

*Natural Sciences Tripos*  
*Part IA Mathematics - Course A*  
**Mathematical Methods I**  
*Examples Sheet 2: Solutions*

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 Michaelmas Term 2023

## Chapters 4-6

H6. (a)  $\sin x + x \cos x$ . (b)  $2(\cos \theta + \theta \sin \theta) / \cos^2 \theta$ . (c)  $t(1 + 2 \ln t)$ . (d)  $e^y(\cos y - \sin y)$ .  
 (e)  $\cosh 2x$ . (f)  $2xe^{x^2+2}$ .

H7.

$$\frac{dy}{dx} = -\frac{(y + e^y \sin y)^2}{1 + e^y(\sin y + \cos y)}.$$

H9.  $\ln 2$

H10. (a)  $\frac{1}{6} \cos^6 x + c$ . (b)  $\ln |\tan x| + c$ . (c)  $\ln \left| \frac{x+1}{x-1} \right| + c$ . (d)  $\ln |x^3 + 2x - 1| + c$ .  
 (e)  $(1+x)(1+2x)^{-1/2} + c$ . (f)  $-\ln |\cos x| + c$ . (g)  $\frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c$ .  
 (h)  $-\cos x + \frac{1}{3} \cos^3 x + c$ .

H11. (a)  $2\sqrt{\sec \theta} + c$ . (b)  $\sin^{-1} e^x + c$ . (c)  $2(6x - x^2)^{1/2} + 2 \sin^{-1}(\frac{1}{3}x - 1) + c$ .  
 (d)  $\ln |\tan \frac{1}{2}x| + c$ . (e)  $\sec x + c$ .

H12. (a)  $3x(\ln x - 1) + c$ . (b)  $x[(\ln x)^3 - 3(\ln x)^2 + 6 \ln x - 6] + c$ .

H13. (a)  $(2 - y^2) \cos y + 2y \sin y - 2$ . (b)  $\frac{1}{2}y^2(\ln y - \frac{1}{2}) + \frac{1}{4}$ . (c)  $y \sin^{-1} y + \sqrt{1 - y^2} - 1$ .  
 (d)  $\ln(a^2 + 1) - y^{-1} \ln(a^2 + y^2) - \frac{2}{a}(\tan^{-1} \frac{1}{a} - \tan^{-1} \frac{y}{a})$ .

H14. (b)  $J = \frac{1}{b}(1 - e^{ax} \cos bx) + \frac{a}{b}I$ .

$$(c) I = \frac{ae^{ax} \cos bx + be^{ax} \sin bx - a}{a^2 + b^2}, \quad J = \frac{ae^{ax} \sin bx - be^{ax} \cos bx + b}{a^2 + b^2}.$$

H15. (a)  $I_n = \frac{3n}{3n+1}I_{n-1}$ ,  $I_4 = \frac{243}{455}$ .

$$(b) I_n = n\left(\frac{1}{2}\pi\right)^{n-1} - n(n-1)I_{n-2}, I_6 = \frac{3}{16}\pi^5 - 15\pi^3 + 360\pi - 720.$$

$$(c) I_n = e - nI_{n-1}, I_5 = 120 - 44e.$$

$$H16. (a)(i) \frac{1}{2}\pi, (ii) 1, (iii) 1, (iv) \frac{1}{2}, (v) \frac{1}{1+m}, (vi) \frac{1}{n+1}. (c)(i) \frac{1}{24}, (ii) 8/693, (iii) (7/2048)\pi$$

$$H17. (a) \text{ odd. } (b) \text{ neither (even about } x = a). (c) \text{ odd. } (d) \text{ even. } (e) \text{ neither. } (f) \text{ even.}$$

$$(g) \text{ neither. } (h) \text{ even. } (i) \text{ odd.}$$

$$I1. (a) \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3. (b) \sin \frac{31}{180}\pi \approx 0.515038.$$

$$I2. (a) 1 - \frac{1}{2}x^2. (b) x + \frac{1}{6}x^3. (c) 1 + x. (d) x - \frac{1}{2}x^2.$$

$$I3. (a) \sum_{r=0}^n \frac{n!}{(n-r)!r!}x^r. (b) 1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8.$$

$$I4. (a) 1 + \frac{3}{2}x + \frac{3}{8}x^2. (b) 2 + \frac{3}{4}x - \frac{9}{64}x^2. (c) \frac{1}{3} + \frac{1}{9}x + \frac{1}{27}x^2.$$

$$I5. (a) n > 2 \times 10^{10}. (b) 4\left(\frac{5}{6} - \frac{35}{648} + \frac{55}{7776} - \dots\right).$$

$$I6. (a) \text{ See lecture notes. } (b) x_1 = \frac{14}{5} = 2.8, x_2 = \frac{383}{140} = 2.73571.$$

## Chapter 7

$$J1. P(\{B, B\}) = \frac{2}{9}, P(\{B, G\}) = \frac{4}{9}, P(\{B, Y\}) = \frac{1}{9}, P(\{G, G\}) = \frac{2}{15}, P(\{G, Y\}) = \frac{4}{45}.$$

$$J2. \frac{1}{6}.$$

$$J4. (a) \frac{2}{3}. (b) \frac{5}{8}. (c) \text{ equal chances if first or second.}$$

$$J5. \frac{3}{5}.$$

$$J6. (a) \frac{2}{13}. (b) \frac{53}{13}. (c) \frac{6}{13}. (d) \frac{3253}{169}.$$

$$J7. (a) \alpha/(\alpha + \beta - \alpha\beta). (b) \alpha(1 - \beta)/(\alpha + \beta - \alpha\beta). (c) \alpha(1 - \beta)/(\alpha + \beta - \alpha\beta).$$

$$J8. (a) ({}^{49}C_6)^{-1} \approx 7.15112 \times 10^{-8}. (b). ({}^6C_r)({}^{43}C_{6-r})({}^{49}C_6)^{-1}.$$

$$(c) 6({}^{49}C_6)^{-1} \approx 4.29067 \times 10^{-7}.$$

$$K1. P(X = r) = \frac{1}{3}, \text{ where } r = 1, 2, 3; \langle X \rangle = 2. P(Y = r) = 2^{r-1}3^{-r}; \langle Y \rangle = 3.$$

$$K2. (a) p_r = {}^5C_r\left(\frac{1}{3}\right)^r\left(\frac{2}{3}\right)^{5-r}; \langle x \rangle = \frac{5}{3}; \sigma^2 = \frac{10}{9}.$$

$$(b) P\left(\langle r \rangle - \frac{1}{2}\sigma \leq r \leq \langle r \rangle + \frac{1}{2}\sigma\right) = \frac{80}{243}; P\left(\langle r \rangle - \sigma \leq r \leq \langle r \rangle + \sigma\right) = \frac{160}{243}.$$

$$(d) p_r = \left\{ \frac{32}{243}, \frac{80}{243}, \frac{80}{243}, \frac{40}{243}, \frac{10}{243}, \frac{1}{243} \right\};$$

$$p_r \approx \{0.131687, 0.329218, 0.329218, 0.164609, 0.041152, 0.004115\};$$

$P_1(r) \approx \{0.110244, 0.315038, 0.366023, 0.172897, 0.033205, 0.002593\}$  ( $P_1$  is normalised).

(e) All different, apart from the mean.

(f) Answers are the same, provided the counters are replaced.

K3. (a)  $d = \frac{25}{36}$ . (b) 0.6. (c)  $\frac{71}{120}$ . (d)(i)  $\frac{1}{8}$ , (ii)  $\frac{157}{216}$ .

K4.

$$(a) P = \begin{cases} 0, & y < 1, \\ y - 1, & 1 \leq y \leq 2, \\ 1, & y > 2. \end{cases} \quad (b) P = \begin{cases} 1, & z < 1, \\ (2 - z)^3, & 1 \leq z \leq 2, \\ 0, & z > 2. \end{cases}$$

K5.  $P(\text{diseased}|\text{positive test}) \approx 0.090$ .