Mathematical Tripos Part II Michaelmas 2015
Integrable Systems, hand out 2: Arnold-Liouville [sketch proof] Dr A.C.L. Ashton

Theorem (Arnold-Liouville Theorem)**.** *Let (M, H) be an integrable $2n$-dimensional Hamiltonian system with first integrals*

$$(H = f_1, f_2, \ldots, f_n).$$

For a constant vector $\mathbf{c} \in \mathbf{R}^n$ define

$$M_{\mathbf{c}} = \{(\mathbf{q}, \mathbf{p}) \in M : f_i(\mathbf{q}, \mathbf{p}) = c_i, \quad i = 1, \ldots, n\}.$$

Then:

1) *$M_{\mathbf{c}}$ defines a smooth n-dimensional surface in M. If $M_{\mathbf{c}}$ is compact and connected, it is diffeomorphic to a torus*

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ copies}}$$

2) *Locally there exists a canonical coordinate transformation $(\mathbf{q}, \mathbf{p}) \mapsto (\boldsymbol{\phi}, \mathbf{I})$ (called 'action-angle' coordinates)*

$$(\boldsymbol{\phi}, \mathbf{I}) = (\phi_1, \ldots, \phi_n, I_1, \ldots, I_n) \in T^n \times \mathbf{R}^n$$

such that the angles $\{\phi_k\}_{k=1}^n$ are coordinates on $M_{\mathbf{c}}$, the actions $\{I_k\}_{k=1}^n$ are first integrals and $H(\mathbf{q}, \mathbf{p}) = \tilde{H}(\mathbf{I})$. In particular, Hamilton's equations

$$\dot{\mathbf{I}} = 0, \quad \dot{\boldsymbol{\phi}} = \frac{\partial \tilde{H}}{\partial \mathbf{I}} = \text{const}$$

can be integrated up.

The full proof is a little beyond the scope of this course. In what follows we give a reasonable sketch of the ideas – you should try to understand the main points but don't worry too much about the details.

Sketch Proof:
1 : *The surface $M_{\mathbf{c}}$.* The fact that $M_{\mathbf{c}}$ is a (smooth) n-dimensional surface in M is a consequence of the independence of the gradients $\{\partial_{\mathbf{x}} f_i\}$ and the implicit function theorem. Let's consider the Hamiltonian vector fields

$$V_{f_i} = J\partial_{\mathbf{x}} f_i.$$

Then these vector fields are tangent to the surface $M_{\mathbf{c}}$ because the first integrals $\{f_j\}$ don't change in the direction of any of the V_{f_i}:

$$(V_{f_i} \cdot \partial_{\mathbf{x}}) f_j = \partial_{\mathbf{x}} f_j \cdot J\partial_{\mathbf{x}} f_i = \{f_j, f_i\} = 0,$$

since the first integrals are in involution. So each of the vector fields V_{f_i} give rise to a flow map g_i^t which maps $M_{\mathbf{c}}$ to itself. What is more, since these vector fields commute

$$[V_{f_i}, V_{f_j}] = -V_{\{f_i, f_j\}} = 0,$$

we know that the flow maps also commute, $g_i^t g_j^s = g_j^s g_i^t$. If we set $g^{\mathbf{t}} = g_1^{t_1} \cdots g_n^{t_n}$ for $\mathbf{t} \in \mathbf{R}^n$ then we have $g^{\mathbf{t}_1} g^{\mathbf{t}_2} = g^{\mathbf{t}_1 + \mathbf{t}_2}$, by the commutivity of the flows $\{g_i\}$. For fixed $\mathbf{x} \in M_{\mathbf{c}}$ define $\text{Stab}(\mathbf{x}) = \{\mathbf{t} \in \mathbf{R}^n : g^{\mathbf{t}}\mathbf{x} = \mathbf{x}\}$ and consider the map

$$\varphi : \mathbf{R}^n/\text{Stab}(\mathbf{x}) \to M_{\mathbf{c}} \quad \mathbf{t} \mapsto \varphi(\mathbf{t}) = g^{\mathbf{t}}\mathbf{x}.$$

Then φ is surjective (see example sheet 1) and is also injective, since $\varphi(\mathbf{t}_1) = \varphi(\mathbf{t}_2)$ iff $g^{\mathbf{t}_1 - \mathbf{t}_2}\mathbf{x} = \mathbf{x}$ iff $\mathbf{t}_1 - \mathbf{t}_2 \in \text{Stab}(\mathbf{x})$, i.e. $\mathbf{t}_1 = \mathbf{t}_2$ in $\mathbf{R}^n/\text{Stab}(\mathbf{x})$. So φ gives us our (smooth) bijection

$$M_{\mathbf{c}} \simeq \mathbf{R}^n/\text{Stab}(\mathbf{x}).$$

It can be shown (and is intuitively clear) that $\text{Stab}(\mathbf{x}) \simeq \mathbf{Z}^k$ for some $1 \leq k \leq n$ and so the diffeomorphism reads $M_{\mathbf{c}} \simeq \mathbf{R}^n/\mathbf{Z}^k \simeq \mathbf{R}^{n-k} \times \mathbf{R}^k/\mathbf{Z}^k \simeq \mathbf{R}^{n-k} \times T^k$. But if $M_{\mathbf{c}}$ is compact we must have $n = k$, so $M_{\mathbf{c}} \simeq T^n$.

2 : *The action-angle coordinates.* Let us prove this in the case $n = 2$. The general case is exactly the same but requires more technical language (exterior calculus). We have the constraints $f_i(\mathbf{q}, \mathbf{p}) = c_i$ for $i = 1, 2$. Let us assume that we can locally solve these equations for $\mathbf{p}$, i.e. $\det(\partial f_i/\partial p_j) \neq 0$. So on (locally) on $M_\mathbf{c}$ we can write

$$\mathbf{p} = \mathbf{p}(\mathbf{q}, \mathbf{c}).$$

Since $f_i(\mathbf{q}, \mathbf{p}(\mathbf{q}, \mathbf{c})) = c_i$ hold identically for $i = 1, 2$ and we can differentiate them

$$\frac{\partial f_i}{\partial q_k} + \frac{\partial f_i}{\partial p_l}\frac{\partial p_l}{\partial q_k} = 0,$$

where summation convention is implied. We use this in the condition that the first integrals $\{f_i\}$ are in involution.

$$\begin{aligned} 0 &= \{f_i, f_j\} \\ &= \frac{\partial f_i}{\partial q_k}\frac{\partial f_j}{\partial p_k} - \frac{\partial f_i}{\partial p_k}\frac{\partial f_j}{\partial q_k} \\ &= \left[-\frac{\partial f_i}{\partial p_l}\frac{\partial p_l}{\partial q_k}\right]\frac{\partial f_j}{\partial p_k} - \frac{\partial f_i}{\partial p_k}\left[-\frac{\partial f_j}{\partial p_l}\frac{\partial p_l}{\partial q_k}\right] \\ &= \left[-\frac{\partial f_i}{\partial p_k}\frac{\partial p_k}{\partial q_l}\right]\frac{\partial f_j}{\partial p_l} - \frac{\partial f_i}{\partial p_k}\left[-\frac{\partial f_j}{\partial p_l}\frac{\partial p_l}{\partial q_k}\right] \\ &= \frac{\partial f_i}{\partial p_k}\left[\frac{\partial p_l}{\partial q_k} - \frac{\partial p_k}{\partial q_l}\right]\frac{\partial f_j}{\partial p_l}. \end{aligned}$$

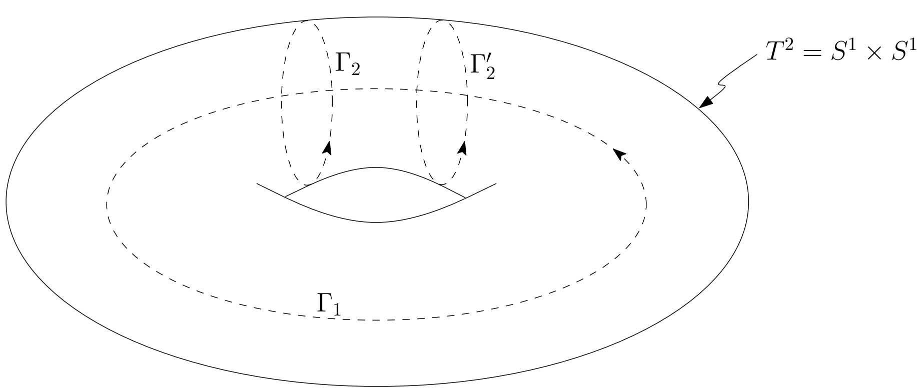
Again, using the fact that $\det(\partial f_i/\partial p_j) \neq 0$ it means the matrices outside the square brackets are invertible. We conclude that the term in the square brackets must vanish identically, so

$$\frac{\partial p_2}{\partial q_1} - \frac{\partial p_1}{\partial q_2} = 0$$

everywhere on $M_\mathbf{c}$. This fact will turn out to be very useful. Let us define define

$$I_j = \frac{1}{2\pi}\oint_{\Gamma_j} \mathbf{p}\cdot \mathrm{d}\mathbf{q}\Big|_{M_\mathbf{c}}, \quad j = 1, 2.$$

where Γ_j is the j-th cycle on $M_\mathbf{c} \simeq T^2$. It is independent of *which* representative for the basic cycle we choose because of our useful fact. For example, on the diagram we see two potential representatives for the second cycle, labelled Γ_2 and Γ_2'.



Let A denote the region on $M_\mathbf{c}$ bound between the two cycles. Then by Green's theorem[1]

$$\oint_{\Gamma_2'} \mathbf{p}\cdot \mathrm{d}\mathbf{q}\Big|_{M_\mathbf{c}} - \oint_{\Gamma_2} \mathbf{p}\cdot \mathrm{d}\mathbf{q}\Big|_{M_\mathbf{c}} = \oint_{\partial A} \mathbf{p}\cdot \mathrm{d}\mathbf{q} = \iint_A \left(\frac{\partial p_2}{\partial q_1} - \frac{\partial p_1}{\partial q_2}\right) \mathrm{d}q_1\,\mathrm{d}q_2 = 0.$$

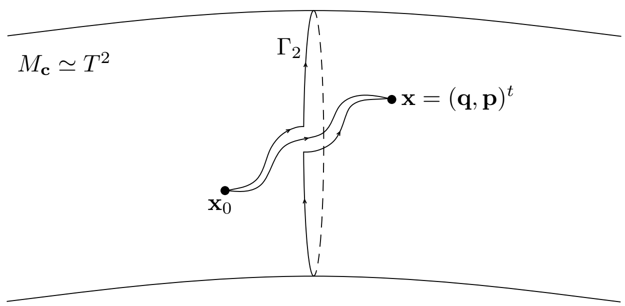
[1] Remember IA vector calculus? $\oint_{\partial A}(P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_A (\partial Q/\partial x - \partial P/\partial y)\,\mathrm{d}x\,\mathrm{d}y$.

Note also that each I_j depends only on $M_{\mathbf{c}}$, hence $I_j = I_j(\mathbf{c})$. Assuming we can invert the relationship between $\mathbf{c}$ and $\mathbf{I} = (I_1, \ldots, I_n)$ we can write $\mathbf{c} = \mathbf{c}(\mathbf{I})$. This means the value of $\mathbf{I}$ completely determines the surface $M_{\mathbf{c}}$. Fix $\mathbf{x}_0 \in M_{\mathbf{c}}$ and define

$$S(\mathbf{q}, \mathbf{I}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{p}' \cdot \mathrm{d}\mathbf{q}' \Big|_{M_{\mathbf{c}(\mathbf{I})}}, \quad \mathbf{x} = (\mathbf{q}, \mathbf{p})^t, \quad \text{where } \mathbf{p} = \mathbf{p}(\mathbf{q}, \mathbf{c}(\mathbf{I})).$$

We will use S as a generating function for a canonical transformation. Note that S is a multi-valued function: it depends on the path we take from $\mathbf{x}_0$ to $\mathbf{x}$. Small deformations in the path from $\mathbf{x}_0$ to $\mathbf{x}$ wont change the function S since integrals round any small contractable loop on $M_{\mathbf{c}}$ will vanish. However, if two paths differ by a cycle Γ_j (which can't be contracted) then the value of the function changes

$$S(\mathbf{q}, \mathbf{I}) \mapsto S(\mathbf{q}, \mathbf{I}) + \int_{\Gamma_j} \mathbf{p} \cdot \mathrm{d}\mathbf{q} = S(\mathbf{q}, \mathbf{I}) + 2\pi I_j.$$



So the derivatives $\partial S / \partial I_j$, $j = 1, 2$ are well-defined functions modulo 2π, i.e. they're angles! So we define

$$\boldsymbol{\phi} = \frac{\partial S}{\partial \mathbf{I}}$$

which corresponds to the local angle coordinate on $M_{\mathbf{c}} \simeq T^2$. Also, by the fundamental theorem of calculus[2]

$$\frac{\partial S}{\partial \mathbf{q}} = \mathbf{p}.$$

Now we pause for a second: all our computations have been performed *on the level surface* $M_{\mathbf{c}}$, so that we have only constructed the values of the functions $(\boldsymbol{\phi}, \mathbf{I})$ on the surface $M_{\mathbf{c}}$. But since our computations work for *any* level surface $M_{\mathbf{c}}$, they work for the whole phase space M! In particular, since the coordinate $\mathbf{I}$ only depends on the level surface $M_{\mathbf{c}}$ we happen to be on, it only depends on the value of the first integrals $\mathbf{f} = (f_1, \ldots, f_n)$. So in general $\mathbf{I} = \mathbf{I}(\mathbf{f})$ and hence $\dot{\mathbf{I}} = 0$. In summary, the function $S = S(\mathbf{q}, \mathbf{I})$ has given rise to a transformation $(\mathbf{q}, \mathbf{p}) \mapsto (\boldsymbol{\phi}, \mathbf{I})$ defined implicitly by

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}, \quad \boldsymbol{\phi} = \frac{\partial S}{\partial \mathbf{I}}.$$

This is the same form of transformation (born of a generating function S) that appeared in handout 1, so the transformation $(\mathbf{q}, \mathbf{p}) \mapsto (\boldsymbol{\phi}, \mathbf{I})$ is canonical. So Hamilton's equations are preserved. Since $\dot{\mathbf{I}} = 0$ we have $H(\mathbf{q}, \mathbf{p}) = \tilde{H}(\mathbf{I})$ and so Hamilton's equations are simply

$$\dot{\boldsymbol{\phi}} = \frac{\partial \tilde{H}}{\partial \mathbf{I}} = \text{const}, \quad \dot{\mathbf{I}} = 0,$$

which we can integrate up to give $\boldsymbol{\phi}(t) = \boldsymbol{\phi}_0 + \boldsymbol{\Omega} t$ and $\mathbf{I}(t) = \mathbf{I}_0$ where $\boldsymbol{\Omega} = \partial \tilde{H}_{\mathbf{I}}(\mathbf{I}_0)$. ■

Moral of the story: if you want to integrate up an integrable Hamiltonian system, first identify the cycles on $M_{\mathbf{c}}$ and compute

$$I_j = \frac{1}{2\pi} \oint_{\Gamma_j} \mathbf{p} \cdot \mathrm{d}\mathbf{q} \Big|_{M_{\mathbf{c}}}, \quad j = 1, \ldots, n.$$

[2]If this isn't clear, set $\mathbf{F} = (\mathbf{p}', 0)^t$ so $S = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} \cdot \mathrm{d}\mathbf{x}'$ and $\partial_{\mathbf{x}} S = \mathbf{F}$. Hence $\partial_{\mathbf{q}} S = \mathbf{p}$.

Then invert to get $\mathbf{c} = \mathbf{c}(\mathbf{I})$. Define S as in the proof and compute $\partial S/\partial I_j$ for $j = 1, \ldots, n$. Identifying the cycles is the tricky bit, but we know they exist because part (**1**) of the theorem guarantees $M_{\mathbf{c}} \simeq T^n$.

Worked example: We will again study the one dimensional harmonic oscillator with Hamiltonian

$$H(q,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2.$$

We already know some action angle coordinates for this Hamiltonian system (see the example in your notes). Here we will reconstruct them using the formula for I and ϕ given in the proof of the Arnold-Liouville theorem. This will involve running through the individual steps in part (**2**) of the proof.

The level surface M_c is defined by $M_c = \{(q,p) : \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 = c > 0\}$. This is certainly compact and connected and since $n = 1$ so we know that

$$M_c \simeq T^1 = S^1.$$

But this is obviously true since M_c is an ellipse! On this level surface we can write $p = p(q,c)$ where

$$p(q,c) = \pm\sqrt{2c - \omega^2 q^2}.$$

To construct the value of $I = I(c)$ on this surface we must compute

$$I = \frac{1}{2\pi}\oint p(q,c)\,\mathrm{d}q$$

where the integral is taken around the (only) cycle on M_c. In this integral we take the positive sign for p when we're on the top half of the ellipse and the negative sign when we're on the bottom. Since the ellipse hits the q-axis at $q = \pm\sqrt{2c}/\omega$ we have (traversing the curve clockwise)

$$\begin{aligned} I &= \frac{1}{2\pi}\int_{\frac{\sqrt{2c}}{\omega}}^{-\frac{\sqrt{2c}}{\omega}} \left(-\sqrt{2c - \omega^2 q^2}\right)\mathrm{d}q + \frac{1}{2\pi}\int_{-\frac{\sqrt{2c}}{\omega}}^{\frac{\sqrt{2c}}{\omega}} \sqrt{2c - \omega^2 q^2}\,\mathrm{d}q \\ &= \frac{1}{\pi}\int_{-\frac{\sqrt{2c}}{\omega}}^{\frac{\sqrt{2c}}{\omega}} \sqrt{2c - \omega^2 q^2}\,\mathrm{d}q \\ &= \frac{c}{\omega}. \end{aligned}$$

As promised, we can invert this relationship to get $c(I) = \omega I$. Since $c = H(q,p)$ on this curve and the above computation works for any $c > 0$ we have $I = \frac{1}{\omega}H(q,p)$. Now consider

$$S(q,I) = \int_{x_0}^{x} p'\,\mathrm{d}q'\Big|_{M_c}, \quad x = (q,p)^t, \quad \text{where } p = p(q, c(I)).$$

Taking $x_0 = (q_0, p_0) = (0, \sqrt{2c})$ we have

$$S(q,I) = \int_0^q \sqrt{2c(I) - \omega^2 q'^2}\,\mathrm{d}q' = \int_0^q \sqrt{2I\omega - \omega^2 q'^2}\,\mathrm{d}q'.$$

Hence

$$\phi = \frac{\partial S}{\partial I} = \omega\int_0^q \frac{\mathrm{d}q'}{\sqrt{2I\omega - \omega^2 q'^2}} = \int_0^{q\sqrt{\frac{\omega}{2I}}} \frac{\mathrm{d}y}{\sqrt{1-y^2}} = \arcsin\left(q\sqrt{\frac{\omega}{2I}}\right).$$

Note that the left hand side is only defined modulo 2π as predicted! In summary, we have constructed the new coordinates

$$\phi(q,p) = \arcsin\left(q\sqrt{\frac{\omega}{2I(q,p)}}\right), \quad I(q,p) = \frac{1}{\omega}\left(\frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2\right).$$

Inverting this transformation gives

$$q(\phi, I) = \sqrt{\frac{2I}{\omega}}\sin\phi, \quad p(\phi, I) = \sqrt{2I\omega}\cos\phi$$

which is precisely the coordinate transformation we used in motivating example in your notes. However in this case we used Arnold-Liouville construction, rather than divine inspiration.