BAYESIAN INVERSE PROBLEMS – EXAMPLES 2 Lent 2019

Please email comments, corrections to: hnk22@cam.ac.uk

Return answers to questions 2. and 6. by Friday 1 March 4pm to DAMTP pigeonhole.

1. Show that the definition of Hellinger distance does not depend on the choice of the dominating measure μ.

2. Show that the total variation and Hellinger metrics are related by the inequalities

$$\frac{1}{\sqrt{2}} d_{TV}(\mu, \mu') \leq d_{Hell}(\mu, \mu') \leq \sqrt{d_{TV}(\mu, \mu')}.$$

3. Assume that the measures μ' and μ are equivalent, that is, $\mu' \ll \mu$ and $\mu \ll \mu'$. The Kullback–Leibler divergence between μ' and μ is defined as

$$D_{KL}(\mu' || \mu) = \int \log \Big(\frac{d\mu'}{d\mu} \Big) d\mu'.$$

Is D_{KL} a metric? Let $\mu_1 = \mathcal{N}(\theta_1, \sigma_1^2)$ and $\mu_2 = \mathcal{N}(\theta_2, \sigma_2^2)$ be two Gaussian densities on $\mathbb{R}$. Show that

$$D_{KL}(\mu_1 || \mu_2) = \log \left(\frac{\sigma_2}{\sigma_1} \right) + \frac{1}{2} \left(\frac{\sigma_1^2}{\sigma_2^2} - 1 \right) + \frac{(\theta_1 - \theta_2)^2}{2\sigma_2^2}.$$

4. Assume that the measures μ' and μ are equivalent. Show that

$$d_{Hell}(\mu, \mu')^2 \leq \frac{1}{2} D_{KL}(\mu || \mu').$$

5. Let $\mathcal{H}$ be a separable Hilbert space and $\Sigma : \mathcal{H} \to \mathcal{H}$ be the covariance operator of measure μ defined by $\mathbb{E}^{\mu}\big(\langle \phi, (x - \theta) \rangle \langle \psi, (x - \theta) \rangle\big) = \langle \Sigma \phi, \psi \rangle$ for all $\phi, \psi \in \mathcal{H}$ (we can the use Riesz's representation theorem to identify $\mathcal{H}$ with its dual). Show that Σ is a trace class operator and

$$\int_{\mathcal{H}} \|x\|^2 d\mu(x) = \mathrm{Tr}(\Sigma).$$

6. (Proof of theorem 2.18) Let Σ be a self-adjoint, positive semi-definite, trace class operator in a Hilbert space $\mathcal{H}$, and let $\theta \in \mathcal{H}$. Using the Karhunen–Loève expansion show that there exists a Gaussian measure with mean θ and covariance operator Σ.

Matlab exercises

1. Sample from ℓ^1, Cauchy and Gaussian priors using Matlab. Plot the samples as a 2D image.

2. Assume that we have a posterior distribution with density

$$\pi(x, y) = \exp\big(- 10(x^2 - y)^2 - (y - 1/4)^4\big).$$

Write a Metropolis–Hastings algorithm to sample from π using the pseudocode given in Example 1.25. Try your code with different choices of γ and plot the first co-ordinates of the samples. What do you notice? What percentage of the suggested moves is accepted?