Can everything be computed? - On the Solvability Complexity Index and Towers of Algorithms

ANDERS C. HANSEN & JONATHAN BEN-ARTZI

Department of Applied Mathematics and Theoretical Physics, University of Cambridge,

Questions on computing have fascinated mathematicians for centuries. Arguably, the most famous of these questions is the problem of computing the zeros of a polynomial. In particular, can one compute the zeros using only finitely many arithmetic operations and radicals of the coefficients? The negative answer to this problem for the quintic has had an enormous impact on the theory of computations, as mathematicians had to search for other approaches to compute zeros of polynomials. In particular one had to resort to approximations and then take limits. This was the birth of computational mathematics.

This project addresses some of the fundamental barriers in the theory of computations. This is done via the concept of the Solvability Complexity Index (SCI) [2] and towers of algorithms [1]. The SCI is the smallest number of limits needed in order to compute a desired quantity (spectra of operators, roots of polynomials, solutions to linear equations etc.). Intriguingly, several of the fundamental problems in computations have SCI greater than one (i.e., more than one limit is needed). While this may come as a surprise, this fact touches onto the fundamental boundaries of computational mathematics. In several cases (spectral problems, inverse problems) one can provide sharp results on the SCI, and it is therefore possible to establish the absolute barriers for what can be achieved computationally.

Fascinatingly, there are several fundamental problems in computational mathematics that are still open, for example the problem of computing spectra of Schrödinger operators

$$H = -\Delta + V.$$

Indeed, it is unknown whether or not there exists an algorithm that can compute the spectra of Schrödinger operators on the line, even when one has bounds on the total variation of the potential function! This is quite intriguing, especially since it is roughly 80 years since the Nobel prizes were awarded to Heisenberg, Schrödinger and Dirac for their contributions in quantum mechanics, yet we still don't know how to compute spectra of general Schrödinger operators.

This project will be about such fundamental problems in computations. Although the project is about computational mathematics, it is rather theoretical. Mathematical prerequisites are first and foremost functional analysis and operator theory. Some spectral theory background can be useful, as well as some knowledge of numerical analysis, but this is not essential. There is a wide range of problems to look at, so the project can be tailored to fit the student's background. Also, it is possible to implement some of the newly developed towers of algorithms if one is interested in also doing some coding.

References

[1] P. Doyle and C. McMullen. Solving the quintic by iteration. *Acta Math.*, 163(3-4):151–180, 1989.

[2] A. Hansen. On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators. *J. Amer. Math. Soc.*, 24(1):81–124, 2011.