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Inverse Problems

Example sheet **2** Presentation **14 February 2018**, **2-3pm**, **MR15**.

Exercise 1 (Right-shift operator - submit)
The right-shift operator $K : \ell^2 \to \ell^2$, $\{u_j\}_{j\in\mathbb{N}} \mapsto \{f_j\}_{j\in\mathbb{N}}$, is given by

$$f_j = (Ku)_j := \begin{cases} 0 & j = 1, \\ u_{j-1} & j \geq 2. \end{cases}$$

a) Compute the range $\mathcal{R}(K)$ and the kernel $\mathcal{N}(K)$ of K.

b) Prove or falsify: "The Moore-Penrose inverse of K continuous." Argue only with the definition of the operator and your results of a).

c) Compute the Moore-Penrose inverse of K. State the domain and the range of $K^\dagger$.

Exercise 2 (Inverse problem of differentiation - submit)
We consider the problem of differentiation, formulated as the inverse problem of finding u from $Ku = f$ with the integral operator $K : L^2([0, 1]) \to L^2([0, 1])$ defined as

$$(Ku)(y) := \int_0^y u(x)\, dx\,.$$

a) Let f be given by

$$f(x) := \begin{cases} 0 & x < \frac{1}{2}, \\ 1 & x > \frac{1}{2}. \end{cases}$$

Show that $f \in \overline{\mathcal{R}(K)}$.

b) Let f be given as in a). Show that $f \in \overline{\mathcal{R}(K)} \setminus \mathcal{R}(K)$.
Hint: Consider the Picard criterion.

c) Prove or falsify: "The Moore-Penrose inverse of K continuous."

Please turn over!

Exercise 3 (Differential quotient operator)
As in exercise 2, we consider the inverse problem of differentiation. As an approximation to $K^\dagger$ we are interested in studying the following differential quotient operator $R_\alpha : L^2([0,1]) \to L^2([0,1])$ with

$$(R_\alpha f)(x) := \frac{1}{\alpha}\begin{cases} f(x+\alpha) - f(x) & x \in \left[0, \frac{1-\alpha}{2}\right[\\ f(x+\frac{\alpha}{2}) - f(x-\frac{\alpha}{2}) & x \in \left[\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right[\\ f(x) - f(x-\alpha) & x \in \left[\frac{1+\alpha}{2}, 1\right] \end{cases}$$

for $\alpha \in]0, 1/2[$. Further, let $H^2([0,1])$ denote the Hilbert space

$$H^2([0,1]) = \left\{ f \in L^2([0,1]) \mid f'', f' \in L^2([0,1]) \right\} .$$

We consider the case of a noisy measurement, i.e. we observe $f^\delta \in L^2([0,1])$ for which

$$\|f - f^\delta\|_{L^2([0,1])} \leq \delta$$

holds true, for the exact data $f \in \mathcal{D}(K^\dagger)$.

a) Assume that $f \in H^2([0,1])$ and $\|f''\|_{L^2([0,1])} \leq c$. Verify the following estimate for the overall L^2-error between $u^\dagger$ and $R_\alpha f^\delta$:

$$\|K^\dagger f - R_\alpha f^\delta\|_{L^2([0,1])} \leq \frac{\sqrt{6}}{\alpha}\delta + \frac{\sqrt{17}}{4}\alpha c \tag{1}$$

b) Show that $R_\alpha : L^2([0,1]) \to L^2([0,1])$ is a convergent regularisation method and determine a corresponding a-priori parameter choice rule.

c) Discretise R_α by evaluating R_α at $2n$ discrete points $x_k := (k-1)\frac{\alpha}{2}$, $k \in \{1, \dots, 2n\}$, for $\alpha = \frac{1}{n-1}$ and $n \in \mathbb{N} \setminus \{1\}$. This way we obtain a mapping $\tilde{R}_\alpha : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Implement a Matlab function `diffquot` that takes a vector $\tilde{f} = (f(x_1), f(x_2), \dots, f(x_{2n}))^T \in \mathbb{R}^{2n}$ as an input argument and returns the output $\tilde{R}_\alpha \tilde{f}$.

d) Test your function for $\alpha = 2^{-k}$, $k \in \{2, 4, \dots, 8\}$, and

   i. $f(x) = \cos(\pi x)$ for $x \in [0,1]$;

   ii. $f(x) = \begin{cases} 0 & x \in \left[0, \frac{1}{3}\right[, \\ x - \frac{1}{3} & x \in \left[\frac{1}{3}, \frac{2}{3}\right[, \\ \frac{1}{3} & x \in \left[\frac{2}{3}, 1\right] , \end{cases}$

   and plot the maximum error $\|\tilde{R}_\alpha \tilde{f} - (f'(x_1), f'(x_2), \dots, f'(x_{2n}))^\top\|_\infty$ in dependence of α.

Exercise 4 (Deconvolution)
Let $\Omega := [0,1]^2$, $k \in L^2(\Omega)$ and $\tilde{k}$ be the periodic extension of k with

$$\tilde{k}(z) = \begin{cases} k(z) & z \in \Omega \\ k(\mathrm{mod}(z,1)) & z \in \mathbb{R}^2 \setminus \Omega \end{cases},$$

where $\mathrm{mod}(z,1) = (\mathrm{mod}(z_1,1), \mathrm{mod}(z_2,1))$ and consider the convolution operator $K : L^2(\Omega) \to L^2(\Omega)$ with

$$(Ku)(x) := \int_\Omega \tilde{k}(x-y)u(y)\,dy\,.$$

a) Compute the singular value decomposition of K.
Hint: you can represent a function $v \in L^2(\Omega)$ as $v = \sum_{m,n\in\mathbb{Z}} \langle v, \varphi_{m,n}\rangle \varphi_{m,n}$ with $\varphi_{m,n}(x_1,x_2) = \exp(-i2\pi(mx_1 + nx_2))$.

b) Argue empirically with the singular values whether the inverse problem is ill-posed or not, for the specific choices

i. $k(x_1,x_2) = \frac{1}{h^2}\chi_{[-\frac{1}{2},\frac{1}{2}]}\left(\frac{x_1-1/2}{h}\right)\chi_{[-\frac{1}{2},\frac{1}{2}]}\left(\frac{x_2-1/2}{h}\right)$ for $0 < h < 1$.

ii. $k(x_1,x_2) = \varphi(x_1)\varphi(x_2)$ with $\varphi(x) := \begin{cases} \exp\left(-\frac{1}{1/4-(x-1/2)^2}\right) & x \in]0,1[\\ 0 & \text{else} \end{cases}.$

Is the ill-posedness mild or severe?

c) Implement the deconvolution as in exercise 4 of example sheet 1. Regularise the problem using

i. Truncated singular value decomposition,

ii. Tikhonov regularisation.

How does the latter relate to exercise 4 c) on example sheet 1?

Please turn over!

Exercise 5 (The Radon transform)

a) The Matlab command `f = radon(u, phi);` computes a discretised two-dimensional radon transform of a discrete image `u` for a vector of angles `phi`. Use this command to set up a matrix `R` that maps the column-vector representation of `u` into the column-vector representation of the sinogram `f` for an arbitrary image `u` $\in \mathbb{R}^{64\times 64}_{\geq 0}$ and angles `phi` with `phi(j) = j` for $j \in \{0, 2, \dots, 178\}$.

b) Create a noisy sinogram by applying `R` to a down-sampled version of the Shepp-Logan phantom (built-in in Matlab; use the command `phantom`) and subsequently adding non-negative, random numbers to the sinogram. Create multiple versions with different noise levels.

c) Compute a singular value decomposition of `R` via the Matlab command `svd` and visualise selected singular vectors of your choice.

d) Create a 'pseudo'-inverse of `R` by constructing an appropriate matrix with inverted singular values and apply this matrix to the column-vector representations of your noisy sinograms. Regularise the Moore-Penrose inverse using

    i. Truncated singular value decomposition;

    ii. Tikhonov regularisation.