Department of Applied Mathematics and Theoretical Physics
Matthias J. Ehrhardt, Lukas F. Lang

Inverse Problems

Example sheet **4** Presentation **11 March 2018, 2-3pm, MR15**
Submission **9 March 2018, 12pm, MR14**

Please submit exercises 3 and 4.

Exercise 1 (Convexity of a sum)
Let $\alpha \geq 0$ and $E, F \colon \mathcal{U} \to \mathbb{R}_\infty$ be two convex functions. Prove the following two statements.

a) The sum of the two functions $E + \alpha F \colon \mathcal{U} \to \mathbb{R}_\infty$ is convex

b) If, in addition $\alpha > 0$ and F is strictly convex, then $E + \alpha F$ is strictly convex.

Exercise 2 (Convexity of data term)
Let $\mathcal{U}, \mathcal{V}$ be normed spaces. Furthermore, let $K \in \mathcal{L}(\mathcal{U}, \mathcal{V}), f \in \mathcal{V}$ and $D \colon \mathcal{V} \to \mathbb{R}_\infty$ be defined as $D(u) := \frac{1}{2}\|Ku - f\|_{\mathcal{V}}^2$. Prove the following two statements.

a) Show that D is convex.

b) Show that D may not be strictly convex in general, even if K is injective.

c) Let $\mathcal{V}$ be an inner product space and $\lambda \in (0,1)$, $u, v \in \mathcal{U}$. Show that

$$D(\lambda u + (1-\lambda)v) = \lambda D(u) + (1-\lambda)D(v) - \frac{\lambda(1-\lambda)}{2}\|K(u-v)\|_{\mathcal{V}}^2 .$$

Exercise 3 (Convex conjugate)
Let $\mathcal{U}$ be a Banach space and let $E \colon \mathcal{U} \to \mathbb{R}_\infty$ be proper, lower semi-continuous and convex. Then the *Fenchel conjugate* or *convex conjugate* of E is defined to be the mapping $E^* \colon \mathcal{U}^* \to \mathbb{R}_\infty$ with

$$E^*(v) := \sup_{u \in \mathcal{U}} \Big\{ \langle v, u \rangle - E(u) \Big\} .$$

a) Compute the convex conjugates of the following functionals.

   i. $E \colon \mathbb{R} \to \mathbb{R}, E(u) = \frac{1}{p}|u|^p$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.
   ii. $E(u) = \frac{1}{2}\|u\|^2$ for a Hilbert space $\mathcal{U}$.
   iii. $F(u) = E(\alpha u - a) + \langle b, x \rangle + \beta$ for $\alpha \neq 0, \beta \in \mathbb{R}, a \in \mathcal{U}, b \in \mathcal{U}^*$.

b) Let $\mathcal{U}$ be a Hilbert space and $E : \mathcal{U} \to \mathbb{R}_\infty$ a proper, lower semi-continuous and convex functional. Show that

$$p \in \partial E(u) \quad \Leftrightarrow \quad u \in \partial E^*(p)$$

for all $u, p \in \mathcal{U}$.
Hint: You may exploit the fact that under the stated assumptions $E = E^{**}$ holds true.

Exercise 4 (Differentiation)
Let $\mathcal{U}$ be a Banach space and $E : \mathcal{U} \to \mathbb{R}$ be a convex functional that is Fréchet-differentiable in $u \in \mathcal{U}$. Then

$$\partial E(u) = \{E'(u)\}\,.$$

Exercise 5 (Bregman iteration)
Let $f \in \mathcal{V}, K \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $J : \mathcal{U} \to \mathbb{R}_\infty$ be a functional. The iteration

$$\begin{aligned} u^{k+1} &\in \arg\min_{u \in \mathcal{U}} \left\{ \frac{1}{2} \|Ku - f\|_{\mathcal{V}}^2 + D_J^{p^k}(u, u^k) \right\} \\ p^{k+1} &= p^k + K^*(f - Ku^{k+1}) \end{aligned} \tag{1}$$

with $u^0 = p^0 = 0$ and $p^k \in \partial J(u^k)$ for all $k \in \mathbb{N}$ is known as *Bregman iteration*. In this exercise we will analyse properties of the Bregman iteration which can be seen as an iterative regularisation method.

a) Show that the Bregman iterates monotonically decrease the data fidelity, i.e. they satisfy $\|Ku^{k+1} - f\|_{\mathcal{V}} \leq \|Ku^k - f\|_{\mathcal{V}}$.

b) If $\mathcal{V}$ is a Hilbert space, show that the Bregman iteration can also be written as

$$\begin{aligned} u^{k+1} &\in \arg\min_{u \in \mathcal{U}} \left\{ \frac{1}{2} \|Ku - f^k\|_{\mathcal{V}}^2 + J(u) \right\} \\ f^{k+1} &= f^k + f - Ku^{k+1} \end{aligned}$$

for $u^0 = 0, f^0 = f$.

c) Show that if there exists a $k_* \in \mathbb{N}$ such that u^{k_*} satisfies $Ku^{k_*} = f$, $u^k = u^{k_*}$ for all $k \geq k_*$ and u^{k_*} is a J-minimising solution of all elements in the set $\{u \in \mathcal{U} \mid Ku = f\}$.

d) Consider now the noisy case where u^k are the iterates for noisy data, i.e. f^δ replaces f in the definition of the iteration and $\|f - f^\delta\| \leq \delta$ for some $f \in \mathcal{R}(K)$. Let $u^* \in \mathcal{U}$ such that $Ku^* = f$. Show that u^k progressively approximates u^* in a Bregman sense, i.e.

$$D_J^{p^{k+1}}(u^\dagger, u^{k+1}) \leq D_J^{p^k}(u^\dagger, u^k)\,,$$

as long as $\|Ku^{k+1} - f^\delta\|_{\mathcal{V}} \geq \delta$. Thus the Bregman iteration with Morozov's discrepancy principle as a stopping criterion is a useful strategy to find J-minimising solutions.

Note that this also implies $D_J^{p^{k+1}}(u^\dagger, u^{k+1}) \leq D_J^{p^k}(u^\dagger, u^k)$ for all $k \in \mathbb{N}$ in case of $\delta = 0$.

e) Consider again the noise-free case, i.e. $\delta = 0$. Show the estimate

$$D_J^{p^k}(u^\dagger, u^k) \leq \frac{\|w\|_{\mathcal{V}}^2}{2k}$$

for $k \geq 1$ on the speed of the convergence under a source condition $p^\dagger := K^* w \in \partial J(u^\dagger)$ for some $u^\dagger \in \mathcal{U}$ with $Ku^\dagger = f$.