Forms and algebras in supergravity

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Outline

- Introduction
- General idea: from superspace to Borcherds algebras
- Example: 1/2 MSG in D=3
- Borcherds to KM
- Summary

Jesper Greitz and PH.

Introduction

A key feature of supergravity theories is the fact that they have hidden (duality) symmetries.

In particular, for MSG we have E_6, E_7, E_8 for $D = 5, 4, 3$ and E_9 in $D = 2$.

Cremmer,Scherk,Schwarz | Cremmer, Julia | Marcus, Schwarz | Nicolai, Warner

For $D = 1$ the algebra E_{10} has been proposed and the final possibility would seem to be E_{11}.

Damour, Henneaux, Julia, Nicolai, | Julia; West

These enhanced algebras would imply that there are new states in the theory, including forms.

On the other hand, there are algebraic structures that arise directly from the forms themselves together with their spacetime duals.

Cremmer, Julia, Lü, Pope; Lavrinenko,Lü, Pope,Stelle

It was later realised that these structures could be viewed as truncated Borcherds algebras. *Henry-Labordére,Julia,Paulot*

As well as the physical forms there are their duals and the $(D-2)$-form duals of the scalars.

Addionally, there can be non-physical $(D-1)$ and D-forms. (Or de-forms and top forms).

These are related to massive deformations and space-filling branes.

These forms, and the representations of the duality group that they transform under, are predicted by E_{11}.

Riccioni,West; Bergshoeff,de Roo,Kerstan,Riccioni

and also by Borcherds algebras. The relation between the two was studied in

Henneaux,Julia,Levie

Introduction III

These forms also arise in the tensor hierarchies associated with gauged supergravities.

These are constructed by gauging a subgroup G_0 of the duality group G and involve the introduction of a mass parameter.

In the construction it is found to be necessary to introduce higher-degree forms to maintain covariance, a procedure which leads to a hierarchy of forms.

de Wit,Samtleben,Trigiante;de Wit, Nicolai, Samtleben

In this talk a very simple account of forms and algebras in supergravity theories will be given using only supersymmetry and duality in a superspace setting.

The use of superspace offers some advantages:

- Supersymmetry is manifest.
- We can work in terms of field strengths rather than potentials so that gauge invariance is also manifest.
- The forms can have arbitrary degrees so there is no need to truncate.

This is because the odd basis forms in superspace commute.

The last point implies, in particular, that the top forms can be studied in terms of their $(D+1)$-form field strengths, F_{D+1}, and can therefore be treated on the same footing as the other forms in a covariant approach.

These $(D+1)$-form field strengths do not vanish in superspace.

The use of field-strength forms leads directly to an algebraic structure.

It is also possible that there could be further forms beyond the spacetime limit. These could be called “OTT” forms.

For example, there is a non-vanishing $(D+2)$-form field strength in IIA.

General idea

The physical forms generate further forms through the determination of all consistent Bianchi Identities (CBIs) of the form

$$\mathcal{I}_{\ell+2} = (dF_{\ell+1} - \sum_{m+n=\ell} F_{m+1}F_{n+1}) = 0$$

where consistency means

$$d\mathcal{I} = 0 \qquad \mod \ \mathcal{I}$$

Clearly, the new Fs will transform under reps of the duality group.

In addition, each CBI must be soluble.

In fact, the whole set of forms can be generated from a subset of the physical ones.

General idea II

Using superspace cohomology it can be shown, rather easily, that the set of CBIs are automatically satisfied if those for the generating set are.

The only obstruction that arises is when there is non-trivial cohomology; this leads to the supersymmetry constraint on the representations that can arise for certain forms. This needs to be imposed to ensure solubility.

When this has been implemented the only non-trivial components of the CBIs determine the non-vanishing components of the higher-degree field-strength forms in terms of the physical fields.

In particular, for MSG in $3 \leq D \leq 7$ and 1/2 MSG in $D = 3$ the generating forms are level one (1-form potentials) and the susy constraint is at level 2.

General idea III

A set of consistent BIs determines a Lie (super) coalgebra whose dual can be identified with (a positive part of) a Borcherds (super)algebra.

A Lie (super) coalgebra is a (super) vector space $\mathcal{A}$ with a linear map $d\mathcal{A} \to \mathcal{A} \wedge \mathcal{A}$ (antisymmetry) which extends to a graded derivation of the exterior algebra of $\mathcal{A}$ and which is nilpotent $d^2 = 0$ (Jacobi). Generically ∞-dimensional in SG context.

$$\mathcal{A} = \oplus_{\ell \in \mathbb{Z},\, \ell \geq 1} \mathcal{A}_\ell = \mathcal{A}^+ \oplus \mathcal{A}^-$$

where ℓ is the degree of the potential form and the even and odd parts of $\mathcal{A}$ correspond to even and odd ℓ.

The definition of a Borcherds (or GKM) (super)algebra starts with a generalised symmetric Cartan matrix which generalises the KM case in that the diagonal elements can be positive, negative or zero.

Borcherds;Ray

For the purposes of this talk we will only consider superalgebras that are obtained from Lie algebras by the addition of a single odd node in the Dynkin diagram. This means that we have a set of $3(r+1)$ generators $\{h_I, e_I, f_I\}$ where r is the rank of the Lie algebra $\mathfrak{g}$, and where $I = (0, i);\ i = 1, \ldots r$. The Cartan matrix has the form:

$$A_{IJ} = \begin{pmatrix} A_{00} & A_{0i} \\ A_{i0} & A_{ij} \end{pmatrix},$$

where $A_{ij} = a_{ij}$ is the Cartan matrix for $\mathfrak{g}$, symmetrised if necessary, and where

$$A_{00} = 0\,.$$

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Borcherds algebras

The relations satisfied by the generators are

$$
\begin{aligned}
[h_I, h_J] &= 0 \\
[h_I, e_J] &= A_{IJ} e_J, \qquad [h_I, f_J] = -A_{IJ} f_J, \qquad [e_I, f_J] = \delta_{IJ} h_I \\
(\mathrm{ad}\, e_I)^{1-\frac{2A_{IJ}}{A_{II}}} e_J &= 0, \qquad \text{for } A_{II} > 0 \text{ and } I \neq J \\
[e_I, e_J] &= 0 \qquad \text{when } A_{IJ} = 0 \,,
\end{aligned}
$$

with the last two conditions remaining valid if e_I, e_J are replaced by f_I, f_J. The generators h_I are even, and the generator f_I is even or odd if e_I is. If $A_{II} > 0$ the integer $\frac{2A_{IJ}}{A_{II}}$ is negative.

Borchherds algebras

In a Borcherds algebra there is still a triangular decomposition of the form $\mathcal{B} = \mathcal{N}^- \oplus \mathcal{H} \oplus \mathcal{N}^+$, and it is still possible to define roots as in the Kac-Moody case.

The Borcherds algebra can be decomposed into finite-dimensional representations of the subalgebra $\mathfrak{g}$.

The sub-algebra generated by $\{f_0, h_0, e_0\}$ is isomorphic to the Heisenberg super-algebra.

Example: Half-MSG in D=3

$N = 8$ susy in $D = 3$. The models under consideration are sigma models with the scalar fields in the cosets $(SO(8) \times SO(n))\backslash SO(8, n)$.

Marcus,Schwarz; de Wit,Nicolai,Tollsten

The forms and algebraic structures (for all half-maximal theories) were investigated by *Bergshoeff,Gomis,Nutma,Roest*

There is an induced (super)geometry, but the physical degrees of freedom are just given by the scalars and spinors of the sigma model. Strictly speaking there are no physical forms but we can start with the vectors dual to the scalars. They can be taken to transform under the adjoint of $G = SO(8, n)$, and satisfy the BI

$$dF_2 = 0 \, .$$

The level two BI is $dF_3 = F_2 \wedge F_2$.

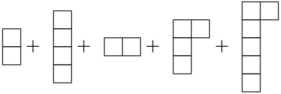
The possible 3-form field strengths are therefore in the following reps:

$$\left(\ydiagram{1,1} \otimes \ydiagram{1,1}\right)_{\mathrm{sym}} = \ydiagram{1,1,1,1} \oplus \ydiagram{2} \oplus 1 \oplus \ydiagram{2,2} .$$

It is not difficult to show that the dimension-zero cpt of this BI cannot be solved for the Weyl representation.

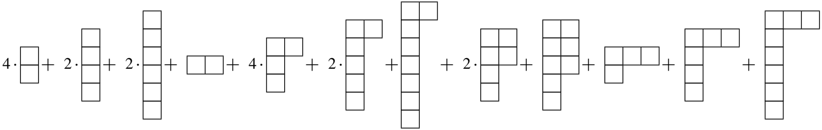
Half-MSG in D=3 IV

For all higher degree forms there are no obstructions to solving the CBIs for cohomological reasons. As a result we find that the level three representations (top forms) are



Bergshoeff,Gomis,Nutma,Roest

while at level four (OTT) they are



with the indicated multiplicities.

We have included level four here as 5-form field strengths can be non-zero in $D = 3$. This is because they have dimension-zero cpts $F_{3,2}$:

$$F^X_{abc,\alpha I\beta J} \sim \varepsilon_{abc}\varepsilon_{\alpha\beta}F^X_{IJ}$$

This will be non-zero if the $SO(8,n)$ representation X contains the adjoint representation of $SO(8)$ times the singlet of $SO(n)$.

These 5-forms appear in the covariant approach to the hierarchy of forms in gauged supergravity (i.e. using field strengths and BIs).

de Wit,Herger,Samtleben| Nicolai, Samtleben; de Wit,Samtleben| Bergshoeff,Hohm,Nutma

We can now construct the Borcherds algebra for 1/2MSG in $D = 3$. The algebra of forms is graded according to the degrees of the potential forms ℓ and at each level the forms transform under representations $\mathcal{R}_\ell$ of the duality Lie algebra $\mathfrak{g}$ with $\mathcal{R}_{\ell+1} \subset \mathcal{R}_\ell \otimes \mathcal{R}_1$.

Since we can create representations of $\mathfrak{g}$ by acting on a lowest weight state with the raising operators $\{e_i\}$, it is natural to assume that there is a fermionic element e_0 (because the lowest forms are one-forms). For e_0 to be a lowest-weight state we must have $[f_i, e_0] = 0$.

Since e_0 is a lowest-weight state for the adjoint representation we must have $[h_i, e_0] = -p_i e_0$, where p_i are the corresponding Dynkin labels, $p_i = (0, 1, 0, \ldots)$.

Half-MSG in D=3: Borcherds algebra II

To construct the full algebra we include $\mathfrak{g}$ and a negative odd generator f_0 which is taken to be a highest weight state for the adjoint representation, $[e_i, f_0] = 0$ $[h_i, f_0] = p_i f_0$. We then define $h_0 = [e_o, f_0]$.

The Cartan matrix A_{IJ} will have the form

$$A_{IJ} = \begin{pmatrix} A_{00} & A_{0i} \\ A_{i0} & A_{ij} \end{pmatrix}$$

where we can take $A_{0i} = A_{i0} = -(0, 1, 0, \ldots)$ and $A_{ij} = a_{ij}$, the latter being the Cartan matrix for $\mathfrak{g}$.

It remains to determine A_{00}. At level two it is easy to show that $[e_0, e_0]$ is the lowest weight state for the Weyl representation and so must vanish. This is turn leads to $A_{00} = 0$.

It is then straightforward to check that all the Borcherds superalgebra conditions are satisfied.

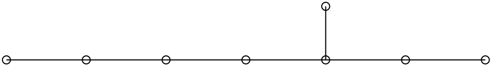
We have seen that the set of soluble CBIs in $D = 3$ 1/2MSG determines a Borcherds super-algebra with one set of odd generators. From this one can then show that the form representations agree with those predicted by some very extended version of the duality group.

Palmkvist

In the maximal case it was shown that the representations agree level by level, up to dimension D, with the forms predicted by E_{11}, for $3 \leq D \leq 7$, and it was subsequently shown that this result can be extended to other groups.

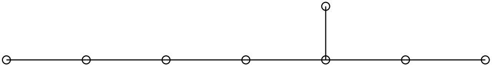
We shall illustrate this diagrammatically for the case of MSG in $D = 3$ for which $G = E_8$.

*E*8

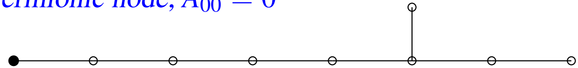


Borchers to Kac-Moody

$E8$

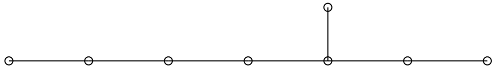


Borcherds: fermionic node, $A_{00} = 0$

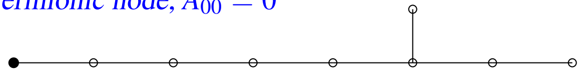


Borchers to Kac-Moody

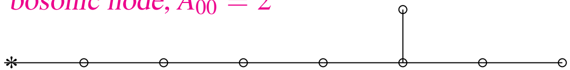
E8



Borcherds: fermionic node, $A_{00} = 0$

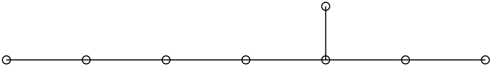


Intermediate: bosonic node, $A_{00} = 2$

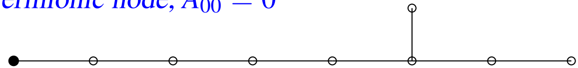


Borcherds to Kac-Moody

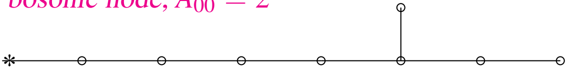
E8



Borcherds: fermionic node, $A_{00} = 0$



Intermediate: bosonic node, $A_{00} = 2$



E11

Summary I

- The superspace approach to supergravity forms is based on consistent, soluble sets of Bianchi identities.
- It is manifestly supersymmetric and gauge invariant.
- The CBIs determine a Lie (super)coalgebra which is dual to (a positive sector of) a Borcherds (super)algebra.
- There is no limit to the degrees of the forms, so that the algebra does not need to be truncated.
- The form representations determined by the Borcherds algebra can be shown to be equivalent to those predicted by a related KM algebra.
- Gauged supergravity can be studied by deforming the CBIs

Summary II

- There can also be OTT forms.
- In supergravity the field-strength forms can be non-zero up to degree $D + 2$, and even the $(D + 4)$-form BIs are not completely trivial. (Because $(F \wedge F)_{D,4}$ has dimension zero.)
- In the covariant approach to gauged supergravity OTT forms are needed at the end of the hierarchy.
- In addition, there is some indication that some of these OTT forms could be non-zero when α' corrections are turned on.
- *If* one were to take the OTT forms seriously then one would need to go beyond E_{11} on the KM side in MSG.

  e.g. $(D + 1)$-forms in MSG would require E_{12}.