Natural Sciences Tripos Part IB

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0.0.1 Taylor series for analytic functions

If a function of a complex variable is analytic in a region $\mathcal{R}$ of the complex plane, not only is it differentiable everywhere in $\mathcal{R}$, it is also differentiable any number of times. It follows that if $f(z)$ is analytic at $z = z_0$, it has an infinite Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \text{where} \quad a_n = \frac{1}{n!} f^{(n)}(z_0) \equiv \frac{1}{n!} \frac{\mathrm{d}^n f}{\mathrm{d}z^n}(z_0)\,. \tag{1}$$

As discussed in § 0.2, this series converges within some neighbourhood of z_0.

Alternative definition of analyticity. An alternative definition of the analyticity of a function $f(z)$ at $z = z_0$ is that $f(z)$ has a Taylor series expansion about $z = z_0$ with a *non-zero radius of convergence*.

0.1 Zeros, Poles and Essential Singularities

0.1.1 Zeros of complex functions

Definition: Order. The zeros of $f(z)$ are the points $z = z_0$ in the complex plane where $f(z_0) = 0$. A zero is of order N if

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(N-1)}(z_0) = 0 \qquad \text{but} \quad f^{(N)}(z_0) \neq 0 \tag{2a}$$

The first non-zero term in the Taylor series of $f(z)$ about $z = z_0$ is then proportional to $(z - z_0)^N$. Indeed

$$f(z) \sim a_N (z - z_0)^N \qquad \text{as} \quad z \to z_0 \tag{2b}$$

A *simple zero* is a zero of order 1. A *double zero* is one of order 2, etc.

Examples.

1. $f(z) = z$ has a simple zero at $z = 0$.
2. $f(z) = (z - \mathrm{i})^2$ has a double zero at $z = \mathrm{i}$.
3. $f(z) = z^2 - 1 = (z - 1)(z + 1)$ has simple zeros at $z = \pm 1$.

Worked exercise. Find and classify the zeros of $f(z) = \sinh z$.

Answer.

$$\sinh z = \tfrac{1}{2}(\mathrm{e}^z - \mathrm{e}^{-z}) = 0$$

if

$$\mathrm{e}^z = \mathrm{e}^{-z} \quad \Rightarrow \quad \mathrm{e}^{2z} = 1 \quad \Rightarrow \quad z = n\pi\mathrm{i}, \quad n \in \mathbb{Z}\,.$$

Since $f'(z) = \cosh z = \cos(n\pi) \neq 0$ at these points, all the zeros are simple zeros.

0.1.2 Poles of complex functions

Definition: Order. Suppose $g(z)$ is analytic and non-zero at $z = z_0$. Consider the function

$$f(z) = (z - z_0)^{-N} g(z)\,, \tag{3a}$$

in which case

$$f(z) \sim g(z_0)(z - z_0)^{-N} \qquad \text{as} \quad z \to z_0 \,. \tag{3b}$$

$f(z)$ is not analytic at $z = z_0$, and we say that $f(z)$ has a *pole of order* N. We refer to a pole of order 1 as a *simple pole*, a pole of order 2 as a *double pole*, etc.

Expansion of $f(z)$ near a pole. Because $g(z)$ is analytic, from (1) it has a Taylor series expansion at z_0:

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n \qquad \text{with} \quad b_0 \neq 0 \,. \tag{4a}$$

Hence

$$f(z) = (z - z_0)^{-N} g(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n \,, \tag{4b}$$

with $a_n = b_{n+N}$, and $a_{-N} \neq 0$. This is not a Taylor series because it includes negative powers of $z - z_0$, and $f(z)$ is not analytic at $z = z_0$.

Remarks.

1. If $f(z)$ has a zero of order N at $z = z_0$, then $1/f(z)$ has a pole of order N there, and vice versa.

2. If $f(z)$ is analytic and non-zero at $z = z_0$ and $g(z)$ has a zero of order N there, then $f(z)/g(z)$ has a pole of order N there.

0.1.3 Laurent series and essential singularities

Definition: Laurent series. It can be shown that any function that is analytic (and single-valued) throughout an annulus $\alpha < |z - z_0| < \beta$ centred on a point $z = z_0$ has a unique *Laurent series*,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \,, \tag{5}$$

which converges for all values of z within the annulus.

If $\alpha = 0$, then $f(z)$ is analytic throughout the disk $|z - z_0| < \beta$ except possibly at $z = z_0$ itself, and the Laurent series determines the behaviour of $f(z)$ near $z = z_0$. There are three possibilities:

1. If the first non-zero term in the Laurent series has $n \geqslant 0$, then $f(z)$ is analytic at $z = z_0$ and the series is just a Taylor series.

2. If the first non-zero term in the Laurent series has $n = -N < 0$, then $f(z)$ has a pole of order N at $z = z_0$.

3. Otherwise, if the Laurent series involves an infinite number of terms with $n < 0$, then $f(z)$ has an *essential singularity* at $z = z_0$.

Example of a essential singularity. An example of an essential singularity is $f(z) = \mathrm{e}^{1/z}$ at $z = 0$, where the Laurent series can be generated from a Taylor series in $1/z$:

$$\mathrm{e}^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n \tag{6}$$

Remark. The behaviour of a function near an essential singularity is remarkably complicated. *Picard's theorem* states that, in any neighbourhood of an essential singularity, the function takes all possible complex values (possibly with one exception) at infinitely many points. In the case of $f(z) = \mathrm{e}^{1/z}$, the exceptional value 0 is never attained.

0.1.4 Behaviour at infinity

We can examine the behaviour of a function $f(z)$ as $z \to \infty$ by defining a new variable $\zeta = 1/z$ and a new function $g(\zeta) = f(z)$. Then $z = \infty$ maps to a single point $\zeta = 0$, the *point at infinity*.

If $g(\zeta)$ has a zero, pole or essential singularity at $\zeta = 0$, then we can say that $f(z)$ has the corresponding property at $z = \infty$.

Examples.

1.
$$f_1(z) = \mathrm{e}^z = \mathrm{e}^{1/\zeta} = g_1(\zeta) \tag{7a}$$

has an essential singularity at $z = \infty$.

2. $$f_2(z) = z^2 = 1/\zeta^2 = g_2(\zeta) \tag{7b}$$

has a double pole at $z = \infty$.

3. $$f_3(z) = \mathrm{e}^{1/z} = \mathrm{e}^{\zeta} = g_3(\zeta) \tag{7c}$$

is analytic at $z = \infty$.

Remark. It can be shown that all entire functions $f(z)$ have essential singularities at $z = \infty$ unless they are polynomials, and all polynomials have poles at $z = \infty$ unless they are constant.

0.2 Power Series of a Complex Variable

0.2.1 Convergence of Power Series

A power series about $z = z_0$ of a complex variable has the form

$$f(z) = \sum_{r=0}^{\infty} a_r (z - z_0)^r \quad \text{where} \quad a_r \in \mathbb{C}\,. \tag{8}$$

Hence the Taylor series for an analytic function, (1), is a power series.

Many of the tests of convergence for real series can be generalised for complex series. Indeed, we have already noted that if the sum of the absolute values of a complex series converges, i.e. if $\sum |u_r|$ converges, then so does the series, i.e. $\sum u_r$. Hence if $\sum |a_r(z-z_0)^r|$ converges, so does $\sum a_r(z-z_0)^r$.

0.2.2 Radius of convergence

If the power series (8) converges for $z = z_1$, then the series converges absolutely for all z such that $|z - z_0| < |z_1 - z_0|$.

Proof. Since $\sum a_r(z_1 - z_0)^r$ converges, then from the necessary condition for convergence,

$$\lim_{r\to\infty} a_r(z_1 - z_0)^r = 0\,. \tag{9a}$$

Hence for a given ε there exists $N \equiv N(\varepsilon)$ such that if $r > N$ then

$$|a_r(z_1 - z_0)^r| < \varepsilon\,. \tag{9b}$$

Thus for $r > N$

$$|a_r(z-z_0)^r| = |a_r(z_1-z_0)^r| \left|\frac{z-z_0}{z_1-z_0}\right|^r$$
$$< \varepsilon\varrho^r \quad \text{where} \quad \varrho = \left|\frac{z-z_0}{z_1-z_0}\right|. \tag{9c}$$

Thus, by means of a comparison with a geometric series, $\sum a_r(z-z_0)^r$ converges for $\varrho < 1$, i.e. for $|z-z_0| < |z_1-z_0|$.

Corollary. If the sum diverges for $z = z_1$ then it diverges for all z such that $|z-z_0| > |z_1-z_0|$. For suppose that it were to converge for some such $z = z_2$ with $|z_2-z_0| > |z_1-z_0|$, then it would converge for $z = z_1$ by the above result; this is in contradiction to the hypothesis.

Definition: Radius and circle of convergence. These results imply there must exist a real, non-negative number R such that

$$\begin{array}{r} \sum a_r(z-z_0)^r \text{ converges for } |z-z_0| < R \\ \sum a_r(z-z_0)^r \text{ diverges for } |z-z_0| > R \end{array} . \tag{10}$$

R is called the *radius of convergence*, and $|z - z_0| = R$ is called the *circle of convergence*, within which the series converges and outside of which it diverges.

Remarks.

1. The radius of convergence may be be zero (exceptionally), positive or infinite.
2. On the circle of convergence, the series may either converge or diverge.
3. The radius of convergence of the Taylor series of a function $f(z)$ about the point $z = z_0$ is equal to the distance of the nearest singular point of the function $f(z)$ from z_0. Since a convergent power series defines an analytic function, no singularity can lie inside the circle of convergence.

0.2.3 Determination of the radius of convergence

Without loss of generality take $z_0 = 0$, so that (8) becomes

$$f(z) = \sum_{r=0}^{\infty} u_r \quad \text{where} \quad u_r = a_r z^r \,. \tag{11}$$

Use D'Alembert's ratio test. *If* the limit exists, then

$$\lim_{r\to\infty} \left|\frac{a_{r+1}}{a_r}\right| = \frac{1}{R}\,. \tag{12a}$$

Proof. We have that

$$\lim_{r\to\infty} \left|\frac{u_{r+1}}{u_r}\right| = \lim_{r\to\infty} \left|\frac{a_{r+1}}{a_r}\right| |z| = \frac{|z|}{R} \quad \text{by hypothesis (12a).}$$

Hence the series converges absolutely by D'Alembert's ratio test if $|z| < R$. On the other hand if $|z| > R$, then

$$\lim_{r\to\infty} \left|\frac{u_{r+1}}{u_r}\right| = \frac{|z|}{R} > 1\,. \tag{12b}$$

So the series does not converge. It follows that R is the radius of convergence.

Remark. The limit (12a) may not exist, e.g. if $a_r = 0$ for r odd then $\left|\frac{a_{r+1}}{a_r}\right|$ is alternately 0 or ∞.

Use Cauchy's test (unlectured). *If* the limit exists, then

$$\lim_{r\to\infty} |a_r|^{1/r} = \frac{1}{R} \,. \tag{13a}$$

Proof. We have that

$$\lim_{r\to\infty} |u_r|^{1/r} = \lim_{r\to\infty} |a_r|^{1/r}|z| = \frac{|z|}{R} \quad \text{by hypothesis.} \tag{13b}$$

Hence the series converges absolutely by Cauchy's test if $|z| < R$.

On the other hand if $|z| > R$, choose τ with $1 < \tau < |z|/R$. Then there exists $M \equiv M(\tau)$ such that

$$|u_r|^{1/r} > \tau > 1 \,, \quad \text{i.e.} \quad |u_r| > \tau^r > 1 \,, \quad \text{for all} \quad r > M \,.$$

Thus, since $u_r \not\to 0$ as $r \to \infty$, $\sum u_r$ must diverge. It follows that R is the radius of convergence.

0.2.4 Examples

1. Suppose that $a_r = 1$ for all r, then $f(z)$ is the geometric series

$$f(z) = \sum_{r=0}^{\infty} z^r . \tag{14a}$$

Both D'Alembert's ratio test, (12a), and Cauchy's test, (13a), give $R = 1$:

$$\left|\frac{a_{r+1}}{a_r}\right| = 1 \quad \text{and} \quad |a_r|^{1/r} = 1 \quad \text{for all } r. \tag{14b}$$

Hence the series converges for $|z| < 1$. In fact

$$f(z) = \frac{1}{1-z}, \tag{14c}$$

where we note that it is the *singularity* at $z = 1$ which determines the radius of convergence.

2. Suppose next that $a_r = (-1)^{r-1}/r$ for all r, then $f(z)$ is the geometric

series

$$f(z) = -\sum_{r=1}^{\infty} \frac{(-z)^r}{r} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots . \tag{15a}$$

D'Alembert's ratio test gives

$$\frac{1}{R} = \lim_{r\to\infty} \left| \frac{a_{r+1}}{a_r} \right| = \lim_{r\to\infty} \frac{r}{r+1} = 1 . \tag{15b}$$

For Cauchy's test, we first note that

$$\lim_{r\to\infty} \log |a_r|^{\frac{1}{r}} = \lim_{r\to\infty} \frac{1}{r} \log \frac{1}{r} = 0 , \tag{15c}$$

and thence, as for D'Alembert's ratio test,

$$\frac{1}{R} = \lim_{r\to\infty} |a_r|^{1/r} = 1 . \tag{15d}$$

Remark. The series converges to $\log(1+z)$ for $|z| < 1$, where the singularity at $z = -1$ limits the radius of convergence. In fact it can be shown that the series converges on the circle $|z| = 1$ except at the point $z = -1$.

3. If $a_r = \frac{1}{r!}$ for all r, then $f(z)$ is the series

$$f(z) = \sum_{r=0}^{\infty} \frac{z^r}{r!}\,. \tag{16a}$$

D'Alembert's ratio test gives an infinite radius of convergence:

$$\frac{1}{R} = \lim_{r\to\infty} \left|\frac{a_{r+1}}{a_r}\right| = \lim_{r\to\infty} \frac{1}{r+1} = 0\,. \tag{16b}$$

For Cauchy's test, we first note, using Stirling's formula,[1] that

$$\log |a_r|^{\frac{1}{r}} = -\frac{1}{r}\log r! \sim -\log r \quad \text{as} \quad r \to \infty\,, \tag{16c}$$

and thence we confirm an infinite radius of convergence:

$$\frac{1}{R} = \lim_{r\to\infty} |a_r|^{1/r} = 0\,. \tag{16d}$$

[1] Stirling's formula states that

$$\log r! \sim r\log r - r + \tfrac{1}{2}\log(2\pi r) \quad \text{as} \quad r \to \infty\,.$$

The series converges to e^z for all finite z, which is an entire function.

4. Instead consider $a_r = r!$. This has *zero radius of convergence* since by D'Alembert's ratio test

$$\frac{a_{r+1}}{a_r} = r + 1 \to \infty \quad \text{as} \quad r \to \infty\,. \tag{17}$$

This conclusion can be confirmed using Cauchy's test. The series $\sum_{r=0}^{\infty} r! z^r$ fails to define a function since it does not converge for any non-zero z.

5. Finally consider

$$z \sum_{r=0}^{\infty} \frac{1}{2r+1}(-z^2)^r = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots = \arctan z\,. \tag{18}$$

Thought of as a power series in $(-z^2)$, this has $|a_{r+1}/a_r| = (2r+1)/(2r+3) \to 1$ as $r \to \infty$. Therefore $R = 1$ in terms of $(-z^2)$. But since $|-z^2| = 1$ is equivalent to $|z| = 1$, the series converges for $|z| < 1$ and diverges for $|z| > 1$.

0.2.5 Why Do We Have To Do This Again?

You already know the 'definition'

$$\int_a^b f(t)\mathrm{d}t = \lim_{N\to\infty} \sum_{j=1}^{N} f(a+jh)\,h \quad \text{where} \quad h = (b-a)/N\,, \tag{19}$$

so why are mathematicians not really content with it?

One answer is that while (19) is OK for OK functions, consider Dirichlet's function

$$f = \begin{cases} 0 & \text{on irrationals,} \\ 1 & \text{on rationals.} \end{cases} \tag{20}$$

If

- $a = 0$ and $b = \pi$, then (19) evaluates to 0,
- $a = 0$ and $b = p/q$, where p/q is a rational approximation to π (e.g. $22/7$ or better), then (19) evaluates to p/q.

0.2.6 Properties of the Riemann Integral

It is possible to show for integrable functions f and g, $a < c < b$, and $k \in \mathbb{R}$, that

$$\int_a^b f(t)\,\mathrm{d}t = -\int_b^a f(t)\,\mathrm{d}t\,, \tag{21}$$

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t\,, \tag{22}$$

$$\int_a^b kf(t)\,\mathrm{d}t = k\int_a^b f(t)\,\mathrm{d}t\,, \tag{23}$$

$$\int_a^b (f(t) + g(t))\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t + \int_a^b g(t)\,\mathrm{d}t\,, \tag{24}$$

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \int_a^b |f(t)|\,\mathrm{d}t\,. \tag{25}$$

It is also possible to deduce that if f and g are integrable then so if fg.

Schwarz's Inequality. For integrable functions f and g

$$\left(\int_a^b fg\,\mathrm{d}t\right)^2 \leqslant \left(\int_a^b f^2\mathrm{d}t\right)\left(\int_a^b g^2\mathrm{d}t\right). \tag{26}$$

Proof. Using the above properties it follows that

$$0 \leqslant \int_a^b (\lambda f + g)^2\mathrm{d}t = \lambda^2 \int_a^b f^2\mathrm{d}t + 2\lambda \int_a^b fg\,\mathrm{d}t + \int_a^b g^2\mathrm{d}t\,. \tag{27}$$

- If $\int_a^b f^2\mathrm{d}t = 0$ then

$$2\lambda \int_a^b fg\,\mathrm{d}t + \int_a^b g^2\mathrm{d}t \geqslant 0\,.$$

This can only be true for all λ if $\int_a^b fg\,\mathrm{d}t = 0$; the [in]equality follows.

- If $\int_a^b f^2\mathrm{d}t \neq 0$ then choose

$$\lambda = -\frac{\displaystyle\int_a^b fg\,\mathrm{d}t}{\displaystyle\int_a^b f^2\mathrm{d}t}, \tag{28}$$

and the inequality again follows.

Remark. This will not be the last time that we will find an analogy between scalar/inner products and integrals.

0.2.7 The Fundamental Theorems of Calculus

Suppose f is integrable. Define

$$F(x) = \int_a^x f(t)\,\mathrm{d}t\,. \tag{29}$$

F is continuous. F is a continuous function of x since

$$\begin{aligned} |F(x+h) - F(x)| &= \left| \int_x^{x+h} f(t)\,\mathrm{d}t \right| \\ &\leqslant \int_x^{x+h} |f(t)|\,\mathrm{d}t \\ &\leqslant \left(\max_{x \leqslant t \leqslant x+h} |f(t)| \right) h\,, \end{aligned}$$

and hence

$$\lim_{h\to 0} |F(x+h) - F(x)| = 0\,.$$

The First Fundamental Theorem of Calculus. This states that

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x)\,, \tag{30}$$

i.e. *the derivative of the integral of a function is the function*.

Proof. Suppose that

$$m = \min_{x\leqslant t\leqslant x+h} f(t) \quad \text{and} \quad M = \max_{x\leqslant t\leqslant x+h} f(t)\,.$$

We can show from the definition of a Riemann integral that for $h > 0$

$$mh \leqslant \int_x^{x+h} f(t)\,\mathrm{d}t \leqslant Mh\,,$$

so

$$m \leqslant \frac{F(x+h) - F(x)}{h} \leqslant M\,.$$

But if f is continuous, then as $h \to 0$ both m and M tend to $f(x)$. We can similarly 'sandwich' $(F(x+h) - F(x))/h$. (30) then follows from the definition of a derivative.

The Second Fundamental Theorem of Calculus. This essentially states that *the integral of the derivative of a function is the function*, i.e. if g is differentiable then

$$\int_a^x \frac{\mathrm{d}g}{\mathrm{d}t}\,\mathrm{d}t = g(x) - g(a)\,. \tag{31}$$

Proof. Define $f(x)$ by

$$f(x) = \frac{\mathrm{d}g}{\mathrm{d}x}(x)\,,$$

and then define F as in (29). Then using (30) we have that

$$\frac{\mathrm{d}}{\mathrm{d}x}(F - g) = 0\,.$$

Hence from integrating and using the fact that $F(a) = 0$ from (29),

$$F(x) - g(x) = -g(a)\,.$$

Thus using the definition (29)

$$\int_a^x \frac{\mathrm{d}g}{\mathrm{d}t}\,\mathrm{d}t = g(x) - g(a)\,. \tag{32}$$

The Indefinite Integral. Let f be integrable, and suppose $f = F'(x)$ for some function F. Then, based on the observation that the lower limit a in (29), etc. is arbitrary, we define the *indefinite integral* of f by

$$\int^x f(t)\,\mathrm{d}t = F(x) + c \tag{33}$$

for any constant c.