Infinite-Dimensional Compressed Sensing and Frames in Medical Imaging

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In Magnetic Resonance Imaging (MRI) the mathematical reconstruction problem is as follows: Given

$$f = \mathcal{F}g, \qquad g \in L^2(\mathbb{R}^N),$$

reconstruct g, where $\mathcal{F}$ is the Fourier transform. The MRI machine will only acquire point samples of f and due to physical constraints one can only acquire a limited amount of samples per time unit. The mathematical task is therefore to reconstruct g as accurately as possible with the given point samples of f. The standard way of doing reconstruction is via the Shannon Sampling Theorem [4] that assures that if g is compactly supported then g can be reconstructed perfectly from (countably many) point samples of f at a certain rate depending on the support of g. As one can only sample finitely many points, this type of reconstruction is based on approximating g with a trigonometric polynomial. However, the speed of convergence is based on how well g can be approximated by trigonometric polynomials. In many cases there will be other bases or frames that are more suitable (typically wavelets or shearlets), and this is the motivation for a new generalized sampling theory [1] that has just been launched. This theory allows one to reconstruct using arbitrary bases and frames, and therefore one can often improve the quality of the reconstruction dramatically.

Combining these techniques with the novel ideas of Compressed Sensing [2] allows one to improve the reconstruction even further and in particular it allows one to speed up the acquisition time. The traditional way of doing Compressed Sensing is by subsampling Fourier samples uniformly at random. This technique turns out to be less than optimal as it requires the sampling basis (trigonometric polynomials) and the reconstruction basis (wavelets) to be *incoherent* [3] (the inner products of the sampling basis and the reconstruction basis should be small). By introducing a new type of semi-random sampling scheme one can break this *incoherence barrier* and show that this technique dramatically outperforms the traditional approach.

The key to the improvements of these new techniques is to use a "continuous/analog" model as opposed to a "discrete" model. In particular, one ends up with an infinite-dimensional reconstruction problem as opposed to the finite-dimensional method that is traditionally applied. Although the reconstruction can be dramatically improved using these techniques, both the analysis and the actual computations are more difficult than for the more traditional methods.

There are a lot of open questions, in particular related to reconstruction in frames such as Curvelets, Shearlets, Contourlets etc as well as Total Variation approaches. The project can be made as pure or applied as one wants. In particular, there is a great need for "theorem/proof"-type of research as well as actual coding, algorithm design and experiments with real life MRI data. (The pure part of the project requires functional analysis, some wavelet theory and probability and the applied part requires numerical analysis and programming skills.)

References

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