)$ and $\overrightarrow{AC} = (2, -1, -5)$,
   (a) calculate the vector, $\overrightarrow{OC}$.
   (b) Using the cosine rule, calculate the angle between vectors $\overrightarrow{OA}$ and $\overrightarrow{OC}$.
6. An aeroplane has an air speed of 125 km/hr and is flying North. How fast will the plane travel over the Earth, and in what direction, if the wind is:
   (a) From the North at speed 40 km/hr?
   (b) From the South-East at speed 80km/hr?
7. The structure of diamond may be described as a repeating unit cell. The unit cell is a cube with sides of length a, with carbon atoms at each vertex and at the centre of each face (i.e. face-centred cubic). There are additional carbon atoms displaced by $\frac{1}{4}a(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$ from each of these atoms, where $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$ are the unit vectors along the cube axes.

   (a) Draw the structure of the unit cell

   (b) What are the position vectors of the four nearest neighbours to the atom at $\frac{1}{4}a(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$?

   (c) What are the vectors joining $\frac{1}{4}a(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$ to its four nearest neighbours?

8. Two rabbits (Peter and Bugsy) and a fox are positioned in a field as shown in the diagram below. Each square has sides of length 10m. Bugsy spies the fox and begins to run first with a velocity vector (3,4) m/s, Peter starts 10 seconds later with velocity vector (5,2) m/s. The fox starts chasing 10 seconds later than Peter (20 seconds after Bugsy) with velocity vector (5,3) m/s.

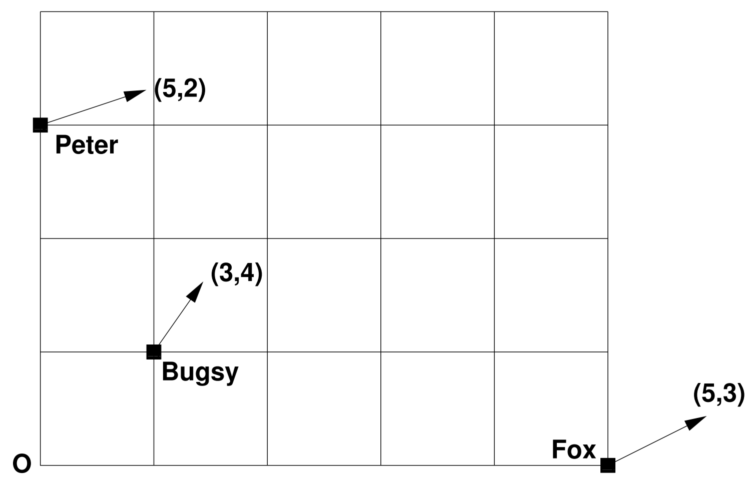
   (a) Does the fox catch a rabbit?

   If so,

   (b) Which one?

   (c) How long does it take the fox?

   (d) How far have each of the animals travelled in that time?



9. ABC is a triangle whose vertices have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$.

   (a) What is the position vector $\boldsymbol{d}$ of the mid-point of BC?

   (b) Write down the position vector $\boldsymbol{p}$ of a point P on the *median* joining A to the mid-point of BC so that it lies a fraction λ along it.

   (c) Write down similar expressions for points on the other two medians.

   (d) By guessing a suitable value of λ (or otherwise) show that the three medians all meet at one point. What is its position vector?

10. (a) Given two position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ write down the equation of any point, $\boldsymbol{r}$, lying on the straight line that passes through $\boldsymbol{a}$ and $\boldsymbol{b}$ in term of $\boldsymbol{a}$, $\boldsymbol{b}$ and a scale factor λ.

    (b) If $\boldsymbol{a} = (1, 1, 0)$ and $\boldsymbol{b} = (3, 1, 4)$, use the equation above to establish which of the following points lie on the straight line passing through $\boldsymbol{a}$ and $\boldsymbol{b}$.

    (i) $\boldsymbol{c} = (-2, 0, -4)$,

    (ii) $\boldsymbol{d} = (5, 1, 8)$,

    (iii) $\boldsymbol{e} = (1, 1, -4)$,

    (iv) $\boldsymbol{f} = (-1/2, 1/2, 2)$.

11. A straight line passes through the points $\boldsymbol{a} = (1, 1, 1)$ and $\boldsymbol{b} = (4, 2, 5)$. A second straight line passes through $\boldsymbol{c} = (3, 1, 6)$ and $\boldsymbol{d} = (2, 0, 7)$.

    Write down their equations,

    (a) using the form $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$,

    (b) using the form $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$.

    Calculate the position vector of the point where the two lines intersect. Show that both equations found in parts (a) and (b) give the same answer.

B. Scalar Product

Basic Skills

1. The vectors $\boldsymbol{a} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$. Calculate the scalar products $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{b} \cdot \boldsymbol{a}$ directly and verify that they are equal.

2. Suppose $\boldsymbol{c} = (2, 1, 4)$ and $\boldsymbol{d} = (3, -2, 1)$.

    (a) Calculate the angle between the vectors $\boldsymbol{c}$ and $\boldsymbol{d}$.

    (b) Find a vector perpendicular to $\boldsymbol{c}$ and calculate the angle it makes with $\boldsymbol{d}$.

    (c) Find a vector perpendicular to $\boldsymbol{d}$ and calculate the angle it makes with $\boldsymbol{c}$.

3. If $\boldsymbol{e}_1 = (1, 1, 0)$, $\boldsymbol{e}_2 = (1, a, 1)$ and $\boldsymbol{e}_3 = (1, b, -2)$,

    (a) find the values of a and b such that $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ form an orthogonal basis;

    (b) calculate the unit vectors $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ of this basis.

Main Questions

4. The vectors $\boldsymbol{A} = (3, 2, 1)$ and $\boldsymbol{B} = (1, 1, 0)$ and the point $(1, 1, 1)$ lie in a plane.

    (a) Calculate the *unit* normal $\hat{\boldsymbol{n}}$ of the plane.

    (b) Write down the equation of the plane in the form $(\boldsymbol{r} - \boldsymbol{a}) \cdot \hat{\boldsymbol{n}} = 0$.

    (c) Calculate the perpendicular distance of the plane from the origin $(0, 0, 0)$.

5. Consider a plane containing the point with position vector $\boldsymbol{r}$, possessing the unit normal $\hat{\boldsymbol{n}}$, and located a perpendicular distance d from the origin.

    (a) Derive an expression for the shortest distance between the plane and the point P (described by the position vector $\boldsymbol{p}$). Your final result should contain d, $\boldsymbol{p}$, and $\hat{\boldsymbol{n}}$.

    (b) Calculate the shortest distances between the plane $3x - y + 2z = 6$ and the points S, with position vector $(1, -1, 3)$, and T, with position vector $(1, 0, -1)$.

    (c) Do both points S and T lie on the same side of the plane?

6. Consider a plane containing the points with position vectors $\boldsymbol{r}$ and $\boldsymbol{a}$.

   (a) Write down an expression for the unit normal if the the plane is described by $\boldsymbol{b} \cdot (\boldsymbol{r} - \boldsymbol{a}) = 0$?

   A point C lies off the plane with position vector $\boldsymbol{c}$, calculate the shortest distance from the point C to the plane

   (b) Calculate the projection of $\boldsymbol{c}$ onto the plane.

7. Show that the line of intersection of the two planes $x+2y+3z = 0$ and $3x+2y+z = 0$

   (a) is equally inclined to the x and z axes,

   (b) and makes an angle $\cos^{-1}(-\sqrt{2/3})$ with the y-axis.

8. Find the acute angle at which two diagonals of a cube intersect.

9. Four points A, B, C, D are such that AD is perpendicular to BC, and BD is perpendicular to AC. Show that CD is perpendicular to AB.

10. Identify the surfaces:

    (a) $|\mathbf{r}| = d$,

    (b) $\mathbf{r} \cdot \mathbf{u} = e$,

    (c) $\mathbf{r} \cdot \mathbf{u} = f|\mathbf{r}|$,

    (d) $|\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}| = g$

    where d, e, f and g are fixed scalars and $\boldsymbol{u}$ is a fixed unit vector.

C. Vector Product

Basic Skills

1. Let $\boldsymbol{a} = (2, 1, 3)$, $\boldsymbol{b} = (6, 0, 5)$, and $\boldsymbol{c} = (5, 3, 1)$. Evaluate the following:

   (a) $\boldsymbol{a} \wedge \boldsymbol{b}$ and show that it is perpendicular to both vectors.

   (b) $\boldsymbol{b} \wedge \boldsymbol{a}$ and show that it is perpendicular to both vectors.

   (c) $(\boldsymbol{a} + \boldsymbol{b}) \wedge \boldsymbol{c}$ and show that it is equal to $\boldsymbol{a} \wedge \boldsymbol{c} + \boldsymbol{b} \wedge \boldsymbol{c}$.

   (d) $(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c}$ show that it is not equal to $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$.

   (e) The angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ using the vector product, and check you get the same answer with the scalar product.

   (f) The angle between $\boldsymbol{b}$ and $\boldsymbol{c}$ using the vector product, and check you get the same answer with the scalar product.

(g) The angle between $\boldsymbol{c}$ and $\boldsymbol{a}$ using the vector product, and check you get the same answer with the scalar product.

(h) $\boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c})$.

(i) $\boldsymbol{c} \cdot (\boldsymbol{a} \wedge \boldsymbol{b})$.

(j) $\boldsymbol{b} \cdot (\boldsymbol{a} \wedge \boldsymbol{c})$.

(k) $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$ and show that it is equal to $(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$.

Main Questions

2. Find the angle between the position vectors of the points $(2, 1, 1)$ and $(3, -1, -5)$ and find the direction cosines of a vector perpendicular to both.

3. A parallelepiped has one vertex at the origin and the nearest three vertices to the origin at the points A, B, and C, with associated position vectors $(3, 0, 0)$, $(0, 2, 0)$, and $(1, 1, 5)$.

   (a) Calculate the area of the side containing the origin and the points A and B.

   (b) Calculate the volume of the parallelepiped.

4. Show, for any three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, that $(\boldsymbol{b} - \boldsymbol{a}) \wedge (\boldsymbol{c} - \boldsymbol{a}) = \boldsymbol{a} \wedge \boldsymbol{b} + \boldsymbol{b} \wedge \boldsymbol{c} + \boldsymbol{c} \wedge \boldsymbol{a}$.

5. A line is inclined at equal angles to the x, y, z axes and passes through the origin. A second line passes through the points $(1, 2, 4)$ and $(0, 0, 1)$. Find the minimum distance between the two lines using the vector product.

6. You need to drill a straight hole in a piece of metal at right angles to a flat surface containing the points: $(1, 0, 0)$, $(1, 1, 1)$, and $(0, 2, 0)$; and you want the hole to end at the point $(2, 1, 0)$.

   (a) How long a drill must you use?

   (b) Where in the plane given by the flat surface must you start drilling?

7. Suppose $\boldsymbol{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

   (a) Draw a two-dimensional Cartesian basis set on the plane and mark the points represented by the position vectors $\boldsymbol{c}$ and $\boldsymbol{d}$. Add to this diagram the non-orthogonal basis set represented by $\boldsymbol{a}$ and $\boldsymbol{b}$.

   (b) Write the vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ in terms of the new basis set.

8. Set $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

   Show that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ form a non-orthogonal basis set.

(b) Write the vector $\boldsymbol{d} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in terms of this basis by using the scalar triple product.

D. Coordinate Systems

Basic Skills

1. By drawing a 2D diagram calculate:
   (a) the x co-ordinate of a point in terms of its plane polar co-ordinates r and θ.
   (b) the y co-ordinate of a point in terms of its plane polar co-ordinates r and θ.
2. By drawing a 2D diagram calculate:
   (a) the r co-ordinate of a point in terms of its Cartesian co-ordinates x and y.
   (b) the θ co-ordinate of a point in terms of its Cartesian co-ordinates x and y.
3. Describe the following loci (given in plane polar co-ordinates),
   (a) $\theta = a$,
   (b) $r = \theta$,

   where a is a constant.

Main Questions

4. By drawing a sphere in Cartesian co-ordinates derive:
   (a) x in terms of r, θ and ϕ,
   (b) y in terms of r, θ and ϕ,
   (c) z in terms of r, θ and ϕ.
5. By drawing a cylinder in Cartesian co-ordinates derive:
   (a) x in terms of r, θ,
   (b) y in terms of r, θ,
   (c) z in terms of z.
6. A point has Cartesian co-ordinates (1,2,5). What are its co-ordinates in:
   (a) cylindrical polar co-ordinates,
   (b) spherical polar co-ordinates.

E. Vector Area

1. Four points define a rectangle in 3D space. Their position vectors are $(2,0,7)$, $(5,0,3)$, $(5,4,3)$, and $(2,4,7)$.

   (a) Find the unit normal to the plane of the rectangle, $\hat{\boldsymbol{n}}$.

   (b) What is the total vector area $\boldsymbol{S}$ in terms of $A\hat{\boldsymbol{n}}$?

   (c) Calculate the projected area S_x on to the y-z plane from (b) and verify it graphically

   (d) Calculate the projected area S_y on to the x-z plane from (b) and verify it graphically

   (e) Calculate the projected area S_z on to the x-y plane from (b) and verify it graphically.

2. Points O, B, C, D, and E have position vectors $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$ and $(1,1,1)$, respectively. What are the vector areas of:

   (a) the square $OBCD$ projected on to the plane with unit normal $\hat{\boldsymbol{n}} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$,

   (b) the upper surface of the pyramid with base $OBCD$ and vertex E,

   (c) a lampshade (truncated hollow cone) whose base is a horizontal circle of radius 4, and whose top is a horizontal circle of radius 3, located a height 5 above the base.

F. Complex Numbers

Basic Skills

1. If $z_1 = 2 + i$ and $z_2 = i - 3$ plot the following numbers on an Argand diagram:

   (a) z_1 (b) z_2
   (c) z_1^* (d) z_2^*
   (e) $z_2 - z_1$ (f) $z_1 - 2z_2$
   (g) iz_1 (h) $z_1 z_2$
   (i) z_1^2 (j) $|z_2|^2$.

2. If $z_1 = 2 + i$ and $z_2 = i - 3$ write the following in the the form $z = |z|e^{i\theta}$

   (a) z_1 (b) z_2
   (c) z_1^* (d) z_2^*
   (e) $z_2 - z_1$ (f) $z_1 - 2z_2$
   (g) iz_1 (h) $z_1 z_2$
   (i) z_1^2 (j) $|z_2|^2$.

3. Find the modulus and principal argument of the following:
   (a) $1+\sqrt{5}i$
   (b) $-\sqrt{3}-\dfrac{i}{\sqrt{3}}$
   (c) $4i-3$.

4. If $z = x + iy$, find the real and imaginary parts of the following functions in terms of x and y:
   (a) z^2
   (b) iz
   (c) $(1+i)z$
   (d) $z^2(z-1)$.

Main Questions

5. Find the real and imaginary parts of the following complex numbers:
   (a) i^3 (b) i^{4n}
   (c) $\left(\dfrac{i+1}{\sqrt{2}}\right)^2$ (d) $\left(\dfrac{1-i}{\sqrt{2}}\right)^2$
   (e) $\left(\dfrac{1}{2}+\dfrac{\sqrt{3}i}{2}\right)^3$ (f) $\dfrac{1+i}{2-5i}$
   (g) $\left(\dfrac{1+i}{1-i}\right)^2$ (h) $i^{1/3}$
   (i) $(3i-2)^{1/4}$ (j) i^i.

6. Factorise the following expressions:
   (a) z^2+1
   (b) z^2-2z+2
   (c) z^2+i
   (d) $z^2+(1-i)z-i$.

7. Find all the solutions to the following equations, and mark their positions in the Argand diagram.
   (a) $z^3=-1$
   (b) $z^4=1$
   (c) $z^2=i$
   (d) $z^3=-i$.

8. Write down the following in the form $a + ib$ where a and b are real.
   (a) $e^{-i\pi/2}$ (b) $e^{-i\pi}$
   (c) $e^{i\pi/4}$ (d) e^{1+i}
   (e) $\exp(2e^{i\pi/4})$ (f) $\exp(re^{i\theta})$, where r and θ are real.

9. Sketch on an Argand diagram the set of points described by the equations:
   (a) $|z| = 4$ (b) $|z - 1| = 3$
   (c) $|z - i| = 2$ (d) $|z - (1 - 2i)| = 3$
   (e) $|z^* - 1| = 1$ (f) $|z^* - i| = 1$
   (g) $|z - 2| = |z + i|$ (h) $|z - 2| = |z^* + i|$
   (i) $|z| = 2|z - 2|$ (j) $\arg(z) = \dfrac{\pi}{2}$
   (k) $\arg(z^*) = \dfrac{\pi}{4}$ (l) $\arg(z) = |z|$.

10. By expressing the following in terms of $|z|e^{i\theta}$ find the natural logarithms of:
    (a) $-1 + 0.0001i$ (b) $-1 - 0.0001i$
    (c) i (d) $(1 + i)$
    (e) $(x + iy)$ (f) $(x + iy)^*$
    (g) $ire^{i\theta}$.

11. (a) Write down $e^{i\theta}$ in terms of trigonometric functions.
    (b) By considering $e^{in\theta}$ derive *De Moivre's* theorem.
    (c) Use *De Moivre's* theorem to find $\cos 2\theta$ in terms of $\cos\theta$
    (d) Check your result using the trigonometric identity
    $$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

12. Using *De Moivre's* theorem find:
    (a) $\cos 3\theta$ in terms of powers of $\cos\theta$
    (b) $\sin^5\theta$ in terms of $\sin\theta$, $\sin 2\theta$, $\sin 3\theta$, etc.

13. Calculate the following sums:
    (a) $\sum_{n=1}^{5} \sin(n\theta)$
    (b) $\sum_{n=1}^{N} \cos(n\theta)$.
    [Hint: Consider $\sum \exp(in\theta)$.]

14. The position of a particle moving in simple harmonic motion in one dimension is $x = 7 + 24\cos 3t + 7\sin 3t$.
    (a) Show that the displacement from the centre of motion can be written as $\Re(X)$ where $X = (24 - 7i)e^{3it}$.
    (b) What is the amplitude of the motion?
    (c) Find the time of the first two passages through the centre (after $t = 0$) by writing X in the form $Ae^{i\phi}$ where A and ϕ are real.
    (d) Also find the distance of the stationary points of the particle from the origin.

15. A mass on a spring moves in simple harmonic motion with a frequency, ω. At $t = 0$ it is a distance x_0 from the centre of motion and is moving with a velocity v_0. Find the complex amplitude A in the expression $Ae^{i\omega t}$ that represents the sinusoidal variation in distance of the mass from the centre of the motion.

G. Hyperbolic Functions

1. By writing the trigonometric function in terms of exponential functions (or otherwise) prove the following:
   (a) $\tanh(ix) = i\tan(x)$
   (b) $\text{sech}(ix) = \sec(x)$
   (c) $\cosh^2 x - \sinh^2 x = 1$
   (d) $1 - \tanh^2 x = \text{sech}^2 x$.

2. Find the following trigonometric identities:
   (a) $\cosh(x + y)$ in terms of cosh and sinh functions,
   (b) $\sinh(x + y)$ in terms of cosh and sinh functions,
   (c) $\tanh(x + y)$ in terms of tanh functions.

3. Sketch the graphs of the following functions:
   (a) e^x (b) $\cosh x$
   (c) $\tanh x$ (d) $\log x$
   (e) $\sinh^{-1} x$ (f) $\tanh^{-1} x$.

4. Consider the set of points in the x, y plane described by:
   (a) $(x, y) = (a\cos\theta, b\sin\theta)$,
   (b) $(x, y) = (a\cosh\theta, b\sinh\theta)$,

   where a and b are real and fixed and θ is a real parameter. In each case, find the Cartesian equation of the locus and sketch it.

5. By writing cosh and sinh in terms of exponentials express the following inverse hyperbolic functions in closed form:
   (a) $\sinh^{-1} x$
   (b) $\cosh^{-1} x$
   (c) $\tanh^{-1} x$.

N.B: Further hyperbolic function questions will be included in the calculus section.

Numerical Answers to Basic Skills

A.Vector Addition & Subtraction

1. (a) $(5,6)$ (b) $(-1,4)$ (c) $(1,11)$
   (d) $(3,-1)$ (e) $(9,16)$ (f) $(2,-1.5)$
   (g) $(9,10.5)$ (h) $(6,2,-7)$ (i) $3\hat{\boldsymbol{\imath}} + 7\hat{\boldsymbol{k}}$

2. $(-6,-9)$

3. (a) $\overrightarrow{OA} = (4,7)$, $\overrightarrow{OB} = (7,2)$, $\overrightarrow{OC} = (12,10)$
   (b) $\overrightarrow{AB} = (3,-5)$ $\overrightarrow{BC} = (5,8)$, $\overrightarrow{AC} = (8,3)$
   (c) Show to be true numerically

4. $\overrightarrow{BC} = (-3,1,3)$

B. Scalar Product

1. (a) 2
   (b) Show on diagram
   (c) Evaluate numerically and indicate on diagram

2. (a) 62.2^o
   (b) E.g. $(-1,2,0)$. Angle $= 146.8^o$.
   (c) E.g. $(0,1,2)$. Angle $= 26.6^o$.

3. (a) $a = -1$, $b = -1$
   (b) $\hat{\boldsymbol{e}}_1 = \frac{1}{\sqrt{2}}(1,1,0)$, $\hat{\boldsymbol{e}}_2 = \frac{1}{\sqrt{3}}(1,-1,1)$, $\hat{\boldsymbol{e}}_3 = \frac{1}{\sqrt{6}}(1,-1,-2)$

C. Vector Product

1. (a) $(5,8,-6)$ (b) $(-5,-8,6)$ (c) $(-23,32,19)$
   (d) $(26,-35,-25)$ (e) 22.5^o (f) 40.8^o
   (g) 43.7^o (h) 43 (i) 43
   (j) -43 (k) $(-39,-81,53)$

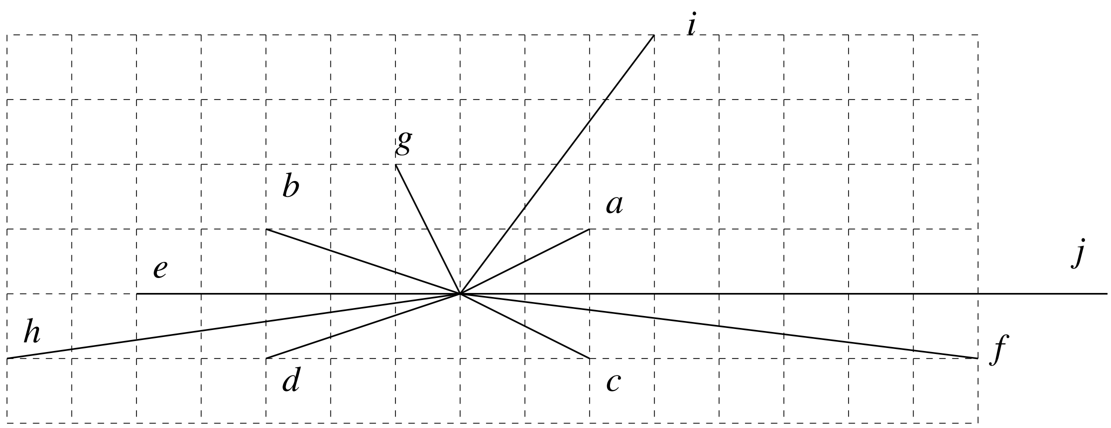
D. Coordinate Systems

1. See lecture notes

2. See lecture notes

3. (a) Straight line at angle a to the$x$ axis.
   (b) Spiral

F. Complex Numbers



1.

2.

(a) modulus $= \sqrt{5}$, arg $= 26.6^o$

(b) modulus $= \sqrt{10}$, arg $= 161.6^o$

(c) modulus $= \sqrt{5}$, arg $= -26.6^o$

(d) modulus $= \sqrt{10}$, arg $= -161.6^o$

(e) modulus $= 5$, arg $= 180^o$

(f) modulus $= \sqrt{65}$, arg $= -7.1^o$

(g) modulus $= \sqrt{5}$, arg $= 116.6^o$

(h) modulus $= \sqrt{50}$, arg $= -171.9^o$

(i) modulus $= 5$, arg $= 53.1^o$

(j) modulus $= 10$, arg $= 0^o$

3. (a) modulus $= \sqrt{6}$, arg $= 65.9^o$

(b) modulus $= \sqrt{10/3}$, arg $= -161.6^o$

(c) modulus $= 5$, arg $= 126.9^o$

4. (a) Re $= x^2 - y^2$, Im $= 2xy$

(b) Re $= -y$, Im$= x$

(c) Re $= x - y$, Im$= x + y$

(d) Re $= x^3 - 3xy^2 - x^2 + y^2$, Im $= 3x^2y - 2xy - y^3$