



Department of Applied Mathematics
and Theoretical Physics

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Inverse Problems in Imaging

Example sheet 1 Presentation 31 October 2018, 1-2pm, MR14.

Please submit after the lecture on 25 October 2018.

Exercise 1 (Integral operators - submit)

For $\Omega = [0, 1]^2$ and $\mathcal{X} = L^2(\Omega)$, we consider the integral operator $A : \mathcal{X} \rightarrow \mathcal{X}$ with

$$(Au)(y) := \int_{\Omega} k(x, y)u(x) dx,$$

for $k \in L^2(\Omega \times \Omega)$. Show that

- (a) A is linear with respect to u ,
- (b) A is a bounded linear operator, i.e. $\|Au\|_{\mathcal{X}} \leq \|A\|_{\mathcal{L}(\mathcal{X}, \mathcal{X})}\|u\|_{\mathcal{X}}$. Give also an estimate for $\|A\|_{\mathcal{L}(\mathcal{X}, \mathcal{X})}$,
- (c) the adjoint A^* is given via

$$(A^*v)(y) = \int_{\Omega} k(y, x)v(x) dx.$$

Exercise 2 (Inverse problem of differentiation - submit)

We consider the problem of differentiation, formulated as the inverse problem of finding u from $Au = f$ with the integral operator $A : L^2([0, 1]) \rightarrow L^2([0, 1])$ defined as

$$(Au)(y) := \int_0^y u(x) dx.$$

- (a) Let f be given by

$$f(x) := \begin{cases} 0 & x < \frac{1}{2}, \\ 1 & x > \frac{1}{2}. \end{cases}$$

Show that $f \in \overline{\mathcal{R}(A)}$.

- (b) Let f be given as in a). Show that $f \in \overline{\mathcal{R}(A)} \setminus \mathcal{R}(A)$.

Hint: Consider the Picard criterion.

- (c) Prove or falsify: “The Moore-Penrose inverse of A continuous.”

Please turn over!

Exercise 3 (Generalised inverse)

(a) Let $m, n \in \mathbb{N}$ with $m \geq n \geq 2$. Compute the Moore-Penrose inverses of the following matrices:

(i) $A = (1, 1, \dots, 1) \in \mathbb{R}^{1 \times n}$

(ii) $A = \text{diag}(a_1, \dots, a_n) \in \mathbb{R}^{n \times n}$ with $a_j \in \mathbb{R}$ for $j \in \{1, \dots, n\}$

(iii) $A \in \mathbb{R}^{m \times n}$ with $A^T A = I_n$

(b) Let $a, b \in \mathbb{R}$ with $a < b$. Compute the Moore-Penrose inverse of the operator $A : L^2([a, b]) \rightarrow \mathbb{R}$ with

$$Au = \int_a^b u(x) dx.$$

Exercise 4 (Convolution)

Many forward problems are either modelled as convolutions or they are modelled as the composition of several components one of which is a convolution. Therefore convolutions play an important role in inverse problems. As in Exercise 1, let $\Omega = [0, 1]^2$ be the unit square and let $\mathcal{X} = L^2(\Omega)$. A convolution is the special case of an integral operator $A : \mathcal{X} \rightarrow \mathcal{X}$ where the kernel has a simple structure:

$$(Au)(y) := \int_{\Omega} k(y-x)u(x) dx,$$

for $k \in L^2(\Omega)$. It follows easily from Exercise 1 that A is linear and bounded.

(a) Although shown in general in Exercise 1, give an explicit form of the adjoint of the convolution.

(b) Let $f = Au$. It follows from the convolution theorem that a convolution can be inverted by means of the Fourier transform

$$u = (2\pi)^{-\frac{n}{2}} \mathcal{F}^{-1} \left(\frac{\mathcal{F}(f)}{\mathcal{F}(k)} \right), \quad (1)$$

where \mathcal{F} is the Fourier transform and \mathcal{F}^{-1} its inverse. Implement this formula in MATLAB to deblur (deconvolve) the blurry tree image f generated by the script `ex4b_generate_data.m`, which is provided online¹. Note that the script also outputs $\mathcal{F}(k)$. Add some noise to the data and show that the inversion formula is ill-conditioned.

Hint: Make use of the MATLAB commands `fft2` and `ifft2`.

(c) Reformulate equation (1) so that the denominator is non-negative and give a stable approximation of this formula. Implement this formula in MATLAB and empirically show that it is stable.

Hint: Make use of the MATLAB command `conj`.

¹http://store.maths.cam.ac.uk/DAMTP/yk362/teaching/201819_Michaelmas_InverseProblemsImaging/ex4b_generate_data.zip

Exercise 5 (The Radon transform)

- (a) The Matlab command `f = radon(u, phi)`; computes a discretised two-dimensional radon transform of a discrete image `u` for a vector of angles `phi`. Use this command to set up a matrix `R` that maps the column-vector representation of `u` into the column-vector representation of the sinogram `f` for an arbitrary image $u \in \mathbb{R}_{\geq 0}^{64 \times 64}$ and angles `phi` with `phi(j) = j` for $j \in \{0, 2, \dots, 178\}$.
- (b) Create a noisy sinogram by applying `R` to a down-sampled version of the Shepp-Logan phantom (built-in in Matlab; use the command `phantom`) and subsequently adding non-negative, random numbers to the sinogram. Create multiple versions with different noise levels.
- (c) Compute a singular value decomposition of `R` via the Matlab command `svd` and visualise selected singular vectors of your choice.
- (d) Create a 'pseudo'-inverse of `R` by constructing an appropriate matrix with inverted singular values and apply this matrix to the column-vector representations of your noisy sinograms. Regularise the Moore-Penrose inverse using
 - (i) Truncated singular value decomposition;
 - (ii) Tikhonov regularisation.