Introduction to Optimal Transport

Matthew Thorpe

F2.08, Centre for Mathematical Sciences
University of Cambridge
Email: m.thorpe@maths.cam.ac.uk

Lent 2018
Current Version: Thursday 8th March, 2018

Foreword

These notes have been written to supplement my lectures given at the University of Cambridge in the Lent term 2017-2018. The purpose of the lectures is to provide an introduction to optimal transport. Optimal transport dates back to Gaspard Monge in 1781 [11], with significant advancements by Leonid Kantorovich in 1942 [8] and Yann Brenier in 1987 [4]. The latter in particular lead to connections with partial differential equations, fluid mechanics, geometry, probability theory and functional analysis. Currently optimal transport enjoys applications in image retrieval, signal and image representation, inverse problems, cancer detection, texture and colour modelling, shape and image registration, and machine learning, to name a few. The purpose of this course is to introduce the basic theory that surrounds optimal transport, in the hope that it may find uses in people's own research, rather than focus on any specific application.

I can recommend the following references. My lectures and notes are based on *Topics in Optimal Transportation* [15]. Two other accessible introductions are *Optimal Transport: Old and New* [16] (also freely available online) and *Optimal Transport for Applied Mathemacians* [12] (also available for free online). For a more technical treatment of optimal transport I refer to *Gradient Flows in Metric Spaces and in the Space of Probability Measures* [2]. For a short review of applications in optimal transport see the article *Optimal Mass Transport for Signal Processing and Machine Learning* [9].

Please let me know of any mistakes in the text. I will also be updating the notes as the course progresses.

Some Notation:

1. $C_b^0(Z)$ is the space of all continuous and bounded functions on Z.

2. A sequence of probability measures $\pi_n \in \mathcal{P}(Z)$ converges weak* to π, and we write $\pi_n \stackrel{*}{\rightharpoonup} \pi$, if for any $f \in C_b^0(Z)$ we have $\int_Z f \, \mathrm{d}\pi_n \to \int_Z f \, \mathrm{d}\pi$.

3. A *Polish space* is a separable completely metrizable topological space (i.e. a complete metric space with a countable dense subset).

4. $\mathcal{P}(Z)$ is the set of probability measures on Z, i.e. the subset of $\mathcal{M}_+(Z)$ with unit mass.

5. $\mathcal{M}_+(Z)$ is the set of positive Radon measures on Z.

6. $P^X : X \times Y \to X$ is the projection onto X, i.e. $P(x, y) = x$, similarly $P^Y : X \times Y \to Y$ is given by $P^Y(x, y) = y$.

7. A function $\Theta : E \to \mathbb{R} \cup \{+\infty\}$ is *convex* if for all $(z_1, z_2, t) \in E \times E \times [0,1]$ we have $\Theta(tz_1+(1-t)z_2) \leq t\Theta(z_1)+(1-t)\Theta(z_2)$. A convex function Θ is *proper* if $\Theta(z) > -\infty$ for all $z \in E$ and there exists $z^\dagger \in E$ such that $\Theta(z^\dagger) < +\infty$.

8. If E is a normed vector space then E^* is its dual space, i.e. the space of all bounded and linear functions on E.

9. For a set A in a topological space Z the *interior* of A, which we denote by $\mathrm{int}(A)$, is the set of points $a \in A$ such that there exists an open set $\mathcal{O}$ with the property $a \in \mathcal{O} \subseteq A$.

10. All vector spaces are assumed to be over $\mathbb{R}$.

11. The *closure* of a set A in a topological space Z, which we denote by $\overline{A}$, is the set of all points $a \in Z$ such that for any open set $\mathcal{O}$ with $a \in \mathcal{O}$ we have $\mathcal{O} \cap A \neq \emptyset$.

12. The *graph* of a function $\varphi : X \to \mathbb{R}$ which we denote by $\mathrm{Gra}(\varphi)$, is the set $\{(x,y) : x \in X, y = \varphi(x)\}$.

13. The k^{th} *moment* of $\mu \in \mathcal{P}(X)$ is defined as $\int_X |x|^k \,\mathrm{d}\mu(x)$.

14. The *support* of a probability measure $\mu \in \mathcal{P}(X)$ is the smallest closed set A such that $\mu(A) = 1$.

15. $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^d$ (the dimension d should be clear by context).

16. We write $\mu\lfloor_A$ for the measure μ restricted to A, i.e. $\mu\lfloor_A(B) = \mu(A \cap B)$ for all measurable B.

17. Given a probability measure μ we say a property holds μ-almost surely if it holds on a set of probability one. If μ is the Lebesgue measure we will just say that it holds almost surely.

Contents

Chapter 1

Formulation of Optimal Transport

There are two ways to formulate the optimal transport problem: the Monge and Kantorovich formulations. We explain both these formulations in this chapter. Historically the Monge formulation comes before Kantorovich which is why we introduce Monge first. The Kantorovich formulation can be seen as a generalisation of Monge. Both formulations have their advantages and disadvantages. My experience is that Monge is more useful in applications, whilst Kantorovich is more useful theoretically. In a later chapter (see Chapter 4) we will show sufficient conditions for the two problems to be considered equivalent. After introducing both formulations we give an existence result for the Kantorovich problem; existence results for Monge are considerably more difficult. We look at special cases of the Monge and Kantorovich problems in the next chapter, a more general treatment is given in Chapters 3 and 4.

1.1 The Monge Formulation

Optimal transport gives a framework for comparing measures μ and ν in a Lagrangian framework. Essentially one pays a cost for transporting one measure to another. To illustrate this consider the first measure μ as a pile of sand and the second measure ν as a hole we wish to fill up. We assume that both measures are probability measures on spaces X and Y respectively. Let $c : X \times Y \to [0, +\infty]$ be a cost function where $c(x, y)$ measures the cost of transporting one unit of mass from $x \in X$ to $y \in Y$. The optimal transport problem is how to transport μ to ν whilst minimizing the cost c.[1] First, we should be precise about what is meant by transporting one measure to another.

Definition 1.1. *We say that $T : X \to Y$ transports $\mu \in \mathcal{P}(X)$ to $\nu \in \mathcal{P}(Y)$, and we call T a* transport map*, if*

$$\nu(B) = \mu(T^{-1}(B)) \qquad \text{for all } \nu\text{-measurable sets } B. \tag{1.1}$$

[1] Some time ago I either read or was told that the original motivation for Monge was how to design defences for Napoleon. In this case the pile of sand is a wall and the hole a moat. Obviously one wishes to to make the wall using the dirt dug out to form the moat. In this context the optimal transport problem is how to transport the dirt from the moat to the wall.

To visualise the transport map see Figure 1.1. For greater generality we work with the inverse of T rather than T itself; the inverse is treated in the general set valued sense, i.e. $x \in T^{-1}(y)$ if $T(x) = y$, if the function T is injective then we can equivalently say that $\nu(T(A)) = \mu(A)$ for all μ-measurable A. What Figure 1.1 shows is that for any ν-measurable B, and $A = \{x \in X : T(x) \in B\}$ that $\mu(A) = \nu(B)$. This is what we mean by T transporting μ to ν. As shorthand we write $\nu = T_\# \mu$ if (1.1) is satisfied.

Proposition 1.2. *Let* $\mu \in \mathcal{P}(X)$, $T : X \to Y$, $S : Y \to Z$ *and* $f \in L^1(Y)$. *Then*

1. **change of variables formula**

$$\int_Y f(y) \, \mathrm{d}(T_\#\mu)(y) = \int_X f(T(x)) \, \mathrm{d}\mu(x); \tag{1.2}$$

2. **composition of maps**

$$(S \circ T)_\# \mu = S_\#(T_\#\mu).$$

Proof. Recall that, for non-negative $f : Y \to \mathbb{R}$

$$\int_Y f(y) \, \mathrm{d}(T_\#\mu)(y) := \sup \left\{ \int_Y s(y) \, \mathrm{d}(T_\#\mu)(y) : 0 \leq s \leq f \text{ and } s \text{ is simple} \right\}.$$

Now if $s(y) = \sum_{i=1}^N a_i \delta_{U_i}(y)$ where $a_i = s(y)$ for any $y \in U_i$ then

$$\int_Y s(y) \, \mathrm{d}(T_\#\mu)(y) = \sum_{i=1}^N a_i T_\#\mu(U_i) = \sum_{i=1}^N a_i \mu(V_i) = \int_X r(x) \, \mathrm{d}\mu(x)$$

for $V_i = T^{-1}(U_i)$ and $r = \sum_{i=1}^N a_i \delta_{V_i}$. For $x \in V_i$ we have $T(x) \in U_i$ and therefore $r(x) = a_i = s(T(x)) \leq f(T(x))$. From this it is not hard to see that

$$\sup_{0 \leq s \leq f} \int_Y s(y) \, \mathrm{d}(T_\#\mu)(y) = \sup_{0 \leq r \leq f \circ T} \int_X r(x) \, \mathrm{d}\mu(x)$$

where both supremums are taken over simple functions. Hence (1.2) holds for non-negative functions. By treating signed functions as $f = f^+ - f^-$ we can prove the proposition for $f \in L^1(Y)$.

For the second statement let $A \subset Z$ and observe that $T^{-1}(S^{-1}(A)) = (S \circ T)^{-1}(A)$. Then

$$S_\#(T_\#\mu)(A) = T_\#\mu(S^{-1}(A)) = \mu(T^{-1}(S^{-1}(A))) = \mu((S \circ T)^{-1}(A)) = (S \circ T)_\#\mu(A).$$

Hence $S_\#(T_\#\mu) = (S \circ T)_\#\mu$. □

Given two measures μ and ν the existence of a transport map T such that $T_\#\mu = \nu$ is not only non-trivial, but it may also be false. For example, consider two discrete measures $\mu = \delta_{x_1}$, $\nu = \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta_{y_2}$ where $y_1 \neq y_2$. Then $\nu(\{y_1\}) = \frac{1}{2}$ but $\mu(T^{-1}(y_1)) \in \{0, 1\}$ depending on whether $x_1 \in T^{-1}(y_1)$. Hence no transport maps exist.

There are two important cases where transport maps exist:

1. the discrete case when $\mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ and $\nu = \frac{1}{n}\sum_{j=1}^n \delta_{y_j}$;
2. the absolutely continuous case when $\mathrm{d}\mu(x) = f(x)\,\mathrm{d}x$ and $\mathrm{d}\nu(y) = g(y)\,\mathrm{d}y$.

It is important that in the discrete case that μ and ν are supported on the same number of points; the supports do not have to be the same but they do have to be of the same size. We will revisit both cases (the discrete case in the next chapter, the absolutely continuous case in Chapter 4).

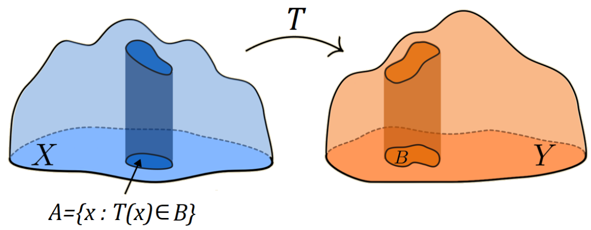

Figure 1.1: Monge's transport map, figure modified from Figure 1 in [9].

With this notation we can define Monge's optimal transport problem as follows.

Definition 1.3. ***Monge's Optimal Transport Problem:*** *given $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$,*

$$\textit{minimise } \mathbb{M}(T) = \int_X c(x, T(x))\,\mathrm{d}\mu(x)$$

over μ-measurable maps $T : X \to Y$ subject to $\nu = T_{\#}\mu$.

Monge originally considered the problem with L^1 cost, i.e. $c(x, y) = |x - y|$. This problem is significantly harder than with L^2 cost, i.e. $c(x, y) = |x-y|^2$. In fact the first correct proof with L^1 cost dates back only a few years to 1999 (see Evans and Gangbo [6]) and required stronger assumptions than the L^2 cost, Sudakov thought to have proven the result in 1979 [14] however this was found to contain a mistake which was later fixed by Ambrosio and Pratelli [1, 3].

In general Monge's problem is difficult due to the non-linearity in the constraint (1.1). If we assume that μ and ν are absolutely continuous with respect to the Lebegue measure on $\mathbb{R}^d$, i.e. $\mathrm{d}\mu(x) = f(x)\,\mathrm{d}x$ and $\mathrm{d}\nu(y) = g(y)\,\mathrm{d}y$, and assume T is a C^1 diffeomorphism (T is bijective and T, T^{-1} are differentiable) then one can show that (1.1) is equivalent to

$$f(x) = g(T(x))\,|\det(\nabla T(x))|\,.$$

The above constraint is highly non-linear and difficult to handle with standard techniques from the calculus of variations.

1.2 The Kantorovich Formulation

Observe that in the Monge formulation mass is mapped $x \mapsto T(x)$. In particular, this means that mass is not split. In the discrete case this causes difficulties concerning the existence of maps T such that $T_{\#}\mu = \nu$, see the example $\mu = \delta_{x_1}$, $\nu = \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta_{y_2}$ in the previous section. Observe that if we allow mass to be split, i.e. half of the mass from x_1 goes to y_1 and half the mass goes to y_2, then we have a natural relaxation. This is in effect what the Kantorovich formulation does. To formalise this we consider a measure $\pi \in \mathcal{P}(X \times Y)$ and think of $\mathrm{d}\pi(x,y)$ as the amount of mass transferred from x to y; this way mass can be transferred from x to multiple locations. Of course the total amount of mass removed from any measurable set $A \subset X$ has to equal to $\mu(A)$, and the total amount of mass transferred to any measurable set $B \subset Y$ has to be equal to $\nu(B)$. In particular, we have the constraints:

$$\pi(A \times Y) = \mu(A), \qquad \pi(X \times B) = \nu(B) \quad \text{for all measurable sets } A \subseteq X,\ B \subseteq Y.$$

We say that any π which satisfies the above has first marginal μ and second marginal ν, we denote the set of such π by $\Pi(\mu,\nu)$. We will call $\Pi(\mu,\nu)$ the set of *transport plans* between μ and ν. Note that $\Pi(\mu,\nu)$ is never non-empty (in comparison with the set of transport plans) since $\mu \otimes \nu \in \Pi(\mu,\nu)$ which is the trivial transport plan which transports every grain of sand at x to y proportional to $\nu(y)$. We can now define Kantorovich's formulation of optimal transport.

Definition 1.4. ***Kantorovich's Optimal Transport Problem:*** *given $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$,*

$$\textit{minimise } \mathbb{K}(\pi) = \int_{X\times Y} c(x,y)\,\mathrm{d}\pi(x,y)$$

over $\pi \in \Pi(\mu,\nu)$.

By the example with discrete measures, where we showed there did not exist any transport maps, we know that Kantorovich's and Monge's optimal transport problems do not always coincide. However, let us assume that there exists a transport map $T^\dagger : X \to Y$ that is optimal for Monge, then if we define $\mathrm{d}\pi(x,y) = \mathrm{d}\mu(x)\delta_{y=T^\dagger(x)}$ a quick calculation shows that $\pi \in \Pi(\mu,\nu)$:

$$\pi(A \times Y) = \int_A \delta_{T^\dagger(x)\in Y}\,\mathrm{d}\mu(x) = \mu(A)$$
$$\pi(X \times B) = \int_X \delta_{T^\dagger(x)\in B}\,\mathrm{d}\mu(x) = \mu((T^\dagger)^{-1}(B)) = T^\dagger_{\#}\mu(B) = \nu(B).$$

Since,

$$\int_{X\times Y} c(x,y)\,\mathrm{d}\pi(x,y) = \int_X\int_Y c(x,y)\delta_{y=T^\dagger(x)}\,\mathrm{d}y\,\mathrm{d}\mu(x) = \int_X c(x,T^\dagger(x))\,\mathrm{d}\mu(x)$$

it follows that

$$\inf \mathbb{K}(\pi) \leq \inf \mathbb{M}(T). \tag{1.3}$$

In fact one does not need minimisers of Monge's problem to exist. If $\mathbb{M}(T^\dagger) \leq \min \mathbb{M}(T) + \varepsilon$ for some $\varepsilon > 0$ then $\inf \mathbb{K}(\pi) \leq \inf \mathbb{M}(T) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary then (1.3) holds.

When the optimal plan $\pi^\dagger$ can be written in the form $\mathrm{d}\pi^\dagger(x,y) = \mathrm{d}\mu(x)\delta_{y=T(x)}$ it follows that T is an optimal transport map and $\inf \mathbb{K}(\pi) = \inf \mathbb{M}(T)$. Conditions sufficient for such a condition will be explored in Chapter 4. For now we just say that if $c(x, y) = |x - y|^2$, μ, ν both have finite second moments, and μ does not give mass to small sets[2] then there exists an optimal plan $\pi^\dagger$ which can be written as $\mathrm{d}\pi^\dagger(x, y) = \mathrm{d}\mu(x)\delta_{y=T^\dagger(x)}$ where $T^\dagger$ is an optimal map.

Let us observe the advantages of both Monge and Kantorovich formulation. Transport maps give a natural method of interpolation between two measures, in particular we can define

$$\mu_t = ((1-t)\mathrm{Id} + tT^\dagger)_{\#}\mu$$

then μ_t interpolates between μ and ν. In fact this line of reasoning will lead us directly to geodesics that we consider in greater detail in Chapter 5. In Figure 1.2 we compare the optimal transport interpolation with the Euclidean interpolation defined by $\mu_t^E = (1-t)\mu + t\nu$. In many applications the Lagrangian nature of optimal transport will be more realistic than Euclidean formulations.

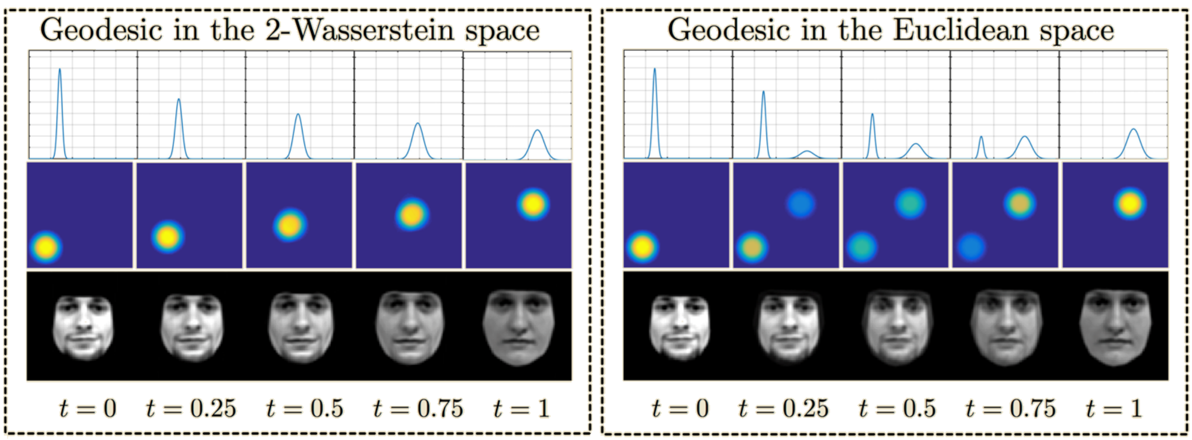

Figure 1.2: Interpolation in the optimal transport framework (left) and Euclidean space (right), figure modified from Figure 2 in [9].

Notice that the Kantorovich problem is convex (the constraints are convex and one usually has that the cost function $c(x, y) = d(x - y)$ where d is convex). In particular let us consider the Kantorovich problem between discrete measures $\mu = \sum_{i=1}^m \alpha_i \delta_{x_i}$, $\nu = \sum_{j=1}^n \beta_j \delta_{y_j}$ where $\sum_{i=1}^m \alpha_i = 1 = \sum_{j=1}^n \beta_j$, $\alpha_i \geq 0, \beta_j \geq 0$. Let $c_{ij} = c(x_i, y_j)$ and $\pi_{ij} = \pi(x_i, x_j)$. Then the Kantorovich problem is to solve

$$\text{minimise } \sum_{i=1}^m \sum_{j=1}^n c_{ij}\pi_{ij} \text{ over } \pi \text{subject to } \pi_{ij} \geq 0, \sum_{i=1}^m \pi_{ij} = \beta_j, \sum_{j=1}^n \pi_{ij} = \alpha_i.$$

This is a linear programme! In fact Kantorovich is considered as the inventor of linear programming. Not only does this provide a method for solving optimal transport problems (either through off the shelf linear programming algorithms, or through more recent advances such as

[2] $\mu \in \mathcal{P}(\mathbb{R}^d)$ does not give mass to small sets if for all sets A of Hausdorff dimension at most $d-1$ that $\mu(A) = 0$.

entropy regularised approaches see [5]) but the dual formulation:

$$\inf_{\pi\geq 0,\mathcal{C}^\top\pi=(\mu^\top,\nu^\top)^\top} c\cdot\pi = \sup_{\mathcal{C}(\varphi^\top,\phi^\top)^\top\leq c} (\mu\cdot\varphi+\nu\cdot\phi)$$

is an important stepping stone in establishing important properties such as the characterisation of optimal transport maps and plans. We study the dual formulation in the Chapter 3. In the next section we prove the existence of transport plans under fairly general conditions.

1.3 Existence of Transport Plans

Section references: Proposition 1.5 is taken from [15, Proposition 2.1].

We complete this chapter by proving the existence of a minimizer to Kantorovich's optimal transport problem. For the proof we use the direct method from the calculus of variations. Approximately the direct method is *compactness* plus *lower semi-continuity*. More precisely if we are considering a variational problem $\inf_{v\in V} F(v)$ then we first show that the set V is compact (or at least a set which contains the minimizer is compact). Then, let v_n be a minimising sequence, i.e. $F(v_n)\to\inf F$. Upon extracting a subsequence we can assume that $v_n\to v^\dagger\in V$. This gives a candidate minimizer. If we can show that F is lower semi-continuous then $\lim_{n\to\infty} F(v_n)\geq F(v^\dagger)$ and hence $v^\dagger$ is a minimiser.

Proposition 1.5. *Let $\mu\in\mathcal{P}(X)$, $\nu\in\mathcal{P}(Y)$ where X,Y are Polish spaces, and assume $c: X\times Y\to[0,\infty)$ is lower semi-continuous. Then there exists $\pi^\dagger\in\Pi(\mu,\nu)$ that minimises $\mathbb{K}$ (defined in Definition 1.4) over all $\pi\in\Pi(\mu,\nu)$.*

Proof. Note that $\Pi(\mu,\nu)$ is non-empty. Let us see that $\Pi(\mu,\nu)$ is compact in the weak* topology. Let $\delta>0$ and choose compact sets $K\subset X$, $L\subset Y$ such that

$$\mu(X\setminus K)\leq\delta, \qquad \nu(Y\setminus L)\leq\delta.$$

(Existence of sets follows directly since by definition Radon measures are inner regular.) If $(x,y)\in(X\times Y)\setminus(K\times L)$ then either $x\notin K$ or $y\notin L$, hence $(x,y)\in X\times(Y\setminus L)$ or $(x,y)\in(X\setminus K)\times Y$. So, for any $\pi\in\Pi(\mu,\nu)$

$$\begin{aligned}\pi((X\times Y)\setminus(K\times L)) &\leq \pi(X\times(Y\setminus L))+\pi((X\setminus K)\times Y)\\ &=\nu(Y\setminus L)+\mu(X\setminus K)\\ &\leq 2\delta.\end{aligned}$$

Hence, $\Pi(\mu,\nu)$ is tight. By Prokhorov's theorem the closure of $\Pi(\mu,\nu)$ is sequentially compact in the topology of weak* convergence.[3]

To check that $\Pi(\mu,\nu)$ us (weak*) closed let $\pi_n\in\Pi(\mu,\nu)$ be a sequence weakly* converging to $\pi\in\mathcal{M}(X\times Y)$, i.e.

$$\int_{X\times Y} f(x,y)\,\mathrm{d}\pi_n(x,y)\to\int_{X\times Y} f(x,y)\,\mathrm{d}\pi(x,y) \quad \forall f\in C_b^0(X\times Y).$$

[3] Prokhorov's theorem: if (S,ρ) is a separable metric space then $K\subset\mathcal{P}(S)$ is tight if and only if the closure of K is sequentially compact in $\mathcal{P}(S)$ equipped with the topology of weak* convergence.

We choose $f(x,y) = \tilde{f}(x)$, where $\tilde{f}$ is continuous and bounded. We have,

$$\int_X \tilde{f}(x)\,\mathrm{d}\mu(x) \to \int_{X\times Y} \tilde{f}(x)\,\mathrm{d}\pi(x,y) = \int_X \tilde{f}(x)\,\mathrm{d}P^X_{\#}\pi(x)$$

where $P^X(x,y) = x$ is the projection onto X (so $P^X_{\#}\pi$ is the X marginal). Since this is true for all $\tilde{f} \in C^0_b(X)$ it follows that $P^X_{\#}\pi = \mu$. Similarly, $P^Y_{\#}\pi = \nu$. Hence, $\pi \in \Pi(\mu,\nu)$ and $\Pi(\mu,\nu)$ is weakly* closed.

Let $\pi_n \in \Pi(\mu,\nu)$ be a minimising sequence, i.e. $\mathbb{K}(\pi_n) \to \inf_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi)$. Since $\Pi(\mu,\nu)$ is compact we can assume that $\pi_n \overset{*}{\rightharpoonup} \pi^\dagger \in \Pi(\mu,\nu)$. Our candidate for a minimiser is $\pi^\dagger$. Note that c is lower semi-continuous and bounded from below. Then,

$$\inf_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi) = \lim_{n\to\infty} \int_{X\times Y} c(x,y)\,\mathrm{d}\pi_n(x,y) \geq \int_{X\times Y} c(x,y)\,\mathrm{d}\pi^\dagger(x,y)$$

where we use the Portmanteau theorem which provides equivalent characterisations of weak* convergence. Hence $\pi^\dagger$ is a minimiser. □

Chapter 2

Special Cases

In this section we look at some special cases where we can prove existence and characterisation of optimal transport maps and plans. Generalising these results requires some work and in particular a duality theorem. On the other hand the results in this chapter require less background and are somehow "lower hanging fruit". Chapters 3 and 4 are essentially the results of this chapter generalised to more abstract settings. The two special cases we consider here are when measures μ, ν are on the real line, and when measures μ, ν are discrete. We start with the real line.

2.1 Optimal Transport in One Dimension

Section references: a version of Theorem 2.1 can be found in [15, Theorem 2.18] and [12, Theorem 2.9 and Proposition 2.17], versions of Corollary 2.2 can be found in [15, Remark 2.19] and [12, Lemma 2.8 and Proposition 2.17], Proposition 2.3 can be found in [7, Theorem 2.3].

Let us consider two measures $\mu, \nu \in \mathcal{P}(\mathbb{R})$ with cumulative distribution functions F and G respectively. We recall that

$$F(x) = \int_{-\infty}^{x} \mathrm{d}\mu = \mu((-\infty, x])$$

and F is right-continuous, non-decreasing, $F(-\infty) = 0$ and $F(+\infty) = 1$. We define the generalised inverse of F on $[0, 1]$ by

$$F^{-1}(t) = \inf \{x \in \mathbb{R} \,:\, F(x) > t\}.$$

In general $F^{-1}(F(x)) \geq x$ and $F(F^{-1}(t)) \geq t$. If F is invertible then $F^{-1}(F(x)) = x$ and $F(F^{-1}(t)) = t$. The main result of this section is the following theorem.

Theorem 2.1. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ with cumulative distributions F and G respectively. Assume $c(x, y) = d(x - y)$ where d is convex and continuous. Let $\pi^\dagger$ be the measure on $\mathbb{R}^2$ with cumulative distribution function $H(x, y) = \min\{F(x), G(y)\}$. Then $\pi^\dagger \in \Pi(\mu, \nu)$ and furthermore $\pi^\dagger$ is optimal for Kantorovich's optimal transport problem with cost function c. Moreover the* *optimal transport cost is*

$$\min_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi) = \int_0^1 d\left(F^{-1}(t) - G^{-1}(t)\right)\, \mathrm{d}t.$$

Before proving the theorem we state a corollary.

Corollary 2.2. *Under the assumptions of Theorem 2.1 the following holds.*

1. *If $c(x,y) = |x-y|$ then the optimal transport distance is the L^1 distance between cumulative distribution functions, i.e.*

$$\inf_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi) = \int_{\mathbb{R}} |F(x) - G(x)|\, \mathrm{d}x.$$

2. *If μ does not give mass to atoms then $\min_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi) = \min_{T\,:\,T_\#\mu=\nu} \mathbb{M}(T)$ and furthermore $T^\dagger = G^{-1}\circ F$ is a minimiser to Monge's optimal transport problem, i.e. $T^\dagger_\#\mu = \nu$ and*

$$\inf_{T\,:\,T_\#\mu=\nu} \mathbb{M}(T) = \mathbb{M}(T^\dagger).$$

Proof. For the first part, by Theorem 2.1, it is enough to show that

$$\int_0^1 \left|F^{-1}(t) - G^{-1}(t)\right|\, \mathrm{d}t = \int_{\mathbb{R}} |F(x) - G(x)|\, \mathrm{d}x.$$

Define $\mathcal{A}\subset\mathbb{R}^2$ by

$$\mathcal{A} = \{(x,t)\,:\, \min\{F(x),G(x)\} \leq t \leq \max\{F(x),G(x)\},\, x\in\mathbb{R}\}.$$

From Figure 2.1 we notice that we can equivalently write

$$\mathcal{A} = \left\{(x,t)\,:\, \min\{F^{-1}(t),G^{-1}(t)\} \leq x \leq \max\{F^{-1}(t),G^{-1}(t)\},\, t\in[0,1]\right\}.$$

By Fubini's theorem

$$\mathcal{L}(\mathcal{A}) = \int_{\mathbb{R}} \int_{\min\{F(x),G(x)\}}^{\max\{F(x),G(x)\}} \mathrm{d}t\,\mathrm{d}x = \int_0^1 \int_{\min\{F^{-1}(t),G^{-1}(t)\}}^{\max\{F^{-1}(t),G^{-1}(t)\}} \mathrm{d}x\,\mathrm{d}t$$

where $\mathcal{L}$ is the Lebesgue measure. Since $\max\{a,b\} - \min\{a,b\} = |a-b|$ then

$$\begin{aligned}\int_{\mathbb{R}} \int_{\min\{F(x),G(x)\}}^{\max\{F(x),G(x)\}} \mathrm{d}t\,\mathrm{d}x &= \int_{\mathbb{R}} \min\{F(x),G(x)\} - \max\{F(x),G(x)\}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} |F(x) - G(x)|\, \mathrm{d}x.\end{aligned}$$

Similarly

$$\int_0^1 \int_{\min\{F^{-1}(t),G^{-1}(t)\}}^{\max\{F^{-1}(t),G^{-1}(t)\}} \mathrm{d}x\,\mathrm{d}t = \int_0^1 \left|F^{-1}(t) - G^{-1}(t)\right|\, \mathrm{d}t.$$

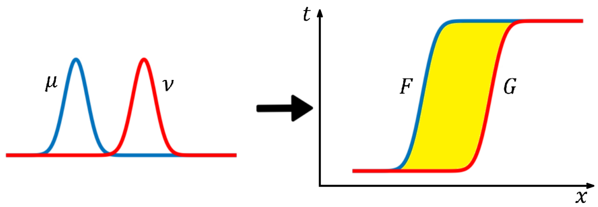

Figure 2.1: Optimal transport distance in 1D with cost $c(x, y) = |x - y|$, figure is taken from [10].

This proves the first part of the corollary.

For the second part we recall by Proposition 1.2 that $T_{\#}\mu = G^{-1}_{\#}(F_{\#}\mu)$. We show that (i) $G^{-1}_{\#}\mathcal{L}\lfloor_{[0,1]} = \nu$ and (ii) $\mathcal{L}\lfloor_{[0,1]} = F_{\#}\mu$. This is enough to show that $T_{\#}\mu = \nu$. For (i),

$$\begin{aligned} G^{-1}_{\#}\mathcal{L}\lfloor_{[0,1]}((-\infty, y]) &= \mathcal{L}\lfloor_{[0,1]}\left(\left\{t \,:\, G^{-1}(t) \leq y\right\}\right) \\ &= \mathcal{L}\lfloor_{[0,1]}(\{t \,:\, G(y) \geq t\}) \\ &= G(y) \\ &= \nu\left((-\infty, y]\right) \end{aligned}$$

where we used $G^{-1}(t) \leq y \Leftrightarrow G(y) \geq t$. For (ii) we note that F is continuous (as μ does not give mass to atoms). So for all $t \in (0, 1)$ the set $F^{-1}([0, t])$ is closed, in particular $F^{-1}([0, t]) = (-\infty, x_t]$ for some x_t with $F(x_t) = t$. Now, for $t \in (0, 1)$,

$$\begin{aligned} F_{\#}\mu\left([0, t]\right) &= \mu\left(\{x \,:\, F(x) \leq t\}\right) \\ &= \mu\left(\{x \,:\, x \leq x_t\}\right) \\ &= F(x_t) \\ &= t. \end{aligned}$$

Hence $F_{\#}\mu = \mathcal{L}_{[0,1]}$.

Now we show that $T^{\dagger}$ is optimal. By Theorem 2.1

$$\begin{aligned} \inf_{\pi \in \Pi(\mu,\nu)} \mathbb{K}(\pi) &= \int_0^1 d\left(F^{-1}(t) - G^{-1}(t)\right) \, \mathrm{d}t \\ &= \int_{\mathbb{R}} d\left(x - G^{-1}(F(x))\right) \, \mathrm{d}\mu(x) \qquad \text{since } F_{\#}\mu = \mathcal{L}\lfloor_{[0,1]} \text{ and by Proposition 1.2} \\ &= \int_{\mathbb{R}} d\left(x - T^{\dagger}(x)\right) \, \mathrm{d}\mu(x) \\ &\geq \inf_{T \,:\, T_{\#}\mu = \nu} \mathbb{M}(T). \end{aligned}$$

Since $\inf_{T \,:\, T_{\#}\mu=\nu} \mathbb{M}(T) \geq \min_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi)$ then the minimum of the Monge and Kantorovich optimal transport problems coincide and $T^{\dagger}$ is an optimal map for Monge. □

Before we prove Theorem 2.1 we give some basic ideas in the proof. The key is the idea of monotonicity. We say that a set $\Gamma \subset \mathbb{R}^2$ is *monotone* (with respect to d) if for all $(x_1, y_1), (x_2, y_2) \in \Gamma$ that

$$d(x_1 - y_1) + d(x_2 - y_2) \leq d(x_1 - y_2) + d(x_2 - y_1).$$

For example, if $\Gamma = \{(x,y) : f(x) = y\}$ and f is increasing, then Γ is monotone (assuming that d is increasing). The definition generalises to higher dimensions and often appears in convex analysis (for example the subdifferential of a convex function satisfies a monotonicity property). As a result, this concept can also be used to prove analogous results to Theorem 2.1 in higher dimensions. The definition should be natural for optimal transport. In particular, let Γ be the support of $\pi^\dagger$, which is a solution of Kantorovich's optimal transport problem. If $\pi^\dagger$ transports mass from x_1 to y_1 and from $x_2 > x_1$ to y_2 we expect $y_2 > y_1$, else it would have been cheaper to transport from x_1 to y_2, and from x_2 to y_1. The following proposition formalises this reasoning.

Proposition 2.3. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$. Assume $\pi^\dagger \in \Pi(\mu,\nu)$ is an optimal transport plan in the Kantorovich sense for cost function $c(x,y) = d(x-y)$ where d is continuous. Then for all $(x_1,y_1),(x_2,y_2) \in \mathrm{supp}(\pi^\dagger)$ we have*

$$d(x_1 - y_1) + d(x_2 - y_2) \leq d(x_1 - y_2) + d(x_2 - y_1).$$

Proof. Let $\Gamma = \mathrm{supp}(\pi^\dagger)$ and $(x_1,y_1),(x_2,y_2) \in \Gamma$. Assume there exists $\eta > 0$ such that

$$d(x_1 - y_1) + d(x_2 - y_2) - d(x_1 - y_2) - d(x_2 - y_1) \geq \eta.$$

Let I_1, I_2, J_1, J_2 be closed intervals with the following properties:

1. $x_i \in I_i$, $y_i \in J_i$, $i = 1,2$;
2. $d(x-y) \geq d(x_i - y_j) - \varepsilon$ for $x \in I_i$, $y \in J_j$, $i,j = 1,2$, where $\varepsilon < \frac{\eta}{4}$;
3. $I_i \times J_j$ are disjoint;
4. $\pi^\dagger(I_1 \times J_1) = \pi^\dagger(I_2 \times J_2) = \delta > 0$.

Properties 1-3 can be satisfied by choosing the intervals I_i, J_j sufficiently small. It may not be possible to satisfy property 4, however since $(x_i, y_i) \in \Gamma$ then we can find set I_i, J_j that satisfy 1-3 and $\pi^\dagger(I_1 \times J_1) > 0$, $\pi^\dagger(I_2 \times J_2) > 0$. It makes the notation in the proof easier to assume that $\pi^\dagger(I_1 \times J_1) = \pi^\dagger(I_2 \times J_2)$ however if not the proof can be adapted and we briefly describe how at the end.

The idea of the proof is to, instead of transferring mass from x_1 to y_1, and from x_2 to y_2, transfer mass from x_1 to y_2, and from x_2 to y_1. To make the argument rigorous we talk about the mass around each of x_i, y_i (hence the need for the intervals I_i, J_i).

Let

$$\tilde{\mu}_1 = P^X_{\#}\pi^\dagger\lfloor_{I_1 \times J_1}, \qquad \tilde{\mu}_2 = P^X_{\#}\pi^\dagger\lfloor_{I_2 \times J_2},$$
$$\tilde{\nu}_1 = P^Y_{\#}\pi^\dagger\lfloor_{I_1 \times J_1}, \qquad \tilde{\nu}_2 = P^Y_{\#}\pi^\dagger\lfloor_{I_2 \times J_2}.$$

And choose any $\tilde{\pi}_{12} \in \Pi(\tilde{\mu}_1, \tilde{\nu}_2)$, $\tilde{\pi}_{21} \in \Pi(\tilde{\mu}_2, \tilde{\nu}_1)$. We define $\tilde{\pi}$ to satisfy

$$\tilde{\pi}(A \times B) = \begin{cases} \pi^\dagger(A \times B) & \text{if } (A \times B) \cap (I_i \times J_j) = \emptyset \text{ for all } i,j \\ 0 & \text{if } A \times B \subseteq I_i \times J_i \text{ for some } i \\ \pi^\dagger(A \times B) + \tilde{\pi}_{12}(A \times B) & \text{if } A \times B \subseteq I_1 \times J_2 \\ \pi^\dagger(A \times B) + \tilde{\pi}_{21}(A \times B) & \text{if } A \times B \subseteq I_2 \times J_1. \end{cases}$$

For sets $(A \times B) \cap (I_i \times J_j) \neq \emptyset$ but $A \times B \not\subseteq (I_i \times J_j)$ then we define $\tilde{\pi}(A \times B)$ by

$$\tilde{\pi}(A \times B) = \tilde{\pi}((A \times B) \cap (I_i \times J_j)) + \tilde{\pi}((A \times B) \cap (I_i \times J_j)^c).$$

By construction, for $B \cap (J_1 \cup J_2) = \emptyset$,

$$\tilde{\pi}(\mathbb{R} \times B) = \pi^\dagger(\mathbb{R} \times B) = \nu(B).$$

If $B \subseteq J_1$ then

$$\begin{aligned}
\tilde{\pi}(\mathbb{R} \times B) &= \tilde{\pi}((\mathbb{R} \setminus (I_1 \cup I_2)) \times B) + \tilde{\pi}(I_1 \times B) + \tilde{\pi}(I_2 \times B) \\
&= \pi^\dagger((\mathbb{R} \setminus (I_1 \cup I_2)) \times B) + 0 + \pi^\dagger(I_2 \times B) + \tilde{\pi}_{21}(I_2 \times B) \\
&= \pi^\dagger((\mathbb{R} \setminus I_1) \times B) + \pi^\dagger(I_1 \times B) \\
&= \pi^\dagger(\mathbb{R} \times B) \\
&= \nu(B)
\end{aligned}$$

since $\tilde{\pi}_{21}(I_2 \times B) = \tilde{\nu}_1(B) = \pi^\dagger(I_1 \times (B \cap J_1)) = \pi^\dagger(I_1 \times B)$. Similarly for $B \subseteq J_2$. Hence we have $\tilde{\pi}(\mathbb{R} \times B) = \nu(B)$ for all measurable B. Analogously $\tilde{\pi}(A \times \mathbb{R}) = \mu(A)$ for all measurable A. Therefore $\tilde{\pi} \in \Pi(\mu, \nu)$.

Now,

$$\begin{aligned}
&\int_{\mathbb{R} \times \mathbb{R}} d(x-y)\, \mathrm{d}\pi^\dagger(x,y) - \int_{\mathbb{R} \times \mathbb{R}} d(x-y)\, \mathrm{d}\tilde{\pi}(x,y) \\
&\quad = \int_{I_1 \times J_1 \cup I_2 \times J_2} d(x-y)\, \mathrm{d}\pi^\dagger(x,y) - \int_{I_1 \times J_2} d(x-y)\, \mathrm{d}\tilde{\pi}_{12}(x,y) \\
&\qquad\quad - \int_{I_2 \times J_1} d(x-y)\, \mathrm{d}\tilde{\pi}_{21}(x,y) \\
&\quad \geq \delta\left(d(x_1 - y_1) - \varepsilon\right) + \delta\left(d(x_2 - y_2) - \varepsilon\right) - \delta\left(d(x_1 - y_2) + \varepsilon\right) - \delta\left(d(x_2 - y_1) + \varepsilon\right) \\
&\quad \geq \delta(\eta - 4\varepsilon) \\
&\quad > 0
\end{aligned}$$

since $\tilde{\pi}_{12}(I_1 \times J_2) = \tilde{\mu}_1(I_1) = \pi^\dagger(I_1 \times J_1) = \delta$, and similarly $\tilde{\pi}_{21}(I_2 \times J_1) = \delta$. This contradicts the assumption that $\pi^\dagger$ is optimal, hence no such η can exist.

Finally we remark that if $\pi^\dagger(I_1 \times J_1) > \pi^\dagger(I_2 \times J_2)$ then one can adapt the constructed plan $\tilde{\pi}$ by transporting some mass with the original plan $\pi^\dagger$. In particular the new constructed plan is chosen to satisfy

$$\tilde{\pi}(A \times B) = \pi^\dagger(A \times B)\left(1 - \frac{\pi^\dagger(I_2 \times J_2)}{\pi^\dagger(I_1 \times J_1)}\right)$$

if $A \times B \subset I_1 \times J_1$, and $\tilde{\mu}_1, \tilde{\nu}_1$ are rescaled:

$$\tilde{\mu}_1 = \frac{\pi^\dagger(I_2 \times J_2)}{\pi^\dagger(I_1 \times J_1)} P^X_\# \pi^\dagger \lfloor_{I_1 \times J_1}, \qquad \tilde{\nu}_1 = \frac{\pi^\dagger(I_2 \times J_2)}{\pi^\dagger(I_1 \times J_1)} P^Y_\# \pi^\dagger \lfloor_{I_1 \times J_1}.$$

All other definitions remain unchanged. One can go through the argument above and reach the same conclusion. □

We now prove Theorem 2.1.

Proof of Theorem 2.1. Assume d is continuous and strictly convex. By Proposition 1.5 there exists $\pi^* \in \Pi(\mu, \nu)$ that is an optimal transport plan in the Kantorovich sense. We will show that $\pi^* = \pi^\dagger$ By Proposition 2.3 $\Gamma = \mathrm{supp}(\pi^*)$ is monotone, i.e.

$$d(x_1 - y_1) + d(x_2 - y_2) \leq d(x_1 - y_1) + d(x_2 - y_1)$$

for all $(x_1, y_1), (x_2, y_2) \in \Gamma$. We claim that for any x_1, x_2, y_1, y_2 satisfying the above and $x_1 < x_2$ that $y_1 \leq y_2$. Assume that $y_2 < y_1$ and let $a = x_1 - y_1$, $b = x_2 - y_2$ and $\delta = x_2 - x_1$. We know that

$$d(a) + d(b) \leq d(b - \delta) + d(a + \delta).$$

Let $t = \frac{\delta}{b-a}$, it is easy to check that $t \in (0, 1)$ and $b - \delta = (1 - t)b + ta$, $a + \delta = tb + (1 - t)a$. Then, by strict convexity of d,

$$d(b - \delta) + d(a + \delta) < (1 - t)d(b) + td(a) + td(b) + (1 - t)d(a) = d(b) + d(a).$$

This is a contradiction, hence $y_2 \geq y_1$.

Now we show that $\pi^\dagger = \pi^*$. More precisely we show that $\pi^* \left((-\infty, x], (-\infty, y]\right) = \min\{F(x), G(y)\}$. Let $A = (-\infty, x] \times (y, +\infty)$, $B = (x, +\infty) \times (-\infty, y]$. We know that if $(x_1, y_1), (x_2, y_2) \in \Gamma$ and $x_1 < x_2$ then $y_1 \leq y_2$. This implies that, if $(x_0, y_0) \in \Gamma$ then

$$\Gamma \subset \{(x, y) \,:\, x \leq x_0, y \leq y_0\} \cup \{(x, y) \,:\, x \geq x_0, y \geq y_0\}.$$

Hence $\pi(A)$ and $\pi(B)$ cannot both be non-zero. In particular

$$\begin{aligned}\pi^* \left((-\infty, x] \times (-\infty, y]\right) = \min \Big\{&\pi^* \left(((-\infty, x] \times (-\infty, y]) \cup A\right), \\ &\pi^* \left(((-\infty, x] \times (-\infty, y]) \cup B\right) \Big\}.\end{aligned}$$

But,

$$\begin{aligned}\pi^* \left(((-\infty, x] \times (-\infty, y]) \cup A\right) = \pi \left((-\infty, x] \times \mathbb{R}\right) = F(x) \\ \pi^* \left(((-\infty, x] \times (-\infty, y]) \cup B\right) = \pi \left(\mathbb{R} \times (-\infty, y]\right) = G(y).\end{aligned}$$

Hence $\pi^* \left((-\infty, x] \times (-\infty, y]\right) = \min\{F(x), G(y)\}$.

Now we generalise to d not strictly convex. Since d is convex it can be bounded below by an affine function. Let $d(x) \geq (ax + b)_+$. One can check that $f(x) = \frac{1}{2}\sqrt{4 + (ax + b)^2} + \frac{1}{2}(ax + b)$ is strictly convex and satisfies $0 \leq f(x) \leq 1 + d(x)$. Then $d_\varepsilon : d + \varepsilon f$ is strictly convex and satisfies $d \leq d_\varepsilon \leq (1 + \varepsilon)d + \varepsilon$. Now let $\pi \in \Pi(\mu, \nu)$, then

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}} d(x - y) \,\mathrm{d}\pi^\dagger(x, y) &\leq \int_{\mathbb{R}\times\mathbb{R}} d_\varepsilon(x - y) \,\mathrm{d}\pi^\dagger(x, y) \\ &\leq \int_{\mathbb{R}\times\mathbb{R}} d_\varepsilon(x - y) \,\mathrm{d}\pi(x, y) \\ &\leq (1 + \varepsilon) \int_{\mathbb{R}\times\mathbb{R}} d(x - y) \,\mathrm{d}\pi(x, y) + \varepsilon.\end{aligned}$$

Taking $\varepsilon \to 0$ proves that $\pi^\dagger$ is an optimal plan in the sense of Kantorovich.

Now we show that $\int_{\mathbb{R}\times\mathbb{R}} d(x-y)\,\mathrm{d}\pi^\dagger(x,y) = \int_0^1 d(F^{-1}(t)-G^{-1}(t))\,\mathrm{d}t$. We claim that $\pi^\dagger = (F^{-1},G^{-1})_\#\mathcal{L}\lfloor_{[0,1]}$. Assuming so, then

$$\int_{\mathbb{R}\times\mathbb{R}} d(x-y)\,\mathrm{d}\pi^\dagger(x,y) = \int_{\mathbb{R}\times\mathbb{R}} d(x-y)\,\mathrm{d}\left((F^{-1},G^{-1})_\#\mathcal{L}\right)(x,y) = \int_0^1 d(F^{-1}(t)-G^{-1}(t))\,\mathrm{d}t$$

by the change of variable formula (Proposition 1.2).

To prove the claim we have

$$\begin{aligned}
(F^{-1},G^{-1})_\#\mathcal{L}\lfloor_{[0,1]}((-\infty,x]\times(-\infty,y]) &= \mathcal{L}\lfloor_{[0,1]}\Big((F^{-1},G^{-1})^{-1}\left((-\infty,x]\times(-\infty,y]\right)\Big) \\
&= \mathcal{L}\lfloor_{[0,1]}\left(\left\{t : F^{-1}(t)\le x \text{ and } G^{-1}(t)\le y\right\}\right) \\
&= \mathcal{L}\lfloor_{[0,1]}(\{t : F(x)\ge t \text{ and } G(y)\ge t\}) \\
&= \min\{F(x),G(y)\} \\
&= \pi^\dagger\left((-\infty,x]\times(-\infty,y]\right).
\end{aligned}$$

where we used $F^{-1}(t)\le x \Leftrightarrow F(x)\ge t$. □

Remark 2.4. Note that we actually showed that if d is continuous and strictly convex then $\pi^\dagger$ is unique.

2.2 Existence of Transport Maps for Discrete Measures

Section references: the discrete special case is based on the proof outlined in the introduction to [15]. The proof of the Minkowski-Carathéodory Theorem comes from [13, Theorem 8.11]

Proving the existence of a transport map $T^\dagger$ that are optimal for Monge's optimal transport problem, i.e. $T^\dagger$ minimises $\mathbb{M}(T)$ over all T satisfying $T_\#\mu = \nu$, is difficult and in fact for general measures we will only consider this problem for a specific cost function $c(x,y)=|x-y|^2$. Here we consider general cost functions but restrict to discrete measures $\mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ and $\nu = \frac{1}{n}\sum_{j=1}^n \delta_{y_j}$. Note that since all points $X=\{x_i\}_{i=1}^n, Y=\{y_j\}_{j=1}^n$ have equal mass that the map $T: X\to Y$ defined by $T(x_i)=y_{\sigma(i)}$ where $\sigma:\{1,\dots,n\}\to\{1,\dots,n\}$ is a permutation is a transport map (i.e. satisfies (1.1)). Hence the set of transport maps is non-empty.

For a convex and compact set B in a Banach space M we define the set of extreme points, which we denote by $\mathcal{E}(B)$, as the set of points in B that cannot be written as nontrivial convex combinations of points in B. I.e. if $B\ni\pi=\sum_{i=1}^m\alpha_i\pi_i$ (where $\sum_{i=1}^m\alpha_i=1$, $\alpha_i\ge 0$, $\pi_i\in B$) then $\pi\in\mathcal{E}(B)$ if and only if $\alpha_i\in\{0,1\}$. We recall two results. The first is the Minkowski–Carathéodory Theorem. The theorem is set in Euclidean spaces but can be generalised to Banach spaces where it is known as Choquet's theorem.

Theorem 2.5. Minkowski–Carathéodory Theorem. *Let $B\subset\mathbb{R}^M$ be a non-empty, convex and compact set. Then for any $\pi^\dagger\in B$ there exists a measure η supported on $\mathcal{E}(B)$ such that for any affine function f*

$$f(\pi^\dagger) = \int f(\pi)\,\mathrm{d}\eta(\pi).$$

Furthermore η can be chosen such that the cardinality of the support of η is at most $\dim(B)+1$ and (the support is) independent of $\pi^\dagger$.

Proof. Let $d=\dim(B)$. It is enough to show that there exists $\{a_i\}_{i=0}^d$ such that $\pi^\dagger=\sum_{i=0}^d a_i\pi^{(i)}$ where $\sum_{i=0}^n a_i=1$ and $\{\pi^{(i)}\}_{i=0}^d\subset\mathcal{E}(B)$. We prove the result by induction. The case when $d=0$ is trivial since B is just a point.

Now assume the result is true for all sets of dimension at most $d-1$. Pick $\pi^\dagger\in B$ and assume $\pi^\dagger\notin\mathcal{E}(B)$. Pick $\pi^{(0)}\in\mathcal{E}(B)$ and take the line segment $[\pi^{(0)},\pi]$ and extend it until it intersects with the boundary of B, i.e. let θ parametrise the line then $\{\theta\,:\,(1-\theta)\pi^{(0)}+\theta\pi\in B\}=[0,\alpha]$ for some $\alpha\geq 1$ (where α exists and is finite by convexity and compactness of B). Let $\xi=(1-\alpha)\pi^{(0)}+\alpha\pi$ then $\pi=(1-\theta_0)\xi+\theta_0\pi^{(0)}$ where $\theta_0=1-\frac{1}{\alpha}$. Now since $\xi\in F$ for some proper face F of B[1] then by the induction hypothesis there exists $\{\pi^{(i)}\}_{i=1}^d$ such that $\xi=\sum_{i=1}^n\theta_i\pi^{(i)}$ with $\sum_{i=1}^d\theta_i=1$. Hence, $\pi=\sum_{i=1}^d(1-\theta_0)\theta_i\pi^{(i)}+\theta_0\pi^{(0)}$. Since $(1-\theta_0)\sum_{i=1}^d\theta_i+\theta_0=1$ then π is a convex combination of $\{\pi^{(i)}\}_{i=0}^d$. Note that we chose $\pi^{(0)}$ independently of $\pi^\dagger$. □

Theorem 2.6. Birkhoff's theorem. *Let B be the set of $n\times n$ bistochastic matrices, i.e.*

$$B=\left\{\pi\in\mathbb{R}^{n\times n}\,:\,\forall ij,\,\pi_{ij}\geq 0;\,\forall j,\,\sum_{i=1}^n\pi_{ij}=1;\forall i,\,\sum_{j=1}^n\pi_{ij}=1\right\}.$$

Then the set of extremal points $\mathcal{E}(B)$ of B is exactly the set of permutation matrices, i.e.

$$\mathcal{E}(B)=\left\{\pi\in\{0,1\}^{n\times n}\,:\,\forall j,\,\sum_{i=1}^n\pi_{ij}=1;\forall i,\,\sum_{j=1}^n\pi_{ij}=1\right\}.$$

Proof. We start by showing that every permutation matrix is an extremal point. Let $\pi_{ij}=\delta_{j=\sigma(i)}$ where σ is a permutation. Assume that $\pi\notin\mathcal{E}(B)$. Then there exists $\pi^{(1)},\pi^{(2)}\in B$, with $\pi^{(1)}\neq\pi\neq\pi^{(2)}$, and $t\in(0,1)$ such that $\pi=t\pi^{(1)}+(1-t)\pi^{(2)}$. Let ij be such that $0=\pi_{ij}\neq\pi_{ij}^{(1)}$, then

$$0=\pi_{ij}=t\pi_{ij}^{(1)}+(1-t)\pi_{ij}^{(2)}\quad\Longrightarrow\quad\pi_{ij}^{(2)}=-\frac{\pi_{ij}^{(1)}}{1-t}<0.$$

This contradicts $\pi_{ij}^{(2)}\geq 0$, hence $\pi\in\mathcal{E}(B)$.

Now we show that every $\pi\in\mathcal{E}(B)$ is a permutation matrix. We do this in two parts: we (i) show that $\pi\in\mathcal{E}(B)$ implies that $\pi_{ij}\in\{0,1\}$, then (ii) show $\pi=\delta_{j=\sigma(i)}$ for a permutation σ.

For (i) let $\pi\in\mathcal{E}(B)$ and assume there exists i_1j_1 such that $\pi_{i_1j_1}\in(0,1)$. Since $\sum_{i=1}^n\pi_{ij_1}=1$ then there exists $i_2\neq i_1$ such that $\pi_{i_2j_1}\in(0,1)$. Similarly, since $\sum_{j=1}^n\pi_{i_2j}=1$ there exists $j_2\neq j_1$ such that $\pi_{i_2j_2}\in(0,1)$. Continuing this procedure until $i_m=i_1$ we obtain two sequences:

$$\mathcal{I}=\{i_kj_k\,:\,k\in\{1,\dots,m-1\}\}\qquad\mathcal{I}^+=\{i_{k+1}j_k\,:\,k\in\{1,\dots,m-1\}\}$$

[1] A face F of a convex set B is any set with the property that if $\pi^{(1)},\pi^{(2)}\in B$, $t\in(0,1)$ and $t\pi^{(1)}+(1-t)\pi^{(2)}\in F$ then $\pi^{(1)},\pi^{(2)}\in F$. A proper face is a face which has dimension at most $\dim(B)-1$. A result we use without proof is that the boundary of a convex set is the union of all proper faces.

with $i_{k+1} \neq i_k$ and $j_{k+1} \neq j_k$. Define $\pi^{(\delta)}$ by the following

$$\pi_{ij}^{(\delta)} = \begin{cases} \pi_{i_k j_k} + \delta & \text{if } ij = i_k j_k \text{ for some } k \\ \pi_{i_{k+1} j_k} - \delta & \text{if } ij = i_{k+1} j_k \text{ for some } k \\ \pi_{ij} & \text{else.} \end{cases}$$

Then,

$$\sum_{i=1}^n \pi_{ij}^{(\delta)} = \sum_{i=1}^n \pi_{ij} + \delta \left|\{ij \in \mathcal{I} \,:\, i \in \{1,\dots,n\}\}\right| - \delta \left|\{ij \in \mathcal{I}^+ \,:\, i \in \{1,\dots,n\}\}\right|.$$

Now if $ij \in \mathcal{I}$ then there exists i' such that $i'j \in \mathcal{I}^+$, and likewise, if $ij \in \mathcal{I}^+$ then there exists i' such that $i'j \in \mathcal{I}$. Hence,

$$\left|\{ij \in \mathcal{I} \,:\, i \in \{1,\dots,n\}\}\right| = \left|\{ij \in \mathcal{I}^+ \,:\, i \in \{1,\dots,n\}\}\right|.$$

It follows that $\sum_{i=1}^n \pi_{ij}^{(\delta)} = 1$ and analogously $\sum_{j=1}^n \pi_{ij}^{(\delta)} = 1$.

Choose $\delta = \min\{\min\{\pi_{ij}, 1-\pi_{ij}\} \,:\, ij \in \mathcal{I} \cup \mathcal{I}^+\} \in (0,1)$. Define $\pi^{(1)} = \pi^{(-\delta)}$, $\pi^{(\delta)}$. We have that $\pi_{ij}^{(1)}, \pi_{ij}^{(2)} \geq 0$ and therefore $\pi^{(1)}, \pi^{(2)} \in B$ with $\pi^{(1)} \neq \pi^{(2)}$. Moreover we have $\pi = \frac{1}{2}\pi^{(1)} + \frac{1}{2}\pi^{(2)}$. Hence, $\pi \notin \mathcal{E}(B)$. The contradiction implies that there does not exist $i_1 j_1$ such that $\pi_{i1j_1} \in (0,1)$. We have shown that if $\pi \in \mathcal{E}(B)$ then $\pi_{ij} \in \{0,1\}$.

We're left to show (ii): that $\pi_{ij} = \delta_{j=\sigma(i)}$. Since $\pi \in B$ then for each i there exists j^* such that $\pi_{ij^*} = 1$ (else $\sum_{j=1}^n \pi_{ij} \neq 1$). We let $\sigma(i) = j^*$ so by construction we have $\pi_{i\sigma(i)} = 1$. We claim σ is a permutation. It is enough to show that σ is injective. Now if $j = \sigma(i_1) = \sigma(i_2)$ where $i_1 \neq i_2$ then

$$1 = \sum_{i=1}^n \pi_{ij} \geq \pi_{i_1 j} + \pi_{i_2 j} = 2.$$

The contradiction implies that $i_1 = i_2$ and therefore σ is injective. □

We now show that the existence of optimal transport maps between discrete measures $\mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ and $\nu = \frac{1}{n}\sum_{j=1}^n \delta_{y_j}$.

Theorem 2.7. *Let $\mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ and $\nu = \frac{1}{n}\sum_{j=1}^n \delta_{y_j}$. Then there exists a solution to Monge's optimal transport problem between μ and ν.*

Proof. Let $c_{ij} = c(x_i, y_j)$ and B be the set of bistochastic $n \times n$ matrices, i.e.

$$B = \left\{ \pi \in \mathbb{R}^{n\times n} \,:\, \forall ij,\, \pi_{ij} \geq 0;\, \forall j,\, \sum_{i=1}^n \pi_{ij} = 1; \forall i,\, \sum_{j=1}^n \pi_{ij} = 1 \right\}.$$

The Kantorovich problem reads as

$$\text{minimise } \frac{1}{n}\sum_{i,j} c_{ij}\pi_{ij} \quad \text{over } \pi \in B.$$

Although, by Proposition 1.5, there exists a minimiser to the Kantorovich optimal transport problem we do not use this fact here. Let M be the minimum of the Kantorovich optimal transport problem, $\varepsilon > 0$ and find an approximate minimiser $\pi^\varepsilon \in B$ such that

$$M \geq \sum_{ij} c_{ij}\pi^\varepsilon - \varepsilon.$$

If we let $f(\pi) = \sum_{ij} c_{ij}\pi_{ij}$ then assuming that B is compact and convex we have that there exists a measure η supported on $\mathcal{E}(B)$ such that

$$f(\pi^\varepsilon) = \int f(\pi)\,\mathrm{d}\eta(\pi).$$

Hence

$$M \geq \int \sum_{ij} c_{ij}\pi_{ij}\,\mathrm{d}\eta(\pi) - \varepsilon \geq \inf_{\pi\in\mathcal{E}(B)} \sum_{ij} c_{ij}\pi_{ij} - \varepsilon \geq M - \varepsilon.$$

Since this is true for all ε it holds that $\inf_{\pi\in\mathcal{E}(B)}\sum_{ij} c_{ij}\pi_{ij} = M$. We claim that $\mathcal{E}(B)$ is compact, in which case there exists a minimiser $\pi^\dagger \in \mathcal{E}(B)$. Note that we have also shown (independently from Proposition 1.5) that there exists a solution to Kantorovich's optimal transport problem.

By Birkhoff's theomem $\pi^\dagger$ is a permutation matrix, that is there exists a permutation $\sigma^\dagger : \{1,\dots,n\} \to \{1,\dots,n\}$ such that $\pi^\dagger_{ij} = \delta_{j=\sigma^\dagger(i)}$. Let $T^\dagger : X \to Y$ be defined by $T^\dagger(x_i) = y_{\sigma(i)}$.

We already know that the set of transport maps is non-empty. Let T be any transport map and define $\pi_{ij} = \delta_{y_j = T(x_i)}$, (it is easy to see that $\pi \in B$) then

$$\sum_{i=1}^n c(x_i, T(x_i)) = \sum_{ij} c_{ij}\pi_{ij} \geq \sum_{ij} c_{ij}\pi^\dagger_{ij} = \sum_{i=1}^n c(x_i, T^\dagger(x_i)).$$

Hence $T^\dagger$ is a solution to Monge's optimal transport problem.

We are left to show that B is compact and convex, and $\mathcal{E}(B)$ is compact. To show B is compact we consider the ℓ^1 norm: $\|\pi\|_1 := \sum_{ij}|\pi_{ij}|$ (since all norms are equivalent on finite dimensional spaces it does not really matter which norm we choose). Clearly B is bounded as for all $\pi \in B$ we have $\|\pi\|_1 \leq n^2$. For closure, we consider a sequence $\pi^{(m)} \in B$ with $\pi^{(m)} \to \pi$. Trivially $\pi^{(m)}_{ij} \to \pi_{ij}$ for all ij and therefore $\pi_{ij} \geq 0$, likewise $\sum_{i=1}^n \pi_{ij} = \lim_{m\to\infty}\sum_{i=1}^n \pi^{(m)}_{ij} = 1$ and $\sum_{j=1}^n \pi_{ij} = 1$. Hence $\pi \in B$ and B is closed. Therefore B is compact.

Convexity of B is easy to check by considering $\pi^{(1)}, \pi^{(2)} \in B$ and $\pi = t\pi^{(1)} + (1-t)\pi^{(2)}$ for $t \in [0,1]$ then clearly $\pi_{ij} \geq 0$,

$$\sum_{i=1}^n \pi_{ij} = t\sum_{i=1}^n \pi^{(1)}_{ij} + (1-t)\sum_{i=1}^n \pi^{(2)}_{ij} = t + (1-t) = 1,$$

and similarly $\sum_{j=1}^n \pi_{ij} = 1$. Hence $\pi \in B$ and B is convex.

For compactness of $\mathcal{E}(B)$ it is enough to show closure. If $\mathcal{E}(B) \ni \pi^{(m)} \to \pi$ then we already know that $\pi \in B$ and by pointwise convergence of $\pi^{(m)}_{ij} \to \pi_{ij}$ we also have $\pi_{ij} \in \{0,1\}$. Hence $\pi \in \mathcal{E}(B)$ and therefore $\mathcal{E}(B)$ is closed. □

Chapter 3

Kantorovich Duality

We saw in the previous chapter how Kantorovich's optimal transport problem resembles a linear programme. It should not therefore be surprising that Kantorovich's optimal transport problem admits a dual formulation. In the following section we state the duality result and give an intuitive but non-rigorous proof. In Section 3.2 we give a general minimax principle upon whoch we can base the proof of Kantorovich duality. In Section 3.3 we can then rigorously prove duality. With additional assumptions such as restricting X, Y to Euclidean spaces we prove the existence of solutions to the dual problem in Section 3.4.

3.1 Kantorovich Duality

Section references: The statement and proof of the main result, Theorem 3.1, come from [15, Theorem 1.3].

We start by stating Kantorovich then give an intuitive proof with one key step missing. The proof is made rigorous in Section 3.3.

Theorem 3.1. Kantorovich Duality. *Let $\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y)$ where X, Y are Polish spaces. Let $c : X \times Y \to [0, +\infty]$ be a lower semi-continuous cost function. Define $\mathbb{K}$ as in Definition 1.4 and $\mathbb{J}$ by*

$$\mathbb{J} : L^1(\mu) \times L^1(\nu) \to \mathbb{R}, \qquad \mathbb{J}(\varphi, \psi) = \int_X \varphi \, \mathrm{d}\mu + \int_Y \psi \, \mathrm{d}\nu. \tag{3.1}$$

Let Φ_c be defined by

$$\Phi_c = \left\{ (\varphi, \psi) \in L^1(\mu) \times L^1(\nu) \,:\, \varphi(x) + \psi(y) \leq c(x, y) \right\}$$

where the inequality is understood to hold for μ-almost every $x \in X$ and ν-almost every $y \in Y$. Then,

$$\min_{\pi \in \Pi(\mu,\nu)} \mathbb{K}(\pi) = \sup_{(\varphi,\psi) \in \Phi_c} \mathbb{J}(\varphi, \psi).$$

Let us give an informal interpretation of the result which originally comes from Caffarelli and I take from Villani [15]. Consider the *shippers problem*. Suppose we own a number of coal mines and a number of factories, we wish to transport the coal from mines to factories. The amount each mine produces and each factory requires is fixed (and we assume equal). The cost for you to transport from mine x to factory y is $c(x, y)$. The total optimal cost is the solution to Kantorovich's optimal transport problem. Now a clever shipper comes to you and says they will ship for you and you just pay a price $\varphi(x)$ for loading and $\psi(y)$ for unloading. To make it in your interest the shipper makes sure that $\varphi(x) + \psi(y) \leq c(x, y)$ that is the cost is no more than what you would have spent transporting the coal yourself. Kantorovich duality tells us that one can find φ and ψ such that this price scheme costs just as much as paying for the cost of transport yourself.

We now give an informal proof that will subsequently be made rigorous. Let $M = \inf_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi)$. Observe that

$$M = \inf_{\pi\in\mathcal{M}_+(X\times Y)} \sup_{(\varphi,\psi)} \left(\int_{X\times Y} c(x, y) \,\mathrm{d}\pi + \int_X \varphi \,\mathrm{d}\left(\mu - P^X_\# \pi\right) + \int_Y \psi \,\mathrm{d}\left(\nu - P^Y_\# \pi\right) \right) \tag{3.2}$$

where we take the supremum on the right hand side over $(\varphi, \psi) \in C^0_b(X) \times C^0_b(Y)$. This follows since

$$\sup_{\varphi\in C^0_b(X)} \int_X \varphi \,\mathrm{d}\left(\mu - P^X_\# \pi\right) = \begin{cases} +\infty & \text{if } \mu \neq P^X_\# \pi \\ 0 & \text{else.} \end{cases}$$

Hence, the infimum over π of the right hand side of (3.2) is on the set where $P^X_\# \pi = \mu$ and, similarly, $P^Y_\# \pi = \nu$ (which means that $\pi \in \Pi(\mu, \nu)$). We can rewrite (3.2) more conveniently as

$$M = \inf_{\pi\in\mathcal{M}_+(X\times Y)} \sup_{(\varphi,\psi)} \left(\int_{X\times Y} c(x, y) - \varphi(x) - \psi(y) \,\mathrm{d}\pi + \int_X \varphi \,\mathrm{d}\mu + \int_Y \psi \,\mathrm{d}\nu \right).$$

Assuming a minimax principle we switch the infimum and supremum to obtain

$$M = \sup_{(\varphi,\psi)} \left(\int_X \varphi \,\mathrm{d}\mu + \int_Y \psi \,\mathrm{d}\nu + \inf_{\pi\in\mathcal{M}_+(X\times Y)} \int_{X\times Y} c(x, y) - \varphi(x) - \psi(y) \,\mathrm{d}\pi \right). \tag{3.3}$$

Now if there exists $(x_0, y_0) \in X \times Y$ and $\varepsilon > 0$ such that $\varphi(x_0) + \psi(y_0) - c(x_0, y_0) = \varepsilon > 0$ then by letting $\pi_\lambda = \lambda\delta_{(x_0,y_0)}$ for some $\lambda > 0$ we have

$$\inf_{\pi\in\mathcal{M}_+(X\times Y)} \int_{X\times Y} c(x, y) - \varphi(x) - \psi(y) \,\mathrm{d}\pi \leq -\lambda\varepsilon \to -\infty \quad \text{as } \lambda \to \infty.$$

Hence the infimum on right hand side of (3.3) can be restricted to when $\varphi(x) + \psi(y) \leq c(x, y)$ for all $(x, y) \in X \times Y$, i.e. $(\varphi, \psi) \in \Phi_c$ (this heuristic argument actually used $(\varphi, \psi) \in C^0_b(X) \times C^0_b(Y)$ not $L^1(\mu) \times L^1(\nu)$ and there is a difference between the constraint $\varphi(x) + \psi(y) \leq c(x, y)$ holding everywhere and holding *almost* everywhere, these are technical details that are not important at this stage). When $(\varphi, \psi) \in \Phi_c$ then

$$\inf_{\pi\in\mathcal{M}_+(X\times Y)} \int_{X\times Y} c(x, y) - \varphi(x) - \psi(y) \,\mathrm{d}\pi = 0$$

which is achieved for $\pi \equiv 0$ for example. Hence,

$$\inf_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi) = \sup_{(\varphi,\psi)\in\Phi_c} \int_X \varphi(x)\,\mathrm{d}\mu(x) + \int_Y \psi(y)\,\mathrm{d}\nu(y).$$

This is the statement of Kantorovich duality. To complete this argument we need to make the minimax principle rigorous. In the next section we prove a minimax principle, in the section after we apply it to Kantorovich duality and provide a complete proof.

3.2 Fenchel-Rockafeller Duality

Section references: I take the duality theorem (Theorem 3.2) from [15, Theorem 1.9]. Lemma 3.3 is hopefully obvious and the Hahn-Banach theorem is well known.

To rigorously prove the Kantorovich duality theorem we need a minimax principle, i.e. conditions sufficient to interchange the infimum and supremum when we introduced the Lagrange multipliers φ, ψ in (3.2). The minimax principle is specific to convex functions; at this stage it is perhaps not clear how to apply it to Kantorovich's optimal transport problem when we made no convexity assumption on c. We define the Legendre-Fenchel transform for a convex function $\Theta : E \to \mathbb{R} \cup \{+\infty\}$ where E is a normed vector space by

$$\Theta^* : E^* \to \mathbb{R} \cup \{+\infty\}, \qquad \Theta^*(z^*) = \sup_{z\in E} \left(\langle z^*, z\rangle - \Theta(z)\right).$$

Convex analysis will play a greater role in the sequel, in particular in Chapter 4 where we will provide a more in-depth review. We now state the minimax principle taken from Villani [15].

Theorem 3.2. Fenchel-Rockafellar Duality. *Let E be a normed vector space and $\Theta, \Xi : E \to \mathbb{R} \cup \{+\infty\}$ two convex functions. Assume there exists $z_0 \in E$ such that $\Theta(z_0) < \infty$, $\Xi(z_0) < \infty$ and Θ is continuous at z_0. Then,*

$$\inf_E (\Theta + \Xi) = \max_{z^*\in E^*} \left(-\Theta^*(-z^*) - \Xi^*(z^*)\right).$$

In particular the supremum on the right hand side is attained.

We recall a couple of preliminary results (that we do not prove) before we prove the theorem.

Lemma 3.3. *Let E be a normed vector space.*

1. *If $\Theta : E \to \mathbb{R} \cup \{+\infty\}$ is convex then so is the epigraph A defined by*

$$A = \{(z,t) \in E \times \mathbb{R} \,:\, t \geq \Theta(z)\}.$$

2. *If $\Theta : E \to \mathbb{R} \cup \{+\infty\}$ is concave then so is the hypograph B defined by*

$$B = \{(z,t) \in E \times \mathbb{R} \,:\, t \leq \Theta(z)\}.$$

3. *If $C \subset E$ is convex then $\mathrm{int}(C)$ is convex.*

4. *If* $D \subset E$ *is convex and* $\text{int}(D) \neq \emptyset$ *then* $\overline{D} = \overline{\text{int}(D)}$.

The following theorem, the Hahn-Banach theorem can be stated in multiple different forms. The most convenient form for us is in terms of separation of convex sets.

Theorem 3.4. Hahn-Banach Theorem. *Let* E *be a topological vector space. Assume* A, B *are convex, non-empty and disjoint subsets of* E*, and that* A *is open. Then there exists a closed hyperplane separating* A *and* B*.*

We now prove Theorem 3.2.

Proof of Theorem 3.2. By writing

$$-\Theta^*(-z^*) - \Xi^*(z^*) = \inf_{x,y \in E} \left(\Theta(x) + \Xi(y) + \langle z^*, x - y \rangle\right)$$

and choosing $y = x$ on the right hand side we see that

$$\inf_{x \in E} \left(\Theta(x) + \Xi(x)\right) \geq \sup_{z^* \in E^*} \left(-\Theta^*(-z^*) - \Xi^*(z^*)\right).$$

Let $M = \inf\left(\Theta + \Xi\right)$, and define the sets A, B by

$$\begin{aligned} A &= \{(x, \lambda) \in E \times \mathbb{R} \,:\, \lambda \geq \Theta(x)\} \\ B &= \{(y, \sigma) \in E \times \mathbb{R} \,:\, \sigma \leq M - \Xi(y)\}. \end{aligned}$$

By Lemma 3.3 A and B are convex. By continuity and finiteness of Θ at z_0 the interior of A is non-empty and by finiteness of Ξ at z_0 B is non-empty. Let $C = \text{int}(A)$ (which is convex by Lemma 3.3. Now, if $(x, \lambda) \in C$ then $\lambda > \Theta(x)$, therefore $\lambda + \Xi(x) > \Theta(x) + \Xi(x) \geq M$. Hence $(x, \lambda) \notin B$. In particular $B \cap C = \emptyset$. By the Hahn-Banach theorem there exists a hyperplane $H = \{\Phi = \alpha\}$ that separates B and C, i.e. if we write $\Phi(x, \lambda) = f(x) + k\lambda$ (where f is linear) then

$$\begin{aligned} &\forall (x, \lambda) \in C, \qquad f(x) + k\lambda \geq \alpha \\ &\forall (x, \lambda) \in B, \qquad f(x) + k\lambda \leq \alpha. \end{aligned}$$

Now if $(x, \lambda) \in A$ then there exists a sequence $(x_n, \lambda_n) \in C$ such that $(x_n, \lambda_n) \to (x, \lambda)$. Hence $f(x) + k\lambda \leftarrow f(x_n) + k\lambda_n \geq \alpha$. Therefore

$$\forall (x, \lambda) \in A, \qquad f(x) + k\lambda \geq \alpha \tag{3.4}$$

$$\forall (x, \lambda) \in B, \qquad f(x) + k\lambda \leq \alpha. \tag{3.5}$$

We know that $(z_0, \lambda) \in A$ for λ sufficiently large, hence $k \geq 0$. We claim $k > 0$. Assume $k = 0$. Then

$$\begin{aligned} &\forall (x, \lambda) \in A, \quad f(x) \geq \alpha \qquad \Longrightarrow \qquad f(x) \geq \alpha \quad \forall x \in \text{Dom}(\Theta) \\ &\forall (x, \lambda) \in B, \quad f(x) \leq \alpha \qquad \Longrightarrow \qquad f(x) \leq \alpha \quad \forall x \in \text{Dom}(\Xi). \end{aligned}$$

As $\mathrm{Dom}(\Xi) \ni z_0 \in \mathrm{Dom}(\Theta)$ then $f(z_0) = \alpha$. Since Θ is continuous at z_0 there exists $r > 0$ such that $B(z_0, r) \subset \mathrm{Dom}(\Theta)$, hence for all z with $\|z\| < r$ and $\delta \in \mathbb{R}$ with $|\delta| < 1$ we have

$$f(z_0 + \delta z) \geq \alpha \quad \Longrightarrow \quad f(z_0) + \delta f(z) \geq \alpha \quad \Longrightarrow \quad \delta f(z) \geq 0.$$

This is true for all $\delta \in (-1, 1)$ and therefore $f(z) = 0$ for $z \in B(0, r)$. Hence $f \equiv 0$ on E. It follows that $\Phi \equiv 0$ which is clearly a contradiction (either $H = E \times \mathbb{R}$ if $\alpha = 0$ or $H = \emptyset$). It must be that $k > 0$.

By (3.4) we have

$$\begin{aligned} \Theta^* \left(-\frac{f}{k} \right) &= \sup_{z \in E} \left(-\frac{f(z)}{k} - \Theta(z) \right) \\ &= -\frac{1}{k} \inf_{z \in E} \left(f(z) + k\Theta(z) \right) \\ &\leq -\frac{\alpha}{k} \end{aligned}$$

since $(z, \Theta(z)) \in A$. Similarly, by (3.5) we have

$$\begin{aligned} \Xi^* \left(\frac{f}{k} \right) &= \sup_{z \in E} \left(\frac{f(z)}{k} - \Xi(z) \right) \\ &= -M + \frac{1}{k} \sup_{z \in E} \left(f(z) + k(M - \Xi(z)) \right) \\ &\leq -M + \frac{\alpha}{k} \end{aligned}$$

since $(z, M - \Xi(z)) \in B$. It follows that

$$M \geq \sup_{z^* \in E^*} \left(-\Theta^*(-z^*) - \Xi^*(z^*) \right) \geq -\Theta^* \left(-\frac{f}{k} \right) - \Xi^* \left(\frac{f}{k} \right) \geq \frac{\alpha}{k} + M - \frac{\alpha}{k} = M.$$

So

$$\inf_{x \in E} \left(\Theta(x) + \Xi(x) \right) = M = \sup_{z^* \in E^*} \left(-\Theta^*(-z^*) - \Xi^*(z^*) \right).$$

Furthermore $z^* = \frac{f}{k}$ must achieve the supremum. □

3.3 Proof of Kantorovich Duality

Section references: The two lemmas in this section together prove the Kantorovich duality theorem, both lemmas come from [15].

Finally we can prove Kantorovich dualiy as stated in Theorem 3.1. We break the theorem into two parts.

Lemma 3.5. *Under the same conditions as Theorem 3.1 we have*

$$\sup_{(\varphi, \psi) \in \Phi_c} \mathbb{J}(\varphi, \psi) \leq \inf_{\pi \in \Pi(\mu, \nu)} \mathbb{K}(\pi).$$

Proof. Let $(\varphi, \psi) \in \Phi_C$ and $\pi \in \Pi(\mu, \nu)$. Let $A \subset X$ and $B \subset Y$ be sets such that $\mu(A) = 1$, $\nu(B) = 1$ and

$$\varphi(x) + \psi(y) \le c(x, y) \qquad \forall (x, y) \in A \times B.$$

Now $\pi(A^c \times B^c) \le \pi(A^c \times Y) + \pi(X \times B^c) = \mu(A^c) + \nu(B^c) = 0$. Hence,

$$\begin{aligned}
\pi(A \times B) &= \pi(X \times B) - \pi(A^c \times B) \\
&= \nu(B) - \pi(A^c \times Y) + \pi(A^c \times B^c) \\
&= 1 - \mu(A^c) + \pi(A^c \times B^c) \\
&= 1.
\end{aligned}$$

So it follows that $\varphi(x) + \psi(y) \le c(x, y)$ for π-almost every (x, y). Then,

$$\mathbb{J}(\varphi, \psi) = \int_X \varphi \, \mathrm{d}\mu + \int_Y \psi \, \mathrm{d}\nu = \int_{X \times Y} \varphi(x) + \psi(y) \, \mathrm{d}\pi(x, y) \le \int_{X \times Y} c(x, y) \, \mathrm{d}\pi(x, y).$$

The result of the lemma follows by taking the supremum over $(\varphi, \psi) \in \Phi_c$ on the right hand side and the infimum over $\pi \in \Pi(\mu, \nu)$ on the left hand side. □

To complete the proof of Theorem 3.1 we need to show that the opposite inequality in Lemma 3.5 is also true.

Lemma 3.6. *Under the same conditions as Theorem 3.1 we have*

$$\sup_{(\varphi, \psi) \in \Phi_c} \mathbb{J}(\varphi, \psi) \ge \inf_{\pi \in \Pi(\mu, \nu)} \mathbb{K}(\pi).$$

Proof. The proof is completed in three steps in increasing generality:

1. we assume X, Y are compact and c is continuous;
2. the assumption that X, Y are compact is relaxed, c is still continuous;
3. c is only assumed to be lower semi-continuous.

1. Let $E = C_b^0(X \times Y)$ equipped with the supremum norm. The dual space of E is the space of Radon measures $E^* = \mathcal{M}(X \times Y)$ (by the Riesz–Markov–Kakutani representation theorem). Define

$$\Theta(u) = \begin{cases} 0 & \text{if } u(x, y) \ge -c(x, y) \\ +\infty & \text{else,} \end{cases}$$

$$\Xi(u) = \begin{cases} \int_X \varphi(x) \, \mathrm{d}\mu(x) + \int_Y \psi(y) \, \mathrm{d}\nu(y) & \text{if } u(x, y) = \varphi(x) + \psi(y) \\ +\infty & \text{else.} \end{cases}$$

Note that although the representation $u(x, y) = \varphi(x) + \psi(y)$ is not unique (φ and ψ are only unique upto a constant) Ξ is still well defined. We claim that Θ and Ξ are convex. For Θ

consider u, v with $\Theta(u), \Theta(v) < +\infty$, then $u(x,y) \geq -c(x,y)$ and $v(x,y) \geq -c(x,y)$, hence $tu(x,y) + (1-t)v(x,y) \geq c(x,y)$ for any $t \in [0,1]$. It follows that

$$\Theta(tu + (1-t)v) = 0 = t\Theta(u) + (1-t)\Theta(v).$$

If either $\Theta(u) = +\infty$ or $\Theta(v) = +\infty$ then clearly

$$\Theta(tu + (1-t)v) \leq t\Theta(u) + (1-t)\Theta(v).$$

Hence Θ is convex. For Ξ if either $\Xi(u) = +\infty$ or $\Xi(v) = +\infty$ then clearly

$$\Xi(tu + (1-t)v) \leq t\Xi(u) + (1-t)\Xi(v).$$

Assume $u(x,y) = \varphi_1(x) + \psi_1(y)$, $v(x,y) = \varphi_2(x) + \psi_2(y)$ then

$$tu(x,y) + (1-t)v(x,y) = t\varphi_1(x) + (1-t)\varphi_2(x) + t\psi_1(y) + (1-t)\psi_2(y)$$

and therefore

$$\Xi(tu + (1-t)v) = \int_X t\varphi_1 + (1-t)\varphi_2 \,\mathrm{d}\mu + \int_Y t\psi_1 + (1-t)\psi_2 \,\mathrm{d}\nu = t\Xi(u) + (1-t)\Xi(v).$$

Hence Ξ is convex.

Let $u \equiv 1$ then $\Theta(u), \Xi(u) < +\infty$ and Θ is continuous at u. By Theorem 3.2

$$\inf_{u \in E} \left(\Theta(u) + \Xi(u)\right) = \max_{\pi \in E^*} \left(-\Theta^*(-\pi) - \Xi^*(\pi)\right). \tag{3.6}$$

First we calculate the left hand side of (3.6). We have

$$\inf_{u \in E} \left(\Theta(u) + \Xi(u)\right) \geq \inf_{\substack{\varphi(x)+\psi(y) \geq -c(x,y) \\ \varphi \in L^1(\mu), \psi \in L^1(\nu)}} \int_X \varphi(x) \,\mathrm{d}\mu(x) + \int_Y \psi(y) \,\mathrm{d}\nu(y) = -\sup_{(\varphi,\psi) \in \Phi_c} \mathbb{J}(\varphi,\psi).$$

We now consider the right hand side of (3.6). To do so we need to find the convex conjugates of Θ and Ξ. For Θ^* we compute

$$\Theta^*(-\pi) = \sup_{u \in E} \left(-\int_{X \times Y} u \,\mathrm{d}\pi - \Theta(u)\right) = \sup_{u \geq -c} -\int_{X \times Y} u \,\mathrm{d}\pi = \sup_{u \leq c} \int_{X \times Y} u \,\mathrm{d}\pi.$$

Then we find

$$\Theta^*(-\pi) = \begin{cases} \int_{X \times Y} c(x,y) \,\mathrm{d}\pi & \text{if } \pi \in \mathcal{M}_+(X \times Y) \\ +\infty & \text{else.} \end{cases}$$

For Ξ^* we have

$$\begin{aligned}
\Xi^*(\pi) &= \sup_{u \in E} \left(\int_{X \times Y} u \,\mathrm{d}\pi - \Xi(u)\right) \\
&= \sup_{u(x,y) = \varphi(x) + \psi(y)} \left(\int_{X \times Y} u \,\mathrm{d}\pi - \int_X \varphi(x) \,\mathrm{d}\mu - \int_Y \psi(y) \,\mathrm{d}\nu\right) \\
&= \left(\sup_{u(x,y) = \varphi(x) + \psi(y)} \int_X \varphi \,\mathrm{d}(P_\#^X - \mu) + \int_Y \psi \,\mathrm{d}(P_\#^Y - \nu)\right) \\
&= \begin{cases} 0 & \text{if } \pi \in \Pi(\mu,\nu) \\ +\infty & \text{else.} \end{cases}
\end{aligned}$$

Hence, the right hand side of (3.6) reads

$$\max_{\pi\in E^*}\left(-\Theta^*(-\pi)-\Xi^*(\pi)\right)=-\min_{\pi\in\Pi(\mu,\nu)}\int_{X\times Y}c(x,y)\,\mathrm{d}\pi=-\min_{\pi\in\Pi(\mu,\nu)}\mathbb{K}(\pi).$$

This completes the proof of part 1.

Parts 2 and 3 and more complicated (part 2 takes some work, part 3 is actually quite straghtforward) and are omitted; both parts can be found in [15, pp 28-32]. □

3.4 Existence of Maximisers to the Dual Problem

Section references: Theorem 3.7 is adapted from the special case $X = Y = \mathbb{R}^n$, $c(x,y) = |x-y|^2$ in [15, Theorem 2.9], the other results in this section, Lemmas 3.8 and 3.9 are adapted from [15, Lemma 2.10].

The objective of this section is to prove the existence of a maximiser to the dual problem. We state the theorem before giving a preliminary result followed by the proof of the theorem.

Theorem 3.7. *Let $\mu\in\mathcal{P}(X)$, $\nu\in\mathcal{P}(Y)$, where X and Y are Polish, and $c : X\times Y\to[0,\infty)$. Assume that there exists $c_X\in L^1(\mu)$, $c_Y\in L^1(\nu)$ such that $c(x,y)\le c_X(x)+c_Y(y)$ for μ-almost every $x\in X$ and ν-almost every $y\in Y$. In addition, assume that*

$$M := \int_X c_X(x)\,\mathrm{d}\mu(x)+\int_Y c_Y(y)\,\mathrm{d}\nu(y)<\infty. \tag{3.7}$$

Then there exists $(\varphi,\psi)\in\Phi_c$ such that

$$\sup_{\Phi_c}\mathbb{J}=\mathbb{J}(\varphi,\psi).$$

Furthermore we can choose $(\varphi,\psi)=(\eta^{cc},\eta^c)$ for some $\eta\in L^1(\mu)$, where η^c is defined below.

The condition that $M<\infty$ is effectively a moment condition on μ and ν. In particular, if $c(x,y)=|x-y|^p$ then $c(x,y)\le C(|x|^p+|y|^p)$ and the requirement that $M<\infty$ is exactly the condition that μ,ν have finite p^{th} moments.

The proof relies on similar concepts as the proof of duality, in particular, for $\varphi : X\to\overline{\mathbb{R}}$, the c-transforms φ^c, φ^{cc} defined by

$$\begin{aligned}\varphi^c &: Y\to\overline{\mathbb{R}}, &\qquad \varphi^c(y)&=\inf_{x\in X}\left(c(x,y)-\varphi(x)\right)\\ \varphi^{cc} &: X\to\overline{\mathbb{R}}, &\qquad \varphi^{cc}(x)&=\inf_{y\in Y}\left(c(x,y)-\varphi^c(y)\right)\end{aligned}$$

for $\varphi : X\to\mathbb{R}$ are key; one should compare this to the Legendre-Fenchel transform defined in the previous section. We first give a result which implies we only need to consider c-transform pairs.

Lemma 3.8. *Let $\mu\in\mathcal{P}(X)$, $\nu\in\mathcal{P}(Y)$. For any $a\in\mathbb{R}$ and $(\tilde\varphi,\tilde\psi)\in\Phi_c$ we have $(\varphi,\psi)=(\tilde\varphi^{cc}-a,\tilde\varphi^c+a)$ satisfies $\mathbb{J}(\varphi,\psi)\ge\mathbb{J}(\tilde\varphi,\tilde\psi)$ and $\varphi(x)+\psi(y)\le c(x,y)$ for μ-almost every $x\in X$ and ν-almost every $y\in Y$.*

Furthermore, if $\mathbb{J}(\tilde\varphi,\tilde\psi)>-\infty$, $M<+\infty$ (where M is defined by (3.7)), and there exists $c_X\in L^1(\mu)$ and $c_Y\in L^1(\nu)$ such that $\varphi\le c_X$ and $\psi\le c_Y$, then $(\varphi,\psi)\in\Phi_c$.

Proof. Clearly $\mathbb{J}(\varphi - a, \psi + a) = \mathbb{J}(\varphi, \psi)$ for all $a \in \mathbb{R}$, $\varphi \in L^1(\mu)$ and $\nu \in L^1(\nu)$, so it is enough to show that $\varphi = \tilde{\varphi}^{cc} \geq \tilde{\varphi}$, $\psi = \tilde{\varphi}^c \geq \tilde{\psi}$, $\varphi(x) + \psi(y) \leq c(x,y)$.

Note that

$$\psi(y) = \inf_{x \in X} \left(c(x,y) - \tilde{\varphi}(x) \right) \geq \tilde{\psi}(y)$$

since $\tilde{\varphi}(x) + \tilde{\psi}(y) \leq c(x,y)$, and

$$\varphi(x) = \inf_{y \in Y} \sup_{z \in X} \left(c(x,y) - c(z,y) + \tilde{\varphi}(z) \right) \geq \tilde{\varphi}(x)$$

by choosing $z = x$.

We easily see that

$$\varphi(x) + \psi(y) = \inf_{z \in Y} \left(c(x,z) - \tilde{\varphi}^c(z) + \tilde{\varphi}^c(y) \right) \leq c(x,y)$$

by choosing $z = y$.

For the furthermore part of the lemma it is left to show integrability of φ, ψ. Note that

$$\int_X \varphi(x) - c_X(x) \, \mathrm{d}\mu(x) + \int_Y \psi(y) - c_Y(y) \, \mathrm{d}\nu(y) = \mathbb{J}(\varphi, \psi) - M \geq \mathbb{J}(\tilde{\varphi}, \tilde{\psi}) - M$$

and since $\varphi - c_X \leq 0$, $\psi - c_Y \leq 0$ then both integrals on the left hand side are negative. In particular

$$\begin{aligned} \|\varphi - c_X\|_{L^1(\mu)} + \|\psi - c_Y\|_{L^1(\nu)} &= -\int_X \varphi(x) - c_X(x) \, \mathrm{d}\mu(x) - \int_Y \psi(y) - c_Y(y) \, \mathrm{d}\nu(y) \\ &\leq M - \mathbb{J}(\tilde{\varphi}, \tilde{\psi}). \end{aligned}$$

Hence $\varphi - c_X \in L^1(\mu)$, $\psi - c_Y \in L^1(\nu)$ from which it follows $\varphi \in L^1(\mu)$, $\psi \in L^1(\nu)$. □

The next result gives an upper bound on maximising sequences.

Lemma 3.9. *Let $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, and $c : X \times Y \to \mathbb{R}$. Assume that $c(x,y) \leq c_X(x) + c_Y(y)$ where $c_X \in L^1(\mu)$ and $c_Y \in L^1(\nu)$. Furthermore, assume that M given by* (3.7) *satisfies $M < \infty$. Then there exists a sequence $(\varphi_k, \psi_k) \in \Psi_c$ such that $\mathbb{J}(\varphi_k, \psi_k) \to \sup_{\Phi_c} \mathbb{J}$ and satisfying the bounds*

$$\begin{aligned} \varphi_k(x) &\leq c_X(x) \quad \forall x \in X, \forall k \in \mathbb{N} \\ \psi_k(y) &\leq c_Y(y) \quad \forall y \in Y, \forall k \in \mathbb{N}. \end{aligned}$$

Proof. Let $(\tilde{\varphi}_k, \tilde{\psi}_k) \in \Phi_c$ be a maximising sequence. Notice that since $0 \leq \sup_{\Phi_c} \mathbb{J} \leq \inf_{\Pi(\mu,\nu)} \mathbb{K} \leq M < \infty$ that $\tilde{\varphi}_k, \tilde{\psi}_k$ must be proper functions (in fact not equal to $\pm\infty$ anywhere). Let $(\varphi_k, \psi_k) = (\tilde{\varphi}_k^{cc} - a_k, \tilde{\varphi}_k^c + a_k)$ where we will choose

$$a_k = \inf_{y \in Y} \left(c_Y(y) - \tilde{\varphi}_k^c(y) \right).$$

By Lemma 3.8 $(\varphi_k, \psi_k) \in \Phi_c$ and (φ_k, ψ_k) is a maximising sequence once we have shown that $\varphi_k \leq c_X$ and $\psi_k \leq c_Y$.

We start by showing $a_k \in (-\infty, +\infty)$. Since $(\tilde{\varphi}_k, \tilde{\psi}_k) \in \Phi_c$ then $\tilde{\varphi}_k(x) \leq c(x, y) - \tilde{\psi}_k(y)$ for all $y \in Y$. Hence there exists $y_0 \in Y$ and $b_0 \in \mathbb{R}$ (possibly depending on k) such that $\tilde{\varphi}_k(x) \leq c(x, y_0) + b_0$. Then,

$$\tilde{\varphi}_k^c(y_0) = \inf_{x \in X} (c(x, y_0) - \tilde{\varphi}_k(x)) \geq -b_0.$$

Hence, $a_k \leq c_Y(y_0) - \varphi_k^c(y_0) \leq c_Y(y_0) + b_0 < \infty$. We also have

$$c_Y(y) - \tilde{\varphi}_k^c(y) = \sup_{x \in X} (c_Y(y) - c(x, y) + \tilde{\varphi}_k(x)) \geq \sup_{x \in X} (\tilde{\varphi}_k(x) - c_X(x)) \geq \tilde{\varphi}_k(x_0) - c_X(x_0)$$

for any $x_0 \in X$. Hence, $a_k \geq \tilde{\varphi}(x_0) - c_X(x_0)$ which is greater than $-\infty$. We have shown that $a_k \in (-\infty, +\infty)$ and the pair (φ_k, ψ_k) are well defined.

Clearly $\psi_k(y) = \tilde{\varphi}_k^c(y) + a_k \leq c_Y(y)$. And,

$$\varphi_k(x) - c_X(x) = \inf_{y \in Y} (c(x, y) - \tilde{\varphi}_k^c(y) - a_k - c_X(x)) \leq \inf_{y \in Y} (c_Y(y) - \tilde{\varphi}_k^c(y) - a_k) = 0.$$

So, (φ_k, ψ_k) satisfy the bounds stated in the lemma. □

With the help of the preceding lemma we can prove Theorem 3.7.

Proof of Theorem 3.7. Note that $\inf_{\pi \in \Pi(\mu, \nu)} \mathbb{K}(\pi) \leq M < \infty$ by Lemma 3.5. Let $(\varphi_k, \psi_k) \in \Phi_c$ be a maximising sequence as in Lemma 3.9. Define $\varphi_k^{(\ell)}$, $\psi_k^{(\ell)}$ by

$$\begin{aligned} \varphi_k^{(\ell)}(x) &= \max\{\varphi_k(x) - c_X(x), -\ell\} + c_X(x) \\ \psi_k^{(\ell)}(y) &= \max\{\psi_k(y) - c_Y(y), -\ell\} + c_Y(y). \end{aligned}$$

Note that $\varphi_k \leq \varphi_k^{(\ell)}$, $\psi_k \leq \psi_k^{(\ell)}$,

$$\begin{aligned} -\ell \leq \varphi_k^{(\ell)}(x) - c_X(x) &\leq 0 \qquad \forall x \in X, \forall k \in \mathbb{N}, \forall \ell \in \mathbb{N} \\ -\ell \leq \psi_k^{(\ell)}(y) - c_Y(y) &\leq 0 \qquad \forall y \in Y, \forall k \in \mathbb{N}, \forall \ell \in \mathbb{N} \\ \varphi_k^{(1)} &\geq \varphi_k^{(2)} \geq \dots \\ \psi_k^{(1)} &\geq \psi_k^{(2)} \geq \dots \end{aligned}$$

and

$$\begin{aligned} \varphi_k^{(\ell)}(x) + \psi_k^{(\ell)}(y) &\leq \max\{\varphi_k(x) - c_X(x) + \psi_k(y) - c_Y(y), -\ell\} + c_X(x) + c_Y(y) \\ &\leq \max\{c(x, y) - c_X(x) - c_Y(y), -\ell\} + c_X(x) + c_Y(y). \end{aligned} \tag{3.8}$$

For each ℓ the sequence $\varphi_k^{(\ell)}$ is bounded in L^∞ so $\overline{\{\varphi_k^{(\ell)}\}_{k \in \mathbb{N}}}$ is weakly compact in $L^p(\mu)$ for any $p \in (1, \infty)$ (for reflexive Banach spaces boundedness plus closure is equivalent to weak compactness). Let's choose $p = 2$ then, after extracting a subsequence, we have that $\varphi_k^{(\ell)} \rightharpoonup \varphi^{(\ell)} \in L^1(\mu)$ (since $L^2(\mu) \subset L^1(\mu)$). By a diagonalisation argument we can assume that

$\varphi_k^{(\ell)} \rightharpoonup \varphi^{(\ell)}$ for all $\ell \in \mathbb{N}$. We can apply the same argument to $\psi_k^{(\ell)}$ to imply the existence of weak limits $\psi^{(\ell)} \in L^1(\nu)$. We have that

$$c_X \geq \varphi^{(1)} \geq \varphi^{(2)} \geq \dots$$
$$c_Y \geq \psi^{(1)} \geq \psi^{(2)} \geq \dots$$

Since $\varphi^{(\ell)}, \psi^{(\ell)}$ are bounded above by an L^1 function and monotonically decreasing we can apply the Monotone Convergence Theorem to infer

$$\lim_{\ell\to\infty} \int_X \varphi^{(\ell)}(x) \, \mathrm{d}\mu(x) = \int_X \varphi^\dagger(x) \, \mathrm{d}\mu(x)$$
$$\lim_{\ell\to\infty} \int_Y \psi^{(\ell)}(y) \, \mathrm{d}\nu(y) = \int_Y \psi^\dagger(y) \, \mathrm{d}\nu(y)$$

where $\varphi^\dagger$, $\psi^\dagger$ are the pointwise limits of $\varphi^{(\ell)}$, $\psi^{(\ell)}$:

$$\varphi^\dagger(x) = \lim_{\ell\to\infty} \varphi^{(\ell)}(x), \qquad \psi^\dagger(y) = \lim_{\ell\to\infty} \psi^{(\ell)}(y).$$

The functions $(\varphi^\dagger, \psi^\dagger)$ are our candidate maximisers. We are required to show that $(\varphi^\dagger, \psi^\dagger) \in \Psi_c$ and $\mathbb{J}(\varphi, \psi) \leq \mathbb{J}(\varphi^\dagger, \psi^\dagger)$ for all $(\varphi, \psi) \in \Phi_c$.

Since $\sup_{\Phi_c} \mathbb{J} = \lim_{k\to\infty} \mathbb{J}(\varphi_k, \psi_k) \leq \lim_{k\to\infty} \mathbb{J}(\varphi_k^{(\ell)}, \psi_k^{(\ell)}) = \mathbb{J}(\varphi^{(\ell)}, \psi^{(\ell)})$ for any $\ell \in \mathbb{N}$ then

$$\mathbb{J}(\varphi^\dagger, \psi^\dagger) = \lim_{\ell\to\infty} \mathbb{J}(\varphi^{(\ell)}, \psi^{(\ell)}) \geq \sup_{\Phi_c} \mathbb{J}.$$

Hence $(\varphi^\dagger, \psi^\dagger)$ maximises $\mathbb{J}$.

It follows from taking $\ell \to \infty$ in (3.8) that $\varphi^\dagger(x) + \psi^\dagger(y) \leq c(x, y)$. Now integrability follows from

$$0 \geq \int_X \varphi^\dagger(x) - c_X(x) \, \mathrm{d}\mu(x) + \int_Y \psi^\dagger(y) - c_Y(y) \, \mathrm{d}\nu(y) \geq \sup_{\Phi_c} \mathbb{J} - M.$$

In particular, since $\varphi^\dagger - c_X \leq 0$, $\psi^\dagger - c_Y \leq 0$ then it follows that both integrals are finite and $\varphi^\dagger - c_X \in L^1(\mu)$, $\psi^\dagger - c_Y \in L^1(\nu)$. Hence $\varphi^\dagger \in L^1(\mu)$ and $\psi^\dagger \in L^1(\nu)$.

For the furthermore part of the theorem we use the double c-transform trick as in the proof of Lemma 3.9. For any $a \in \mathbb{R}$ we have, by Lemma 3.8,

$$\mathbb{J}(\varphi^\dagger, \psi^\dagger) \leq \mathbb{J}((\varphi^\dagger)^{cc} - a, (\varphi^\dagger)^c + a) = \mathbb{J}((\varphi^\dagger)^{cc}, (\varphi^\dagger)^c).$$

We only have to show $((\varphi^\dagger)^{cc}, (\varphi^\dagger)^c) \in L^1(\mu) \times L^1(\nu)$. Let $a = \inf_{y\in Y} \big(c_Y(y) - (\varphi^\dagger)^c(y)\big)$ then $a \in \mathbb{R}$ for the same reasons that $a_k \in \mathbb{R}$ in the proof of Lemma 3.9. Clearly $(\varphi^\dagger)^c(y) + a \leq c_Y(y)$ and

$$(\varphi^\dagger)^{cc}(x) = \inf_{y\in Y} \big(c(x, y) - (\varphi^\dagger)^c(y) - a\big) \leq \inf_{y\in Y} \big(c_X(x) + c_Y(y) - (\varphi^\dagger)^c(y) - a\big) \leq c_X(x).$$

Hence, $((\varphi^\dagger)^{cc} - a, (\varphi^\dagger)^c + a) \in L^1(\mu) \times L^1(\nu)$ by Lemma 3.8. Trivially $((\varphi^\dagger)^{cc}, (\varphi^\dagger)^c) \in L^1(\mu) \times L^1(\nu)$. □

Chapter 4

Existence and Characterisation of Transport Maps

Our aim is to characterise the optimal transport plans that arise as a minimiser to the Kantorovich optimal transportation problem and show sufficient conditions for the existence of optimal transport maps. In this chapter we will restrict ourselves to the cost function $c(x,y) = \frac{1}{2}|x-y|^2$. Generalisations to other cost functions is possible but with an additional notational burden.[1] The second restriction we make is to assume that X and Y are subsets of $\mathbb{R}^n$.

The chapter is structured as follows. We first state our objectives and in particular the results we will prove, then give motivating explanations. We will require some results and definitions from convex analysis which we give in Section 4.2 before finally proving the main theorems from the first section.

4.1 Knott-Smith Optimality and Brenier's Theorem

Section references: Theorem 4.1, Theorem 4.2 and Corollary 4.3 form [15, Theorem 2.12].

We will give (1) a characterisation of optimal transport plans and (2) sufficient conditions for the existence of optimal transport maps (and when $\min_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi) = \min_{T_\# \mu = \nu} \mathbb{M}(T)$).

We will restate the theorem in Section 4.3 with a change of notation (more precisely we look at the equivalent problems $\sup_{\pi\in\Pi(\mu,\nu)} \int_{X\times Y} x\cdot y \,\mathrm{d}\pi(x,y)$ and $\inf_{\tilde{\Phi}} \mathbb{J}$ where $\tilde{\Phi}$ is defined in Section 4.3).

The subdifferential $\partial\varphi$ of a convex function φ is defined by

$$\partial\varphi(x) := \left\{ y \,:\, \varphi(z) \geq \varphi(x) + y\cdot(z-x)\,\forall z \in \mathbb{R}^n \right\}.$$

We will review some convex analysis in Section 4.2 which will inform the definition. For now it is enough to know that the subdifferential is a generalisation of the differential which always exists for lower semi-continuous convex functions. Note that the subdifferential is in general a set however if φ is differentiable at x then $\partial\varphi(x) = \{\nabla\varphi(x)\}$.

---

[1] For example it is possible to show the existence of optimal transport maps for strictly convex and superlinear cost functions, and characterise them in terms of c-superdifferentials of c-convex functions.

Theorem 4.1. Knott-Smith Optimality Criterion *Let $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ with $X, Y \subset \mathbb{R}^n$ and assume that μ, ν both have finite second moments. Then $\pi \in \Pi(\mu, \nu)$ is a minimizer of Kantorovich's optimal transport problem with cost $c(x, y) = \frac{1}{2}|x - y|^2$ if and only if there exists an $L^1(\mu)$, convex lower semi-continuous function φ such that $\mathrm{supp}(\pi) \subseteq \mathrm{Gra}(\partial\varphi)$, or equivalently $y \in \partial\varphi(x)$ for π-almost every (x, y). Moreover the pair (φ, φ^*) is a minimiser of the problem $\inf_{\tilde{\Phi}} \mathbb{J}(\varphi, \psi)$ where φ^* is the convex conjugate defined in Definition 4.4, $\mathbb{J}$ is defined by* (3.1)*, and $\tilde{\Phi}$ is defined by*

$$\tilde{\Phi} = \left\{ (\tilde{\varphi}, \tilde{\psi}) \in L^1(\mu) \times L^1(\nu) \,:\, \tilde{\varphi}(x) + \tilde{\psi}(y) \geq x \cdot y \right\}.$$

Why do we expect the optimal plan to have support in the graph of the subdifferential of a convex function? Let us first consider the 1D case and a map $T_{\#}\mu = \nu$. We should expect any map that is 'optimal' to be order preserving, i.e. if $x_1 \leq x_2$ then $T(x_1) \leq T(x_2)$. This is equivalent to saying that T is non-decreasing.

Maps rule out splitting since each $x \mapsto T(x)$. However if we let T set valued (i.e. we are considering plans instead of maps) then the increasing property in some sense should still hold. Let

$$\Gamma = \{(x, y) \,:\, x \in X \text{ and } y \in T(x)\}.$$

We can write the increasing property as: for any $(x_1, y_1), (x_2, y_2) \in \Gamma$ with $x_1 \leq x_2$ then $y_1 \leq y_2$.

In convex analysis this property is called cyclical monotonicity. It can be shown that any cyclically monotone set can be written as the subgradient of a convex function (since any convex function has a non-decreasing derivative). Hence we expect any optimal plan to be supported in the subgradient of a convex function. This turns out to also be true in dimensions greater than one.

The next result specifically gives conditions sufficient for the existence of transport maps.

Theorem 4.2. Brenier's Theorem *Let $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ with $X, Y \subset \mathbb{R}^n$ and assume that μ, ν both have finite second moments and that μ does not give mass to small sets. Then there is a unique solution $\pi^\dagger \in \Pi(\mu, \nu)$ to Kantorovich's optimal transport problem with cost $c(x, y) = \frac{1}{2}|x - y|^2$ which is given by*

$$\pi^\dagger = (\mathrm{Id} \times \nabla\varphi)_{\#}\mu \qquad \textit{or equivalently} \qquad \mathrm{d}\pi^\dagger(x, y) = \mathrm{d}\mu(x)\delta(y = \nabla\varphi(x))$$

where $\nabla\varphi$ is the gradient of a convex function (defined μ-almost everywhere) that pushes μ forward to ν, i.e. $(\nabla\varphi)_{\#}\mu = \nu$.

Any convex function is locally Lipschitz on the interior of its domain. It follows (from Rademacher's theorem) that φ is almost everywhere (in the Lebesgue sense) differentiable on the interior of its domain. The fact that the set of non-differentiability is a small set and π gives zero mass to small sets implies we can talk about the derivative of φ on the support of π as though it exists everywhere.

The following corollary is immediate from Brenier's theorem.

Corollary 4.3. *Under the assumptions of Theorem 4.2 $\nabla\varphi$ is the unique solution to the Monge transportation problem:*

$$\frac{1}{2}\int_X |x - \nabla\varphi(x)|^2 \,\mathrm{d}\mu(x) = \frac{1}{2}\inf_{T_{\#}\mu = \nu} \int_X |x - T(x)|^2 \,\mathrm{d}\mu(x).$$

We sketch the proof of the corollary. From the inequality

$$\min_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi) \leq \inf_{T\,:\,T_{\#}\mu=\nu} \mathbb{M}(T)$$

that was argued in Section 1.2 it is enough to show $T^\dagger = \nabla\varphi$ satisfies $\mathbb{M}(T^\dagger) \leq \min_{\Pi(\mu,\nu)} \mathbb{K}$ ($T^\dagger_{\#}\mu = \nu$ is given in Theorem 4.2). Now, let $\pi^\dagger \in \Pi(\mu,\nu)$ be as in Theorem 4.2,

$$\begin{aligned}\mathbb{M}(T^\dagger) &= \frac{1}{2}\int_X |x - T^\dagger(x)|^2 \,\mathrm{d}\mu(x) \\ &= \frac{1}{2}\int_{X\times Y} |x - T^\dagger(x)|^2 \,\mathrm{d}\pi^\dagger(x,y) \\ &= \frac{1}{2}\int_{X\times Y} |x - y|^2 \,\mathrm{d}\pi^\dagger(x,y) \\ &= \min_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi)\end{aligned}$$

since $T^\dagger(x) = y$ for π-almost every $(x,y) \in X \times Y$. Uniqueness of $T^\dagger$ follows from uniqueness of $\pi^\dagger$.

4.2 Preliminary Results from Convex Analysis

Section references: These results are a subset of the background in convex analsyis given in [15]. In particular Proposition 4.5 is [15, Proposition 2.4] and Proposition 4.7 is [15, Proposition 2.5].

In order to characterise subgradients we will use the convex conjugate defined below. This is essentially a special case of the Legendre-Fenchel transform we defined in Section 3.2. The convex conjugate is also sometimes called the Legendre transform.

Definition 4.4. *The convex conjugate of a proper function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by*

$$\varphi^*(y) = \sup_{x\in\mathbb{R}^n} \left(x \cdot y - \varphi(x)\right).$$

The following proposition characterises the subdifferential.

Proposition 4.5. *Let φ be a proper, lower semi-continuous, convex function on $\mathbb{R}^n$. Then for all $x, y \in \mathbb{R}^n$*

$$x \cdot y = \varphi(x) + \varphi^*(y) \quad \Leftrightarrow \quad y \in \partial\varphi(x).$$

Proof. Since $\varphi^*(y) \geq x \cdot y - \varphi(x)$ for all x, y we have

$$\begin{aligned} x \cdot y = \varphi(x) + \varphi^*(y) &\Leftrightarrow x \cdot y \geq \varphi(x) + \varphi^*(y) \\ &\Leftrightarrow x \cdot y \geq \varphi(x) + y \cdot z - \varphi(z) \quad \forall z \in \mathbb{R}^d \\ &\Leftrightarrow \varphi(z) \geq \varphi(x) + y \cdot (z - x) \quad \forall z \in \mathbb{R}^d \\ &\Leftrightarrow y \in \partial\varphi(x)\end{aligned}$$

which proves the proposition. □

In fact if φ is convex then φ is differentiable almost everywhere, hence we have that $\partial\varphi(x) = \{\nabla\varphi(x)\}$ for almost every x.

Proposition 4.6. *If $\varphi : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is convex then (1) φ is almost everywhere differentiable and (2) whenever φ is differentiable $\partial\varphi(x) = \{\nabla\varphi(x)\}$.*

Proof. Let $x \in \mathrm{int}(\mathrm{Dom}(\varphi))$ and δ^* be such that $\overline{B(x,\delta^*)} \subset \mathrm{int}(\mathrm{Dom}(\varphi))$. We show that φ is Lipschitz continuous on $B(x,\delta^*/4)$. Then, by Rademacher's theorem[2], φ is differentiable almost everywhere on $B(x,\delta^*/4)$, and therefore differentiable almost everywhere on $\mathrm{int}(\mathrm{Dom}(\varphi))$. This will complete the proof of (1).

We show φ is Lipschitz on $B(x,\delta^*/4)$ by first showing that φ is uniformly bounded on $\overline{B(x,\delta^*/2)}$. By the Minkowski-Carathéodory Theorem (Theorem 2.5) there exists $\{x_i\}_{i=0}^n \subset \partial B(x,\delta^*)$ such that for all $y \in B(x,\delta^*)$ there exists $\{\lambda_i\}_{i=0}^n \subset [0,1]$ with $\sum_{i=0}^n \lambda_i = 1$ and $y = \sum_{i=0}^n \lambda_i x_i$. So,

$$\varphi(y) = \varphi(\sum_{i=0}^n \lambda_i x_i) \le \sum_{i=0}^n \lambda_i \varphi(x_i) \le \max_{i=0,\dots,n} |\varphi(x_i)|.$$

Now for $y \in B(x,\delta^*)$ and $y' = x - (y-x) = 2x - y$ we have $y' \in B(x,\delta^*)$ and $x = \frac{1}{2}y' + \frac{1}{2}y$. Therefore $\varphi(x) \le \frac{1}{2}\varphi(y') + \frac{1}{2}\varphi(y)$. In particular,

$$\varphi(y) \ge 2\varphi(x) - \varphi(y') \ge 2\varphi(x) - \max_{i=0,\dots,n} |\varphi(x_i)|.$$

We have shown that

$$2\varphi(x) - \max_{i=0,\dots,n} |\varphi(x_i)| \le \varphi(y) \le \max_{i=0,\dots,n} |\varphi(x_i)| \qquad \forall y \in B(x,\delta^*).$$

Hence

$$\|\varphi\|_{L^\infty(\overline{B(x,\delta^*/2)})} \le \max\left\{ \max_{i=0,\dots,n} |\varphi(x_i)| - 2\varphi(x), \max_{i=0,\dots,n} |\varphi(x_i)| \right\} < \infty.$$

To show φ is Lischitz on $B(x,\delta^*/4)$ let $x_1, x_2 \in B(x,\delta^*/4)$ ($x_1 \neq x_2$) and take x_3 to be the point of intersection of the line through x_1 and x_2 with $\partial B(x,\delta^*/2)$; there are two possibilities for x_3, we choose the option where x_2 lies between x_1 and x_3. Let $\lambda = \frac{|x_2 - x_3|}{|x_1 - x_3|} \in (0,1)$. Now,

$$\begin{aligned}
\lambda x_1 + (1-\lambda)x_3 &= \lambda x_2 + \lambda(x_1 - x_2) + (1-\lambda)x_2 + (1-\lambda)(x_3 - x_2) \\
&= x_2 + \frac{|x_3 - x_2|(x_1 - x_2)}{|x_3 - x_1|} + \frac{(|x_3 - x_1| - |x_3 - x_2|)(x_3 - x_2)}{|x_3 - x_1|} \\
&= x_2 + \frac{1}{|x_3 - x_1|}\left(|x_3 - x_2|(x_1 - x_2) + |x_2 - x_1|(x_3 - x_2)\right) \\
&= x_2
\end{aligned}$$

[2] Rademacher's theorem: if $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$ is Lipschitz continuous then f is differentiable almost everywhere on U.

since $\frac{x_2-x_1}{|x_2-x_1|} = \frac{x_3-x_2}{|x_3-x_2|}$. So by convexity of φ,

$$\begin{aligned}
\varphi(x_2) - \varphi(x_1) &\leq (1-\lambda)\left(\varphi(x_3) - \varphi(x_1)\right) \\
&= \frac{|x_1 - x_3| - |x_2 - x_3|}{|x_1 - x_3|}\left(\varphi(x_3) - \varphi(x_1)\right) \\
&\leq \frac{8M|x_1 - x_2|}{\delta^*}
\end{aligned}$$

where $M = \|\varphi\|_{L^\infty(\overline{B(x,\delta^*/2)})}$ and we use that $|x_1 - x_3| \geq \delta^*/4$. Switching x_1 and x_2 implies that $|\varphi(x_2) - \varphi(x_1)| \leq \frac{8M|x_1-x_2|}{\delta^*}$ hence φ is Lipschitz continuous, with constant $L = \frac{8M}{\delta^*}$ in $B(x, \delta^*/4)$.

For (2) let φ be differentiable at x. Then,

$$\begin{aligned}
\varphi(x) + \nabla\varphi(x) \cdot (z - x) &= \varphi(x) + \lim_{h \to 0^+} \frac{\varphi(x + (z-x)h) - \varphi(x)}{h} \\
&= \varphi(x) + \lim_{h \to 0^+} \frac{\varphi((1-h)x + hz) - \varphi(x)}{h} \\
&\leq \varphi(x) + \lim_{h \to 0^+} \frac{(1-h)\varphi(x) + h\varphi(z) - \varphi(x)}{h} \\
&= \varphi(z).
\end{aligned}$$

Hence $\nabla\varphi(x) \in \partial\varphi(x)$. Now if $y \in \partial\varphi(x)$ then

$$\varphi(x) + y \cdot (z - x) \leq \varphi(z)$$

for all $z \in \mathbb{R}^n$. Let $z = x + hw$ then we can infer that

$$y \cdot w \leq \frac{\varphi(x + hw) - \varphi(x)}{h}$$

for all $h > 0$ and $w \in \mathbb{R}^n$. Letting $h \to 0^+$ we have $y \cdot w \leq \nabla\varphi(x) \cdot w$ for all $w \in \mathbb{R}^n$. Substituting $w \mapsto -w$ we have $y \cdot w = \nabla\varphi(x) \cdot w$ for all $w \in \mathbb{R}^n$. Hence $y = \nabla\varphi(x)$. □

Our final preliminary result gives equivalent conditions for convexity and lower semi-continuity.

Proposition 4.7. *Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper. Then the following are equivalent:*

1. *φ is convex and lower semi-continuous;*
2. *$\varphi = \psi^*$ for some proper function ψ;*
3. *$\varphi^{**} = \varphi$.*

Proof. 3 clearly implies 2. We first show that 2 implies 1. Let $\varphi = \psi^*$ and we will show that φ is convex and lower semi-continuous. For convexity let $x_1, x_2 \in \mathbb{R}^n, t \in [0,1]$ then

$$\begin{aligned}
\varphi(tx_1 + (1-t)x_2) &= \psi^*(tx_1 + (1-t)x_2) \\
&= \sup_{y \in \mathbb{R}^n} \left((tx_1 + (1-t)x_2) \cdot y - \psi(y)\right) \\
&\leq \sup_{y \in \mathbb{R}^n} (tx_1 \cdot y - t\psi(y)) + \sup_{y \in \mathbb{R}^n} ((1-t)x_2 \cdot y - (1-t)\psi(y)) \\
&= t\psi^*(x_1) + (1-t)\psi^*(x_2) \\
&= t\varphi(x_1) + (1-t)\varphi(x_2).
\end{aligned}$$

For lower semi-continuity let $x_m \to x$, then

$$\liminf_{m\to\infty} \varphi(x_m) = \liminf_{m\to\infty} \sup_{y\in\mathbb{R}^n} (x_m \cdot y - \psi(y)) \geq \lim_{m\to\infty} (x_m \cdot y - \psi(y)) = x \cdot y - \psi(y)$$

for any $y \in \mathbb{R}^n$. Taking the supremum over $y \in \mathbb{R}^n$ implies $\liminf_{m\to\infty} \varphi(x_n) \geq \varphi(x)$ as required.

Finally we show that 1 implies 3. Let φ be lower semi-continuous and convex. Fix $x \in \mathbb{R}^n$, we want to show that $\varphi(x) = \varphi^{**}(x)$. Since $\varphi^*(y) \geq x \cdot y - \varphi(x)$ for all $y \in \mathbb{R}^n$ then

$$\varphi(x) \geq \sup_{y\in\mathbb{R}^n} (x \cdot y - \varphi^*(y)) = \varphi^{**}(x).$$

We are left to show $\varphi(x) \leq \varphi^{**}(x)$.

Let $x \in \mathrm{int}(\mathrm{Dom}(\varphi))$, then since φ can be bounded below by an affine function passing through $\varphi(x)$ (since φ is convex) then $\partial\varphi(x) \neq \emptyset$. Let $y_0 \in \partial\varphi(x)$. By Proposition 4.5 $x \cdot y_0 = \varphi(x) + \varphi^*(y_0)$ then

$$\varphi(x) = x \cdot y_0 - \varphi^*(y_0) \leq \sup_{y\in\mathbb{R}^n} (x \cdot y - \varphi^*(y)) = \varphi^{**}(x).$$

Hence we have proved the Proposition for any φ with $\mathrm{int}(\mathrm{Dom}(\varphi)) = \mathbb{R}^d$.

For all other φ we define $\psi_\varepsilon(x) = \frac{|x|^2}{\varepsilon}$ and

$$\varphi_\varepsilon(x) = \inf_{y\in\mathbb{R}^d} (\varphi(x-y) + \psi_\varepsilon(y)) = \inf_{y\in\mathbb{R}^d} (\varphi(y) + \psi_\varepsilon(x-y)).$$

In order to show $\varphi_\varepsilon = \varphi_\varepsilon^{**}$ on $\mathbb{R}^d$ it is enough to show that φ_ε is convex, lower semi-continuous and $\mathrm{int}(\mathrm{Dom}(\varphi_\varepsilon)) = \mathbb{R}^d$. For convexity we note,

$$\begin{aligned}
\varphi_\varepsilon(tx_1 + (1-t)x_2) &= \inf_{y\in\mathbb{R}} (\varphi(tx_1 + (1-t)x_2 - y) + \psi_\varepsilon(y)) \\
&= \inf_{y_1,y_2\in\mathbb{R}^d} (\varphi(t(x_1-y_1) + (1-t)(x_2-y_2)) + \psi_\varepsilon(ty_1 + (1-t)y_2)) \\
&\leq \inf_{y_1,y_2\in\mathbb{R}^d} (t(\varphi(x_1-y_1) + \psi_\varepsilon(y_1)) + (1-t)(\varphi(x_2-y_2) + \psi_\varepsilon(y_2))) \\
&= t\varphi_\varepsilon(x_1) + (1-t)\varphi_\varepsilon(x_2).
\end{aligned}$$

The pointwise limit of an arbitrary collection of lower semi-continuous functions is lower semi-continuous, hence φ_ε is lower semi-continuous. Now let $x \in \mathbb{R}^d$ and since φ is proper there exists $y_0 \in \mathbb{R}^d$ such that $\varphi(y_0) < \infty$, hence $\varphi_\varepsilon(x) \leq \varphi(y_0) + \psi_\varepsilon(x - y_0)$. Since ψ is everywhere finite then it follows that $\varphi_\varepsilon(x)$ is finite hence $x \in \mathrm{Dom}(\varphi_\varepsilon)$. In particular $\mathrm{int}(\mathrm{Dom}(\varphi_\varepsilon)) = \mathbb{R}^d$.

We now show that $\liminf_{\varepsilon\to 0} \varphi_\varepsilon(x) \geq \varphi(x)$ (we actually show that $\lim_{\varepsilon\to 0} \varphi_\varepsilon(x) = \varphi(x)$ but the former statement is all we really need). Fix $x \in \mathbb{R}^n$ and note that $\varphi_\varepsilon(x) \leq A + \frac{B}{\varepsilon}$ Since φ is convex then it is bounded below by an affine function, say $\varphi(z) \geq a \cdot z + b$ for all $z \in \mathbb{R}^n$. Let y_ε be a minimising sequence, i.e. $\varphi_\varepsilon(x) \geq \varphi(x - y_\varepsilon) + \psi_\varepsilon(y_\varepsilon) - \varepsilon$. Then

$$A + \frac{B}{\varepsilon} \geq \varphi_\varepsilon(x) \geq a \cdot (x - y_\varepsilon) + \frac{|y_\varepsilon|^2}{\varepsilon} - \varepsilon \geq -\frac{(1+\varepsilon)|a|^2}{\varepsilon} - \frac{|x|^2}{2} + \frac{|y_\varepsilon|^2}{2\varepsilon} - \varepsilon.$$

This implies $|y_\varepsilon| = O(1)$.

Now let $\varepsilon_n \to 0$ be a subsequence with $\liminf_{\varepsilon\to 0} \varphi_\varepsilon(x) = \lim_{n\to\infty} \varphi_{\varepsilon_n}(x)$. Since y_{ε_n} is bounded then there exists a further subsequence (which we relabel) and some $y \in \mathbb{R}^n$ such that $y_{\varepsilon_n} \to y$. Furthermore,

$$\lim_{n\to\infty} \varphi_{\varepsilon_n}(x) = \lim_{n\to\infty} \left(\varphi(x - y_{\varepsilon_n}) + \psi_{\varepsilon_n}(y_{\varepsilon_n})\right) \geq \begin{cases} \varphi(x) & \text{if } y = 0 \\ +\infty & \text{else.} \end{cases}$$

In both cases the right hand side is greater than $\varphi(x)$ hence $\lim_{n\to\infty} \varphi_{\varepsilon_n}(x) \geq \varphi(x)$.

On the other hand, since $\varphi(x) \geq \varphi_\varepsilon(x)$ then

$$\varphi^{**}(x) = \sup_{y\in\mathbb{R}^n} \inf_{z\in\mathbb{R}^n} \left(y \cdot (x - z) + \varphi(z)\right) \geq \sup_{y\in\mathbb{R}^n} \inf_{z\in\mathbb{R}^n} \left(y \cdot (x - z) + \varphi_\varepsilon(z)\right) = \varphi^{**}(x).$$

Hence,

$$\varphi^{**}(x) \geq \liminf_{\varepsilon\to 0} \varphi_\varepsilon^{**}(x) = \liminf_{\varepsilon\to 0} \varphi_\varepsilon(x) \geq \varphi(x).$$

Which concludes the proof. □

4.3 Proof of the Knott-Smith Optimality Criterion

Section references: The proof of the Knott-Smith optimality condition is based on Villani's proof in [15, Theorem 2.12].

Before proving the theorem we manipulate the Kantorovich optimal transport problem and the dual problem. Let $(\varphi, \psi) \in \Phi_c$ (where $c(x, y) = \frac{1}{2}|x - y|^2$) and define $\tilde{\varphi}(x) = \frac{1}{2}|x|^2 - \varphi(x)$, $\tilde{\psi}(y) = \frac{1}{2}|y|^2 - \psi(y)$. Clearly $\tilde{\varphi} \in L^1(\mu)$, $\tilde{\psi} \in L^1(\nu)$ whenever μ, ν have finite second moments. Furthermore

$$\tilde{\varphi}(x) + \tilde{\psi}(y) = \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 - \varphi(x) - \psi(y) \geq \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 - \frac{1}{2}|x - y|^2 \geq x \cdot y.$$

In fact, $(\varphi, \psi) \in \Phi_c \Leftrightarrow (\tilde{\varphi}, \tilde{\psi}) \in \tilde{\Phi}$ where

$$\tilde{\Phi} = \left\{ (\tilde{\varphi}, \tilde{\psi}) \in L^1(\mu) \times L^1(\nu) \, : \, \tilde{\varphi}(x) + \tilde{\psi}(y) \geq x \cdot y \right\}.$$

Furthermore $\mathbb{J}(\tilde{\varphi}, \tilde{\psi}) = M - \mathbb{J}(\varphi, \psi)$ where

$$M = \frac{1}{2}\int_X |x|^2 \,\mathrm{d}\mu(x) + \frac{1}{2}\int_Y |y|^2 \,\mathrm{d}\nu(y).$$

And for $\pi \in \Pi(\mu, \nu)$,

$$\mathbb{K}(\pi) = \frac{1}{2}\int_{X\times Y} |x-y|^2 \,\mathrm{d}\pi(x,y) = M - \int_{X\times Y} x\cdot y \,\mathrm{d}\pi(x,y).$$

Hence,

$$M - \mathbb{J}(\tilde{\varphi}, \tilde{\psi}) = \mathbb{J}(\varphi, \psi) \le \mathbb{K}(\pi) = M - \int_{X\times Y} x\cdot y \,\mathrm{d}\pi(x,y).$$

Or more conveniently,

$$\mathbb{J}(\tilde{\varphi}, \tilde{\psi}) \ge \int_{X\times Y} x\cdot y \,\mathrm{d}\pi(x,y).$$

Kantorovich duality (Theorem 3.1) implies that

$$\min_{(\tilde{\varphi},\tilde{\psi})\in\tilde{\Phi}} \mathbb{J}(\tilde{\varphi}, \tilde{\psi}) = \max_{\pi\in\Pi(\mu,\nu)} \int_{X\times Y} x\cdot y \,\mathrm{d}\pi(x,y). \tag{4.1}$$

We notice that if $\pi^\dagger \in \Pi(\mu,\nu)$ minimises $\mathbb{K}$ then it also maximises $\int_{X\times Y} x\cdot y \,\mathrm{d}\pi(x,y)$, and vice versa. On the other hand, if $(\varphi,\psi)\in\Phi_c$ maximises $\mathbb{J}$, then $(\tilde{\varphi},\tilde{\psi}) = (\frac{1}{2}|\cdot|^2 - \varphi, \frac{1}{2}|\cdot|^2-\psi)\in\tilde{\Phi}$ minimises $\mathbb{J}$, and vice versa.

Existence of maximisers of $\mathbb{J}$ over Φ_c imply there exists $\varphi\in L^1(\mu)$ such that $(\tilde{\varphi},\tilde{\psi}) = (\frac{1}{2}|\cdot|^2-\varphi, \frac{1}{2}|\cdot|^2-\varphi^c)\in\tilde{\Phi}$ and $(\tilde{\varphi},\tilde{\psi})$ minimise $\mathbb{J}$ over $\tilde{\Phi}$. Furthermore,

$$\begin{aligned}\tilde{\psi}(y) &= \frac{1}{2}|y|^2 - \varphi^c(y)\\ &= \sup_{x\in X}\left(\frac{1}{2}|y|^2 - \frac{1}{2}|x-y|^2 + \varphi(x)\right)\\ &= \sup_{x\in X}\left(x\cdot y - \tilde{\varphi}(x)\right)\\ &= \tilde{\varphi}^*(y)\end{aligned}$$

where $\tilde{\varphi}^*$. We also have,

$$\begin{aligned}\tilde{\varphi}(x) &= \frac{1}{2}|x|^2 - \varphi^{cc}(x)\\ &= \sup_{y\in Y}\left(\frac{1}{2}|x|^2 - \frac{1}{2}|x-y|^2 + \varphi^c(y)\right)\\ &= \sup_{y\in Y}\left(\frac{1}{2}|x|^2 - \frac{1}{2}|x-y|^2 + \frac{1}{2}|y|^2 - \tilde{\varphi}^*(y)\right)\\ &= \sup_{y\in Y}\left(x\cdot y - \tilde{\varphi}^*(y)\right)\\ &= \tilde{\varphi}^{**}(x).\end{aligned}$$

Hence minimisers of $\mathbb{J}$ over $\tilde{\Phi}$ take the form $(\tilde{\varphi}^{**}, \tilde{\varphi}^*)$.

Let $\tilde{\eta} = \tilde{\varphi}^{**}$ then by Proposition 4.7 $\tilde{\eta}$ is convex and lower semi-continuous. Furthermore, again by Proposition 4.7 $\tilde{\eta}^* = \tilde{\varphi}^{***} = \tilde{\varphi}^*$. Hence there exists minimisers of $\mathbb{J}$ over $\tilde{\Phi}$ with the form $(\tilde{\eta}, \tilde{\eta}^*)$ where $\tilde{\eta}$ is a proper, convex and lower semi-continuous function.

Proof of Theorem 4.1. Let $\pi^\dagger \in \Pi(\mu, \nu)$ minimise $\mathbb{K}$ over $\Pi(\mu, \nu)$ and $\tilde{\varphi}$ be the proper lower semi-continuous function such that the pair $(\tilde{\varphi}, \tilde{\varphi}^*)$ minimise $\mathbb{J}$ over $\tilde{\Phi}$. By Kantorovich duality (in particular (4.1)) we have

$$\int_X \tilde{\varphi}(x)\,\mathrm{d}\mu(x) + \int_Y \tilde{\varphi}^*(y)\,\mathrm{d}\nu(y) = \int_{X\times Y} x\cdot y\,\mathrm{d}\pi^\dagger(x,y).$$

Equivalently,

$$\int_{X\times Y} \tilde{\varphi}(x) + \tilde{\varphi}^*(y) - x\cdot y\,\mathrm{d}\pi^\dagger(x,y) = 0.$$

By definition of the convex conjugate $\tilde{\varphi}(x) + \tilde{\varphi}^*(y) \geq x\cdot y$ and therefore the integrand is non-negative. We must have $\tilde{\varphi}(x) + \tilde{\varphi}^*(y) = x\cdot y$ for $\pi^\dagger$-almost every (x,y) and therefore by Proposition 4.5 $y \in \partial\tilde{\varphi}(x)$ for $\pi^\dagger$-almost every (x,y).

Conversely, suppose $y \in \partial\tilde{\varphi}(x)$ for $\pi^\dagger$-almost every (x,y) where $\tilde{\varphi}$ is a $L^1(\mu)$ proper, lower semi-continuous and convex function. Then by Proposition 4.5,

$$\int_{X\times Y} \tilde{\varphi}(x) + \tilde{\varphi}^*(y) - x\cdot y\,\mathrm{d}\pi^\dagger(x,y) = 0.$$

Notice that by definition of the Legendre-Fenchel transform we have that $\tilde{\varphi}(x) + \tilde{\varphi}^*(y) \geq x\cdot y$. We will show integrability of $\tilde{\varphi}^*$ shortly, for now it is assumed then $(\tilde{\varphi}, \tilde{\varphi}^*) \in \tilde{\Phi}$. Hence,

$$\min_{\tilde{\Phi}} \mathbb{J} \leq \mathbb{J}(\tilde{\varphi}, \tilde{\varphi}^*) = \int_{X\times Y} x\cdot y\,\mathrm{d}\pi^\dagger(x,y) \leq \max_{\pi\in\Pi(\mu,\nu)} \int_{X\times Y} x\cdot y\,\mathrm{d}\pi(x,y).$$

By duality (i.e. (4.1)) it follows that $(\tilde{\varphi}, \tilde{\varphi}^*)$ in $\tilde{\Phi}$ achieves the minimum of $\mathbb{J}$ and $\pi^\dagger$ achieves the maximum of $\int_{X\times Y} x\cdot y\,\mathrm{d}\pi(x,y)$ is $\Pi(\mu,\nu)$. Hence $\pi^\dagger$ is an optimal plan in the Kantorovich sense.

The last detail we have to show is $\tilde{\varphi}^* \in L^1(\nu)$. Since $\tilde{\varphi}$ is convex then $\tilde{\varphi}^*$ can be bounded below by an affine function, in particular there exists $x_0 \in X$ such that $\tilde{\varphi}^*(y) \geq x_0\cdot y - \tilde{\varphi}(x_0) \geq x_0\cdot y - b_0 =: f(y)$. So the integral,

$$\|\tilde{\varphi}^* - f\|_{L^1(\nu)} = \int_Y \tilde{\varphi}^*(y) - f(y)\,\mathrm{d}\nu(y) \leq \mathbb{J}(\tilde{\varphi}, \tilde{\varphi}^*) + \|\tilde{\varphi}\|_{L^1(\mu)} + \frac{1}{2}|x_0|^2 + \frac{1}{2}\int_Y |y|^2\,\mathrm{d}\nu(y) + b_0$$

is finite. Hence $\tilde{\varphi}^* - f \in L^1(\nu)$, and since $f \in L^1(\nu)$ then $\tilde{\varphi}^* \in L^1(\nu)$ as required. □

4.4 Proof of Brenier's Theorem

Section references: The proof of the Brenier's theorem is based on Villani's proof in [15, Theorem 2.12].

Proof of Theorem 4.2. Let $\pi^\dagger$ be a minimiser of Kantorovich's optimal transport problem. If we write (by disintegration of measures)

$$\pi^\dagger(A \times B) = \int_A \pi^\dagger(B|x)\,\mathrm{d}\mu(x),$$

for some family $\{\pi^\dagger(\cdot|x)\}_{x\in X} \subset \mathcal{P}(Y)$, then $\mathrm{supp}(\pi^\dagger(\cdot|x)) \subseteq \partial\varphi(x)$ for μ-a.e. $x \in X$ by Theorem 4.1. By Proposition 4.6, $\partial\varphi(x) = \{\nabla\varphi(X)\}$ for $\mathcal{L}$-a.e. $x \in X$ (and therefore μ-a.e. $x \in X$). Hence $\mathrm{supp}(\pi^\dagger(\cdot|x)) \subset \{\nabla\varphi(x)\}$ for μ-a.e. $x \in X$. This implies $\pi^\dagger(\cdot|x) = \partial_{\nabla\varphi(x)}$ for μ-a.e. $x \in X$. We have shown that there exists an optimal $\pi^\dagger$ that can be written as

$$\pi^\dagger = (\mathrm{Id} \times \nabla\varphi)_{\#}\mu$$

and

$$\begin{aligned}
\nu(B) &= \pi^\dagger(\mathbb{R}^n \times B) \\
&= (\mathrm{Id} \times \nabla\varphi)_{\#}\mu(\mathbb{R}^n \times B) \\
&= \mu\left((\mathrm{Id} \times \nabla\varphi)^{-1}(\mathbb{R}^n \times B)\right) \\
&= \mu\left(\{x \,:\, (\mathrm{Id} \times \nabla\varphi)(x) \in \mathbb{R}^n \times B\}\right) \\
&= \mu\left(\{x \,:\, \nabla\varphi(x) \in B\}\right) \\
&= (\nabla\varphi)_{\#}\mu(B)
\end{aligned}$$

then we also have that $(\nabla\varphi)_{\#}\mu = \nu$.

We are left to show uniqueness. Assume $\bar{\varphi}$ is another convex function with $(\nabla\bar{\varphi})_{\#}\mu = \nu$. We will show that $\nabla\varphi = \nabla\bar{\varphi}$ upto μ null sets.

By Theorem 4.1 we know that $(\mathrm{Id} \times \nabla\bar{\varphi})_{\#}\mu$ is an optimal transport plan and the pair $(\bar{\varphi}, \bar{\varphi}^*)$ minimize $\mathbb{J}$ over $\tilde{\Phi}$. So,

$$\int_X \bar{\varphi}\,\mathrm{d}\mu + \int_Y \bar{\varphi}^*\,\mathrm{d}\nu = \int_X \varphi\,\mathrm{d}\mu + \int_Y \varphi^*\,\mathrm{d}\nu.$$

The above implies that

$$\begin{aligned}
\int_{X\times Y} \bar{\varphi}(x) + \bar{\varphi}^*(y)\,\mathrm{d}\pi^\dagger(x,y) &= \int_{X\times Y} \varphi(x) + \varphi^*(y)\,\mathrm{d}\pi^\dagger(x,y) \\
&= \int_{X\times Y} x \cdot y\,\mathrm{d}\pi^\dagger(x,y) \\
&= \int_{X\times Y} x \cdot y\,\mathrm{d}(\mathrm{Id} \times \nabla\varphi)_{\#}\mu(x,y) \\
&= \int_X x \cdot \nabla\varphi(x)\,\mathrm{d}\mu(x)
\end{aligned}$$

where the second line follows as $y \in \partial\varphi(x)$ for π-a.e. (x, y) and by Proposition 4.5. Also,

$$\int_{X\times Y} \bar{\varphi}(x) + \bar{\varphi}^*(y)\,\mathrm{d}\pi^\dagger(x,y) = \int_X \bar{\varphi}(x) + \bar{\varphi}^*(\nabla\varphi(x))\,\mathrm{d}\mu(x).$$

Hence

$$\int_X \left(\bar{\varphi}(x) + \bar{\varphi}^*(\nabla\varphi(x)) - x \cdot \nabla\varphi(x)\right) \, \mathrm{d}\mu(x) = 0.$$

In particular, $\bar{\varphi}(x) + \bar{\varphi}^*(\nabla\varphi(x)) - x \cdot \nabla\varphi(x) = 0$ for μ-almost every x. By Proposition 4.5 this implies that $\nabla\varphi(x) \in \partial\bar{\varphi}(x)$ for μ-almost every x and therefore $\nabla\varphi(x) = \nabla\bar{\varphi}(x)$ for μ-almost every x. □

Chapter 5

Wasserstein Distances

Eulerian based costs, such as L^p, define a metric based on "pointwise differences". This has some notable disadvantages, for example consider in 1D two indicator functions $f(x) = \chi_{[0,1]}(x)$ and $f_\delta(x) = \chi_{[\delta,\delta+1]}(x)$. Notice that in L^p,

$$\|f - f_\delta\|_{L^p}^p = \begin{cases} 2\delta & \text{if } |\delta| < 1 \\ 2 & \text{else.} \end{cases}$$

In particular, we notice that once $|\delta| \geq 1$ the L^p distance is constant. In more general examples, where f and f_δ are not necessarily indicator functions, the L^p distance will be the sum of the L^p norms whenever the supports of f and f_δ are disjoint.

Why do we care? Say we are trying to fit a parametrised curve f_δ to f. Then say we start from a bad initialisation where the support of f_δ is disjoint from the support of f. In this regime the derivative $\frac{\mathrm{d}}{\mathrm{d}\delta}\|f_\delta - f\|_{L^p} = 0$, this is a problem for gradient based optimisation.

On the other hand, we would hope that a transport based distance would do a better job. In particular, in the elementary example $f(x) = \chi_{[0,1]}(x)$ and $f_\delta(x) = \chi_{[\delta,\delta+1]}(x)$ the OT cost would be

$$\min_{T_\# f = f_\delta} \int_0^1 |x - T(x)|^p \, \mathrm{d}x = |\delta|^p$$

where the cost is $c(x,y) = |x-y|^p$ and with an abuse of notation we associate f and f_δ with the measures with density f and f_δ respectively. Note that the OT cost now strictly increases as a function of $|\delta|$.

The objective of this section is to understand how the optimal transport can be used to define a metric and some of the metric properties. In particular, we will define the Wasserstein distance (also sometimes known as the earth movers distance) in the next section. In Section 5.2 we look at the topology of Wasserstein spaces and show that the Wasserstein distance metrizes the weak* convergence. Finally we will look at geodesics and the relation to fluid dynamics.

Throughout this chapter we will assume that $c(x,y) = |x-y|^p$ for $p \in [1, +\infty)$ and X, Y are subsets of $\mathbb{R}^d$.

Before proceeding to the Wasserstein distance, let us note one other important example that can be posed as an optimal transport problem. Let $c(x,y) = \mathbb{I}_{x \neq y}$, i.e. $c(x,y) = 0$ if $x = y$ and $c(x,y) = 1$ otherwise. Then the optimal transport problem coincides with the total variation distance between measures.

Proposition 5.1. *Let $\mu, \nu \in \mathcal{P}(X)$ where X is a Polish space and $c(x,y) = \mathbb{I}_{x\neq y}$ then*

$$\inf_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi) = \frac{1}{2}\|\mu-\nu\|_{\mathrm{TV}}$$

where

$$\|\mu\|_{\mathrm{TV}} := 2\sup_A |\mu(A)|.$$

Proof. We prove the proposition in 4 steps, in the first two steps we only assume that c is a metric, in particular we use that c is lower semi-continuous, symmetric and satisfies the triangle inequality. In the third step we use the specific form of c, but this can also be avoided, see Remark 5.2.

1. Let $\varphi \in L^1(\mu)$, we claim that $|\varphi^c(x) - \varphi^c(y)| \leq c(x,y)$ for almost every $(x,y) \in X \times X$. Indeed,

$$\begin{aligned}
\varphi^c(x) - \varphi^c(y) &= \inf_{z_1\in X}\sup_{z_2\in X} \left(c(x,z_1) - c(y,z_2) - \varphi(z_1) - \varphi(z_2)\right)\\
&\leq \sup_{z_2\in X}\left(c(x,z_2) - c(y,z_2)\right) \quad \text{choosing } z_1 = z_2\\
&\leq \sup_{z_2\in X}\left(c(x,y) + c(y,z_2) - c(y,z_2)\right) \quad \text{by the triangle inequality}\\
&\leq c(x,y).
\end{aligned}$$

By switching x and y we have that $|\varphi^c(x) - \varphi^c(y)| \leq c(x,y)$.

2. We claim $\varphi^{cc} = -\varphi^c$. By part 1 we have $c(x,y) - \varphi^c(y) \geq -\varphi^c(x)$ and therefore

$$\varphi^{cc}(x) = \inf_{y\in X}\left(c(x,y) - \varphi^c(y)\right) \geq -\varphi^c(x).$$

On the other hand

$$\varphi^{cc}(x) = \inf_{y\in X}\left(c(x,y) - \varphi^c(y)\right) \leq -\varphi^c(x)$$

by choosing $y = x$. Hence $\varphi^{cc} = -\varphi^c$ as claimed.

3. Since $|\eta^c(x) - \eta^c(y)| \leq 1$ we can, without loss of generality, assume that $\eta^c(x) \in [0,1]$ in the following. By Theorem 3.1 and Theorem 3.7,

$$\begin{aligned}
\min_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi) &= \sup_{\eta\in L^1} \mathbb{J}(\eta^{cc},\eta^c) = \sup_{\eta\in L^1(\mu)} \mathbb{J}(-\eta^c,\eta^c)\\
&\leq \sup_{0\leq f\leq 1} \mathbb{J}(-f,f) \leq \sup_{\Phi_c} \mathbb{J}(\varphi,\psi) = \min_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi)
\end{aligned}$$

where the penultimate inequality follows as $(-f,f) \in \Phi_c$, to see this we only need to show $-f(x) + f(y) \leq c(x,y)$. Clearly $-f(x) + f(y) \leq 1 = c(x,y)$ for $x \neq y$ and $-f(x) + f(y) = 0 = c(x,y)$ for $x = y$. It follows that

$$\min_{\pi\in\Pi(\mu,\nu)} \mathbb{K}(\pi) = \sup_{0\leq f\leq 1} \mathbb{J}(-f,f).$$

4. Now let $\nu - \mu = (\nu - \mu)_+ - (\nu - \mu)_-$ be the decomposition of $\nu - \mu$ where $(\nu - \mu)_\pm \in \mathcal{M}(X)$ are singular. It follows that

$$\|\mu - \nu\|_{\mathrm{TV}} = 2(\nu - \mu)_+(X).$$

And,

$$\sup_{0 \le f \le 1} \mathbb{J}(-f, f) = \sup_{0 \le f \le 1} \int_X f \,\mathrm{d}(\nu - \mu) = (\nu - \mu)_+(X).$$

Hence $\min_{\pi \in \Pi(\mu,\nu)} \mathbb{K}(\pi) = \sup_{0 \le f \le 1} \mathbb{J}(-f, f) = \frac{1}{2}\|\mu - \nu\|_{\mathrm{TV}}$. □

Remark 5.2. In step 3 of the above proof we showed that

$$\min_{\pi \in \Pi(\mu,\nu)} \mathbb{K}(\pi) = \sup_{(\varphi,\psi) \in \Phi_c} \mathbb{J}(\varphi, \psi) = \sup_{0 \le f \le 1} \int_X f \,\mathrm{d}(\nu - \mu).$$

This is actually a special case of the Kantorovich-Rubinstein Theorem (see [15, Theorem 1.14]) which states that, when c is a metric,

$$\min_{\pi \in \Pi(\mu,\nu)} \mathbb{K}(\pi) = \sup \left\{ \int_X \varphi \,\mathrm{d}(\mu - \nu) \;:\; \varphi \in L^1(|\mu - \nu|), \|\varphi\|_{\mathrm{Lip}} \le 1 \right\}$$

where

$$\|\varphi\|_{\mathrm{Lip}} = \sup_{x \ne y} \frac{|\varphi(x) - \varphi(y)|}{c(x, y)}.$$

5.1 Wasserstein Distances

Section references: The proof of that the Wasserstein distance is a metric, Proposition 5.4, comes from [12, Proposition 5.1 and Lemma 5.4], with the preliminary result, Lemma 5.5, coming from [12, Lemma 5.5]. The equivalence of Wasserstein norms is from [12, Section 5.1].

We will work on the space of probability measures on $X \subset \mathbb{R}^d$ with bounded p^{th} moment, i.e.

$$\mathcal{P}_p(X) := \left\{ \mu \in \mathcal{P}(X) \;:\; \int_X |x|^p \,\mathrm{d}\mu(x) < +\infty \right\}.$$

Of course, if X is bounded then $\mathcal{P}_p(X) = \mathcal{P}(X)$. We now define the Wasserstein distance, it will be the objective of this section to prove that the Wasserstein distance is a metric.

Definition 5.3. *Let $\mu, \nu \in \mathcal{P}_p(X)$, then the Wasserstein distance is defined as*

$$d_{W^p}(\mu, \nu) = \min_{\pi \in \Pi(\mu,\nu)} \left(\int_{X \times X} |x - y|^p \,\mathrm{d}\pi(x, y) \right)^{\frac{1}{p}}.$$

The Wasserstein distance is the p^{th} root of the minimum of the Kantorovich optimal transport problem for cost function $c(x, y) = |x - y|^p$. The motivation is that this cost resembles an L^p distance (in fact we use properties of L^p distances to prove the triangle inequality). One could also consider an analogous distance for cost function $c(x, y) = d(x, y)$ where d is a metric on X. This type of distance is known as the earth movers distance. Notice that when $p = 1$ the

Wasserstein distance is also a earth movers distance. We will not focus on earth movers distances here.

Let us note here that $\mu, \nu \in \mathcal{P}_p(X)$ is enough to guarantee $d_{W^p}(\mu,\nu) < +\infty$. In particular,

$$d_{W^p}^p(\mu,\nu) \leq p \inf_{\pi \in \Pi(\mu,\nu)} \int_{X\times X} |x|^p + |y|^p \,\mathrm{d}\pi(x,y) = p \int_X |x|^p \,\mathrm{d}\mu(x) + p \int_X |y|^p \,\mathrm{d}\nu(y).$$

We now state the result that d_{W^p} is a metric. The proof, minus the triangle inequality, is given below.

Proposition 5.4. *The distance $d_{W^p} : \mathcal{P}_p(X) \times \mathcal{P}_p(X) \to [0,\infty)$ is a metric on $\mathcal{P}_p(X)$.*

Proof. We give the proof of all the required criteria with the exception of the triangle inequality which will require some preliminary results. Firstly, it is clear that $d_{W^p}(\mu,\nu) \geq 0$ for all $\mu,\nu \in \mathcal{P}(X)$ and by symmetry of the cost function $c(x,y) = |x-y|^p$ and $\pi \in \Pi(\mu,\nu) \Leftrightarrow S_{\#}\pi \in \Pi(\nu,\mu)$ where $S(x,y) = (y,x)$ we have symmetry of d_{W^p}. Now if $\mu = \nu$ then we can take $\pi(x,y) = \delta_x(y)\mu(x)$ so that

$$d_{W^p}^p(\mu,\nu) \leq \int_{X\times X} |x-y|^p \,\mathrm{d}\pi(x,y) = 0$$

as $x = y$ π-almost everywhere. Now if $d_{W^p}(\mu,\nu) = 0$ then there exists $\pi \in \Pi(\mu,\nu)$ such that $x = y$ π-almost everywhere. Hence for any test function $f : X \to \mathbb{R}$,

$$\int_X f(x) \,\mathrm{d}\mu(x) = \int_{X\times X} f(x) \,\mathrm{d}\pi(x,y) = \int_{X\times X} f(y) \,\mathrm{d}\pi(x,y) = \int_X f(y) \,\mathrm{d}\nu(y).$$

As this holds for all test functions f then $\mu = \nu$. □

The following lemma is known as the gluing lemma and we will use it to "glue" two transport plans $\pi_1 \in \Pi(\mu,\nu)$ and $\pi_2 \in \Pi(\nu,\omega)$. The triangle inequality then follows from the triangle inequality for L^p distances.

Lemma 5.5. *Given measures $\mu \in \mathcal{P}(X), \nu \in \mathcal{P}(Y), \omega \in \mathcal{P}(Z)$ and transport plans $\pi_1 \in \Pi(\mu,\nu)$ and $\pi_2 \in \Pi(\nu,\omega)$ there exists a measure $\gamma \in \mathcal{P}(X\times Y\times Z)$ such that $P^{X,Y}_{\#}\gamma = \pi_1$ and $P^{Y,Z}_{\#}\gamma = \pi_2$ where $P^{X,Y}(x,y,z) = (x,y)$ and $P^{Y,Z}(x,y,z) = (y,z)$ are the projections onto the two first and two last variables respectively.*

Proof. By the disintegration of measures we can write

$$\pi_1(A\times B) = \int_B \pi_1(A|y) \,\mathrm{d}\nu(y)$$

for some family of probability measures $\pi_1(\cdot|y) \in \mathcal{P}(X)$, and similarly for π_2,

$$\pi_2(B\times C) = \int_B \pi_2(C|y) \,\mathrm{d}\nu(y).$$

Define $\gamma \in \mathcal{M}(X \times Y \times Z)$ by

$$\gamma(A \times B \times C) = \int_B \pi_1(A|y)\pi_2(C|y)\,\mathrm{d}\nu(y).$$

Then,

$$\gamma(A \times B \times Z) = \int_B \pi_1(A|y)\pi_2(Z|y)\,\mathrm{d}\nu(y) = \int_B \pi_1(A|y)\,\mathrm{d}\nu(y) = \pi_1(A \times B).$$

Similarly, $\gamma(X \times B \times C) = \pi_2(B \times C)$. Therefore, $P^{X,Y}_{\#}\gamma = \pi_1$ and $P^{Y,Z}_{\#}\gamma = \pi_2$ as required. □

We are now in a position to complete the proof of Proposition 5.4.

Proof of Proposition 5.4 (the triangle inequality). Let $\mu, \nu, \omega \in \mathcal{P}_p(X)$ and assume $\pi_{XY} \in \Pi(\mu, \nu)$, $\pi_{YZ} \in \Pi(\nu, \omega)$ are optimal, i.e.

$$d^p_{W^p}(\mu, \nu) = \int_{X \times X} |x - y|^p \,\mathrm{d}\pi_{XY}(x, y)$$
$$d^p_{W^p}(\nu, \omega) = \int_{X \times X} |y - z|^p \,\mathrm{d}\pi_{YZ}(y, z).$$

Let $\gamma \in \mathcal{P}(X \times X \times X)$ be such that $P^{X,Y}_{\#}\gamma = \pi_{XY}$ and $P^{Y,Z}_{\#}\gamma = \pi_{YZ}$ (such γ exists by Lemma 5.5). Let $\pi_{XZ} = P^{X,Z}_{\#}\gamma$. Then,

$$\begin{aligned}
\pi_{XZ}(A \times X) &= P^{X,Z}_{\#}\gamma(A \times X) \\
&= \gamma\left(\left\{(x, y, z) \,:\, P^{X,Z}(x, y, z) = (x, z) \in A \times X\right\}\right) \\
&= \gamma\left(\{(x, y, z) \,:\, x \in A\}\right) \\
&= \gamma(A \times X \times X) \\
&= \pi_{XZ}(A \times X) \\
&= \mu(A).
\end{aligned}$$

Similarly $\pi_{XZ}(X \times B) = \omega(B)$. So, $\pi_{XZ} \in \Pi(\mu, \omega)$.

Now,

$$\begin{aligned}
d_{W^p}(\mu, \omega) &\leq \left(\int_{X \times X} |x - z|^p \,\mathrm{d}\pi_{XZ}(x, z)\right)^{\frac{1}{p}} \\
&= \left(\int_{X \times X \times X} |x - z|^p \,\mathrm{d}\gamma(x, y, z)\right)^{\frac{1}{p}} \\
&\leq \left(\int_{X \times X \times X} |x - y|^p \,\mathrm{d}\gamma(x, y, z)\right)^{\frac{1}{p}} + \left(\int_{X \times X \times X} |y - z|^p \,\mathrm{d}\gamma(x, y, z)\right)^{\frac{1}{p}} \\
&= \left(\int_{X \times X} |x - y|^p \,\mathrm{d}\pi_{XY}(x, y)\right)^{\frac{1}{p}} + \left(\int_{X \times X} |y - z|^p \,\mathrm{d}\pi_{YZ}(y, z)\right)^{\frac{1}{p}} \\
&= d_{W^p}(\mu, \nu) + d_{W^p}(\nu, \omega).
\end{aligned}$$

This proves the triangle inequality. □

One can also prove the triangle inequality using an transport maps and an approximation argument. Slightly more precisely, if μ, ν and ω all have densities with respect to Lebesgue then we know there exits transport maps T and S with $T_{\#}\mu = \nu$ and $S_{\#}\nu = \omega$. The map $S \circ T$ then pushes μ onto ω. One can argue, similarly to our proof, that $d_{W^p}(\mu,\nu) + d_{W^p}(\nu,\omega) \geq d_{W^p}(\mu,\omega)$. To extend the argument to arbitrary probability measures μ, ν and ω one uses mollifiers to define $\tilde{\mu}_\varepsilon = \mu * J_\varepsilon$, analogously for $\tilde{\nu}_\varepsilon, \tilde{\omega}_\varepsilon$, where $J_\varepsilon = \frac{1}{\varepsilon^d} J(\cdot/\varepsilon)$ and J is a standard mollifier. The measures $\tilde{\mu}, \tilde{\nu}, \tilde{\omega}$ have densities with respect to the Lebesgue measure and one can show $d_{W^p}(\tilde{\mu}_\varepsilon, \tilde{\nu}_\varepsilon) \to d_{W^p}(\mu,\nu)$ as $\varepsilon \to 0$. We refer to [12, Lemma 5.2 and Lemma 5.3] for full details.

Our final result of the section gives sufficient conditions for equivalence of Wasserstein distances.

Proposition 5.6. *For every $p \in [1, +\infty)$ and any $\mu, \nu \in \mathcal{P}_p(X)$ we have $d_{W^p}(\mu,\nu) \geq d_{W^1}(\mu,\nu)$. Furthermore, if $X \subset \mathbb{R}^n$ is bounded then $d^p_{W^p}(\mu,\nu) \leq \mathrm{diam}(X)^{p-1} d_{W^1}(\mu,\nu)$.*

Proof. By Jensen's inequality, for $\pi \in \Pi(\mu,\nu)$, we have

$$\left(\int_{X\times X} |x-y|^p \,\mathrm{d}\pi(x,y) \right)^{\frac{1}{p}} \geq \int_{X\times X} |x-y| \,\mathrm{d}\pi(x,y).$$

Hence, $d_{W^p}(\mu,\nu) \geq d_{W^1}(\mu,\nu)$.

Now if X is bounded, then $\forall x,y \in X$,

$$|x-y|^p \leq (\max_{w,z\in X} |w-z|^{p-1})|x-y| = (\mathrm{diam}(X))^{p-1}|x-y|.$$

Hence,

$$\int_{X\times X} |x-y|^p \,\mathrm{d}\pi(x,y) \leq (\mathrm{diam}(X))^{p-1} \int_{X\times X} |x-y| \,\mathrm{d}\pi(x,y).$$

From which it follows $d^p_{W^p}(\mu,\nu) \leq \mathrm{diam}(X)^{p-1} d_{W^1}(\mu,\nu)$. □

In fact the above is also true for $p = +\infty$, however we do not consider (or even define) d_{W^∞} here and instead refer to [12, Section 5.5.1] for more information on the ∞-Wasserstein distance.

5.2 The Wasserstein Topology

Section references: The two results regarding the relationship between convergence in Wasserstein distance and weak convergence can be found in [12, Theorem 5.10 and Theorem 5.11].*

In this section we prove the relationship of convergence in Wasserstein distance with weak* convergence. We start with when $X \subset \mathbb{R}^n$ is compact.

Theorem 5.7. *Let $X \subset \mathbb{R}^n$ be compact, and $\mu_m, \mu \in \mathcal{P}(X)$. Then $\mu_m \overset{*}{\rightharpoonup} \mu$ if and only if $d_{W^p}(\mu_m,\mu) \to 0$.*

Proof. By Proposition 5.6 it is enough to show the result for $p = 1$. Assume $d_{W^1}(\mu_m,\mu) \to 0$. By the Kantorovich-Rubinstein theorem, see Remark 5.2, we can write

$$d_{W^1}(\mu,\nu) = \sup\left\{ \int_X \varphi \,\mathrm{d}(\mu-\nu) \;:\; \varphi \in L^1(|\mu-\nu|), |\varphi(x)-\varphi(y)| \leq |x-y| \right\}.$$

Let φ be a Lipschitz function with $\mathrm{Lip}(\varphi) > 0$ then $\tilde{\varphi} = \frac{1}{\mathrm{Lip}(\varphi)}\varphi$ is a 1-Lipschitz function and therefore

$$\frac{1}{\mathrm{Lip}(\varphi)} \int_X \varphi \, \mathrm{d}(\mu_m - \mu) = \int_X \tilde{\varphi} \, \mathrm{d}(\mu_m - \mu) \leq d_{W^1}(\mu_n, \mu) \to 0.$$

By substituting $\varphi \mapsto -\varphi$ we have that

$$\int_X \varphi \, \mathrm{d}\mu_m \to \int_X \varphi \, \mathrm{d}\mu$$

for any Lipschitz function φ. By the Portmanteau theorem $\mu_m \overset{*}{\rightharpoonup} \mu$.

For the converse statement we assume that $\mu_m \overset{*}{\rightharpoonup} \mu$ and let m_k be the subsequence such that

$$\lim_{k\to\infty} d_{W^1}(\mu_{m_k}, \mu) = \limsup_{m\to\infty} d_{W^1}(\mu_m, \mu).$$

Let $\tilde{\varphi}_{m_k}$ be 1-Lipschitz and such that

$$d_{W^1}(\mu_{m_k}, \mu) \leq \int_X \tilde{\varphi}_{m_k} \, \mathrm{d}(\mu_{m_k} - \mu) + \frac{1}{k}.$$

Pick $x_0 \in \mathrm{supp}(\mu)$. Note that, for any $\varphi \in L^1(\nu)$ where $\nu \in \mathcal{M}(X)$, $c \in \mathbb{R}$ that

$$\int_X (\varphi + c) \, \mathrm{d}\nu = \int_X \varphi \, \mathrm{d}\nu$$

if $\nu(X) = 0$. Hence if we let $\varphi_{m_k}(x) = \tilde{\varphi}_{m_k}(x) - \tilde{\varphi}_{m_k}(x_0)$ then

$$d_{W^1}(\mu_{m_k}, \mu) \leq \int_X \varphi_{m_k} \, \mathrm{d}(\mu_{m_k} - \mu) + \frac{1}{k},$$

φ_{m_k} are 1-Lipschitz (in particular equicontinuous) and bounded. By the Arzelà-Ascoli theorem there exists a further subsequence (relabelled) such that $\varphi_{m_k} \to \varphi$ uniformly. In particular, φ is 1-Lipschitz. Hence,

$$\begin{aligned}
\limsup_{m\to\infty} d_{W^1}(\mu_m, \mu) &\leq \limsup_{k\to\infty} \left(\int_X \varphi_{m_k} \, \mathrm{d}(\mu_{m_k} - \mu) + \frac{1}{k} \right) \\
&\leq \limsup_{k\to\infty} \left(\int_X (\varphi_{m_k} - \varphi) \, \mathrm{d}\mu_{m_k} + \int_X \varphi \, \mathrm{d}(\mu_{m_k} - \mu) \right) \\
&\leq \limsup_{k\to\infty} \|\varphi_{m_k} - \varphi\|_{L^\infty} + \limsup_{k\to\infty} \int_X \varphi \, \mathrm{d}(\mu_{m_k} - \mu) \\
&= 0.
\end{aligned}$$

Hence, $d_{W^1}(\mu_m, \mu) \to 0$ as $m \to \infty$. □

We now generalise to unbounded domains.

Theorem 5.8. *Let $\mu_m, \mu \in \mathcal{P}_p(\mathbb{R}^n)$. Then $d_{W^p}(\mu_m, \mu) \to 0$ if and only if $\int_{\mathbb{R}^n} |x|^p \, \mathrm{d}\mu_m \to \int_{\mathbb{R}^n} |x|^p \, \mathrm{d}\mu$ and $\mu_m \overset{*}{\rightharpoonup} \mu$.*

Proof. Let $d_{W^p}(\mu_m,\mu)\to 0$. Then by Proposition 5.6 we have $d_{W^1}(\mu_m,\mu)\to 0$. Analogously to the proof of Theorem 5.7 we have $\int_X \varphi \,\mathrm{d}(\mu_m-\mu)\to 0$ for all Lipschitz functions φ. Hence, by the Portmanteau theorem $\mu_m \overset{*}{\rightharpoonup} \mu$.

To show $\int_{\mathbb{R}^n}|x|^p\,\mathrm{d}\mu_m \to \int_{\mathbb{R}^n}|x|^p\,\mathrm{d}\mu$ we note that

$$\int_{\mathbb{R}^n}|x|^p\,\mathrm{d}\mu_m = d^p_{W^p}(\mu_m,\delta_0) \quad \text{and} \quad \int_{\mathbb{R}^n}|x|^p\,\mathrm{d}\mu = d^p_{W^p}(\mu,\delta_0).$$

Now,

$$d_{W^p}(\mu_m,\delta_0)\le d_{W^p}(\mu_m,\mu)+d_{W^p}(\mu,\delta_0)\to d_{W^p}(\mu,\delta_0)$$

and

$$d_{W^p}(\mu_m,\delta_0)\ge d_{W^p}(\mu,\delta_0)-d_{W^p}(\mu_m,\mu)\to d_{W^p}(\mu,\delta_0)$$

Hence $\int_{\mathbb{R}^n}|x|^p\,\mathrm{d}\mu_m \to \int_{\mathbb{R}^n}|x|^p\,\mathrm{d}\mu$.

For the converse statement let $\mu_m \overset{*}{\rightharpoonup} \mu$ and $\int |x|^p\,\mathrm{d}\mu \to \int |x|^p\,\mathrm{d}\mu$. For any $R>0$ let $\phi_R(x)=(|x|\wedge R)^p=(\min\{|x|,R\})^p$ which is continuous and bounded. We have

$$\int_{\mathbb{R}^n}\left(|x|^p-\phi_R(x)\right)\,\mathrm{d}\mu_n \to \int_{\mathbb{R}^n}\left(|x|^p-\phi_R(x)\right)\,\mathrm{d}\mu \tag{5.1}$$

by weak* convergence and convergence of p^{th} moments. Now

$$\int_{\mathbb{R}^n}\left(|x|^p-\phi_R(x)\right)\,\mathrm{d}\mu(x)=\int_{|x|>R}|x|^p-R^p\,\mathrm{d}\mu\le\int_{|x|>R}|x|^p\,\mathrm{d}\mu<\infty.$$

In particular, we let $\varepsilon>0$ and choose $R>0$ such that

$$\int_{\mathbb{R}^n}\left(|x|^p-\phi_R(x)\right)\,\mathrm{d}\mu(x)<\frac{\varepsilon}{2}.$$

By (5.1) we also have $\int_{\mathbb{R}^n}\left(|x|^p-\phi_R(x)\right)\,\mathrm{d}\mu_m(x)<\varepsilon$ for m sufficiently large.

For $a>b>0$ and $p\ge 1$ we have $(a+b)^p=a^p+pb\xi^{p-1}$ for some $\xi\in[a,a+b]$. Hence, $(a+b)^p\ge a^p+pa^{p-1}b\ge a^p+b^p$.

Using the above, for $|x|>R$ we have $(|x|-R)^p\le |x|^p-R^p=|x|-\phi_R(x)$. So for n sufficiently large,

$$\int_{|x|>R}(|x|-R)^p\,\mathrm{d}\mu_m<\varepsilon \quad \text{and} \quad \int_{|x|>R}(|x|-R)^p\,\mathrm{d}\mu<\varepsilon.$$

Let $P_R:\mathbb{R}^n\to\overline{B(0,R)}$ be the projection onto the ball $\overline{B(0,R)}$, i.e.

$$P_R(x)=\begin{cases} x & \text{if } x\in B(0,R)\\ \operatorname{argmin}_{y\in\partial B(0,R)}|y-x| & \text{else.}\end{cases}$$

The map P_R is continuous and equal to the identity on $\overline{B(0,R)}$. For all $x \notin \overline{B(0,R)}$ we have $|x - P_R(x)| = |x| - R$. Hence,

$$\begin{aligned} d_{W^p}(\mu, (P_R)_{\#}\mu) &\leq \left(\int_{\mathbb{R}^n} |x - P_R(x)|^p \,\mathrm{d}\mu(x) \right)^{\frac{1}{p}} \\ &= \left(\int_{|x|>R} |x - P_R(x)|^p \,\mathrm{d}\mu(x) \right)^{\frac{1}{p}} \\ &= \left(\int_{|x|>R} (|x| - R)^p \,\mathrm{d}\mu(x) \right)^{\frac{1}{p}} \\ &\leq \varepsilon^{\frac{1}{p}}, \end{aligned}$$

and similarly,

$$d_{W^p}(\mu_m, (P_R)_{\#}\mu_m) \leq \varepsilon^{\frac{1}{p}}.$$

For any $\varphi \in C_b^0(\mathbb{R}^n)$ we have

$$\int \varphi \,\mathrm{d}(P_R)_{\#}\mu_m = \int \varphi(P_R(x)) \,\mathrm{d}\mu_m \to \int \varphi(P_R(x)) \,\mathrm{d}\mu = \int \varphi \,\mathrm{d}(P_R)_{\#}\mu$$

since $\varphi \circ P_R$ is continuous and bounded. Hence, $(P_R)_{\#}\mu_m \overset{*}{\rightharpoonup} (P_R)_{\#}\mu$.

Now, $(P_R)_{\#}\mu_m$, $(P_R)_{\#}\mu$ have support in $\overline{B(0,R)}$ (a compact set) so by Theorem 5.7 we have $d_{W^p}((P_R)_{\#}\mu_m, (P_R)_{\#}\mu) \to 0$. Hence,

$$\begin{aligned} \limsup_{m\to\infty} d_{W^p}(\mu_m, \mu) &\leq \limsup_{m\to\infty} \Big(d_{W^p}(\mu_m, (P_R)_{\#}\mu_m) + d_{W^p}((P_R)_{\#}\mu_m, (P_R)_{\#}\mu) \\ &\qquad + d_{W^p}((P_R)_{\#}\mu, \mu) \Big) \\ &\leq 2\varepsilon^{\frac{1}{p}}. \end{aligned}$$

Letting $\varepsilon \to 0$ implies $\lim_{m\to\infty} d_{W^p}(\mu_m, \mu) = 0$ as required. □

5.3 Geodesics in the Wasserstein Space

Section references: The result that the Wasserstein space is a geodesic space (Theorem 5.12) can be found in [12, Theorem 5.27].

The aim of this section is to show that the Wasserstein space $(\mathcal{P}_p(X), d_{W^p})$ is a geodesic space. We start with some definitions. The definitions are given in terms of a metric space $(\mathcal{Z}, d)$, of course we have in mind that this will later be the Wasserstein space.

Definition 5.9. *Let $(\mathcal{Z}, d)$ be a metric space and $\omega : [0,1] \to \mathcal{Z}$ a curve in $\mathcal{Z}$. We say ω is absolutely continuous if there exists $g \in L^1([0,1])$ such that $d(\omega(t_0), \omega(t_1)) \leq \int_{t_0}^{t_1} g(s) \,\mathrm{d}s$ for any $0 \leq t_0 < t_1 \leq 1$. We denote the set of absolutely continuous curves on $\mathcal{Z}$ by $\mathrm{AC}(\mathcal{Z})$.*

Definition 5.10. *Let $(\mathcal{Z}, d)$ be a metric space and $\omega : [0,1] \to \mathcal{Z}$ a curve in $\mathcal{Z}$. We define the length of ω by*

$$\mathrm{Len}(\omega) := \sup \left\{ \sum_{k=0}^{n-1} d(\omega(t_k), \omega(t_{k+1})) \, : \, n \geq 1, 0 = t_0 < t_1 < \cdots < t_n = 1 \right\}.$$

A curve $\omega : [0,1] \to \mathcal{Z}$ is said to be a geodesic between $z_0 \in \mathcal{Z}$ and $z_1 \in \mathcal{Z}$ if

$$\omega \in \operatorname{argmin} \left\{ \mathrm{Len}(\tilde{\omega}) \, : \, \tilde{\omega} : [0,1] \to \mathcal{Z}, \, \tilde{\omega}(0) = z_0, \, \tilde{\omega}(1) = z_1 \right\}.$$

A curve $\omega : [0,1] \to \mathcal{Z}$ is said to be a constant speed geodesic between $z_0 \in \mathcal{Z}$ and $z_1 \in \mathcal{Z}$ if

$$d(\omega(t), \omega(s)) = |t-s| d(\omega(0), \omega(1)).$$

Note that if $\omega : [0,1] \to \mathcal{Z}$ is a constant speed geodesic then it is a geodesic. Indeed, assume that $\omega : [0,1] \to \mathcal{Z}$ and $\tilde{\omega} : [0,1] \to \mathcal{Z}$ satisfy $\omega(0) = z_0 = \tilde{\omega}(0)$, $\omega(1) = z_1 = \tilde{\omega}(1)$,

$$d(\omega(t), \omega(s)) = |t-s| d(z_0, z_1) \quad \forall 0 \leq t < s \leq 1, \qquad \text{and} \qquad \mathrm{Len}(\tilde{\omega}) < \mathrm{Len}(\omega).$$

I.e. we assume that ω is a constant speed geodesic but not a geodesic. Then there exists $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n = 1$ such that

$$\mathrm{Len}(\tilde{\omega}) < \sum_{k=0}^{n-1} d(\omega(t_k), \omega(t_{k+1})) = d(z_0, z_1) \sum_{k=0}^{n-1} (t_{k+1} - t_k) = d(z_0, z_1).$$

This implies $\mathrm{Len}(\tilde{\omega}) < d(z_0, z_1)$. Clearly this is a contradiction (choosing $n = 1$ in the definition of $\mathrm{Len}(\tilde{\omega})$ implies $\mathrm{Len}(\tilde{\omega}) \geq d(z_0, z_1)$).

Note also that if $d(\omega(t), \omega(s)) = |t-s| d(z_0, z_1)$ then $\omega \in \mathrm{AC}(\mathcal{Z})$ with $g(s) = d(z_0, z_1)$.

Definition 5.11. *Let $(\mathcal{Z}, d)$ be a metric space. We say $(\mathcal{Z}, d)$ is a length space if*

$$d(x,y) = \inf \left\{ \mathrm{Len}(\omega) \, : \, \omega \in \mathrm{AC}(\mathcal{Z}), \, \omega(0) = x, \, \omega(1) = y \right\}.$$

We say $(\mathcal{Z}, d)$ is a geodesic space if

$$d(x,y) = \min \left\{ \mathrm{Len}(\omega) \, : \, \omega \in \mathrm{AC}(\mathcal{Z}), \, \omega(0) = x, \, \omega(1) = y \right\}.$$

We now show that the Wasserstein space $(\mathcal{P}_p(X), d_{W^p})$ is a geodesic space.

Theorem 5.12. *Let $p \geq 1$, $X \subseteq \mathbb{R}^n$ be convex and define $P_t : X \times X \to X$ by $P_t(x,y) = (1-t)x + ty$. Let $\mu, \nu \in \mathcal{P}_p(X)$ and assume $\pi \in \Pi(\mu, \nu)$ minimises $\mathbb{K}$ over $\Pi(\mu, \nu)$. Then the curve $\mu_t = (P_t)_{\#}\pi$ is a constant speed geodesic in (X, d_{W^p}) connecting μ and ν. In particular, if $\pi = (\mathrm{Id} \times T)_{\#}\mu$ for some transport map $T : X \to X$ that pushes forwards μ to ν, i.e. $T_{\#}\mu = \nu$ (that is T is a solution to the Monge optimal transport problem), then $\mu_t = ((1-t)\mathrm{Id} + tT)_{\#}\mu$.*

Proof. Note that $P_0 = P^X$ and $P_1 = P^Y$. Therefore, $\mu_0 = (P_0)_\#\pi = \mu$, $\mu_1 = (P_1)_\#\pi = \nu$, so μ_t connects μ and ν. To show $d_{W^p}(\mu_s, \mu_t) = |t - s| d_{W^p}(\mu, \nu)$ it is enough to prove that $d_{W^p}(\mu_s, \mu_t) \leq |t - s| d_{W^p}(\mu, \nu)$. Indeed assuming this is true, then if $d_{W^p}(\mu_s, \mu_t) < |t - s| d_{W^p}(\mu, \nu)$ for any $0 \leq s < t \leq 1$ we have

$$\begin{aligned} d_{W^p}(\mu, \nu) &\leq d_{W^p}(\mu, \mu_s) + d_{W^p}(\mu_s, \mu_t) + d_{W^p}(\mu_t, \nu) \\ &< (s + (t - s) + (1 - t)) d_{W^p}(\mu, \nu) \\ &= d_{W^p}(\mu, \nu) \end{aligned}$$

a contradiction.

To show $d_{W^p}(\mu_s, \mu_t) \leq |t - s| d_{W^p}(\mu, \nu)$ let $\pi_{s,t} = (P_s, P_t)_\#\pi$. Then for any (measurable) $A \subseteq X$,

$$\begin{aligned} \pi_{s,t}(A \times X) &= \pi\left(\{(x, y) \,:\, (1 - s)x + sy \in A,\, (1 - t)x + ty \in X\}\right) \\ &= \pi\left(\{(x, y) \,:\, (1 - s)x + sy \in A\}\right) \\ &= (P_s)_\#\pi(A) \\ &= \mu_s(A). \end{aligned}$$

Hence $P^X_\# \pi_{s,t} = \mu_s$. Similarly, $P^Y_\# \pi_{s,t} = \mu_t$ so $\pi_{s,t} \in \Pi(\mu_s, \mu_t)$. Now,

$$\begin{aligned} d_{W^p}(\mu_s, \mu_t) &\leq \left(\int_{X \times X} |x - y|^p \, \mathrm{d}\pi_{s,t}(x, y) \right)^{\frac{1}{p}} \\ &= \left(\int_{X \times X} |P_s(x, y) - P_t(x, y)|^p \, \mathrm{d}\pi(x, y) \right)^{\frac{1}{p}} \\ &= \left(\int_{X \times X} |(t - s)x - (t - s)y|^p \, \mathrm{d}\pi(x, y) \right)^{\frac{1}{p}} \\ &= |t - s| \left(\int_{X \times X} |x - y|^p \, \mathrm{d}\pi(x, y) \right)^{\frac{1}{p}} \\ &= |t - s| d_{W^p}(\mu, \nu) \end{aligned}$$

as required.

If $\pi = (\mathrm{Id} \times T)_\#\mu$ where T is as in the statement of the theorem, then for $A \subset X$ (measurable) we have

$$\begin{aligned} \mu_t(A) &= (P_t)_\#\pi(A) \\ &= \pi\left(\{(x, y) \,:\, (1 - t)x + ty \in A\}\right) \\ &= (\mathrm{Id} \times T)_\#\mu\left(\{(x, y) \,:\, (1 - t)x + ty \in A\}\right) \\ &= \mu\left(\{x \,:\, (1 - t)x + tT(x) \in A\}\right) \\ &= ((1 - t)\mathrm{Id} + tT)_\#\mu(A) \end{aligned}$$

which shows $\mu_t = ((1 - t)\mathrm{Id} + tT)_\#\mu$. □

Bibliography

[1] L. Ambrosio. *Mathematical Aspects of Evolving Interfaces: Lectures given at the C.I.M.-C.I.M.E. joint Euro-Summer School held in Madeira, Funchal, Portugal, July 3-9, 2000*, chapter Lecture Notes on Optimal Transport Problems, pages 1–52. Springer Berlin Heidelberg, 2003.

[2] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

[3] L. Ambrosio and A. Pratelli. *Optimal Transportation and Applications: Lectures given at the C.I.M.E. Summer School, held in Martina Franca, Italy, September 2-8, 2001*, chapter Existence and stability results in the L1 theory of optimal transportation, pages 123–160. Springer Berlin Heidelberg, 2003.

[4] Y. Brenier. Décomposition polaire et réarrangement monotone des champs de vecteurs. *Comptes rendus de l'Académie des Sciences, Paris, Série I*, 305:805–808, 1987.

[5] M. Cuturi. Sinkhorn distances: Lightspeed computation of optimal transport. In *Advances in Neural Information Processing Systems (NIPS)*, pages 2292–2300, 2013.

[6] L. C. Evans and W. Gangbo. *Differential equations methods for the Monge-Kantorovich mass transfer problem*, volume 653. American Mathematical Society, 1999.

[7] W. Gangbo and R. J. McCann. The geometry of optimal transportation. *Acta Mathematica*, 177(2):113–161, 1996.

[8] L. V. Kantorovich. On translation of mass (in Russian), C. R. Doklady. *Proceedings of the USSR Academy of Sciences*, 37:199–201, 1942.

[9] S. Kolouri, S. R. Park, M. Thorpe, D. Slepčev, and G. K. Rohde. Optimal mass transport: Signal processing and machine-learning applications. *IEEE Signal Processing Magazine*, 34(4):43–59, 2017.

[10] S. Kolouri and G. K. Rohde. Optimal transport a crash course. IEEE ICIP 2016 Tutorial Slides: Part 1, 2016.

[11] G. Monge. *Mémoire sur la théorie des déblais et des remblais*. De l'Imprimerie Royale, 1781.

[12] F. Santambrogio. *Optimal transport for applied mathematicians*. Birkäuser Springer, Basel, 2015.

[13] B. Simon. *Convexity: An Analytic Viewpoint*. Cambridge University Press, 2011.

[14] V. N. Sudakov. Geomtric problems in the theory of infinite-dimensional probability distributions. *Proceedings of the Steklov Institute of Mathematics*, 141:1–178, 1979.

[15] C. Villani. *Topics in Optimal Transportation*. American Mathematical Society, 2003.

[16] C Villani. *Optimal transport: old and new*, volume 338. Springer Science & Business Media, 2008.