*Natural Sciences Tripos*
*Part IA Mathematics - Course A*

# Mathematical Methods I

*Examples Sheet 1: Solutions*

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## Chapter 1

A5. (a) $(3, 1, -2)$, (b) $\cos^{-1}(1/14)$

A6. (a) 85 km/hr, due North. (b) 190.2 km/hr, 17.3° West of North.

A7. (b) $\mathbf{0}$, $\frac{1}{2}a(1,1,0)$, $\frac{1}{2}a(1,0,1)$, $\frac{1}{2}a(0,1,1)$.
(c) $\frac{1}{4}a(-1,-1,-1)$, $\frac{1}{4}a(1,1,-1)$, $\frac{1}{4}a(1,-1,1)$, $\frac{1}{4}a(-1,1,1)$.

A8. (a) Yes. (b) Peter. (c) 70s (50s after fox starts).
(d) Bugsey 350m, Peter $60\sqrt{29}$ m, fox $50\sqrt{34}$ m.

A9. (a) $\mathbf{d} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$.
(b) $\mathbf{p} = (1-\lambda)\mathbf{a} + \frac{1}{2}\lambda\mathbf{b} + \frac{1}{2}\lambda\mathbf{c}$.
(c) $\mathbf{q} = (1-\mu)\mathbf{b} + \frac{1}{2}\mu\mathbf{c} + \frac{1}{2}\mu\mathbf{a}$, $\mathbf{r} = (1-\gamma)\mathbf{c} + \frac{1}{2}\gamma\mathbf{a} + \frac{1}{2}\gamma\mathbf{b}$
(d) $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

A10. (a) $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$. (b) Passes through $\mathbf{d}$.

A11. (a) $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$, $\mathbf{s} = \mathbf{c} + \mu(\mathbf{d} - \mathbf{c})$.
(b) $\frac{x-1}{3} = \frac{y-1}{1} = \frac{z-1}{4}$, $\frac{x-3}{-1} = \frac{y-1}{-1} = \frac{z-6}{1}$. (c) (4,2,5).

B4. (a) $\hat{\mathbf{n}} = (-1,1,1)/\sqrt{3}$. (b) $[\mathbf{r} - (1,1,1)] \cdot (-1,1,1)/\sqrt{3} = (-x+y+z-1)/\sqrt{3}$. (c) $1/\sqrt{3}$.

B5. (a) $|\mathbf{p} \cdot \hat{\mathbf{n}} - d|$. (b) $S : 4/\sqrt{14}$, $T : -5/\sqrt{14}$. (c) Opposite sides.

B6. (a) $\hat{\mathbf{n}} = \mathbf{b}/|\mathbf{b}|$. (b) $(\mathbf{c} - \mathbf{a}) \cdot \hat{\mathbf{n}}$. (c) $\mathbf{c} - (\mathbf{c} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$.

B8. $\cos^{-1}(1/3)$.

B10. (a) Sphere of radius $d$, centred on the origin.
(b) Plane normal to $\mathbf{u}$ at a distance $e$ from the origin
(c) Half-cone with apex at the origin, axis along $\mathbf{u}$ and opening angle $\cos^{-1} f$.
(d) Cylinder of radius $g$, with axis in the $\mathbf{u}$ direction.

C2. $90°$; $(-4, 13, -5)/\sqrt{210}$.

C3. (a) 6. (b) 30.

C5. $1/\sqrt{6}$.

C6. (a) $\sqrt{3/2}$. (b) $(1, \frac{1}{2}, \frac{1}{2})$.

C7. (b) $\mathbf{c} = 0\mathbf{a} + 1\mathbf{b}$, $\mathbf{d} = 3\mathbf{a} + 2\mathbf{b}$.

C8. (b) $\frac{1}{5}(3\mathbf{a} + \mathbf{b} + \mathbf{c})$.

D4. (a) $x = r \sin\theta \cos\phi$. (b) $y = r \sin\theta \sin\phi$. (c) $z = r \cos\theta$.

D5. (a) $x = r \cos\theta$. (b) $y = r \sin\theta$. (d) $z = z$.

D6. (a) $(r, \theta, z) = \left(\sqrt{5}, \tan^{-1} 2, 5\right)$. (b) $(r, \theta, \phi) = \left(\sqrt{30}, \cos^{-1}\sqrt{5/6}, \tan^{-1} 2\right)$.

E1. (a) $\frac{1}{5}(4, 0, 3)$. (b) $4(4, 0, 3)$. (c) 16. (d) 0. (e) 12.

E2. (a) $-2\mathbf{j} + 2\mathbf{k}$. (b) $4\mathbf{k}$. (c) $7\pi\mathbf{k}$.

# Chapters 2 and 3

F5. (a) $-i$. (b) 1. (c) $i$. (d) $-i$. (e) $-1$. (f) $\frac{1}{29}(-3 + 7i)$. (g) $-1$. (h) $\frac{1}{2}(\sqrt{3} + i)$.
(i) $13^{1/8}(\cos\theta + i \sin\theta)$, where $\theta = \frac{1}{4}(-\tan^{-1}\frac{3}{2} + (2n + 1)\pi)$ and $n = 1, 2, 3, 4$.
(j) $\exp\left[-\left(2n + \frac{1}{2}\right)\pi\right]$.

F6. (a) $(z + i)(z - i)$. (b) $(z - 1 - i)(z - 1 + i)$. (c) $\left(z - \frac{1-i}{\sqrt{2}}\right)\left(z + \frac{1-i}{\sqrt{2}}\right)$. (d) $(z - i)(z + 1)$.

F7. (a) $-1$, $\frac{1}{2}(1 \pm i\sqrt{3})$. (b) $\pm 1, \pm i$. (c) $\pm\frac{1}{\sqrt{2}}(1 + i)$. (d) $i$, $\frac{1}{2}(\pm\sqrt{3} - i)$.

F8. (a) $-i$. (b) $-1$. (c) $\frac{1}{\sqrt{2}}(1 + i)$. (d) $e(\cos 1 + i \sin 1)$. (e) $\exp(\sqrt{2})\left(\cos\sqrt{2} + i \sin\sqrt{2}\right)$.
(f) $\exp(r\cos\theta)\left[\cos(r \sin\theta) + i \sin(r \sin\theta)\right]$.

F9. (a) Circle of radius 4, centred on $z = 0$. (b) Circle of radius 3, centred on $z = 1$.
(c) Circle of radius 2, centred on $z = i$. (d) Circle of radius 3, centred on $z = 1 - 2i$.
(e) Circle of radius 1 centred on $z = 1$. (f) Circle of radius 1, centred on $z = -i$.
(g) Perpendicular bisector of points $z = -i$ and $z = 2$.
(h) Perpendicular bisector of points $z = i$ and $z = 2$.
(i) Circle of radius 4/3, centred on $z = 3/8$. (j) Half-line along positive imaginary axis.
(k) Half-line at $-45^\circ$ to real axis, with $\mathrm{Re}(z) \geq 0$.
(l) Anticlockwise spiral, through $z = 0$, $z = \frac{1}{2}\pi i$, $z = -\pi$, etc.

F10. (a) $5 \times 10^{-9} + (2n\pi + 3.14149i) \approx 0 + i(2n + 1)\pi$.
(b) $5\times10^{-9}+(2n\pi-3.14149i) \approx 0+i(2n-1)\pi$. (c) $i\left(2n + \frac{1}{2}\right)\pi$. (d) $\frac{1}{2}\ln 2+i\left(2n + \frac{1}{4}\right)\pi$.
(e) $\frac{1}{2}\ln(x^2 + y^2) + i\left[2n\pi + \tan^{-1}(y/x)\right]$. (f) $\frac{1}{2}\ln(x^2 + y^2) + i\left[2n\pi - \tan^{-1}(y/x)\right]$.
(g) $\ln r + i\left[\theta + \left(2n + \frac{1}{2}\right)\pi\right]$.

F11. (a) $\cos\theta + i\sin\theta$. (b) $e^{in\theta} = (e^{i\theta})^n = \cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$.
(c) $\cos 2\theta = 2\cos^2\theta - 1$.

F12. (a) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. (b) $\sin^5\theta = \frac{1}{16}(10\sin\theta - 5\sin 3\theta + \sin 5\theta)$.

F13.

$$\text{(a)}\ \frac{-\sin 6\theta + \sin 5\theta + \sin\theta}{2(1 - \cos\theta)}. \quad \text{(b)}\ \frac{-\cos 6\theta + \cos 5\theta + \cos\theta - 1}{2(1 - \cos\theta)} = \frac{-\cos 6\theta + \cos 5\theta}{2(1 - \cos\theta)} - \frac{1}{2}.$$

F14. (b) 25. (c) $t = \frac{1}{3}(n\pi + \phi)$, where $\phi = -\tan^{-1}\frac{7}{24}$ and $n = 1, 2$. (d) $7 \pm 25$.

F15. $A = x_0 - iv_0/\omega$.

G2. (a) $\cosh x\cosh y + \sinh x\sinh y$. (b) $\cosh x\sinh y + \sinh x\cosh y$. (c) $\frac{\tanh x+\tanh y}{1+\tanh x\tanh y}$.

G4. Ellipse with semi major/minor axes of lengths $a$ and $b$, aligned with the $x$ and $y$ axes, respectively. (b) Hyperbola crossing the $x$ axis at $x = a$ and asymptoting to $y = \pm\frac{b}{a}x$.

G5. (a) $\ln\left(x + \sqrt{1 + x^2}\right)$. (b) $\pm\ln\left(x + \sqrt{x^2 - 1}\right)$. (c) $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.